

Maps to Noncommutative Projective Spaces (w/ Daniel Chan)

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Conventions

- always work over a field k
- unless otherwise stated, work with right modules
- always let C denote a k -linear Hom-finite abelian category.

Part 1

Maps to Projective Spaces

Maps from line bundles

Suppose

- X is a projective variety
- $\mathcal{L}$ is line-bundle on X gen. by $n + 1$ global sections.

Given $(X, \mathcal{L})$, $\exists$ morphism $f : X \rightarrow \mathbb{P}^n$.

Stein factorization of f

f factors as

$$X \xrightarrow{g} \operatorname{Proj} \Gamma_*(X, \mathcal{L}) \xrightarrow{h} \operatorname{Proj} \mathbb{S}(\Gamma(X, \mathcal{L})) = \mathbb{P}^n$$

where g proper, h finite.

Goal

Generalize above construction to produce maps from nc elliptic curves to nc projective spaces.

Examples

Artin-Zhang (1994) and Polishchuk (2005) study nc generalizations of g .

Elliptic curves in noncommutative projective planes

X a smooth elliptic curve. Artin, Tate and Van den Bergh construct closed immersions $f : X \rightarrow \mathbb{P}_{nc}^2$.

Theorem (S.P. Smith (2003))

If A is a loc. finite noetherian $\mathbb{N}$ -graded algebra and J is a graded ideal, then $A \rightarrow A/J$ induces closed immersion of noncommutative spaces

$$\mathrm{Proj} A/J \rightarrow \mathrm{Proj} A.$$

Double covers of $\mathbb{P}^1$

X a smooth elliptic curve. $\mathcal{L} = \text{deg. } 2$ line bundle over X . $\mathcal{L}$ induces double cover $X \rightarrow \mathbb{P}^1$. No (very) nc analogue.

Data associated to $(X, \mathcal{L})$

Given $(X, \mathcal{L})$, we can construct:

- a canonical finite map $\mathbb{S}(\Gamma(X, \mathcal{L})) \rightarrow \Gamma_*(X, \mathcal{L})$,
- an induced finite morphism $\text{Proj } \Gamma_*(X, \mathcal{L}) \xrightarrow{h} \text{Proj } \mathbb{S}(\Gamma(X, \mathcal{L}))$, and
- pullback of Koszul complex over $\mathbb{P}^n$ to X

$$0 \rightarrow \bigwedge^{n+1} V \otimes \mathcal{L}^{-n-1} \rightarrow \cdots \rightarrow \bigwedge^1 V \otimes \mathcal{L}^{-1} \rightarrow \mathcal{O}_X \rightarrow 0$$

is **exact**, where $V = \Gamma(X, \mathcal{L})$.

Koszul Complex

Let $V = \Gamma(\mathbb{P}^n, \mathcal{O}(1))$. $\exists$ exact sequence

$$0 \rightarrow \bigwedge^{n+1} V \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1) \rightarrow \cdots \rightarrow \bigwedge^1 V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow 0$$

An Example

D, E Noncommutative Spaces

$D \xrightarrow{f} E$ denotes adjoint pair (f^*, f_*) in the diagram

$$D \begin{matrix} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{matrix} E$$

Motivation

If $f : Y \rightarrow X$ is a morphism of commutative schemes, (f^*, f_*) is an adjoint pair.

Notion is too general.

Define $\mathrm{Qcoh} \mathbb{P}^0 \xrightleftharpoons{f_*} \mathrm{Qcoh} \mathbb{P}^1$ by $f^* = H^1(\mathbb{P}^1, -)$. Then f^* is not exact on ses of vector-bundles so can't come from a map of schemes.

Part 2

Maps to Noncommutative Projective Spaces

Replacement for $\mathcal{L}$ (Polishchuk (2005))

Let X be a variety, $\mathcal{L}$ a line bundle on X . Recall

$$\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

depends on monoidal structure on $\mathrm{Coh} X$.

