

# Geometry of algebras related to the Weyl groupoid.

Let  $\underline{g}$  be a finite dimensional classical simple Lie superalgebra over  $\mathbb{C}$ .  $\uparrow$  means so reductive.

Let  $\underline{h}$  be a Cartan subalgebra of  $\underline{g}$ .

Consider 2 commutative algebras.

1.  $\mathcal{I}(\underline{h}) \subseteq S(\underline{h})^W$  image of the injective

Harish-Chandra map  $Z(\underline{g}) \rightarrow S(\underline{h})^W$

Mainly deal with case 1.  $\uparrow$  center of  $U(\underline{g})$ .

2. Let  $G$  be an algebraic supergroup with  $\text{Lie } G = \underline{g}$ .

Then  $\mathcal{I}(G)$  is the supercharacter ring of the category of fte. dim<sup>n</sup>  $\mathbb{Z}_2$ -graded reps. of  $G$ .  
( $\mathcal{I}(G)$  is 'almost' the Grothendieck ring)



## Main Themes,

A. The Nullstellensatz gives a 1-1 correspondence between radical ideals of  $\mathbb{I}(\mathfrak{h})$  and their zero loci in  $\mathfrak{h}^*$  (called superalgebraic sets).

Also any superalgebraic set is uniquely a union of irreducible ones.

Note  $\mathbb{I}(\mathfrak{h})$  is usually not Noetherian.

B.  $\text{Spec } \mathbb{I}(\mathfrak{h})$  is a "pre-geometric" quotient of the continuous Weyl groupoid  $\mathcal{W}^c$  on  $\mathfrak{h}^*$ .

The orbits of the usual Weyl groupoid of Sergeev and Veselov are not closed (unless finite).



Classification, Let  $\underline{g}$  be a  $2k$ , dimensional simple Lie superalgebra /  $\mathbb{C}$  with  $g_0$  reductive

Then  $\underline{g}$  is either  $\underline{g}_0 \neq \underline{g}$

1) A Kac-Moody (KM) Lie superalgebra  
(also known as contragredient Lie superalgebras)

$\infty$  series  $A, B, C, D$  + 3 exceptional cases

2) One of 2  $\infty$  series  $p(n), q(n)$ .

For simplicity I will concentrate on the KM case.

Let  $\underline{h}$  be a Cartan subalgebra of  $\underline{g}_0$ .



## Root Space Decomposition

4.

For  $\alpha \in \underline{h}^*$ , set

$\underline{g}^\alpha = \{x \in \underline{g} \mid [h, x] = \alpha(h)x\},$  then  $\underline{g} = \underline{h} \oplus \bigoplus_{\alpha \in \Delta} \underline{g}^\alpha$   
and  $\Delta = \{\alpha \in \underline{h}^* \mid \alpha \neq 0, \underline{g}^\alpha \neq 0\}.$

## Combinatorial Data.

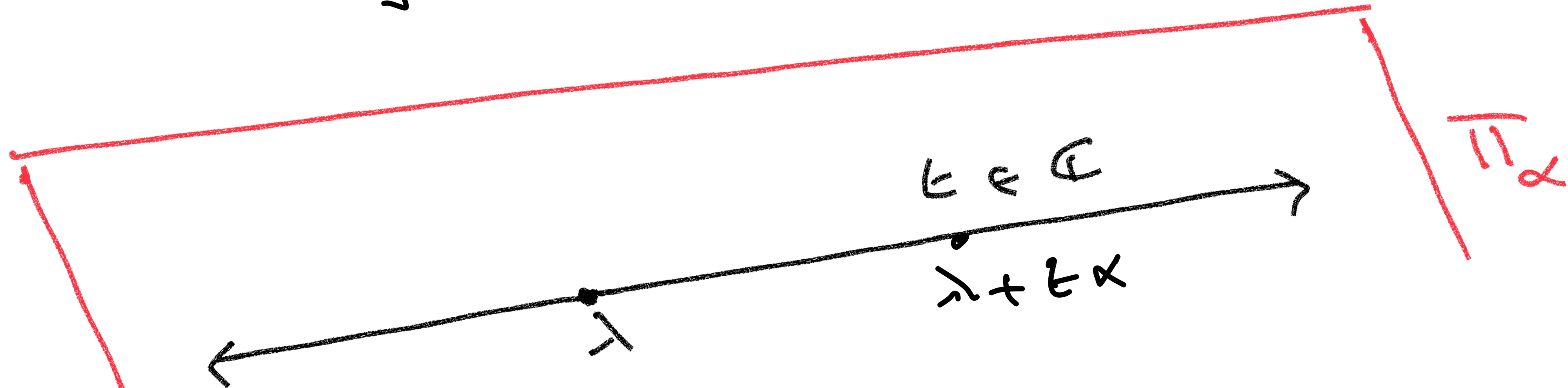
The def<sup>n</sup> of Weyl groupoids depends on

- $\Delta \subseteq \underline{h}^*$  a finite dimensional vsp.
- $(,)$  a non-degen. sym. bilinear form invariant under a finite group  $W$ .
- Isotropic roots  $\Delta_{iso} = \{\alpha \in \Delta \mid (\alpha, \alpha) = 0\} \neq \emptyset$   
 $W$  acts on  $\Delta$  and  $\Delta_{iso}$ .

