The Invariant Theory of Artin-Schelter Regular Algebras

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Recent Advances and New Developments in the Interplay between Noncommutative Algebra and Geometry
University of Washington, Seattle

June 20, 2022

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Classical Invariant Theory:
Group G acting linearly on the algebra A = $\Bbbk[x_1, \dots, x_n]$ and study $\Bbbk[x_1, \dots, x_n]^G = \{a \in A : g.a = a$ for all $g \in G\}$.

Noncommutative Invariant Theory:
Replace:

$\Bbbk[x_1, \dots, x_n]$ with appropriate noncommutative algebra A (Noetherian Artin-Schelter Regular Algebra $\Bbbk = \mathbb{C}$)

G with a group (or Hopf algebra) that acts on A (preserving the grading on A)

to extend classical results.

G a finite group, represented as $n \times n$ matrices, acting on a polynomial ring $A = \Bbbk[x_1, \ldots, x_n]$:

$$g \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ on } \Bbbk[x, y]$$

$$\text{as } g.x = ax + cy \quad g.y = bx + dy$$

Extend to a homomorphism

Invariants = $A^G = \{a \in A : g.a = a \text{ for all } g \in G\}$

Plan: Discuss four ideas used in the extension to noncommutative algebras

- Use trace functions to replace eigenvalues for group actions
- Use the representation theory of a Hopf algebra H to construct algebras A on which H acts
- Use the homological determinant to replace the determinant of a matrix
- Use homological regularities to find bounds on the degrees of the generators of the invariants

Actions – must be well-defined on the algebra

The transposition $g : u \leftrightarrow v$ or $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ acts on $A_q[u, v]$ only for $q = \pm 1$:

$$vu = quv$$

Apply g

$$uv = qvu = q^2uv \qquad \Rightarrow q^2 = 1.$$

Example: $G = S_n$ acts on $A = \Bbbk_{-1}[x_1, \ldots, x_n]$
$x_j x_i = -x_i x_j$ for $i \neq j$

Shephard-Todd-Chevalley Theorem

Let $\mathbb{k}$ be a field of characteristic zero.

Theorem (1954). The ring of invariants $\mathbb{k}[x_1, \cdots, x_n]^G$ under a finite group G is a polynomial ring if and only if G is generated by reflections.

A linear map g on V is called a reflection of V if all but one of the eigenvalues of g are 1,
i.e. dim $V^g =$ dim $V - 1$.

Example: Transposition permutation matrices are reflections, and S_n is generated by reflections.

Noncommutative Generalization?

Under what conditions on G is A^G isomorphic to A?

- S. P. Smith (1989) $A_1(k)$ is not the fixed subring S^G under a finite solvable group of automorphisms of a $\mathbb{C}$-domain S.
- Alev and Polo Rigidity Theorem (1995)
  - Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two semisimple Lie algebras. Let G be a finite group of algebra automorphisms of $U(\mathfrak{g})$ such that $U(\mathfrak{g})^G \cong U(\mathfrak{g}')$. Then G is trivial and $\mathfrak{g} \cong \mathfrak{g}'$.
  - If G is a finite group of algebra automorphisms of $A_n(\mathbb{C})$, then the fixed subring $A_n(\mathbb{C})^G$ is isomorphic to $A_n(\mathbb{C})$ only when G is trivial.
- Akaki Tikaradze (2019)
  The Weyl algebras $A_n(\mathbb{C})$, cannot be a fixed point ring of any $\mathbb{C}$-domain under a nontrivial finite group action.

Definition: A group G is a reflection group for an AS-regular algebra A if A^G is AS regular.

Let $A = \Bbbk_{-1}[u, v]$ $(vu = -uv)$:

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}$ (classical reflection)

$A^g = \Bbbk\langle u^n, v\rangle$ is AS regular.

(b) $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (“mystic reflection")

$A^g = \Bbbk[u^2 + v^2, uv]$ is AS regular.

(c) $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A^g = \Bbbk\langle u + v, u^3 + v^3\rangle$ is NOT AS regular.

I. TRACE FUNCTIONS

The trace of a graded automorphism

The invariant theory of AS regular algebras studied by Jing-Zhang (1997) and Jørgensen-Zhang (2000).

