Parabolic Adjoint Action, Weierstrass Sections and Components of the Nilfibre in Type A

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This talk is based on a work done with Pr. Joseph.

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Richardson's Theorem

Let G be a simple algebraic group, P a parabolic subgroup and $\mathfrak{m}$ the nilradical of its Lie algebra $\mathfrak{p}$.

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A Richardson's theorem implies that $\mathbb{C}[\mathfrak{m}]^{P'}$ is polynomial, its generators are irreducible polynomials .

Let $B \subset P$ be a Borel subgroup of G and $\mathfrak{n}$ the nilradical of its Lie algebra $\mathfrak{b}$.

Orbital Varieties

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Orbital Varieties

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A coadjoint G orbit $\mathscr{O}$ is called nilpotent if $\mathscr{O} \cap \mathfrak{n}$ is not empty,

Its components are called orbital varieties.

Every component is Lagrangian of the form $\overline{B.\mathfrak{n} \cap w(\mathfrak{n})}$.

3. Hypersurface Orbital Varieties

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Thus the generators of $\mathbb{C}[\mathfrak{m}]^{P'}$ exactly correspond to hypersurface orbital varieties.

Terminology

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a collection of columns of heights matching the

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The tableau $\mathscr{T}$ associated to a parabolic P is of shape $\mathscr{D}$ with entries increasing down rows and along columns.

Tableaux

Example: Let a parabolic P of Sl_8 define by $(2, 3, 1, 2)$

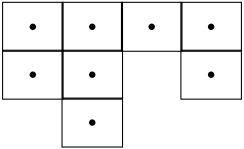

Diagram $\mathscr{D}$ $(2, 3, 1, 2)$.

Example: Let a parabolic P of Sl_8 define by $(2, 3, 1, 2)$

1	3	6	7
2	4		8
	5		

Tableau $\mathscr{T}$ $(2, 3, 1, 2)$.

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Tableau $\mathscr{T}$ $(2, 3, 1, 2)$.

Two columns (resp. blocks) of height s are called neighboring if there are no columns (resp. blocks) of height s between them.

Terminologies example

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Tableau $\mathscr{T}$ $(2, 3, 1, 2)$.

Terminologies example

$$
\begin{pmatrix}
1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} \\
0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} \\
0 & 0 & 1 & 0 & 0 & x_{3,6} & x_{3,7} & x_{3,8} \\
0 & 0 & 0 & 1 & 0 & x_{4,6} & x_{4,7} & x_{4,8} \\
0 & 0 & 0 & 0 & 1 & x_{5,6} & x_{5,7} & x_{5,8} \\
0 & 0 & 0 & 0 & 0 & 1 & x_{6,7} & x_{6,8} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

neighboring blocs of height 2.

In type A.

Let M a $n \times n$ matrix, with entry write $x_{i,j} : i, j \in [1, n]$ the standard matrix units.

Lemma (Benlolo, Sanderson): The generators of the field $\mathbb{C}[\mathfrak{m}]^{P'}$ is a the leading term of the minor located on the lower left hand corner between two blocks of same size s of M restricted to $\mathfrak{m} + Id$.

The generators of the invariant

Joseph & Melnikov (2002) proved that, the number of generators of $\mathbb{C}[\mathfrak{m}]^{P'}$ is the number of neighboring columns (resp. blocks).

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This work was generalized shortly after for all G classical by Elena Perelman.

A component of $\mathscr{N}$

Let $\mathbb{C}[\mathfrak{m}]_+^{P'}$ denote the augmentation of $\mathbb{C}[\mathfrak{m}]^{P'}$ (that it is the subspace spanned by homogeneous invariants of positive degree), let $\mathscr{N}$ denote its zero locus. It is called the nilfibre.

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We established a method to express "the" component of $\mathscr{N}$ as B saturation.

We construct a vector space $\mathfrak{u}$ an interaction of an algebra $\mathfrak{u}_{w_i}$ define as $\mathfrak{n} \cap w_i(\mathfrak{n})$, regarding particular Weyl w_i elements, chosen through pair of neighboring columns.

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Let the tableau $\mathscr{T}$ representing P with a paire of neighboring column of height s, (C, C').

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In cases where there are columns of hight greater than s we slide all the boxes in row greater to s between (C, C') until we reach the column C.

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The Weyl element is the sequence resulting from a reading of the new Tableau from bottom to top and left to right.

