Belavin-Drinfeld Quantum Groups

Timothy Hodges

University of Cincinnati

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Notation: G is a semi-simple complex algebraic group, $\mathfrak{g}$ its Lie algebra; $q \in \mathbb{C}$, generally not a root of unity.

Problem

Extend our understanding of $\mathcal{O}_q(G)$ and $U_q(\mathfrak{g})$ to the more general Belavin-Drinfeld quantum groups $\mathcal{O}_{\pi,q}(G)$ and $U_{\pi,q}(\mathfrak{g})$.

	Quantum Group	Poisson Group	Lie Bialgebra
Standard	$\mathcal{O}_q(G)$	(G, r)	$(\mathfrak{g}, r)$
Multi-parameter	$\mathcal{O}_p(G)$	$(G, (r, u))$	$(\mathfrak{g}, (r, u))$
Belavin-Drinfeld	$\mathcal{O}_{\pi,q}(G)$	$(G, (\pi, u))$	$(\mathfrak{g}, (\pi, u))$

BD Classification of Lie bialgebra structures

Belavin and Drinfeld classified the Lie bialgebra structures on $\mathfrak{g}$ that are given by non-skew symmetric solutions of the CYBE.

Theorem (Belavin-Drinfeld, 1984)

These Lie bialgebra structures on $\mathfrak{g}$ are given by pairs (τ, u) where τ is a BD triple and $u \in \mathfrak{h} \otimes \mathfrak{h}$ is "compatible" with τ.

Definition

A *Belavin–Drinfeld triple* for $\mathfrak{g}$ is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_i \subset \Gamma$ and $\tau \colon \Gamma_1 \to \Gamma_2$ satisfies

1. $(\tau\alpha, \tau\beta) = (\alpha, \beta)$ for all $\alpha, \beta \in \Gamma_1$ (i.e., τ is a graph isomorphism)
2. for all $\alpha \in \Gamma_1$, there exists a k such that $\tau^k \alpha \notin \Gamma_1$

Pairs (τ, u) are called *Belavin-Drinfeld quadruples*. For a BD quadruple π, we denote the corresponding solution of the CYBE by $r_\pi \in \mathfrak{g} \otimes \mathfrak{g}$.

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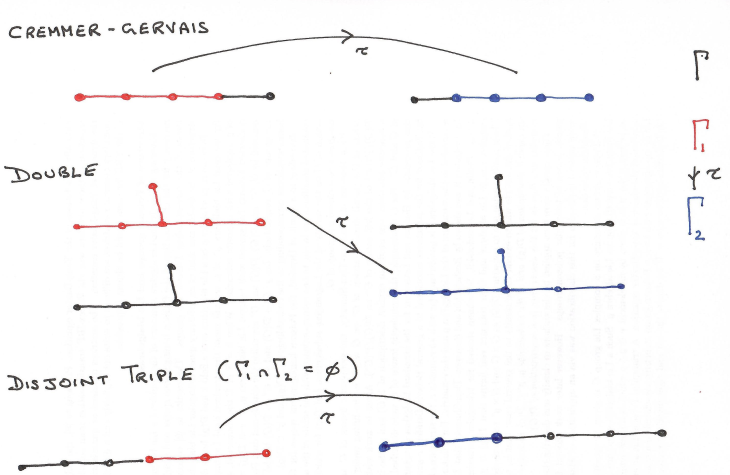
Examples of Belavin-Drinfeld Triples



ESS Quantization

(Note that here we are working over $\mathbb{C}[[h]]$ with $q = e^{h/2}$, rather than $q \in \mathbb{C}$.)

Quantization of Solutions of the CYBE

Given a solution $r \in \mathfrak{g} \otimes \mathfrak{g}$ of the CYBE, find a solution $R \in U_h(\mathfrak{g}) \hat{\otimes} U_h(\mathfrak{g})$ of the QYBE whose linearization is r.

