Quivers and Superpotentials for Semisimple Hopf Actions

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Recent Advances and New Directions in the Interplay of Noncommutative Algebra and Geometry

24th June 2022

Motivation for Noncommutative Invariant Theory

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- See Ellen Kirkman's talk.

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  - ($+$ technical hypotheses on the action of H on A.)
- Can then construct and study the **invariant ring** A^H and the **smash product** $A \# H$.

G-Gradings and Actions of $(\Bbbk G)^*$

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- The invariant ring A^H satisfies $A^H = A_{1_G}$.

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- Let $A = \Bbbk\langle u, v\rangle / \langle u^2 - v^2\rangle$ and $G = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1\rangle$.

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- The pair (A, H) is an example of a **quantum Kleinian singularity**.

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- Let V be a vector space, $\mathrm{w} \in V^{\otimes \ell}$, and $\sigma \in \mathrm{GL}(V)$. Call w a **$\sigma$-twisted superpotential** if it is invariant under

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- Polynomial rings are DQAs. (Exercise: what is w?)

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- We have

$$\begin{aligned} \partial^1 \mathrm{w} \ni (\phi_u \otimes \mathrm{id}^{\otimes 3})(\mathrm{w}) &= uv^2 - v^2u, \\ (\phi_v \otimes \mathrm{id}^{\otimes 3})(\mathrm{w}) &= vu^2 - u^2v, \end{aligned}$$

  so $\mathscr{D}(\mathrm{w}, 1) \cong A$.

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Theorem (Jorgensen–Zhang '00, Kirkman–Kuzmanovich–Zhang '09)

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- Suppose $(\Bbbk G)^* \circlearrowleft A$. Then hdet_A is trivial $\Leftrightarrow \deg_G \mathrm{w} = 1_G$.

The Homological Determinant for Example 1

- $A = \Bbbk\langle u, v\rangle/\langle u^2 - v^2\rangle$ and $G = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1\rangle$, with $\deg_G u = a$ and $\deg_G v = b$.

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- $\deg_G u^2 = 1_G = \deg_G v^2$ so w is G-homogeneous and $\deg_G \mathsf{w} = 1_G$.
- Trivial homological determinant $\Rightarrow A^H$ is Gorenstein by the earlier theorem.

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- A is G-graded via $\deg_G u = a$ and $\deg_G v = b$.
- $\deg_G \mathrm{w} = \deg_G v^2u^2 = b^2a^2 = a^2$ so the homological determinant is nontrivial.

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Moreover, $A^H \cong e_0 \Lambda e_0$.

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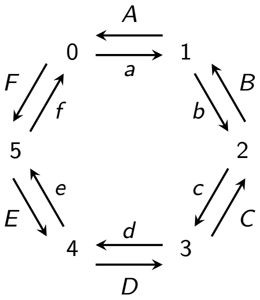

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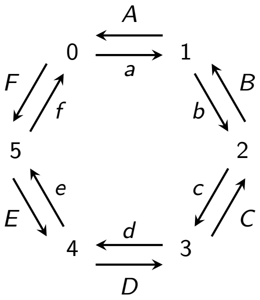

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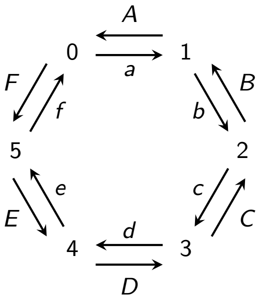

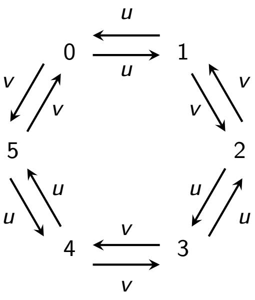

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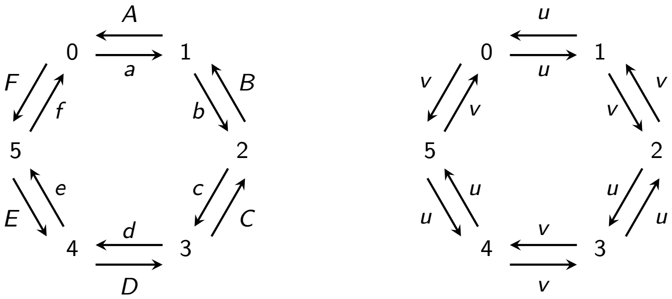