- The trace of a graded automorphism g of a graded ring A, with A_n the elements of degree n, is the formal power series:

$$Tr_A(g,t) = \sum_{n=0}^{\infty} \mathrm{trace}(g|A_n)t^n.$$

- The Hilbert series $H_A(t)$ of the algebra A is the trace of the identity map

$$H_A(t) = \sum_{n=0}^{\infty} \dim(A_n)t^n.$$

- Molien's Theorem: $A = \Bbbk[x_1, \ldots, x_n]$

$$H_{A^G}(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{(1 - \lambda_1 t) \cdots (1 - \lambda_n t)}$$

- Generalization of Molien's theorem
  A connected graded Noetherian $\mathbb{C}$-algebra

$$H_{A^G}(t) = \frac{1}{|G|} \sum_{g \in G} Tr_A(g, t).$$

Reflection of an AS regular algebra

Definition: g is a "reflection" if A^g AS regular

$A = \Bbbk[x_1 \ldots, x_n]$: $n-1$ eigenvalues of g are 1

A AS-regular dimension n: The trace function of g acting on A has a pole of order $n-1$ at $t=1$, where

$$Tr_A(g,t) = \sum_{k=0}^{\infty} \text{trace}(g|A_k)t^k$$

$$= \frac{1}{(1-t)^{n-1}q(t)} \text{ for } q(1) \neq 0.$$

Trace functions determine group reflections.

$G =< g >$ on $A = \Bbbk_{-1}[u,v]$ $(vu = -uv)$:

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}, Tr_A(g,t) = \dfrac{1}{(1-t)(1-\epsilon_n t)}$,
$A^g = \Bbbk\langle u^n, v\rangle$ is AS regular (classical reflection).

(b) $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Tr_A(g,t) = \dfrac{1}{(1-t)(1+t)}$,
$A^g = \Bbbk[u^2+v^2, uv]$ is AS regular ("mystic reflection").

(c) $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Tr_A(g,t) = \dfrac{1}{1+t^2}$,
$A^g = \Bbbk\langle u+v, u^3+v^3\rangle$ is NOT AS regular.
g is NOT a reflection.

[K, Kuzmanovich, Zhang (2010)]
G is a reflection group for $A = \Bbbk_{q_{ij}}[x_1, \cdots, x_n]$ iff G is generated by “reflections”.

Generalization of Shephard-Todd-Chevalley Theorem:

Conjecture: A^G is AS regular if and only if G is generated by “reflections".

Hopf algebra actions

H is a Hopf algebra acting on A:

H is an algebra and a coalgebra

H has a coproduct $\Delta : H \to H \otimes H$

$\Delta(h) = \sum h_i \otimes h'_i$

$h(ab) = \sum (h_i a)(h'_i b)$

$H = \Bbbk G$ is a Hopf algebra with $\Delta(g) = g \otimes g$

Hopf Algebra Actions

H is a Hopf algebra acting on A:

- H is semisimple Hopf algebra
- H acts linearly and homogeneously on A
- A is an H-module algebra
- The action of H on A is inner-faithful

$$A^H = \{a \in A \mid h.a = \epsilon(h)a \text{ for all } h \in H\}.$$

Def: H acts inner-faithfully on A if there is no non-zero Hopf ideal I of H with $IA = 0$ ($\Leftrightarrow IA_1 = 0$).

Proposition: (M. Rieffel (1967))
For H semisimple, H acts inner-faithfully on A if and only if every simple H-module appears in the semisimple decomposition of

$$B = A_1 \oplus (A_1 \otimes A_1) \oplus (A_1 \otimes A_1 \otimes A_1) \oplus \cdots$$

as an H-module.

Hopf Algebra Actions

Etingof and Walton (2013): “No quantum symmetries”.
If

- A is a commutative domain
- H is a semisimple Hopf algebra over a field of characteristic zero
- A is an H-module algebra for an inner faithful action of H on A,

then H is a group algebra.

II. USING REPRESENTATIONS OF H TO DEFINE A

Fix a Hopf algebra H and construct an algebra A that H acts on inner-faithfully.

