Recent results on the Dixmier-Moeglin equivalence

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Primitive ideals

Conventions:

- We'll take k to be an algebraically closed field (not necessary for most things in this talk).
- We'll take A to be a finitely generated noetherian k-algebra.

Goal for this talk: to understand all irreducible representations of A (simple left A-modules).

Is that too ambitious?

Yes, although in some cases it is not too bad:

- When A is commutative.
- When $A = k[G]$ with G a finite group.

In general it is hopeless. Even for the Weyl algebra $\mathbb{C}\{x, y\}/(xy - yx - 1)$ it is very hard.

Dixmier's proposal

Dixmier had a good idea: look at the annihilator ideals of simple (left) modules. These are the (left) primitive ideals of A.

- Primitive ideals are prime ideals.
- We say a ring R is primitive if (0) is a primitive ideal of R.
- We let $\mathrm{Prim}(A)$ denote the subset of primitive ideals in $\mathrm{Spec}(A)$.
- Dixmier's question: Can we characterize this distinguished subset of $\mathrm{Spec}(A)$?

Bergman's example

While still an undergrad, George Bergman found an example of a ring that is left primitive but not right primitive.

His example is highly non-obvious.

For all rings we work with left primitive and right primitive coincide so we'll use the term primitive and not add "left".

Example: Spec of the quantum plane $xy = qyx$ for q not a root fo unity

(x,y-α), $\alpha \neq 0$ (x,y)   (y,x-β), $\beta \neq 0$

| |

(x)   (y)

(0)

(Yes, I am not good at drawing diagrams in Latex.)

The primitive ideals are the maximal ideals at the top of the lattice and the (0) ideal.

Dixmier-Moeglin Equivalence (DME)

Dixmier and Moeglin gave the first satisfying characterization of $\mathrm{Prim}(A) \subseteq \mathrm{Spec}(A)$ in the case that $A = U(\mathcal{L})$, $\mathcal{L}$ a finite-dimensional complex Lie algebra.

Theorem

Let $A = U(\mathcal{L})$. Then for $P \in \mathrm{Spec}(A)$ TFAE:

- *P is primitive;*
- *$\{P\}$ is locally closed in $\mathrm{Spec}(A)$;*
- *P is rational.*

What is locally closed?

Here we just mean that $\{P\}$ has an open neighbourhood U such that $P = \overline{\{P\}} \cap U$.

For us this just means

$$\bigcap_{Q \supsetneq P} Q \supsetneq P.$$

Since our ring is noetherian this means that the poset of primes properly containing P has finitely many atoms.

How about rational?

This one is easy. We look at

$$Z(\mathrm{Frac}(A/P)).$$

P is rational if this is just k (or generally an algebraic extension of the base field).

Implications

If A satisfies the Nullstellensatz then we always have

$$\text{locally closed} \implies \text{primitive} \implies \text{rational}$$

so the interesting direction is rational $\implies$ locally closed.

Goodearl-Letzter/Lorenz

The DME was popularized through the work of Goodearl and Letzter who showed it applied to a large class of quantum algebras. Lorenz looked at extensions to the non-noethrian case, extending much of the Goodearl-Letzter theory.

Spec of the quantum plane for q not a root fo unity, revisited

(x,y-α), $\alpha \neq 0$ (x,y) (y,x-β), $\beta \neq 0$

(x) (y)

(0)

(I am, however, good at recycling poorly crafted diagrams in Latex.)

Recall that the primitive ideals are the maximal ideals at the top of the lattice and the (0) ideal. We have a group of automorphisms isomorphic to $(\mathbb{C}^*)^2$ given by

$$(\lambda, \gamma) \in \mathbb{C}^* \times \mathbb{C}^* \mapsto ((x, y) \mapsto (\lambda x, \gamma y)).$$

Does the DME hold for general noetherian algebras?

No.

What about affine noetherian Hopf algebras?

We'll see that the answer is 'no' in general (even in the cocommutative case) but Zalesski showed that if G is a nilpotent-by-finite group then $k[G]$ satisfies the DME.

Lorenz' example:

Take $G = \mathbb{Z}^2 \rtimes \mathbb{Z}$, where we take an automorphism σ of $\mathbb{Z}^2$ corresponding to an element of $\mathrm{SL}_2(\mathbb{Z})$ with eigenvalues that are not roots of unity. Then

$$\mathbb{C}[G] = \mathbb{C}[x^{\pm 1}, y^{\pm 1}][z^{\pm 1}; \sigma],$$

where if our matrix in $\mathrm{SL}_2(\mathbb{Z})$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then we have $z \cdot x = x^a y^c z$,$z \cdot y = x^b y^d z$.

Here we get (0) is primitive and rational but not locally closed.