Theorem (Etingof-Schedler-Schiffman, 2000)

For each BD quadruple π one can construct a 2-cocycle twist $\mathcal{T} \in U_h(\mathfrak{g}) \hat{\otimes} U_h(\mathfrak{g})$ which yields a new R-matrix of the form

$$R_\pi = \mathcal{T}_{21}^{-1} R \mathcal{T} = 1 + r_\pi h + o(h^2) \in U_h(\mathfrak{g}) \hat{\otimes} U_h(\mathfrak{g})$$

Braided Hopf Algberas and 2-Cocycles

Definition

A braiding (or dual quasi-triangular structure) on a Hopf algebra A is a bilinear pairing $\langle\ ,\ \rangle$ such that for all a, b, and c in A

1. $\sum \langle a_{(1)}, b_{(1)} \rangle b_{(2)} a_{(2)} = \sum \langle a_{(2)}, b_{(2)} \rangle a_{(1)} b_{(1)}$
2. $\langle\ ,\ \rangle$ is invertible in $(A \otimes A)^*$;
3. $\langle a, bc \rangle = \sum \langle a_{(1)}, b \rangle \langle a_{(2)}, c \rangle$.
4. $\langle ab, c \rangle = \sum \langle b, c_{(1)} \rangle \langle a, c_{(2)} \rangle$

Definition

A 2-cocycle on a Hopf algebra A is an invertible pairing $\sigma : A \otimes A \to k$ such that $\sigma(1, 1) = 1$ and for all x, y and z in A,

$$\sum \sigma(x_{(1)}, y_{(1)}) \sigma(x_{(2)} y_{(2)}, z) = \sum \sigma(y_{(1)}, z_{(1)}) \sigma(x, y_{(2)} z_{(2)})$$

Twisting Braided Hopf Algebras

Theorem

Let A be a braided Hopf algebra with braiding β. Let σ be a 2-cocycle on A. One can twist the multiplication on A to get a new Hopf algebra A_σ. The new multiplication is given by

$$x \cdot y = \sum \sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} \sigma^{-1}(x_{(3)}, y_{(3)}).$$

Moreover $\beta_\sigma = \sigma_{21} * \beta * \sigma^{-1}$ (convolution product) is a braiding on A_σ. The categories of comodules over A and A_σ are equivalent as rigid braided monoidal categories.

ESS Quantization in Algebraic Setting ($q \in \mathbb{C}$)

Theorem (Yakimov)

For each such Belavin-Drinfeld quadruple π, we can construct a Hopf 2-cocycle $\mathcal{T}_\pi \in (\mathcal{O}_q(G) \otimes \mathcal{O}_q(G))^*$ yielding a new quantum group $\mathcal{O}_{\pi,q}(G)$.

The algebras $\mathcal{O}_{\pi,q}(G)$ created in this process can be considered as deformations of the Poisson algebra $\mathcal{O}[G]$ corresponding to the quadruple π.

Example

- The Cremmer-Gervais quantum groups $\mathcal{O}_{\mathrm{CG},q}(SL(n))$ (constructed using an explicit R-matrix)
- The double quantum groups $\mathcal{O}_q(D(G)) \cong \mathcal{O}_q(G) \bowtie \mathcal{O}_q(G)$ (constructed using the Drinfeld double construction)
- The quantum groups $\mathcal{O}_{\pi,q}(G)$ constructed from disjoint triples using 2-cocycles induced from a map $\mathcal{O}_{\pi,q}(G) \to \mathcal{O}_q(\tilde{G}) \otimes \mathcal{O}_q(\tilde{G})$.

Construction of $U_{\pi,q}(\mathfrak{g})$ as the FRT dual

For any braided Hopf algebra we have Hopf algebra maps $l^{\pm} : A^{op} \to A^{\circ}$ by

$$l^{+}(a)(b) = \langle a, S(b) \rangle \quad \text{and} \quad l^{-}(a)(b) = \langle b, S(a) \rangle$$

Let $U^{\pm} = \mathfrak{l}^{\pm}(A)$. Define $U(A)$ to be the Hopf subalgebra generated by U^{+} and U^{-}.