Take A_1 an H-module so that

(1) all simple H-modules occur eventually in $A_1 \otimes A_1 \otimes \cdots \otimes A_1$ (Compute $K_0(H)$)

(2) the ideal of relations in A is an H-module.
For example, if A is AS-regular of dimension 2,

$$A = \frac{\Bbbk\langle u, v\rangle}{\langle r\rangle} \text{ where } r \in A_1 \otimes A_1.$$

Kac- Palyutkin 8-dimensional semisimple Hopf algebra (1966)

H_8 is generated by x, y, z with the following relations:

$$x^2 = y^2 = 1, \quad xy = yx, \quad zx = yz,$$

$$zy = xz, \quad z^2 = \frac{1}{2}(1 + x + y - xy).$$

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y,$$

$$\Delta(z) = \frac{1}{2}(z \otimes z + z \otimes xz + yz \otimes z - yz \otimes xz)$$

$$\epsilon(x) = \epsilon(y) = \epsilon(z) = 1$$

Irreducible H_8 representations:

Unique Two Dimensional S:

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Four One Dimensional:

$$T_{1,1,1}, \quad T_{1,1,-1}, \quad T_{-1,-1,i}, \quad T_{-1,-1,-i}$$

Kac-Palyutkin Hopf algebra H_8

Find A AS regular of dimension 2 where H_8 acts inner faithfully:

$$A = \Bbbk\langle u, v\rangle/(r) \quad r \text{ degree } 2.$$

Let ${}_{H_8}A_1 \cong S$.

$$S \otimes S \cong T_1 \oplus T_2 \oplus T_3 \oplus T_4.$$

Take $\langle r\rangle = T_i$. Then H_8 acts on A and action is inner faithful.

$H = H_8$ on dim 2 AS regular algebras

Invariants of H_8 acting on $A = \mathbb{k}\langle u, v\rangle/(r)$.

Relation r	Fixed Ring A^{H_8}
$u^2 + v^2$	comm hypersurface
$u^2 - v^2$	comm poly $\mathbb{k}[u^2, (uv)^2 - (vu)^2]$
$uv + ivu$	comm poly $\mathbb{k}[u^2 + v^2, u^2v^2]$
$uv - ivu$	comm poly $\mathbb{k}[u^2 + v^2, u^2v^2]$

H_8 is a reflection Hopf algebra for
$\mathbb{k}\langle u, v\rangle/(u^2 - v^2) \cong \mathbb{k}_{-1}[u, v]$, and $\mathbb{k}_{\pm i}[u, v]$.
$\mathbb{k}\langle u, v\rangle/(uv + ivu)$ and $\mathbb{k}\langle u, v\rangle/(uv - ivu)$.

with Ferraro, Moore, Won:

Hopf algebras H that act on a quadratic AS regular algebra of dimension 2 or 3 as a Hopf Reflection Algebra:

- (Masuoka) $\mathcal{A}_{4m}$
- (Masuoka) $\mathcal{B}_{4m}$
- (Pansera) H_{2n^2}
- (Kashina) Some of the 16 dimensional Hopf algebras

III. THE HOMOLOGICAL DETERMINANT

$A = \Bbbk[x_1, \ldots x_n]$:

Watanabe's Theorem (1974): If $G \subseteq \mathrm{SL}_n(\mathbb{C})$ then $\Bbbk[x_1, \ldots, x_n]^G$ is a Gorenstein ring.

Generalization of SL$_n(\mathbb{C})$

Homological Determinant (Jørgensen and Zhang (2000)): Scalar associated to a map in local cohomology (the determinant for polynomial rings).

$\mathrm{hdet} : G \to \Bbbk$ a homomorphism.

$\mathrm{hdet}(g)$ can be computed from Tr$_A(g, t)$

Generalized Watanabe's Theorem: A^G is AS-Gorenstein when all elements of G have $\mathrm{hdet}(g) = 1$.

The Homological Determinant of H-action on A

hdet: $H \to \mathbb{k}$ is a 1-dim representation of H.

When A is m-Koszul then $A = T_m(V)/\langle \mathcal{R} \rangle$ where $\mathcal{R} = \delta^{\ell-m}(\mathrm{w})$, where w is the twisted superpotential for A.

(Mori-S.P. Smith)
The hdet of the H action on A satisfies:

$$h \cdot \mathrm{w} = \mathrm{hdet}(\mathrm{h})(\mathrm{w}).$$

Example : H_8 representation on $V = \Bbbk u \oplus \Bbbk v$:

$$x \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

determines the H_8-action on $A = \Bbbk_i[u, v]$.