Categories natural in nc algebraic geometry (e.g. $\mathrm{Mod} R$) may not have a monoidal structure.

Artin-Zhang (1994)

Given $\mathcal{A} \in \mathrm{ob} C$, consider $s^i(\mathcal{A})$ where s is autoequivalence of C .

Bondal-Polishchuk (1993), Polishchuk (2005)

Let $\underline{\mathcal{L}} := (\mathcal{L}_i)_{i \in \mathbb{Z}}$ where $\mathcal{L}_i \in \mathrm{Ob} C$

Question

How do you form a ring from a sequence $(\mathcal{L}_i)_{i \in \mathbb{Z}}$ of objects of C ?

$\mathbb{Z}$ -algebras (Bondal and Polishchuk (1993))

A $\mathbb{Z}$ -algebra is ring A with vector space decomposition $\bigoplus_{i,j \in \mathbb{Z}} A_{ij}$ such that

- $A_{ij}A_{jk} \subset A_{ik}$,
- $A_{ij}A_{kl} = 0$ for $k \neq j$, and
- A_{ii} contains a unit e_i so that $e_i A = \bigoplus_j A_{ij}$.

Periodicity (Sierra (2011))

Periodic $\mathbb{Z}$ -algebras generalize $\mathbb{Z}$ -graded algebras

Let A be a $\mathbb{Z}$ -algebra. Let $A(\ell)$ be the $\mathbb{Z}$ -algebra with

$$A(\ell)_{ij} := A_{i+\ell, j+\ell}$$

A is ℓ -periodic if $A \cong A(\ell)$ as algebras.

Observation (Sierra (2011))

If A is a 1-periodic $\mathbb{Z}$ -algebra, then A is Morita equivalent to a $\mathbb{Z}$ -graded algebra.

Replacement for $\Gamma_*(X, \mathcal{L})$ (Polishchuk (2005))

Let $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$. Let $(B_{\underline{\mathcal{L}}})_{ij} := \text{Hom}_{\mathbb{C}}(\mathcal{L}_{-j}, \mathcal{L}_{-i})$. Then $B_{\underline{\mathcal{L}}}$ with mult. induced by composition, is a $\mathbb{Z}$ -algebra.

The $\mathbb{Z}$ -algebra $B_{\underline{\mathcal{L}}}$ plays the role of $\Gamma_*(X, \mathcal{L})$

Motivation

Let $\mathcal{L}_i := \mathcal{L}^{\otimes i}$. Then $B_{\underline{\mathcal{L}}}$ is 1-periodic and

$$\text{Gr} B_{\underline{\mathcal{L}}} \equiv \text{Gr} \Gamma_*(X, \mathcal{L}).$$

Replacement for $\mathbb{S}(\Gamma(X, \mathcal{L}))$

The noncommutative symmetric algebra of $\underline{\mathcal{L}}$

We define $A_{\underline{\mathcal{L}}}$ to be quadratic part of $B_{\underline{\mathcal{L}}}$.

By construction, there is a morphism of $\mathbb{Z}$ -algebras

$$A_{\underline{\mathcal{L}}} \rightarrow B_{\underline{\mathcal{L}}}.$$

analogous to

$$\mathbb{S}(\Gamma(X, \mathcal{L})) \longrightarrow \Gamma_*(X, \mathcal{L})$$

Relationship to Van den Bergh's $\mathbb{S}^{nc}(V)$

Necessary and sufficient conditions on $\underline{\mathcal{L}}$ are known (N (2019)) to ensure

$$A_{\underline{\mathcal{L}}} \cong \mathbb{S}^{nc}(V).$$

Quadratic Duals (Bondal-Polishchuk (1993))

Definition of $A^!$

- A = locally finite, quadratic $\mathbb{Z}$ -algebra with relns I .
- Define $A^!$ = quadratic $\mathbb{Z}$ -algebra with gens

$$A_{i+1,i}^! := A_{i,i+1}^*$$

with relations the kernel of

$$A_{i+2,i+1}^! \otimes A_{i+1,i}^! \cong (A_{i,i+1} \otimes A_{i+1,i+2})^* \rightarrow I_{i,i+2}^*$$

induced by inclusion $I_{i,i+2} \rightarrow A_{i,i+1} \otimes A_{i+1,i+2}$.

