Tensor triangular geometry in representation theory

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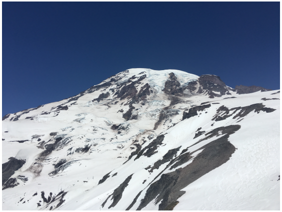

S. Paul Smith' 65th birthday conference
Advances, Directions and Interplay, Seattle, June 20, 2022

G - finite group; k - field
V – vector space over k,
G acts on V via k-linear transformations:

$$G \times V \longrightarrow V.$$

Equivalently, if $\dim_k V = n$,

$$G \longrightarrow \mathrm{Aut}_k(V) \cong \mathrm{GL}_n(k)$$

$$g \longmapsto (a_{ij}).$$

V is a representation of G
$\mathsf{Rep}_k G$ - the category of representations of G over k

V, W - representations of G. $\dim V = n$, $\dim W = m$.

1. Addition. $V \oplus W$, $\dim = n + m$. Action is coordinate-wise. In matrix form: g acts via a matrix A on V, and via a matrix B on W. Then g acts via the block matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ on $V \oplus W$.
2. Multiplication. $V \otimes W$, $\dim = nm$. Action is diagonal:
$$g(v \otimes w) = gv \otimes gw$$
3. "Unit". There is a trivial 1-dimensional representation k.
$$V \otimes k \simeq k \otimes V \simeq V$$
4. "Laws". Associativity, distributivity.

For groups, we also have $V \otimes W \cong W \otimes V$.

$\mathsf{Rep}_k G$ - *symmetric* (finite) tensor category

Another example of a symmetric tensor category: $\mathsf{Qcoh}(X)$.

Theorem (Maschke, 1898)

Let $G \to \mathrm{GL}_n(\mathbb{C})$ be a complex matrix representation of a finite group G, and assume that all matrices corresponding to the elements of the group have the form $\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$, where the dimension of A_1 is a fixed number $r < n$. Then the representation is equivalent to the one of the same form where all submatrices B are equal to 0.

$$\left\{ \begin{array}{c} \text{Maschke's} \\ \text{theorem} \end{array} \right\} \quad \Rightarrow \quad \left\{ \begin{array}{c} \text{any f.d. complex representation} \\ \text{of a finite group } G \text{ is a direct sum} \\ \text{of irreducible representations} \end{array} \right\}$$

$\mathsf{Rep}_k G$ is semi-simple.

Stronger Maschke's thm: $\mathsf{Rep}_k G$ is semi-simple $\Leftrightarrow$ char $k \nmid |G|$.

Regular representation/group algebra kG

kG is generated by $\{g\}_{g\in G}$ as a k-vector space. $\dim kG = |G|$.
Multiplication is extended linearly from G; action of G is by left multiplication.
For $\boxed{k = \mathbb{C}}$

$$kG = \bigoplus S_i^{a_i}$$

where S_i run through **all** irreducible representations of G.

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The structure of $\mathsf{Rep}_k G$:

1. $\mathsf{Rep}_k G$ is semisimple: all representations are direct sums of irreducibles
2. Irreducibles are direct summands of the free module kG ($\Rightarrow$ projective)
3. Every representation is a projective representation

$\mathsf{Rep}_k G$

Char 0 → Irreducible characters

Char p → Character theory - not enough!

Frobenius

Burnside

C. Curtis, “Pioneers of Representation Theory”.

kG - group algebra; generated by $\{g \in G\}$ as k-vector space.
Multiplication

$$kG \otimes kG \to kG \quad g \otimes h \mapsto gh$$

Diagonal map (comultiplication)

$$kG \to kG \otimes kG \quad g \to g \otimes g$$

Inverse

$$kG \to kG \quad g \mapsto g^{-1}$$

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The tensor structure on the category $\mathsf{Rep}_k G$ is given by the diagonal map

$$g \to g \otimes g$$

Hopf algebras

Hopf algebra $\mathfrak{u}$ is a k-vector space with product

$$\mathfrak{u} \otimes \mathfrak{u} \to \mathfrak{u},$$

coproduct

$$\mathfrak{u} \to \mathfrak{u} \otimes \mathfrak{u},$$

coinverse (antipode)

$$\mathfrak{u} \to \mathfrak{u}$$

unit and counit

$$k \to \mathfrak{u}, \quad \mathfrak{u} \to k$$

+ compatibility conditions.