# Action of the continuous Weyl groupoid on $\underline{h}^*$ .

For  $\alpha \in \Delta_{iso}$ , set  $\pi_\alpha = \{ \lambda \in \underline{h}^* \mid (\lambda, \alpha) = 0 \}$ .

$W$  acts everywhere and perpendicular to the  $\pi_\alpha$ .



1. The usual Weyl groupoid moves  $\lambda$  to  $\lambda + n\alpha$   $n \in \mathbb{Z}$

2. If  $\lambda \in \pi_\alpha \cap \pi_\beta$  where  $(\alpha, \beta) = 0$ , the Weyl groupoid can move  $\lambda$  in the direction of  $\alpha$  or  $\beta$ .



## Part A The Nullstellensatz. Certain Ring Extensions.

Assume  $A \subseteq B$  are comm. rings, and  $T \in A$  s.t.  $TB \subseteq A$

Lemma If  $P$  is a radical ideal of  $A$  with  $T \in P$ ,  
Then  $TB \subseteq P$ .

Proof.  $(TB)^2 = T^2 B \subseteq PA = P$ . localizations

Remark If  $T$  is a non-zero divisor, then  $A_T \xrightarrow{\downarrow} B_T$

It is well known, there is a bijection

$$\{P \in \operatorname{Spec} A \mid T \notin P\} \longleftrightarrow \{\operatorname{Spec} A_T\} \quad (*)$$

$$P \mapsto P_T \quad Q \cap A \longleftarrow Q$$

We identify both sides of (\*)

Thm. Let  $\phi: A \rightarrow A/TB = C$  be the natural map. Then

$$\operatorname{Spec} A = \operatorname{Spec} B_T \cup \phi^{-1}(\operatorname{Spec} C) \quad (\text{disjoint union})$$

Cor Suppose  $m \in \text{Max } A = \{\text{max. ideals of } A\}$ , and  $T \notin m$ .  
Then there is a unique  $M \in \text{Max } B$  s.t.  $m = M \cap A$ .  
Also  $M$  is given by  $M = m_T \cap B$ ; and  $mB \neq B$ .  
(Recall  $A_T = B_T$ ).

Cor. With  $A$  and  $C = A/TB$  as above. If  $B_T, C$  have ACC on radical ideals, so does  $A$ .

This implies any radical ideal  $I$   
is a finite intersection of prime ideals.

We can take the primes to be minimal over  $I$ .



Application Define

$$I(\underline{h}) = \{f \in S(\underline{h})^w \mid \text{if } \lambda \in \pi_\alpha \text{ then } f(\lambda) = f(\lambda + t\alpha) \text{ for } \alpha \in \Delta_{iso}, t \in \mathbb{C}\}$$

Previous results apply to  $A = I(\underline{h}) \subseteq B = S(\underline{h})^w$

and  $T = \pi_{\alpha \in \Delta_{iso}} h_\alpha$ . Note  $(, )$  induces  
an iso  $\underline{h}^* \rightarrow \underline{h}$  and  $\alpha \rightarrow h_\alpha$ . ( $T$  vanishes on all  $\pi_\alpha$ )

Maximal ideals of  $A$  containing  $T$  are understood. We need a new ingredient to study  $\text{Max}(A/TB)$ .

if  $\alpha \in \Delta_{iso}$ , any root vector  $x \in \underline{g}^\alpha \subseteq U(\underline{g})$   
satisfies  $x^2 = 0$

$\therefore$  For a  $\underline{g}$ -module  $M$ ,  $xM \subseteq \text{Ker } x|_M$ . Define

$$M_x = \text{Ker } x|_M / xM$$

Applying this to  $M = \underline{g}$ , with the adjoint rep,  
gives a Lie superalgebra  $\underline{g}_x$  of smaller rank,  
and  $M_x$  is a  $\underline{g}_x$ -module.

Thus we have the Duflo-Serganova functor  
 $\underline{g}\text{-mod} \rightarrow \underline{g}_x\text{-mod}$ .  $M \rightarrow M_x$ . The DS-functor

induces maps  $\underline{I}(\underline{h}) \rightarrow \underline{I}(\underline{h}_x)$ .

and  $\underline{J}(G) \rightarrow \underline{J}(G_x)$  (Hoyt - Reif)



Suppose  $A = \mathbb{I}(\underline{h})$   $A_x = \mathbb{I}(\underline{h}_x)$  Then we have a  
 $B = S(\underline{h})^w$   $B_x = S(\underline{h}_x)^{w_x}$  comm. diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad \phi \quad} & A_x \\ \downarrow & & \downarrow \\ B & \xrightarrow{\quad \tilde{\phi} \quad} & B_x \end{array}$$