Motivating Example

In $\mathbb{Z}$ -graded case, we have $\mathbb{S}(V)^! = \bigwedge(V^*)$

The Koszul complex of $\underline{\mathcal{L}}$

$\underline{\mathcal{L}}$ = sequence of objects in $\mathcal{C}$ with $\text{End}(\mathcal{L}_i) = k$ for all i , $A := A_{\underline{\mathcal{L}}}$.
There is a complex of form

$$\cdots \rightarrow A_{j+2,j}^{!*} \otimes \mathcal{L}_{-j-2} \rightarrow A_{j+1,j}^{!*} \otimes \mathcal{L}_{-j-1} \rightarrow A_{j,j}^{!*} \otimes \mathcal{L}_{-j} \rightarrow 0$$

Evaluation is map $\text{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \cong \bigoplus \mathcal{E} \xrightarrow{(f_1, \dots, f_n)} \mathcal{F}$.

Sample map $A_{2,0}^{!*} \otimes \mathcal{L}_{-2} \longrightarrow A_{1,0}^{!*} \otimes \mathcal{L}_{-1}$

$$\begin{aligned} A_{2,0}^{!*} \otimes \mathcal{L}_{-2} &\longrightarrow A_{0,1} \otimes A_{1,2} \otimes \mathcal{L}_{-2} \\ &\xRightarrow{=} A_{0,1} \otimes \text{Hom}(\mathcal{L}_{-2}, \mathcal{L}_{-1}) \otimes \mathcal{L}_{-2} \\ &\xrightarrow{\text{eval}} A_{0,1} \otimes \mathcal{L}_{-1} \\ &\xrightarrow{\cong} A_{1,0}^{!*} \otimes \mathcal{L}_{-1} \end{aligned}$$

Definition of Helix (Chan-N (2022))

A sequence $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ of objects in $\mathcal{C}$ is a **helix of length n** if for all i, j ,

- there exists an $m \geq 0$ such that for all $l \geq m$, $\text{Ext}^j(\mathcal{L}_i, \mathcal{L}_{i+l}) = 0$ for all $j > 0$ (Serre vanishing).
- $\text{End}(\mathcal{L}_i) = k$ (i.e. $\mathcal{L}_i$ is "simple"), and
- there are f.d. vector spaces $V_{j+3,j}, \dots, V_{j+n,j}$ and exact sequences whose right three terms are the Koszul complex

$$0 \longrightarrow V_{j+n,j} \otimes \mathcal{L}_{-j-n} \longrightarrow \cdots \longrightarrow V_{j+3,j} \otimes \mathcal{L}_{-j-3} \longrightarrow$$

$$A_{j+2,j}^{!*} \otimes \mathcal{L}_{-j-2} \xrightarrow{\phi_2} A_{j+1,j}^{!*} \otimes \mathcal{L}_{-j-1} \xrightarrow{\phi_1} A_{j,j}^{!*} \otimes \mathcal{L}_{-j} \longrightarrow 0$$

where $A = A_{\underline{\mathcal{L}}}$.

The map of noncommutative spaces induced by a helix

Let $(B_{\underline{\mathcal{L}}})_{\geq 0} =: B$.