$\mathfrak{u}$ is a finite dimensional Hopf algebra if $\dim_k \mathfrak{u} < \infty$.
$\mathsf{Rep}_k \mathfrak{u}$ - finite tensor category (not necessarily symmetric)

Examples

1. $\mathfrak{u} = kG$, G - finite group (or any group)
2. $\mathcal{U}(\mathfrak{g})$ - universal enveloping algebra of a Lie algebra $\mathfrak{g}$
3. char $k = p$. $\mathfrak{u}(\mathfrak{g})$ - restricted enveloping algebra of a restricted Lie algebra $\mathfrak{g}$; a finite Hopf quotient of $\mathcal{U}(\mathfrak{g})$.
4. char $k = p$. G - finite group scheme over k. $kG := k[G]^*$ - linear dual to the ring of functions of G. Finite dimensional *cocommutative* Hopf algebra, incorporates finite groups and restricted Lie algebras
5. any k. G - finite supergroup scheme. $kG = k[G]^*$ - finite dimensional cocommutative Hopf *superalgebra*. (Slight lie! But allows to add $\Lambda^*(V)$ into the mix).
6. $k = \mathbb{C}$. $\mathcal{U}_q(\mathfrak{g})$ - quantized enveloping algebras (quantum groups)
7. $k = \mathbb{C}$. $\mathfrak{u}_q(\mathfrak{g})$ - Lusztig's small quantum groups: finite dimensional quotients of $\mathcal{U}_q(\mathfrak{g})$ at roots of unity.

stab $\mathfrak{u}$

Want to "measure" the non semisimplicity of $\mathsf{Rep}_k\mathfrak{u}$.
Let $\langle \mathrm{proj}\, \mathfrak{u} \rangle$ be the *additive closure* of all direct summands of $\mathfrak{u}$ (the subcategory of projective $\mathfrak{u}$-modules)

$$\mathrm{stab}\, \mathfrak{u} \quad \text{“} = \text{”} \quad \mathsf{Rep}_k\mathfrak{u}/\langle \mathrm{proj}\, \mathfrak{u} \rangle$$

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Formally, $\mathrm{stab}\,\mathfrak{u}$ is a category with objects - representations of $\mathfrak{u}$; but morphisms are adjusted:

$$\mathrm{Hom}_{\mathrm{stab}\,\mathfrak{u}}(V, W) := \frac{\mathrm{Hom}_{\mathfrak{u}}(V, W)}{\mathrm{PHom}_{\mathfrak{u}}(V, W)}$$

where

$$\mathrm{PHom}_{\mathfrak{u}}(V, W) = \{f : V \to W \mid f \text{ factors through a projective}\}$$

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   sums, tensor products, unit +
   homological operations: syzygies/shifts, extensions/cones

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   homological operations: syzygies/shifts, extensions/cones
4. $\mathfrak{u}$ is a Frobenius algebra (projectives = injectives) $\Rightarrow \operatorname{stab}\mathfrak{u}$ is a tensor triangulated category

- Triangles. $V_1 \to V_2 \to V_3 \to$
  Extensions of length 1 in $\mathrm{stab}\,\mathfrak{u}$.
- Shift Syzygy: $V \hookrightarrow I(V) \twoheadrightarrow \Sigma V$

$$\Sigma : \mathrm{stab}\,\mathfrak{u} \to \mathrm{stab}\,\mathfrak{u}$$

- $V \otimes_k W$, coproduct in $\mathfrak{u}$

$\mathrm{stab}\,\mathfrak{u}$ is a tensor triangulated category $\rightsquigarrow$
Study $\mathrm{stab}\,\mathfrak{u}$ via tensor triangular geometry (tt-geometry)

TT-GEOMETRY

T - essentially small (symmetric) tensor triangulated category. ($T = \mathrm{stab}\, kG$, $T = \mathbb{D}^{\mathrm{perf}}(\mathsf{coh}(X))$)

Thick subcategory $\mathcal{C} \subset$ T: full triangulated subcategory closed under direct summands.

Tensor ideal: $V \in \mathcal{C} \Rightarrow V \otimes W, W \otimes V \in \mathcal{C}$ for any N.