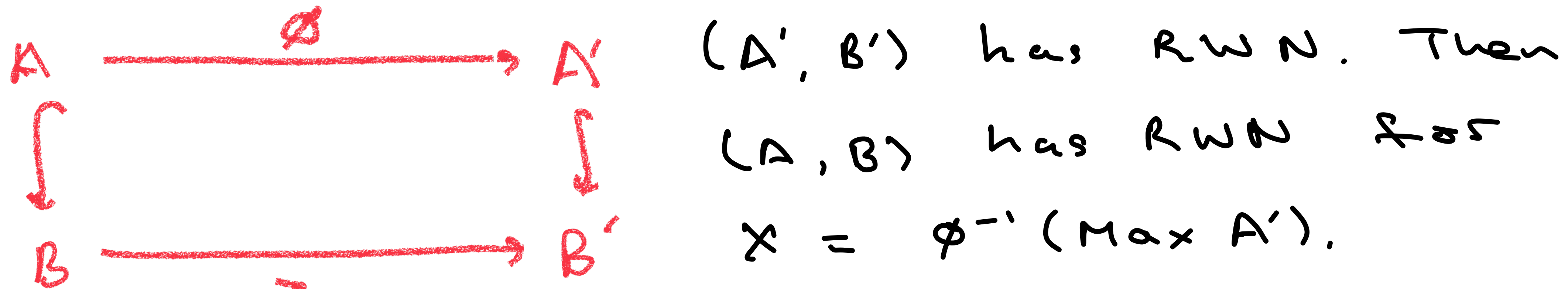
where  $\tilde{\phi}$  is defined  
 by restriction  
 of functions to  $\underline{h}_x^*$

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Thm  $\text{Ker } \phi = \tau B$  and  $\text{Im } \phi = A_x^\sigma$  with  $|\sigma| \leq 2$ .

For a ring extension  $R \subseteq S$ , and  $X \subseteq \text{Max } R$ , say  
 the Relative Weak Nullstellensatz (RW N) holds for  
 $X$  and the pair  $(R, S)$  if for all  $m \in X$ ,  $mS \neq S$ .  
 Say  $(R, S)$  has RW N if it has RW N for  $X = \text{Max } R$ .

Lemma In the diagram, suppose  $\Phi$  is onto and



pf. If  $m \in X$ , then  $m$  is a max. ideal of  $A$  containing  $\ker \phi$ . Then  $\phi(m)$  is a max ideal of  $A'$ . So  $J = \phi(m)B' \neq B'$ . But  $\Phi(mB) = J$  so  $mB \neq B$ .

Lemma Suppose the finite group  $G$  acts on  $S$  and  $|G|$  is a unit in  $S$ . If RWN holds for  $(R, S)$ , then RWN holds for  $(R^G, S^G)$ .



Cor The weak Nullstellensatz.

The maps  $\text{Max } S(h) \rightarrow \text{Max } S(h)^w \rightarrow \text{Max } I(h)$  are onto.

Pf By induction on rank  $\underline{g}$ . The base cases for the induction occur when  $\underline{g}_x$  has no isotropic roots. This occurs when  $\underline{g}_x$  is a reductive Lie algebra or  $\mathfrak{osp}(1, 2n)$ .

In these cases  $A_x = B_x$ .  $\square$

Cor. Any central character of  $U(\underline{g})$  is a central character afforded by a Verma module.

[ A central character is an algebra map  $Z(\underline{g}) \rightarrow \mathbb{C}$  ].  
Any elt. of  $Z(\underline{g})$  acts on a Verma by a scalar. ]

# Thm The Strong Nullstellensatz

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For  $I \subseteq A = \mathbb{C}[\underline{x}]$  consider the zero locus of  $I$ .

$$V(I) = \{x \in \underline{x}^* \mid f(x) = 0 \text{ for all } f \in I\}.$$

Call a set of this form **superalgebraic**.

There is a bijection  $I \rightarrow V(I)$  (inverse  $V \rightarrow I_A(V)$ )

$$\{\text{Radical Ideals of } A\} \leftrightarrow \{\text{Superalgebraic sets}\}$$



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Pf. Key point:  $I_A V(I) \subseteq \text{rad } I$ . If  $G \in I_A V(I)$

the ideal of  $A[z]$  generated by  $I$  and  $Gz-1$ , has no zeros  
so contains 1 by Weak Nullstellensatz, write 1 as a finite  
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so contains 1 by Weak Nullstellensatz. Write 1 as a finite

linear comb. of elts. Mult by  $Y = z^{-1}$ . then by a large power of  $Y$  to clear denominator, and set  $Y = G$ .



## Part B

## Actions of Groupoids on Varieties

For a variety  $X$ , let  $\underline{A}(X)$  is the groupoid with objects all subvarieties of  $X$  and morphisms all isomorphisms between objects

We say a groupoid  $\mathcal{G}$  acts on  $X$  if there is a natural transformation  $\tau: \mathcal{G} \rightarrow \underline{A}(X)$ . In this case call  $X$  a  $\mathcal{G}$ -variety.

## Orbits and the Invariant Ring.

Let  $\tau: \mathcal{G} \rightarrow \underline{A}(X)$  be a natural transformation of groupoids, and consider  $x \in X$ .

Suppose  $g = (g_1, \dots, g_r)$  is an ordered sequence of morphisms, with  $r \geq 1$ . We set

$$gx = \tau(g_1) \dots \tau(g_r) x, \text{ provided the RHS is defined.}$$

The  $\mathcal{G}$ -orbit of  $x$  is

$$\mathcal{G}x = \{gx \mid \text{all } g \text{ as above}\}.$$

The invariant ring is (Assume  $X$  affine).

$$\mathcal{O}(X)^{\mathcal{G}} = \{f \in \mathcal{O}(X) \mid f \text{ is constant on } \mathcal{G}\text{-orbits}\}.$$



## Example

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Assume  $\mathfrak{g}$  is KM with root system  $\Delta$ . Let

$W = W(\Delta)$  the Weyl groupoid of Sergeev & Veselov

There is a natural transf,  $\tau: W \rightarrow \underline{A}(h^*)$ .