Theorem (Chan-N (2022))

If $\underline{\mathcal{L}}$ is a helix of length n , then

- 1 the canonical map

$$\psi : A_{\underline{\mathcal{L}}} \rightarrow B$$

makes Be_j a finitely generated $A_{\underline{\mathcal{L}}}$ -module for all j , and

- 2 the map ψ descends to an adjoint pair

$$\mathrm{Proj} B \rightleftarrows \mathrm{Proj} A_{\underline{\mathcal{L}}}.$$

Recall: $\mathrm{Tors} B$ = full subcategory of objects in $\mathrm{Gr} B$ whose elements generate right-bounded modules.

$$\mathrm{Proj} B := \mathrm{Gr} B / \mathrm{Tors} B.$$

Part 3

Interlude: Noncommutative Elliptic Curves

Conventions for remainder of talk

- $k = \mathbb{C}$
- X is smooth elliptic curve (over k)
- $\text{Coh } X$ is category of coherent sheaves over X

Classification of vector bundles over X (Atiyah (1957))

Let $E(r, d)$ = set of iso. classes of indecomposable vector bundles of rank r and degree d .

Theorem (Atiyah (1957))

For each $r \geq 1$ and each $d \in \mathbb{Z}$, $E(r, d)$ is parameterized by the points of X .

- A bundle $\mathcal{E}$ in $\text{Coh}\mathbb{P}^1$ is simple if and only if $\mathcal{E}$ is a line bundle.
- A bundle $\mathcal{E}$ in $E(r, d)$ is simple if and only if $\gcd(r, d) = 1$.

We will construct helices (of length 2 and 3) whose terms are simple vector bundles over X (not nec. line bundles)

Let A be coherent connected $\mathbb{Z}$ -algebra and let

- $\text{coh}A = \text{cat. of (graded right) coherent modules}$
- $\text{tors}A = \text{full subcat. of right-bounded modules.}$

Definition (Polishchuk (2005))

$$\text{cohproj}A := \text{coh}A / \text{tors}A$$

Remark

If A is noetherian, $\text{cohproj}A \equiv \text{proj}A$.

Noncommutative elliptic curves (Polishchuk (2002))

Theorem (Polishchuk (2002))

For each $\theta \in \mathbb{R}$, $\exists$ t -structure on $D^b(X)$ w/heart C^θ such that

- $D^b(C^\theta) \equiv D^b(X)$,
- C^θ has cohomological dimension 1, and
- if θ is irrational, then every nonzero object in C^θ is nonnoetherian.

Theorem (Polishshcuk (2002))

If $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ is a sequence of simple bundles such that $\mu(\mathcal{L}_m) > \theta$ for all m and $\lim_{m \rightarrow -\infty} \mu(\mathcal{L}_m) = \theta$, then

$$C^\theta \equiv \text{cohproj} B_{\underline{\mathcal{L}}}.$$

Part 4

First Application: Maps to $\mathbb{P}_d^1$

Theorem (Zhang (1998))

If A is connected, gen. in degree 1 and regular of dim 2 then

$$A \cong k\langle x_1, \dots, x_n \rangle / \langle b \rangle$$

where $n \geq 2$, $b = \sum_{i=1}^n x_i \sigma(x_{n-i+1})$ and $\sigma \in \text{Aut } k\langle x_1, \dots, x_n \rangle$. If $n > 2$, A is non-noetherian.

Theorem (Piontkovski (2008))

$n > 2$ implies A is coherent. If $\mathbb{P}_n^1 := \text{cohproj } A$, then $\mathbb{P}_n^1$ depends only on n . Furthermore, $\mathbb{P}_2^1 \equiv \text{Coh } \mathbb{P}^1$.

Example: Maps from Elliptic Curves to Projective Lines

Double cover of $\mathbb{P}^1$

$\mathcal{L}$ = degree 2 line bundle over X .

- $\mathcal{L}$ induces double cover

$$X \cong \operatorname{Proj} \Gamma_*(X, \mathcal{L}) \xrightarrow{h} \operatorname{Proj} \mathbb{S}(\Gamma(X, \mathcal{L})) \cong \mathbb{P}^1$$

ramified at 4 points.

- Pullback of Koszul complex over $\mathbb{P}^1$ takes form

$$0 \longrightarrow \mathcal{L}^{-2} \longrightarrow \operatorname{Hom}(\mathcal{O}_X, \mathcal{L}) \otimes \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Goal

Look for interesting helices over $C = \operatorname{Coh} X$ with the same "shape" as this example.