Example

Example: Let a parabolic P define by $(2, 3, 1, 2)$

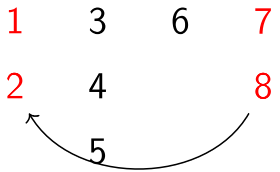

Tableau $\mathscr{T}$ $(2, 3, 1, 2)$.

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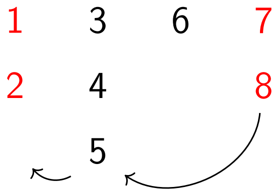

Tableau $\mathscr{T}\ (2,3,1,2)$.

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$$w = (5, 2, 1, 8, 4, 3, 6, 7)$$

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The algebra $\mathfrak{u} = \mathfrak{n} \cap w(\mathfrak{n})$.

Example

Example: Let a parabolic P define by $(2, 3, 1, 2)$

$$\longrightarrow$$

$$\uparrow \quad \begin{array}{cccc} \color{red}{1} & 3 & 6 & \color{red}{7} \\ \color{red}{2} & 4 & & \\ 5 & \color{red}{8} & & \end{array}$$

$$w = (5, 2, 1, 8, 4, 3, 6, 7)$$

The algebra $\mathfrak{u} = \mathfrak{n} \cap w(\mathfrak{n})$.

A component of the Nilfibre, $\mathscr{N} = \overline{B \cdot \mathfrak{u}}$

$B.\mathfrak{u}$ a component

Theorem:

The set $B.\mathfrak{u}$ is include in $\mathscr{N}$.

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The closure of $B.\mathfrak{u}$ is a component of $\mathscr{N}$.

It is called the canonical component.

Weierstrass Sections

A Weierstrass section is a linear subvariety $e + V$ of $\mathfrak{m}$, such that restriction φ of $\mathbb{C}[\mathfrak{m}]^{P'}$ to $\mathbb{C}[e + V]$ gives a isomorphism.

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We obtain a Weierstrass section (e, V) with a canonial construction, forces $e \in \mathscr{N}$.

The Nilfibre

Is the nilfibre $\mathscr{N}$ irreducible? When the nilfibre $\mathscr{N}$ is reducible?

–

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How many the irreducible components of $\mathscr{N}$ and what are their dimension ?

-

Definition

A sequence of the form $(2, \cdots, 1, 1, \cdots, 2)$,

where between the neighboring column of height 2

there are no column of height 2 is called

strongly linear sequence.

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A sequence of the form $(3, \cdots, 2, \cdots, 1, 1, 1, \cdots, 2, \cdots, 3)$, where between the neighboring column of height 3 encircling a strongly linear sequence is called very strongly linear sequence.

Conjecture 1

Let P a parabolic group define by sequence having a a strongly linear subsequence, then the nilfibre $\mathscr{N}$ is reducible.

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A parabolic group define by sequence having no strongly linear subsequence has an irreducible nilfibre $\mathscr{N}$.

Conjecture 2

Let P a parabolic group define by sequence having a

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The numbers of component of nilfibre $\mathscr{N}$ is the product of the number of consecutive columns of height 1 of each strongly linear subsequence.

Conjecture 3

Each of the component can be express as

B saturation.

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The irreducible components $\mathscr{N}$ are equidimensional.

The reducibility nilfibre $\mathscr{N}$

Let a parabolic P define by $(2,1,1,2)$

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The reducibility nilfibre $\mathscr{N}$

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The invariants are the determinant of the matrix encircled by two neighboring blocks

Zoom of the reducibility

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The invariant is the determinant of the matrix

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$(x_{3,1}x_{4,2} - x_{4,1}x_{3,2}) \cdot (x_{5,3}x_{6,4} - x_{6,3}x_{5,4}) = 0.$

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$(x_{3,1}x_{4,2} - x_{4,1}x_{3,2}) \cdot (x_{5,3}x_{6,4} - x_{6,3}x_{5,4}) = 0.$

We have a reducible polynomial.
Consequently, the nilfibre $\mathscr{N}$ is reducible.

Nilfibre $\mathscr{N}$ components

The parabolic P is define by $(2, 1, 1, 2)$.

The number of irreducible is 2 and it is a the number of one of the strongly linear sequence.

Nilfibre $\mathscr{N}$ components

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We construction two algebra $\mathfrak{u}_1$ (canonical) and $\mathfrak{u}_2$ as an intersection of the form $(\mathfrak{n} \cap w_i(\mathfrak{n}))$, such that $\overline{B.\mathfrak{u}_1}$ and $\overline{B.\mathfrak{u}_2}$ are the components of $\mathscr{N}$.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3	4	5
2			6

Tableau $\mathscr{T}$ $(2, 1, 1, 2)$.