So  $W$  acts on  $\underline{h}^*$

Problem with this action:

Orbits are not closed (unless finite).

Therefore instead of giving details of  
we introduce related groupoids next.

## The continuous Weyl-groupoids $W^c$ and $W_*^c$

Let  $\Delta_{iso}$  be the set of isotropic roots in  $\Delta$ .

Define  $I_{iso}^c$  to have base  $\Delta_{iso} \times \mathbb{C}$

$\text{Mor}((\alpha, t) \rightarrow (\beta, s)) \neq \emptyset$  iff  $t = s$  and  $\alpha = \pm \beta$ .

has cardinality  $\leq 1$ .

Denote by  $\sigma_{\alpha, t}$  the unique morphism

$$(\alpha, t) \rightarrow (-\alpha, t)$$

Then  $W$  acts on  $I_{iso}^c$  and we form the

semidirect product groupoid  $W \ltimes I_{iso}^c$ .

Then define  $W^c = W \amalg W \ltimes I_{iso}^c$ .

The def of  $W_*^c$  is similar, but the base is  $\Delta_{iso} \times \mathbb{C}^\times$ .



The nat. trans.  $W^c \rightarrow \underline{A}(h^*)$ .

Recall  $W^c = W \amalg W \rtimes \Delta_{iso}$ .

The base point  $[W]$  maps to the whole space  $\underline{h}^*$ .

For  $\alpha \in \Delta_{iso}$ , set  $\pi_\alpha = \{\lambda \in h^* \mid (\lambda, \alpha) = 0\}$ .

The base element  $(\alpha, t)$  maps to  $\pi_\alpha$ .

$\sigma_{\alpha, t}$  acts by  $\sigma_{\alpha, t}(\lambda) = \lambda + t\alpha$ ,  $\lambda \in \pi_\alpha$ .

Thm  $S(\underline{h})^{W^c} = \underline{I}(\underline{h})$ .

Orbits of  $W^c$  on  $\mathfrak{h}^*$  For  $\lambda \in \mathfrak{h}^*$ , set  $O_\lambda = W^c \lambda$ .

Let  $F(\lambda)$  be a maximal set of mutually orthogonal isotropic roots, all orthogonal to  $\lambda$ .

Thm (a)  $O_\lambda = \bigcup_{w \in W} w(\lambda + \sum_{\alpha \in F(\lambda)} \mathbb{Q}\alpha)$  (indep. of  $F(\lambda)$ )

In partic all orbits are closed

(b)  $\dim O_\lambda = |F(\lambda)|$ .

(c) Let  $M_\lambda = \text{max ideal of } S(\mathfrak{h}) \text{ corres. to } \lambda$   
and  $m_\lambda = M_\lambda \cap I(\mathfrak{h})$ . Then  $V(m_\lambda) = O_\lambda$ .

(d) There is a bijection

$\text{Max } I(\mathfrak{h}) \longleftrightarrow W^c\text{-orbits on } \mathfrak{h}^*.$



Let  $X$  be an affine  $G$ -variety. The inclusion

$$O(X)^G \longrightarrow O(X)$$

corresponds to a map  $\pi: X \rightarrow Y = \operatorname{Spec} O(X)^G$ .

We say that  $\pi$  is a pre-geometric quotient

if all orbits of  $G$  on  $X$  are closed, and  
orbits are in 1-1 correspondence with points  
in  $Y$  (that is with max. ideals in  $O(X)^G$ ).

Cor.  $\underline{h}^* \rightarrow \underline{h}^* / W^c$  is a pre-geometric quotient.

A geometric quotient  $\pi: X \rightarrow Y$  is a pregeometric  
quotient where  $Y$  has the quotient topology

Defn Suppose  $X$  is an affine  $G$ -variety, and  
let  $\mathcal{C}$  be a category. A map  $\pi: X \rightarrow Y$  of varieties,  
is a categorical quotient in  $\mathcal{C}$  if