Kuleshov's Lemma (Kuleshov (1992))

Definition (Kuleshov (1992))

$(\mathcal{E}, \mathcal{F})$ is *simple pair* if $\mathcal{E}$ and $\mathcal{F}$ are simple and exactly one of $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$, $\mathrm{Ext}^1(\mathcal{E}, \mathcal{F})$ is nonzero.

Lemma (Kuleshov (1992))

Let $\mathcal{E}_1$ be a simple bundle. If

$$0 \longrightarrow \mathcal{L} \longrightarrow V \otimes \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

is exact, then TFAE:

- 1 $(\mathcal{L}, \mathcal{E}_1)$ is a simple pair and $V \cong \mathrm{Hom}(\mathcal{L}, \mathcal{E}_1)^*$.
- 2 $(\mathcal{E}_1, \mathcal{E}_2)$ is a simple pair and $V \cong \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$.

Our idea

Use Lemma to construct a helix starting from two simple bundles.
Will need $\mathcal{L} \xrightarrow{\mathrm{coev}} \mathrm{Hom}(\mathcal{L}, \mathcal{E})^* \otimes \mathcal{E}$ to be injective.

Modification of Kuleshov's Lemma

Injective pairs

A simple pair $(\mathcal{E}, \mathcal{F})$ of bundles is an *injective pair* if $\mathrm{Ext}^1(\mathcal{E}, \mathcal{F}) = 0$ and every nonzero map $\mathcal{E} \rightarrow \mathcal{F}$ is injective.

Lemma (Chan-N (2021))

Let $(\mathcal{L}_0, \mathcal{L}_1)$ be an injective pair of bundles such that $d := \dim \mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) > 1$. Then the ses

$$0 \rightarrow \mathcal{L}_0 \xrightarrow{\mathrm{coev}} \mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1)^* \otimes \mathcal{L}_1 \rightarrow \mathcal{L}_2 \rightarrow 0$$

defines an injective pair of bundles $(\mathcal{L}_1, \mathcal{L}_2)$.

Construction of $\underline{\mathcal{L}}$

Start with $\mathcal{L}_0 \in E(1, 0)$, $\mathcal{L}_1 \in E(1, d)$. Lemma gives $(\mathcal{L}_i)_{i \geq 0}$. Do the same starting with the injective pair $(\mathcal{L}_1^*, \mathcal{L}_0^*)$ and use duality to get $(\mathcal{L}_i)_{i < 0}$.

Theorem (Chan-N. (2021))

Let $d > 2$, let $\mathcal{L}_0 \in E(1, 0)$ and let $\mathcal{L}_1 \in E(1, d)$. Then

- ❶ the pair $(\mathcal{L}_0, \mathcal{L}_1)$ extends to a unique helix $\underline{\mathcal{L}}_d$ on $\text{Coh} X$
- ❷ $\text{cohproj} B_{\underline{\mathcal{L}}_d} \equiv C^{\theta_d}$, where

$$\theta_d = -\frac{2d}{d-2+\sqrt{d^2-4}},$$

- ❸ $\text{cohproj} A_{\underline{\mathcal{L}}_d} \equiv \mathbb{P}_d^1$, and
- ❹ the map from Part 2

$$\text{Proj} B_{\underline{\mathcal{L}}_d} \xrightarrow{\sim} \text{Proj} A_{\underline{\mathcal{L}}_d}$$

is a double cover.

Part 5

Second Application: Noncommutative Nonnoetherian $\mathbb{P}^2$'s

Example: Maps from Elliptic Curves to Projective Planes

Embedding of X in $\mathbb{P}^2$

$\mathcal{L}$ = degree 3 line bundle over X , let $V = \mathrm{Hom}(\mathcal{O}_X, \mathcal{L})$.