What's going on with this example?

The prime ideals of $T := R[z^{\pm 1}; \sigma]$ are closely related to σ-invariant ideals of R. In fact, for each σ-invariant ideal I of R we have IT is an ideal of T and if I is σ-prime of R then IT is prime ideal of T.

Notice that in Lorenz' example, we have $R = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ and we get an induced map on $\mathrm{Spec}(R)$. The maximal ideals of R are parametrized by $\mathbb{C}^* \times \mathbb{C}^*$ and (ω_1, ω_2) with ω_1, ω_2 roots of unity is periodic under the map induced by σ.

This shows (0) is not locally closed.

(0) is rational!

Geometrically, rationality is related to the notion of preserving a non-constant fibration (at least in characteristic zero). That is we have an irreducible variety X and a rational self-map τ associated to $R[z^{\pm 1}; \sigma]$ when R is an integral domain.

$$\begin{array}{ccc} X & \rightarrow & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ X & \rightarrow & \mathbb{P}^1 \end{array}$$

We note that for most counterexamples to DME, the behaviour is like that in Lorenz' example where the notions of primitivity and rationality coincide but rational $\not\Longrightarrow$ locally closed.

There are, however, now some pathological examples constructed in similar ways:

- Irving found a noetherian example where primitivity $\not\Longrightarrow$ rational.
- Jordan (with important correction by Brown-Carvalho-Matczuk) gives a ring that is not primitive where (0) is rational and not locally closed.

Zaleski vs. Lorenz: what's the difference?

Recall $\mathbb{C}[G]$ satisfies DME if G is nilpotent-by-finite, but Lorenz gives a polycyclic group G with $\mathbb{C}[G]$ not satisfying DME.

What's the difference?

Growth!

For a finitely generated k-algebra A we can pick a finite-dimensional generating subspace V of A that contains 1. We then get a nested chain:

$$k = V^0 \subseteq V \subseteq V^2 \subseteq \cdots$$

We define the *growth function* of A to be the $d_V(n) = \dim(V^n)$ up to a natural asymptotic equivalence.

- $\exists s \geq 0,\ d_V(n) = \mathrm{O}(n^s) \implies A$ has polynomial growth.
- $\exists C > 1,\ d_V(n) > C^n \implies A$ has exponential growth.

Gromov's Theorem

Bass and Guivarch showed that if G is nilpotent-by-finite then $k[G]$ has polynomial growth.

Theorem

(Gromov) Let G be a finitely generated group. Then $k[G]$ has polynomial growth iff G is nilpotent-by-finite.

Modified question

Does DME hold if we impose a polynomial growth hypothesis?

Spoiler alert: No!

We'll explain why DME fails for a very specific algebra of GK dimension 4 (growth function like n^4).

This goes through the so-called Poisson DME and a question of Brown and Gordon.

Poisson algebras

Intuitively, a Poisson bracket $\{-,-\}$ on a commutative ring can often be thought of as corresponding to a family of deformations of some "classical" object and one can recover the Poisson bracket via semiclassical limits.

As an example, with the quantum plane $xy = qyx$, we can put a Poisson bracket on $\mathbb{C}[x,y]$ $(q=1)$ by defining

$$\{f,g\} = \lim_{q\to 1}[f,g]/(q-1)$$

were we compute $[f,g]$ in the quantum plane.

So

$$\{x,y\} = \lim_{q\to 1}(xy - yx)/(q-1) = \lim_{q\to 1}(qyx - yx)/(q-1) = xy.$$

P−Spec

If R is a commutative ring with a Poisson bracket we can define the Poisson Spectrum of R which is the set of prime ideals closed under Poisson bracket; i.e., prime ideals P such that $\{P, R\} \subseteq P$.

We can define the notions of Poisson primitive, Poisson locally closed, Poisson rational in a natural way for a commutative ring R with a Poisson bracket.

For a Poisson prime ideal P we have the following:

- P Poisson primitive $\iff$ P is the largest Poisson ideal inside some maximal ideal of R.
- Poisson locally closed: same as before.
- Poisson rational: $\mathrm{Frac}(R/P)$ has Poisson centre equal to k; i.e., $\{x, a\} = 0 \ \ \forall a \in R/P \implies x \in k$.

Thus we have a natural notion of Poisson DME and as before the interesting implication is Poisson rational $\implies$ Poisson locally closed.

Example: P−Spec of $\mathbb{C}[x, y]$

$$
\begin{array}{ccccc}
(x,y-\alpha),\ \alpha \neq 0 & & (x,y) & & (y,x-\beta),\ \beta \neq 0 \\
| & & & & | \\
(x) & & & & (y) \\
 & & (0) & &
\end{array}
$$

(I've made peace with this picture.)