a)  $\pi$  is constant on orbits

b)  $\pi$  is universal s.t. a) holds

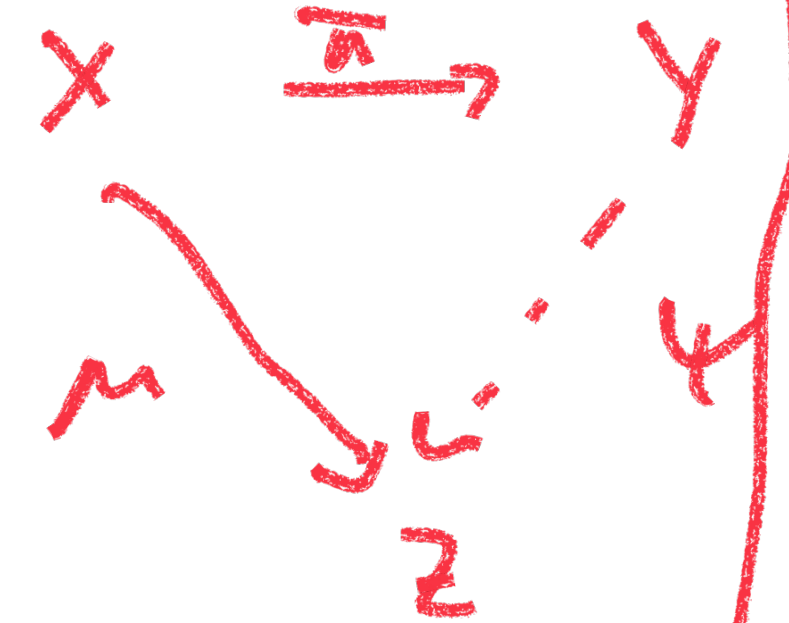


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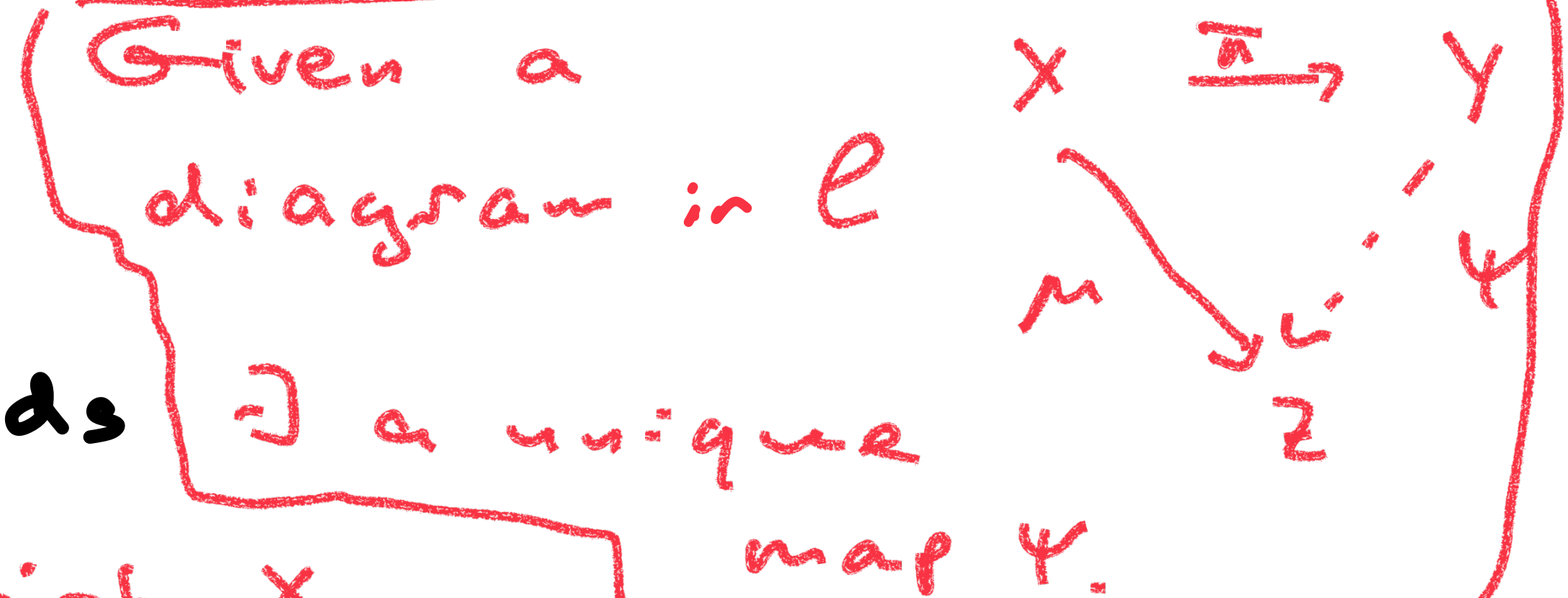
$\exists$  a unique map  $\psi$ .

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Lemma For an affine  $G$ -variety  $X$

$X \rightarrow \text{Spec } \mathcal{O}(X)^G$  is categorical in the category of affine schemes

$$(\text{Affine Schemes})^{\text{op}} = (\text{Comm. Rings})$$

opposite categories.