$\mathrm{w} = vu - iuv$ and the corresponding 1-dim rep of H acting on w is $T_{-1,-1,i}$ = hdet

Generalization of subgroups of $\mathsf{SL}_n(\mathbb{C})$:

The H action on A is trivial if hdet = ϵ.

Morally: Hopf actions on A with $\mathrm{hdet}(h) = \epsilon(h)$ should behave like actions of subgroups of $\mathsf{SL}_n(\mathbb{C})$ on $\Bbbk[x_1, \ldots, x_n]$.

Auslander's Theorem

Let G be a finite subgroup of $\mathrm{SL}_n(\Bbbk)$, and let $A = \Bbbk[x_1, \ldots, x_n]$. Then the skew-group ring $A \# G$ is isomorphic to $\mathrm{End}_{A^G}(A)$ as algebras ($(a \# g)(x) = ag(x)$).

arXiv:2205.07291

Ruipeng Zhu: ”Auslander Theorem for Artin-Schelter Regular PI Algebras"

For actions by $\Bbbk G$ or $\Bbbk G^\circ$ with trivial homological determinant

IV. HOMOLOGICAL REGULARIES AND BOUNDS ON THE DEGREES OF GENERATORS

Bounds on the Degrees of Generators of A^G

$\beta(A^G) =$ maximal degree of a set of minimal generators of A^G

Examples: (A commutative)

(a) $A = \Bbbk[x]$, $G = (\epsilon_n)$, $A^G = \Bbbk[x^n]$.

$$\beta(A^G) = n.$$

(b) $A = \Bbbk[x_1, \ldots, x_n]$ and $G = S_n$,
$A^G = \Bbbk[\sigma_1, \ldots, \sigma_n]$.

$$\beta(A^G) = n.$$

Theorem (E. Noether (1916))
Let $A = \mathbb{k}[x_1, \ldots, x_n]$. Then $\beta(A^G) \leq |G|$ if $\mathbb{k}$ has characteristic zero or $|G| <$ characteristic $\mathbb{k}$.

Göbel's bound – involves the dimension of the representation

Theorem (Göbel (1995)) In any characteristic, G a group of permutations of x_i:

$$\beta(\mathbb{k}[x_1, x_2, \ldots, x_n]^G) \leq \max\{n, \binom{n}{2}\}.$$

Example: Bound is sharp. $\mathbb{k}[x_1, x_2, \ldots, x_n]^{A_n}$ is generated by $\sigma_1, \ldots, \sigma_n, \nabla$, where

$$\nabla = \prod_{i<j}(x_i - x_j).$$

Modular case: $\beta(A^G)$ depends upon dimension of representation

The Noether bound can fail: e.g. in char 2 $|G| = 2$ acting on polynomials in 6 variables with $\beta(A^G) = 3 > |G| = 2$.

Theorem (Symonds (2011))
If G is a finite group of order $|G| > 1$ acting linearly on $A := \mathbb{k}[x_1, \ldots, x_n]$ with $n \geq 2$ then $\beta(A^G) \leq n(|G| - 1)$.

Noncommutative Case is Different

Noether Bound can fail for A noncommutative:

$A = \mathbb{k}_{-1}[x, y]$ and $g : x \leftrightarrow y$.

$A^G = \mathbb{k}\langle x + y, x^3 + y^3 \rangle$ so $\beta(A^G) = 3$.

$\beta(A^G) = 3 > |G| = 2$.

In fact $\beta(A^G) - |G|$ can be arbitrarily large.

Difference $\beta(A^G) - |G|$ is arbitrarily large

Ferraro, K, Moore, Peng (2021):

$A = \mathbb{k}_{-1}[x, y]$ and $G = \langle g \rangle$

$$g = \begin{pmatrix} 0 & \epsilon_n \\ 1 & 0 \end{pmatrix}$$

$$g^2 = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n \end{pmatrix} \text{ so } |G| = 2n.$$

$$n \text{ odd: } \beta(A^G) = 3n$$

Noether's bound fails (for 2 dim rep)

What happens for $A = \Bbbk_{-1}[x_1, \ldots, x_m]$?