Prime tensor ideal: exercise :)

Define

$$\mathrm{Spec}_{Bal}\, \mathsf{T}$$

as the set of prime ideals in T with Zariski topology.

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Balmer theory (2005): T is symmetric
Non symmetric, non braided settings: Buan-Krause-Solberg; Nakano-Vashaw-Yakimov

$\mathrm{Spec}_{Bal}\,\mathsf{T}$ effectively classifies *thick tensor ideals* in T.
There is one-to-one order preserving correspondence:

$$\left\{ \begin{array}{c} \text{Thick tensor} \\ \text{ideals of } \mathsf{T} \end{array} \right\} \sim \left\{ \begin{array}{c} \text{Subsets of } \mathrm{Spec}_{Bal}\,\mathsf{T} \\ \text{closed under specialization} \end{array} \right\}$$

via the universal/abstract notion of support in tt-geometry

$$\mathcal{C} \longmapsto \bigcup_{V \in \mathcal{C}} \mathrm{supp}^{univ}\, V$$

$$\mathcal{C}_Z = \{V \mid \mathrm{supp}^{univ}\, V \subset Z\} \longleftarrow\!\!\dashv Z$$

$\mathrm{supp}^{univ}(V) = \{\mathfrak{P} \in \mathrm{Spec}_{Bal}\,\mathsf{T} \mid V \notin \mathfrak{P}\}$

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$\mathrm{supp}^{univ}(V) = \{\mathfrak{P} \in \mathrm{Spec}_{Bal}\,\mathsf{T} \mid V \notin \mathfrak{P}\}$
For $T = \mathrm{stab}\,\mathfrak{u}$, non semi-simple, $\mathrm{Spec}_{Bal}\,T$ classifies $\mathfrak{u}$-modules not up to direct sums (too hard!) but up to *homological operations* allowed in $\mathrm{stab}\,\mathfrak{u}$.

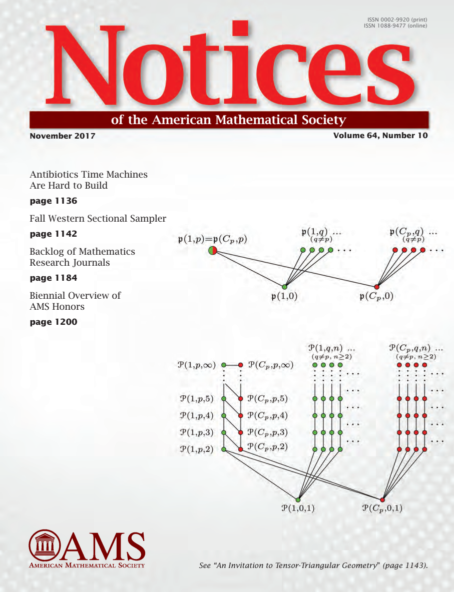
Precursors/motivation:

- Stable Homotopy Theory: Devinatz - Hopkins - Smith ('88)
- Commutative Algebra: $\mathbb{D}^{\text{perf}}(R-\mathsf{mod})$, $\mathbb{D}(R-\mathsf{mod})$, Hopkins ('87), Neeman ('92)
- Algebraic Geometry: $\mathbb{D}^{\text{perf}}(\mathsf{coh}(X))$, Thomason ('97)
- Rep Theory of finite groups: $\operatorname{stab} kG$, Benson - Carlson - Rickard ('97)

Equivariant Balmer spectrum

1. Equivariant stable homotopy theory (Balmer - Sanders' 17)
2. Perfect complexes on a stack $\mathbb{D}^{\mathrm{perf}}([X/G])$: bounded derived category of perfect complexes of G-equivariant vector bundles on a (nice) scheme X. (P. - Smith, A. Krishna ('09))

$$\mathrm{D}^{\mathrm{perf}}([X/G]) \to \mathrm{D}^{\mathrm{b}}([X/G]) \to \mathrm{Sing}[X/G]$$



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See "An Invitation to Tensor-Triangular Geometry" (page 1143).

How to compute Balmer spectrum?
Construct a "good" support theory (lavish)

$V \in \operatorname{stab} \mathfrak{u} \quad \to \quad \operatorname{supp} V \in X$ - realizations of support.