Note that the Poisson prime ideals look exactly like the prime ideals of the quantum plane when q is "generic".

Brown-Gordon

Ken Brown and Iain Gordon asked whether an affine commutative algebra with a Poisson bracket satisfies Poisson DME.

Notice this fits with the philosophy espoused earlier that growth and DME should be connected!

Work with (Léon-Sánchez)-Launois-Moosa

We showed (somewhat surprisingly to me, but perhaps not to Rahim and Omar) that PDME fails for Krull dimension ≥ 4 in general and holds for Krull dimension ≤ 3.

This also gave a noetherian ring of Gelfand-Kirillov dimension four that does not satisfy DME: it is of the form $R[x;\delta]$ with δ a derivation and R an affine commutative domain.

New questions!

Clearly just adding a restricted growth hypothesis was too ambitious. There are, however, variants of this for smaller classes of algebras. The following are all open.

- (Bell-Leung?) Does DME hold for a complex affine noetherian Hopf algebra of polynomial growth?
- (Reichstein?) Does DME hold for a noetherian twisted homogeneous coordinate ring?
- Does DME hold for prime noetherian algebras of GK dimension ≤ 3?

Hopf algebras!

- Wing Hong and I, in a thinly veiled attempt to justify making this conjecture, proved that DME holds in the cocommutative case (finite GK and complex affine noetherian).
- Brown and Goodearl initiated the study of general infinite-dimensional Hopf algebra and Goodearl and Zhang classified GK 2 Hopf domains H with $\mathrm{Ext}^1_H(k_H, k_H) \neq (0)$. They showed DME holds under these hypotheses. (This is a mild hypothesis, but Wang, Zhang, Zhuang showed it doesn't hold in general.)
- Brown, O'Hagan, Zhang, and Zhuang defined iterated Hopf Ore extensions and showed that for small GK iterated Hopf Ore extensions all satisfy DME.

Reichstein's question

A twisted homogeneous coordinate ring is a graded algebra associated to:

- an irreducible complex projective variety X;
- an automorphism $\sigma \in \mathrm{Aut}_{\mathbb{C}}(X)$;
- an ample invertible sheaf $\mathcal{L}$.

The twisted homogeneous coordinate ring associated to these data is noetherian iff it has polynomial growth (Keeler, Stephenson-Zhang). In general, the growth is related to the eigenvalues of the induced action on σ on the finite-dimensional vector space $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

BRS

- In work with Dan Rogalski and Sue Sierra we proved the DME holds for noetherian homogeneous coordinate rings when X is a surface.
- We also proved that it holds if σ lies in an algebraic group (i.e., some iterate is in the connected component of the identity in $\mathrm{Aut}_{\mathbb{C}}(X)$. Equivalently, all eigenvalues of the induced action on $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ are roots of unity and the action is diagonalizable.

Recently with Dragos Ghioca we proved DME holds for noetherian twisted homogeneous coordinate rings of abelian varieties.

Why were we interested in this class? Well, there were two reasons....

As far as we know, there are no affine noetherian counterexamples to DME with GK ≤ 3.

As mentioned before, Goodearl and Zhang showed DME should hold in the Hopf case for GK ≤ 3.

Recently, with two undergrads, Léon Burkhard and Nick Priebe, we've been looking at the DME for iterated Ore extensions of GK dimension ≤ 3.

These are algebras of the form

$$k[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n]$$

with $n \leq 3$ and σ_i automorphisms, δ_i σ_i-derivations, and we have some constraints on the automorphisms to control growth.

Unfortunately, it's hard to understand the prime ideals of a general Ore extension $R[x; \sigma, \delta]$ when σ is not the identity and δ is nonzero. All meaningful work on this difficult problem is due to Goodearl and Goodeal-Letzter.

Goodearl gave a concrete description of prime ideals in $R[x; \sigma, \delta]$ when R is commutative. This can be extended to the case when R is finite over its centre, but extending his work beyond this is hard without some other constraints on σ and δ (e.g., perhaps insisting the commute.)

Bringing back Theorem*

Léon Burkhard and Nick Priebe and I are working on the following theorem.

Theorem*: An iterated Ore extension of GK dimension ≤ 3 over a field of characteristic zero satisfies DME.

Why the $*$?

This result proceeds via classifying all the Ore extensions of GK ≤ 3, which we have done, and then studying DME for them. Using Goodearl's theorem we have been able to handle the case of GK 3 rings of the form $R[x; \sigma, \delta]$ with R PI of GK dimension ≤ 2 and we have handled all but one case in our classification. Hopefully we'll get it soon.

Unfortunately, this conference came too soon so we have to have a star for now. Will it be removed? We'll see!

Thanks!