Theorem (K, Won, Zhang (2021))

Let G be a finite group acting as graded automorphisms on $\Bbbk_{-1}[x_1, \ldots, x_m]$ and suppose that $|G|$ is invertible in $\Bbbk$. Then

$$\beta(\Bbbk_{-1}[x_1, \ldots, x_m]^G) \leq 2|G| + m.$$

Theorem (K, Kuzmanovich, Zhang (2014)):

$A = \Bbbk_{-1}[x_1, \ldots, x_n]$

- $\beta(A^{S_n}) = 2n - 1$
- $\beta(A^{A_n}) = 2n - 3$
- $\beta(A^G) \leq 3n^2/4$ for G permutations.
- $\beta(A^G) \leq n^2$ for G a subgroup of $S_n \rtimes \{\pm 1\}$

Let $\mathbb{k}$ be a field, A be a connected $\mathbb{N}$-graded $\mathbb{k}$-algebra, and X be a complex of graded left A-modules.

Gradings on X:

- Internal degree
- Homological degree

Properties of A reflected in the relationship between these degrees.

A is Koszul $\Leftrightarrow$ $\mathbb{k}$ has a minimal free graded resolution of the form

$$\cdots \to A(-i)^{\beta_i} \to A(-i+1)^{\beta_{i-1}} \to \cdots \to A(-1)^{\beta_1} \to A \to \mathbb{k} \to 0,$$

Homological degree = Internal degree

A is Koszul $\text{Tor}_i^A(\mathbb{k}, \mathbb{k})_j = 0$ for all $j \neq i$.

Measure of growth of degrees of generators in a resolution

A a connected graded algebra.
M a graded A-module.

The *Tor-regularity* of M is $\mathrm{Torreg}(M) :=$

$$\sup\{j - i \mid \text{Tor}_i^A(\Bbbk, M)_j \neq 0 \text{ for } i, j \in \mathbb{Z}\}.$$

A is Koszul if and only if $\mathrm{Torreg}(\Bbbk) = 0$

Jørgensen and Dong-Wu studied the case
A noncommutative

Let M be a nonzero graded left A-module.
The *Castelnuovo–Mumford regularity* of M is

$$\begin{aligned}\mathrm{CMreg}(M) &:= \inf\{p \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(M)_{>p-i} = 0, \forall\, i \in \mathbb{Z}\} \\ &= \sup\{i + \deg(H^i_{\mathfrak{m}}(M)) \mid i \in \mathbb{Z}\}.\end{aligned}$$

$$H^i_{\mathfrak{m}}(M) := \lim_{n \to \infty} \mathrm{Ext}^i_A(A/A_{\geq n}, M).$$

Noncommutative case studied by Jørgensen, Dong-Wu.

(Jørgensen, Dong-Wu) A is a Koszul AS regular algebra if and only if

$$\operatorname{Torreg}(M) = \operatorname{CMreg}(M) \text{ for all } M.$$

Generalization of a result of Symonds
(K, Won, Zhang (2021)):

An graded algebra map $\phi : T \to F$ is called *finite* if ${}_T F$ and F_T are finitely generated.

Suppose there is a finite map $S \to A^H$, where S is a noetherian AS regular algebra.
Let $\delta(A/S) = \mathrm{CMreg}(A) - \mathrm{CMreg}(S)$.
Then $\beta(A^H) \leq \max\{\beta(S), \delta(A/S)\}$

Concavity

$$\Phi(F) := \{T \mid \text{there is a finite map } \phi : T \to F\}$$

where T ranges over connected graded Noetherian AS regular algebras.

The *concavity* of A is defined to

$$c(A) := \inf_{T \in \Phi(A)} \{-\operatorname{CMreg}(T)\} \geq 0.$$

If A is a noetherian AS regular algebra, then $c(A) = 0$ if and only if A is a Koszul.

An Application

Let H be a semisimple Hopf algebra acting on a noetherian AS regular algebra T homogeneously. Let $R = T^H$ denote the invariant subring. Then

$$c(R) \geq \beta(R) - 1$$

where $\beta(R)$ is the maximal degree of a minimal set of generators.

Let $R = \Bbbk[x_1, \cdots, x_n][t]/(t^2 = f(x_1, \cdots, x_n))$ where $\deg x_i = 1$, $\deg t \geq 2$, and f an irreducible homogeneous polynomial in x_i of degree $(2 \deg t)$.

$$0 = c(R) < 1 \leq \deg t - 1 = \beta(R) - 1.$$

Therefore R cannot be isomorphic to T^H.

Thanks for your mathematics

and

Happy Birthday Paul!