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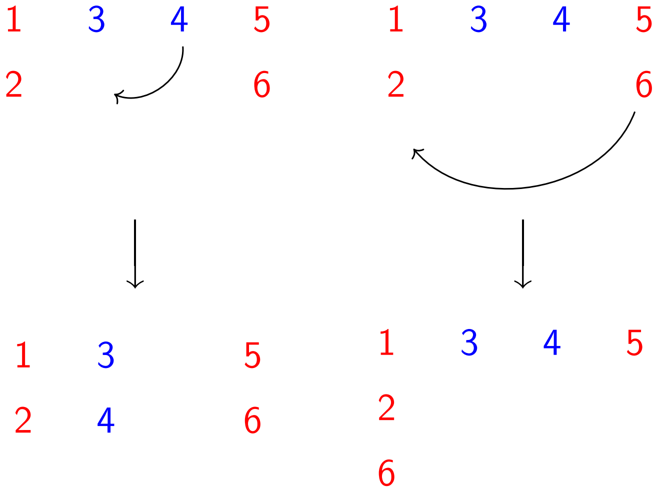


Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3	5
2	4	6

$$w_1 = (2, 1, 4, 3, 6, 5)$$

1	3	4	5
2			
6			

$$w_2 = (6, 2, 1, 3, 4, 5)$$

1	3	5
2	4	6

1	3	4	5
2			
6			

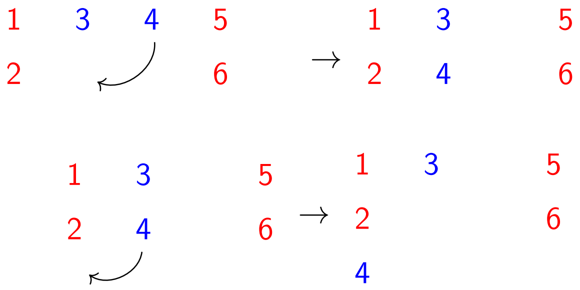
$w_1 = (2, 1, 4, 3, 6, 5) \qquad w_2 = (6, 2, 1, 3, 4, 5)$

These two Weyl elements define for us an vector space

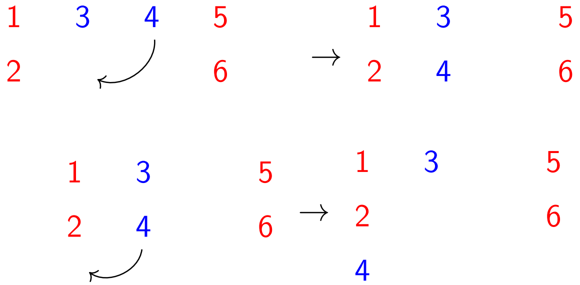
$$\mathfrak{u}_1 = [\mathfrak{n} \cap w_1(\mathfrak{n})] \cap [\mathfrak{n} \cap w_2(\mathfrak{n})] .$$

The (the canonical) component $\overline{B \cdot \mathfrak{u}_1}$.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

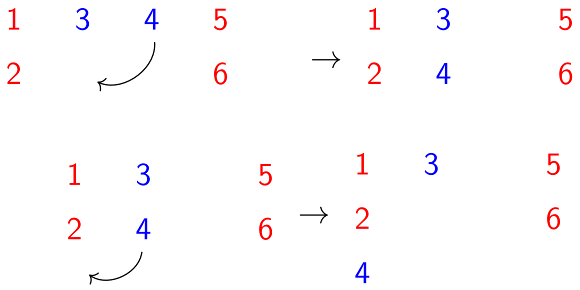


Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra



$w_1' = (2, 1, 4, 3, 6, 5)$ $\quad$ $w_2' = (4, 2, 1, 3, 6, 5)$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra



$w'_1 = (2, 1, 4, 3, 6, 5) \qquad w'_2 = (4, 2, 1, 3, 6, 5)$

These two Weyl elements define the second component

$$\mathfrak{u}_2 = [\mathfrak{n} \cap w'_1(\mathfrak{n})] \cap [\mathfrak{n} \cap w'_2(\mathfrak{n})] .$$

The second component $\overline{B \cdot \mathfrak{u}_2}$.