- $\mathcal{L}$ induces closed immersion

$$X \cong \mathrm{Proj} \Gamma_*(X, \mathcal{L}) \xrightarrow{h} \mathrm{Proj} \mathbb{S}(\Gamma(X, \mathcal{L})) \cong \mathbb{P}^2.$$

- Since $\bigwedge^2 V \cong V^*$, pullback of Koszul complex over $\mathbb{P}^2$ takes form

$$0 \longrightarrow \mathcal{L}^{-3} \longrightarrow V^* \otimes \mathcal{L}^{-2} \longrightarrow V \otimes \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Goal

Look for interesting helices over $C = \mathrm{Coh} X$ with the same "shape" as this example.

Helix construction: main idea

Find sequence $\underline{\mathcal{L}}$ of objects in $\text{Coh}X$ with exact sequences like the Koszul complex

$$0 \rightarrow \mathcal{L}_{i-3} \rightarrow V \otimes \mathcal{L}_{i-2} \rightarrow W \otimes \mathcal{L}_{i-1} \rightarrow \mathcal{L}_i \rightarrow 0$$

Start with *three* simple bundles $(\mathcal{L}_0, \mathcal{L}'_1, \mathcal{L}_1)$. Construct a *new* triple $(\mathcal{L}_1, \mathcal{L}'_2, \mathcal{L}_2)$ as follows:

$$0 \rightarrow \mathcal{L}_0 \rightarrow \text{Hom}(\mathcal{L}_0, \mathcal{L}_1)^* \otimes \mathcal{L}_1 \rightarrow \text{cok} =: \mathcal{L}'_2 \rightarrow 0$$

and

$$0 \rightarrow \mathcal{L}'_1 \rightarrow \text{Hom}(\mathcal{L}'_1, \mathcal{L}_1)^* \otimes \mathcal{L}_1 \rightarrow \text{cok} =: \mathcal{L}_2 \rightarrow 0$$

Would like:

- above sequences to be exact,
- $\mathcal{L}_2, \mathcal{L}'_2$ simple bundles,
- $\mathcal{L}_1 \rightarrow \text{Hom}(\mathcal{L}_1, \mathcal{L}_2)^* \otimes \mathcal{L}_2$ and $\mathcal{L}'_2 \rightarrow \text{Hom}(\mathcal{L}'_2, \mathcal{L}_2)^* \otimes \mathcal{L}_2$ to be injections, etc.

Helix construction: main idea (cont.)

If we can continue to the right, get exact sequences

$$0 \rightarrow \mathcal{L}_{i-3} \rightarrow \mathrm{Hom}(\mathcal{L}_{i-3}, \mathcal{L}_{i-2})^* \otimes \mathcal{L}_{i-2} \rightarrow \mathcal{L}'_{i-1} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{L}'_{i-1} \rightarrow \mathrm{Hom}(\mathcal{L}'_{i-1}, \mathcal{L}_{i-1})^* \otimes \mathcal{L}_{i-1} \rightarrow \mathcal{L}_i \rightarrow 0$$

which fit together to give

$$0 \rightarrow \mathcal{L}_{i-3} \rightarrow V \otimes \mathcal{L}_{i-2} \rightarrow W \otimes \mathcal{L}_{i-1} \rightarrow \mathcal{L}_i \rightarrow 0.$$

where

- $V = \mathrm{Hom}(\mathcal{L}_{i-3}, \mathcal{L}_{i-2})^*$,
- $W = \mathrm{Hom}(\mathcal{L}'_{i-1}, \mathcal{L}_{i-1})^* \cong \mathrm{Hom}(\mathcal{L}_{i-1}, \mathcal{L}_i)$.