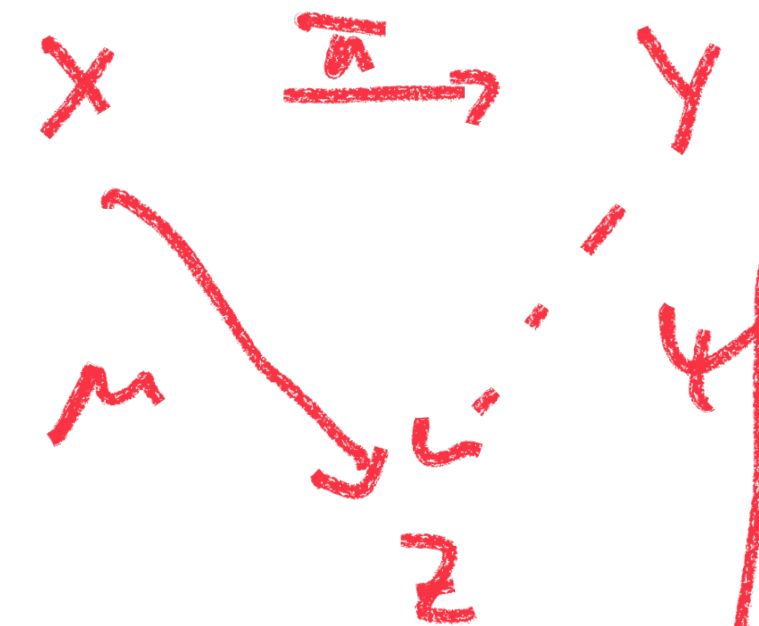


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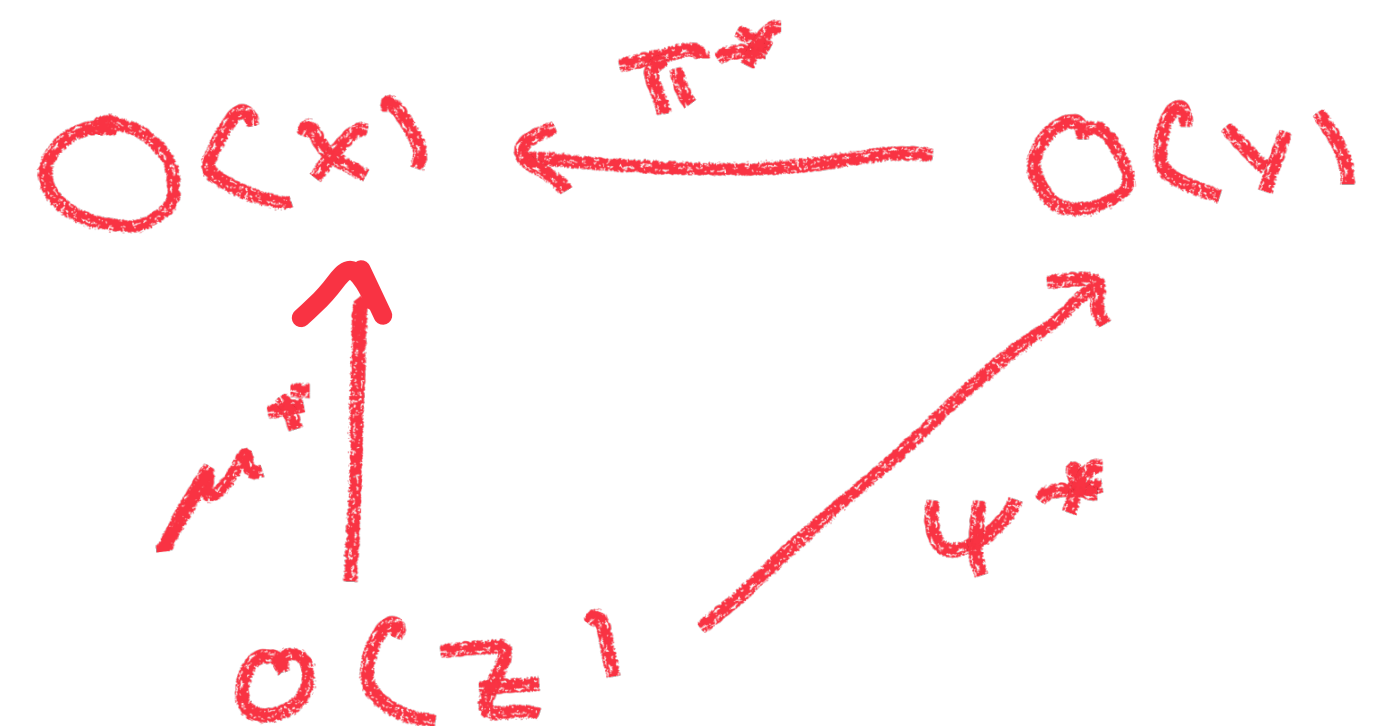
$X \rightarrow \text{Spec } \mathcal{O}(X)^G$  is categorical in the category of affine schemes.

$$\text{PF (Affine Schemes)}^{\text{op}} = (\text{Comm. Rings})$$

opposite categories.

Since  $\mu$  is constant on orbits,

$\text{Image } \mu^* \subseteq \mathcal{O}(X)^G = \mathcal{O}(Y)$ , get diagram



## Two Conjectures.

Conjecture 1. If  $V \subseteq \underline{h}^*$  is  $W^e$ -invariant algebraic set, then  $V$  is superalgebraic.

(The converse statement is easy.)

Conjecture 2. The quotient

$$\pi: \underline{h}^* = \operatorname{Spec} S(\underline{h}) \rightarrow \operatorname{Spec} I(\underline{h}) = \underline{h}^*/W^e$$

is geometric and categorical in the category of

$\mathbb{C}$ -schemes.

I think Conj. 1  $\Rightarrow$  Conj. 2.

Mumford  
- Fogarty  
- Kirwan

Prop 0.1. A geometric quotient is categorical.  
(for group actions).



More generally suppose  $X$  is an affine  $g$ -var.  
and set  $A = \mathcal{O}(X)$ . If  $g \in A^g$ , then

$$X_g = \{x \in X \mid g(x) \neq 0\} = \text{Spec } A_g$$

is open in  $X$ . The inclusion  $A^g \hookrightarrow A$  yields

a map  $\pi: X \rightarrow Y = \text{Spec } A^g$ , and

$$\pi(X_g) = Y_g := \{y \in Y \mid g(y) \neq 0\}.$$

Now suppose  $V = \mathcal{V}(\mathcal{I})$  for  $\mathcal{I} \subseteq A^g$ . Then

$V$  is closed in  $X$  and  $g$ -invariant. Thus

$U = X \setminus V$  is open and  $g$ -invariant.

If Conjecture 1 is true then any  $g$ -invariant  
open set has this form (in the situation of the  
conjecture)

Lemma Suppose  $V = V(\mathcal{I})$  for  $\mathcal{I} \subseteq A^g$ , and  $U = X \setminus V$

Then (a) if  $x \in U$ , then  $x \in X_g \subseteq U$ , some  $g \in \mathcal{I}$ .