Zoom of the reducibility

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The Yellow part is $\mathfrak{u}_1$

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The green part is $\mathfrak{u}_2$

The irreducible component $\overline{B \cdot \mathfrak{u}_1}$ (in yellow) is the annihilator of the polynomials $x_{4,3}$ and $(x_{5,3}x_{6,4} - x_{6,3}x_{5,4})$.

The irreducible component $\overline{B \cdot \mathfrak{u}_2}$ (in green) is the annihilator of the polynomials $x_{4,3}$ and $(x_{3,1}x_{4,2} - x_{4,1}x_{3,2})$.

Theorem. (Spaltenstein): Let $\mathcal{O}$ is a nilpotent orbit, the irreducible components (the orbital varieties) of $(\mathcal{O} \cap \mathfrak{n})$ are equidimensional. Their explicit description causes major combinatorial problems.

We have the important dimension formula

$$\dim(\mathcal{O} \cap \mathfrak{n}) = 1/2 \dim \mathcal{O}.$$

The proof results from the Steinberg triple variety whose construction is mainly based on Bruhat decomposition.

Is knowing a Weierstrass sections (e, V) associated to an irreducible component of $\mathcal{N}$, will allow us pick out the right nilpotent orbit using e?

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NO

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NO

Springer representation.

Thank you for your attention.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

Let P a parabolic group defined by a strong linear sequence $(2, 1, 1, 3, 1, 2)$.

The number of component of $\mathscr{N}$ is $2 \times 1 = 2$.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

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The number of component of $\mathscr{N}$ is 3.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

Let P a parabolic group defined by a strong linear sequence $(2, 1, 1, 1, 2)$.

The number of component of $\mathscr{N}$ is 3.

1	3	4	5	6
2				7

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} & x_{4,7} \\ 0 & 0 & 0 & 0 & 1 & x_{5,6} & x_{5,7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

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$x_{4,3}$ and $x_{5,4}$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} & x_{4,7} \\ 0 & 0 & 0 & 0 & 1 & x_{5,6} & x_{5,7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$x_{4,3}$ and $x_{5,4}$

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$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} & x_{4,7} \\ 0 & 0 & 0 & 0 & 1 & x_{5,6} & x_{5,7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$x_{4,3}$ and $x_{5,4}$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} & x_{4,7} \\ 0 & 0 & 0 & 0 & 1 & x_{5,6} & x_{5,7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$x_{4,3}$ and $x_{5,4}$

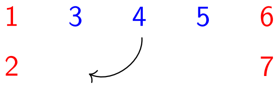
Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} & x_{4,7} \\ 0 & 0 & 0 & 0 & 1 & x_{5,6} & x_{5,7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

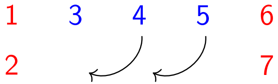
$x_{4,3}$ and $x_{5,4}$,

$x_{5,3}$.

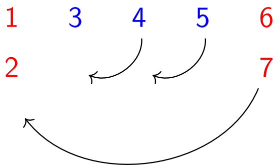
Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra



Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra



Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra



1 3 5 6 1 3 4 6

2 4 7 2 5 7

1 3 4 5 6

2

7

To construct the canonical component.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3	5	6		1	3	4	6
2	4		7		2		5	7

1	3	4	5	6
2				
7				

$$w_1 = (2,1,4,3,5,7,6) \qquad w_2 = (2,1,3,5,4,7,6)$$
$$w_3 = (7,2,1,3,4,5,6)$$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3		5	6	1	3	4		6
2	4			7	2		5		7

1	3	4	5	6
2				
7				

$$w_1 = (2, 1, 4, 3, 5, 7, 6) \qquad w_2 = (2, 1, 3, 5, 4, 7, 6)$$
$$w_3 = (7, 2, 1, 3, 4, 5, 6)$$

The $\mathfrak{u}_1 = [\mathfrak{n} \cap w_1(\mathfrak{n})] \cap [\mathfrak{n} \cap w_2(\mathfrak{n})] \cap [\mathfrak{n} \cap w_3(\mathfrak{n})]$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3	5	6
2	4		7