Properties:

1. $\operatorname{supp}(V) = \operatorname{supp}(\Sigma V)$ for all V in $\mathcal{T}$, and $\operatorname{supp}(0) = \emptyset$.
2. $\operatorname{supp}(V \oplus W) = \operatorname{supp}(V) \cup \operatorname{supp}(W)$.
3. For any exact triangle $V \to W \to V' \to \Sigma V$, we have $\operatorname{supp}(W) \subset (\operatorname{supp}(V) \cup \operatorname{supp}(V'))$
4. exhaustive: All closed subsets $Y \subset X$ are realized as supports of objects in $\mathcal{T}$.
5. "Extend well" to the big stable category $\widetilde{\mathcal{T}} = \operatorname{Stab} \mathfrak{u}$. In particular, faithful: For $V \in \operatorname{Stab} \mathfrak{u}$,

$$\operatorname{supp} V = \emptyset \Leftrightarrow V \cong 0.$$

Tensor product property

- $\operatorname{supp} V \otimes W = \operatorname{supp} V \cap \operatorname{supp} W.$

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Remark

- For general finite dimensional Hopf algebras the tensor product property is known to fail for the *cohomological* realization of supports - see Benson-Witherspoon, Plavnik-Witherspoon
- These counter-examples are in *non-braided* settings, whereas in all my examples of Hopf algebras the representation categories were braided.
- Yet, even in the study of braided categories, the non braided ones often arise naturally. In general, we weaken the "tensor product property" by asking for it to hold when V or W satisfies a certain centralizing condition.

Realizations of support theory

1. Cohomological support. $X = \operatorname{Proj} H^*(\mathfrak{u}, k)$
2. Elementary abelian p-groups: rank varieties (Carlson)
3. Restricted Lie algebras: nullcone (Friedlander-Parshall)
4. cocommutative f.d. Hopf algebras (finite groups schemes) π-supports (Friedlander-P.)
5. finite supergroup schemes: hypersurface support and π-support (Benson-Iyengar-Krause-P.)
6. Lie super algebras $\mathfrak{gl}(n|m)$ in char 0 (Boe-Nakano-Kujawa)
7. Complete intersections (commutative algebra): hypersurface support, support via Matrix factorizations (Eisenbud, Avramov-Buchweitz, Avramov-Iyengar). Applies to various “quantum” situations, in particular via the work on non-commutative MF by Cassidy-Conner-Kirkman-Moore (Negron-P., N. Courts)

Small quantum groups

G - almost simple algebraic group, $\mathfrak{g} = \mathrm{Lie}\, G$,
q - primitive root of unity, $\mathrm{ord}(q) > h$.
$\mathfrak{u}_q(\mathfrak{g})$- Lusztig's small quantum group

Conjecture (Ostrik'98)

$V, W \in \mathsf{Rep}_{\mathbb{C}} \mathfrak{u}_q(\mathfrak{g})$,

$$\mathrm{supp}(M \otimes N) = \mathrm{supp}\, M \cap \mathrm{supp}\, N.$$

$\mathfrak{u}_q(\mathfrak{g})$ - finite dimensional complex Hopf algebra with braided but highly non semi-simple representation category.

Can we compute the Balmer spectrum for $\operatorname{stab} \mathfrak{u}_q(\mathfrak{g})$?

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No. But we can say something about it; via the "equivariant" approach:

$$\mathsf{Rep}\,\mathfrak{u}_q(\mathfrak{g}) \cong \mathsf{Qcoh}[G/G_q]$$

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Let $\mathcal{N} = \mathcal{N}(\mathfrak{g})$ be the nilpotent cone of $\mathfrak{g}$, and let $\pi : \widetilde{\mathcal{N}} = T^*(G/B) \to \mathcal{N}$ be the Springer resolution.

Theorem (Negron-P.)

There is a sequence of surjective maps

$$\mathbb{P}(\widetilde{\mathcal{N}}) \to \operatorname{Spec}_{Bal} \operatorname{stab} \mathfrak{u}_q(\mathfrak{g}) \to \mathbb{P}(\mathcal{N})$$

where the composition is the moment map π.

In particular, there is a support theory for $\operatorname{stab} \mathfrak{u}_q(\mathfrak{g})$ living on $\mathbb{P}(\widetilde{\mathcal{N}})$ which does satisfy the tensor product property for $G = \mathrm{SL}_n$.

THANK YOU