The braiding on A induces a Hopf pairing on $U^{+} \otimes (U^{-})^{op}$ using which one can construct the Drinfeld double $U^{+} \bowtie U^{-}$. The multiplication map $u \otimes v \mapsto uv$ is then a Hopf algebra map

$$\mu : U^{+} \bowtie U^{-} \to U(A)$$

Definition

Define $U_{\pi,q}(\mathfrak{g}) = U(\mathcal{O}_{\pi,q}(G))$.

The braiding on $U_0 = U^+ \cap U^-$

Set

$$U_0 = U^+ \cap U^-$$

The pairing between U^+ and U^- induces a *braiding* on U_0

Thus the FRT dual $U(A)$ of a braided Hopf algebra A contains a canonical braided Hopf subalgebra $U_0(A)$.

Problem

Describe U_0 for the Belavin-Drinfeld quantum group $\mathcal{O}_{\pi,q}(G)$.

Example

For the trivial triple we have $U_0(\mathcal{O}_q(G)) \cong \mathcal{O}(K)$, for K a maximal torus inside G, equipped with the braiding given by the Rosso form.

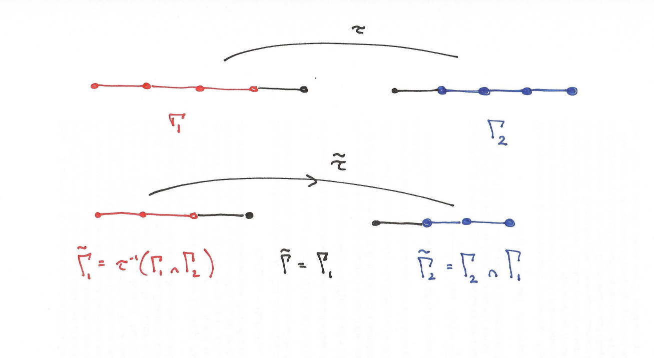
U_0 Conjecture - Derived Triples

Given any triple $(\Gamma_1, \Gamma_2, \tau)$ we have a reduced triple $(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\tau})$ given by restricting τ to Γ_1. That is,

$$\tilde{\Gamma} = \Gamma_1, \quad \tilde{\Gamma}_1 = \tau^{-1}(\Gamma_1 \cap \Gamma_2), \quad \tilde{\Gamma}_2 = \Gamma_1 \cap \Gamma_2,$$

Example

The Cremmer-Gervais triple on A_5 and its derived triple on A_4:



Conjecture

There exists a reductive Lie group $\tilde{G}$ with associated Lie algebra $\tilde{\mathfrak{g}}$ for which $(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\tau})$ is a triple for $\tilde{\mathfrak{g}}$ and such that

$$U_0(\mathcal{O}_{\pi,q}(G)) \cong \mathcal{O}_{\tilde{\pi},q}(\tilde{G})$$

for $\tilde{\pi} = (\tilde{\tau}, \tilde{u})$ suitable choice of continuous parameter $\tilde{u}$.

Example

- For the trivial triple we have $U_0(\mathcal{O}_q(G)) \cong \mathcal{O}(K)$, for K a maximal torus inside G
- For the Cremmer-Gervais quantum group, it was shown that

$$U_0(\mathcal{O}_{CG,q}(SL(n))) \cong \mathcal{O}_{CG,q,u}(GL(n-1)))$$

- For the double quantum groups, we have

$$U_0(\mathcal{O}_q(D(G)) \cong \mathcal{O}_q(G) \otimes \mathcal{O}(K)$$

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Conjecture

Dual to the embedding

$$\mathcal{O}_{\tilde{\pi},q}(\tilde{G}) \hookrightarrow U_{\pi,q}(\mathfrak{g})$$

is a surjective map

$$\mathcal{O}_{\pi,q}(G) \twoheadrightarrow U_{\tilde{\pi},q}(\tilde{g})$$

Problem

Describe the primitive spectrum of $\mathcal{O}_{\pi,q}(G)$

Any such classification would have to include the classification of primitive ideals in $\mathcal{O}_q(G)$ and $U_q(\mathfrak{g})$!