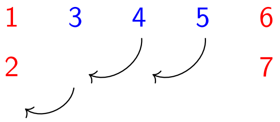
1	3	4	6
2		5	7

1	3	4	5	6
2				
7				

$$w_1 = (2,1,4,3,5,7,6) \qquad w_2 = (2,1,3,5,4,7,6)$$
$$w_3 = (7,2,1,3,4,5,6)$$

The $\mathfrak{u}_1 = [\mathfrak{n} \cap w_1(\mathfrak{n})] \cap [\mathfrak{n} \cap w_2(\mathfrak{n})] \cap [\mathfrak{n} \cap w_3(\mathfrak{n})]$
The first component is defined by $\overline{B.\mathfrak{u}_1}$.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra



To construct the canonical component.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3		5	6	1	3	4		6
2	4			7	2		5		7

1	3		5	6
2				7
4				

$$w_1' = (2,1,4,3,5,7,6) \qquad w_2' = (2,1,3,5,4,7,6)$$
$$w_3' = (4,2,1,3,5,7,6)$$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3	5	6
2	4		7

1	3	4	6
2		5	7

1	3	5	6
2			7
4			

$$w_1' = (2,1,4,3,5,7,6) \qquad w_2' = (2,1,3,5,4,7,6)$$
$$w_3' = (4,2,1,3,5,7,6)$$

The $\mathfrak{u}_2 = [\mathfrak{n} \cap w_1'(\mathfrak{n})] \cap [\mathfrak{n} \cap w_2'(\mathfrak{n})] \cap [\mathfrak{n} \cap w_3'(\mathfrak{n})]$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

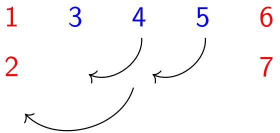
1	3	5	6	1	3	4	6
2	4		7	2		5	7

1	3	5	6
2			7
4			

$$w_1' = (2,1,4,3,5,7,6) \qquad w_2' = (2,1,3,5,4,7,6)$$
$$w_3' = (4,2,1,3,5,7,6)$$

The $\mathfrak{u}_2 = [\mathfrak{n} \cap w_1'(\mathfrak{n})] \cap [\mathfrak{n} \cap w_2'(\mathfrak{n})] \cap [\mathfrak{n} \cap w_3'(\mathfrak{n})]$
The second component is defined by $\overline{B.\mathfrak{u}_2}$.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra



1	3		5	6	1	3	4		6
2	4			7	2		5		7

1	3	4		6
2				7
5				

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3	5	6	1	3	4	6
2	4		7	2		5	7

1	3	4	6
2			7
5			

$$w_1'' = (2, 1, 4, 3, 5, 7, 6) \qquad w_2'' = (2, 1, 3, 5, 4, 7, 6)$$
$$w_3'' = (5, 2, 1, 3, 4, 7, 6)$$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3	5	6
2	4		7

1	3	4	6
2		5	7

1	3	4	6
2			7
5			

$$w_1'' = (2,1,4,3,5,7,6) \qquad w_2'' = (2,1,3,5,4,7,6)$$
$$w_3'' = (5,2,1,3,4,7,6)$$

The $\mathfrak{u}_3 = [\mathfrak{n} \cap w_1''(\mathfrak{n})] \cap [\mathfrak{n} \cap w_2''(\mathfrak{n})] \cap [\mathfrak{n} \cap w_3''(\mathfrak{n})]$

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

1	3	5	6
2	4		7

1	3	4	6
2		5	7

1	3	4	6
2			7
5			

$$w_1'' = (2,1,4,3,5,7,6) \qquad w_2'' = (2,1,3,5,4,7,6)$$
$$w_3'' = (5,2,1,3,4,7,6)$$

The $\mathfrak{u}_3 = [\mathfrak{n} \cap w_1''(\mathfrak{n})] \cap [\mathfrak{n} \cap w_2''(\mathfrak{n})] \cap [\mathfrak{n} \cap w_3''(\mathfrak{n})]$
The third component is defined by $\overline{B.\mathfrak{u}_3}$.

Nilfibre $\mathscr{N}$ component as Borel saturation of an algebra

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} & x_{4,7} \\ 0 & 0 & 0 & 0 & 1 & x_{5,6} & x_{5,7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathfrak{u}_1$

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} & x_{4,7} \\ 0 & 0 & 0 & 0 & 1 & x_{5,6} & x_{5,7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathfrak{u}_2$

$$\begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} \\ 0 & 0 & 0 & 1 & x_{4,5} & x_{4,6} & x_{4,7} \\ 0 & 0 & 0 & 0 & 1 & x_{5,6} & x_{5,7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathfrak{u}_3$

Conclusion

Conclusion

We have a strong convection that these results can be found in the general case.