(b)  $\pi(U)$  is open in  $Y$ .

Pf Let  $m_x = \{f \in A^g \mid f(x) = 0\}$ . If  $\mathcal{I} \subseteq m_x$ ,

then  $x \in V(m_x) \subseteq V(\mathcal{I}) = V$  against hypothesis.

Thus (a) holds for  $g \in \mathcal{I} \setminus m_x$ .

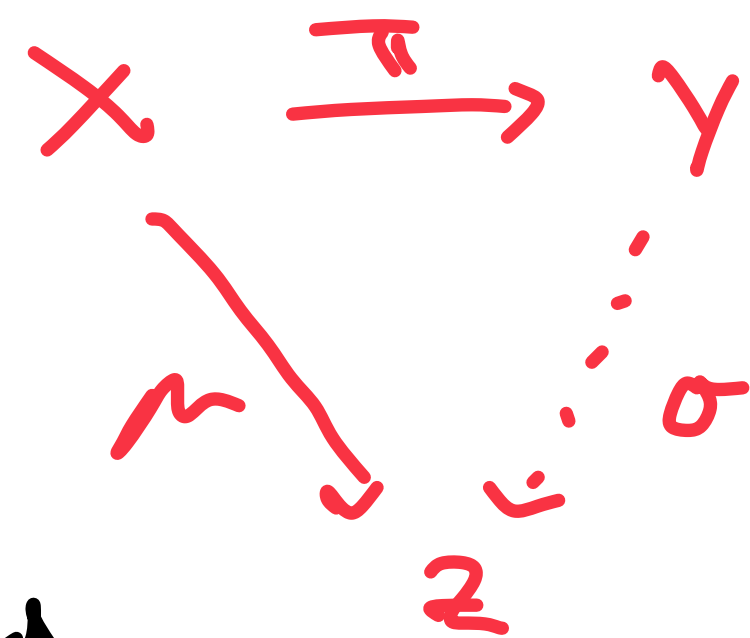
By (a), we can write

$$U = \bigcup_{g \in \mathcal{I}} X_g,$$

then  $\pi(U) = \pi \bigcup X_g = \bigcup Y_g$  is open in  $Y$ .



Prop Let  $X, Y, Z$  be top. spaces s.t.  
the groupoid  $\mathcal{G}$  acts on  $X$ . Suppose  
 $\pi: X \rightarrow Y$ ,  $\mu: X \rightarrow Z$  are continuous and



(a)  $\pi, \mu$  are constant on  $\mathcal{G}$ -orbits

(b)  $\pi$  is onto and a pre-geometric quotient

(c) If  $U$  is a  $\mathcal{G}$ -invariant open set in  $X$ ,  
then  $\pi(U)$  is open in  $Y$ .

Then there is a continuous map  $\sigma: Y \rightarrow Z$   
such that  $\mu = \sigma\pi$ .

Proof Since  $\pi$  is onto, if  $\pi(x) = y$ , define  
 $\sigma(y) = \mu(x)$ . Well defined by assumption. Then  $\mu = \sigma\pi$

Show  $\sigma$  is continuous.

Let  $V$  be open in  $Z$ . Since  $\pi$  is onto we have

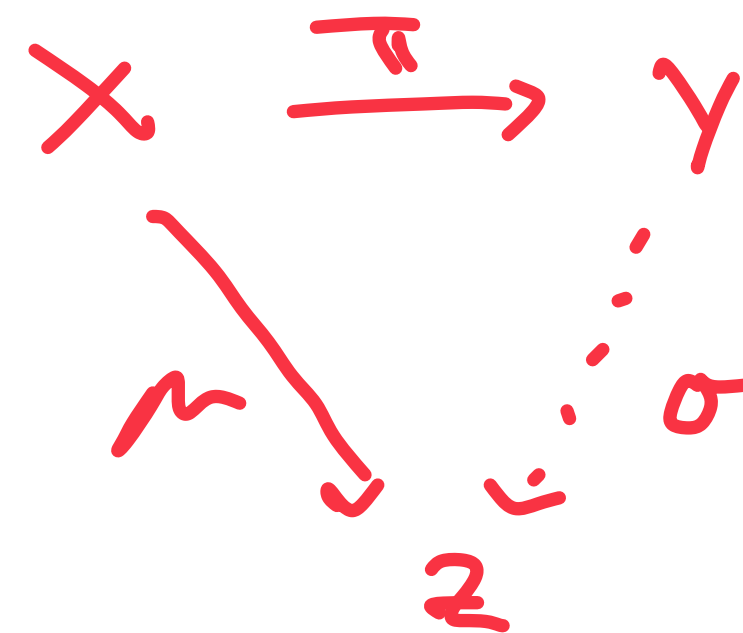
$$\pi \mu^{-1}(V) = \sigma^{-1}(V).$$

By continuity  $U = \mu^{-1}(V)$  is open in  $X$  and by

(a),  $U$  is  $g$ -invariant. Hence by (c)