Definition of $\underline{\mathcal{L}}$

Theorem (Chan-N (2022))

Let $d \geq 3$ be an odd integer. Let $\mathcal{L}_0 \in E(1, 0)$, $\mathcal{L}'_1 \in E(d, 2)$ and let $\mathcal{L}_1 \in E(1, d)$. Then

- 1 the triple $(\mathcal{L}_0, \mathcal{L}'_1, \mathcal{L}_1)$ generates a helix

$$\underline{\mathcal{L}}_d = (\mathcal{L}_i)_{i \in \mathbb{Z}}$$

- 2 the Koszul complex of $\underline{\mathcal{L}}_d$ is exact of length 3, and
- 3 helices $\underline{\mathcal{L}}_3 =$ Bondal-Polishchuk's *elliptic helices of period 3* over X .

Part 3 $\Rightarrow$ we recover all three-dimensional elliptic Artin-Schelter regular algebras over X when $d = 3$.

Theorem (Chan-N (2022))

Let $d > 3$ be an odd integer, and let $\underline{\mathcal{L}}_d$ denote a sequence of the form above. Then

- ❶ $A_{\underline{\mathcal{L}}_d}$ is 3-periodic, Koszul, has global dimension three, and is Gorenstein (with Gorenstein parameter three),
- ❷ $\dim_k(A_{\underline{\mathcal{L}}_d})_{i,i+1} = d$ for all i ,
- ❸ $A_{\underline{\mathcal{L}}_d}$ and $B_{\underline{\mathcal{L}}_d}$ are nonnoetherian,
- ❹ the canonical map $\phi : A_{\underline{\mathcal{L}}_d} \longrightarrow B_{\underline{\mathcal{L}}_d}$ is surjective,
- ❺ ϕ induces $\text{Proj} B_{\underline{\mathcal{L}}_d} \xrightarrow{\sim} \text{Proj} A_{\underline{\mathcal{L}}_d}$,
- ❻ $\text{Proj} B_{\underline{\mathcal{L}}_d} \equiv C^{\tau_d}$, i.e. $\text{Proj} B_{\underline{\mathcal{L}}_d}$ is a noncommutative elliptic curve.

Theorem (Chan-N (2022))

If $\mathcal{E}$ and $\mathcal{F}$ are simple bundles such that

- $\mu(\mathcal{E}) < \mu(\mathcal{F})$ and
- $\text{rank } \mathcal{F} < \text{rank } \mathcal{E} \cdot \dim \text{Hom}(\mathcal{E}, \mathcal{F})$,

then evaluation

$$\text{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F}$$

is surjective, and coevaluation

$$\mathcal{F}^* \rightarrow \text{Hom}(\mathcal{F}^*, \mathcal{E}^*)^* \otimes \mathcal{E}^*$$

is injective.

Thank You!

A bundle in $E(1, 1)$ induces

$$\Phi : E(\gcd(r, d), 0) \xrightarrow{\cong} E(r, d).$$

Using Φ , there exists a distinguished bundle $\mathcal{E}_{r,d} \in E(r, d)$ and every bundle in $E(r, d)$ is

$$\mathcal{L} \otimes \mathcal{E}_{r,d}$$

where $\mathcal{L} \in E(1, 0)$.

Classification

Let $d > 3$ odd. Let $\mathcal{A}_0 \otimes \mathcal{E}_{1,0} \in E(1,0)$, $\mathcal{A}'_1 \otimes \mathcal{E}_{d,2} \in E(d,2)$, $\mathcal{A}_1 \otimes \mathcal{E}_{1,d} \in E(1,d)$.

Question

What are the isomorphism classes of algebras of the form $\mathbb{S}^{nc}(\underline{\mathcal{L}}_d)$?

Rigidify triple by tensoring $\underline{\mathcal{L}}_d$ by $\mathcal{A}_0^*$. Thus, $\mathbb{S}^{nc}(\underline{\mathcal{L}}_d)$ determined by two degree zero line bundles.

Conjecture

The noncommutative symmetric algebra corresponding to $(\mathcal{C}_1, \mathcal{C}_2)$ is isomorphic to that corresponding to $(\mathcal{D}_1, \mathcal{D}_2)$ if and only if $\exists$ an automorphism σ of C^{τ_d} such that $\sigma(\mathcal{C}_i) = \mathcal{D}_i$.