$$\pi(U) = \sigma^{-1}(V) \text{ is open in } Y.$$

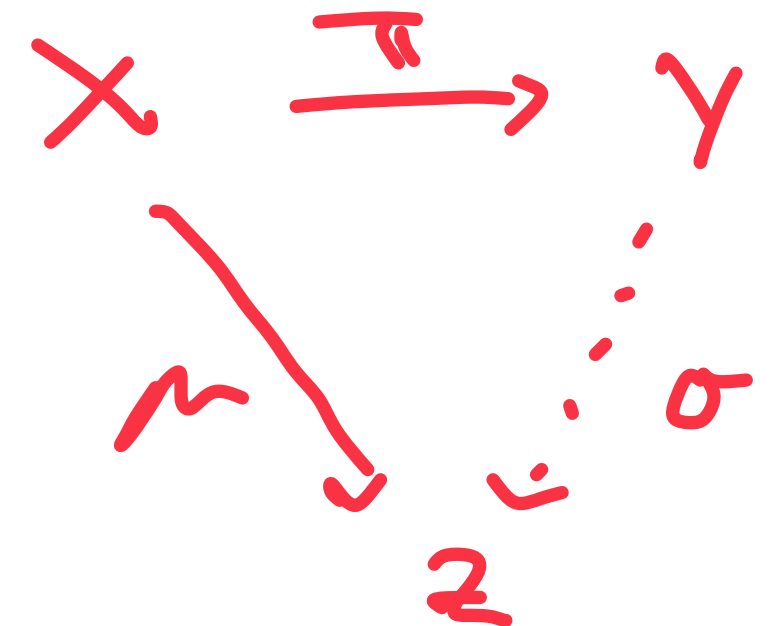
Next apply this to the situation of the Conjecture 2.





Thm (Assume Conj 1). We have a categorical quotient in the category of topological spaces.

$$\pi: X \longrightarrow X/\mathcal{G} = \operatorname{Spec} \mathcal{O}(X)^{\mathcal{G}} = Y.$$



Proof. In the diagram assume

(a)  $\mu, \pi$  constant on  $\mathcal{G}$ -orbits.

(b) Know  $\pi$  is a <sup>pre</sup>geometric quotient and is onto by the Weak Nullstellensatz.

(c)  $U$   $\mathcal{G}$ -invariant open in  $X$

$\Rightarrow \pi(U)$  open in  $Y$  by Lemma.

Action of  $W_*$  on a Torus. For simplicity assume  $\mathfrak{g} \neq \mathfrak{psl}(n, \mathbb{C})$

Let  $\alpha_1, \dots, \alpha_n$  be a basis of simple roots for  $\mathfrak{g}$ . 33

By assumption the Cartan matrix  $(\alpha_i, \alpha_j)$  is non-singular. So  $\exists ! \omega_j \in \mathfrak{h}^*$  s.t.  $(\alpha_i, \omega_j) = \delta_{ij}$

$$\text{Set } Q = \sum_{i=1}^n \mathbb{Z} \alpha_i \subseteq P = \sum_{i=1}^n \mathbb{Z} \omega_i.$$

Let  $G$  be the Lie supergroup with  $\text{Lie } G = \mathfrak{g}$  and  $G_0$  connected, simply connected. Let  $T$  be a maximal torus in  $G_0$ . There is an iso

$$P \rightarrow X(T) = \text{Hom}(T, \mathbb{C}^\times) \text{ written } \alpha \mapsto e^\alpha.$$

Lemma For  $\beta \in P$ ,  $\exists ! c_\beta \in Y(T) = \text{Hom}(\mathbb{C}^\times, T)$  s.t.

$$e^\alpha c_\beta(t) = t^{(\alpha, \beta)} \text{ for all } \alpha \in Q.$$



In phic,  $e^{\alpha_i} c_{\omega_j}(k) = k^{\delta_{i,j}}$ .

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$$e^\alpha c_\beta(t) = t^{(\alpha, \beta)} \quad \text{for all } \alpha \in Q, t \in \mathbb{C}^\times$$

Set  $T_\alpha = \text{Ker } e^\alpha$ . Note that if  $\alpha$  is isotropic

then  $\text{Im } c_\alpha \subseteq T_\alpha$  (analog of a line lying on a hyperplane).

Action of  $W_\alpha^\vee$  on  $T$ .  $W$  acts on  $T$  and  $O(T)$  in

the natural way (multiplicative action).

The morphism  $\sigma_{\alpha, t}$  acts by

$$\sigma_{\alpha, t}(x) \stackrel{\sim}{=} c_\alpha(t)x, \quad x \in T_\alpha.$$

About the fixed ring  $O(T)^{W_c}$

Let  $\mathcal{F}(G)$  be the category of finite dim'  $\mathbb{Z}_2$ -graded  $G$ -modules, with parity change functor  $\pi$

Then let  $J(G)$  be the factor ring of the Grothendieck ring  $K(G)$  of  $\mathcal{F}(G)$  by the ideal generated by  $[M] + [\pi M]$ ,  $M \in \mathcal{F}(G)$ .

Thm (Sergeev-Veselov)

$$O(T)^{W_c} \cong J(G).$$

Analog of Chevalley's Thm: for a simple Lie algebra  $\underline{g}$

$$O(T)^W \cong J(\underline{g}) \quad (\text{see } \underline{\text{Fulton-Harris}}).$$

Actually Sergeev-Veselov prove a version for  $J(\underline{g})$ :  
 $J(\underline{g}) \cong \mathbb{Z}[P]^W$ , but  $P$  need not be finitely generated.



Thank You !!

Happy Birthday Paul !!