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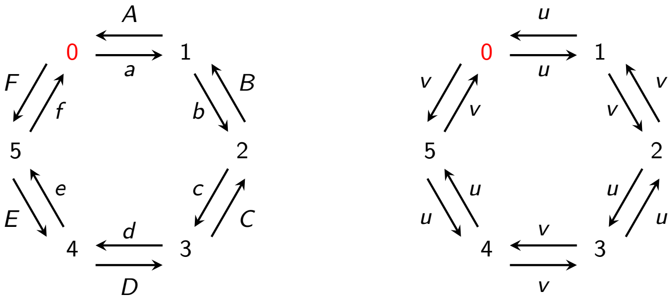

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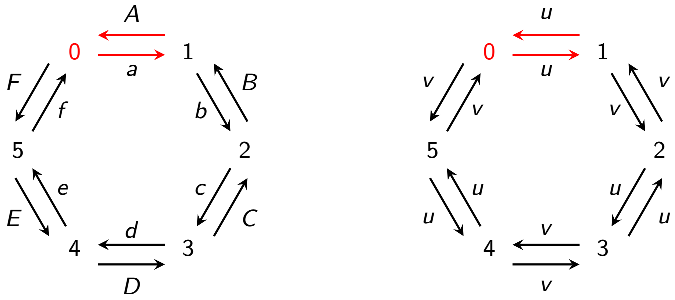

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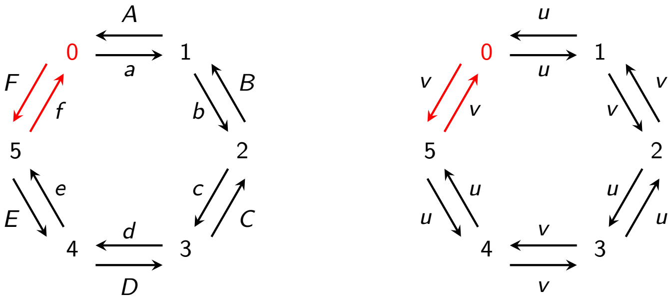

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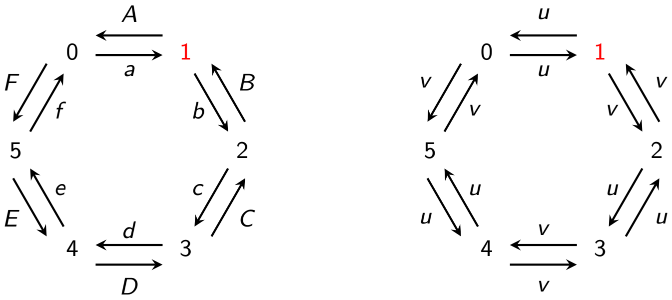

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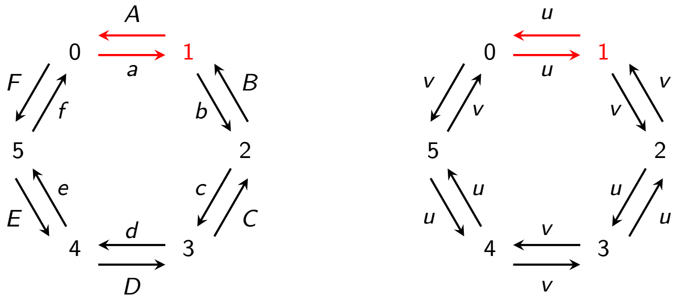

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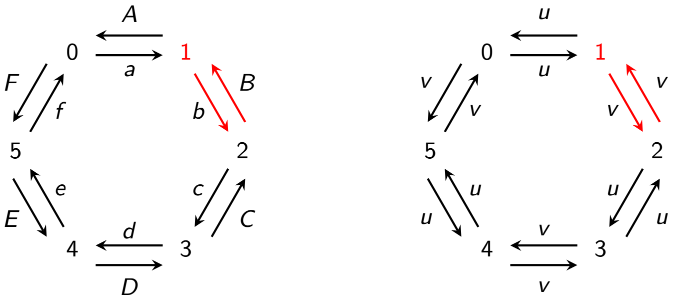

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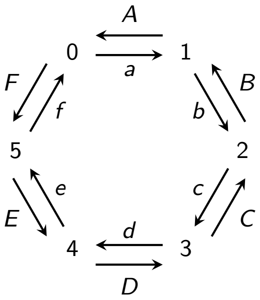

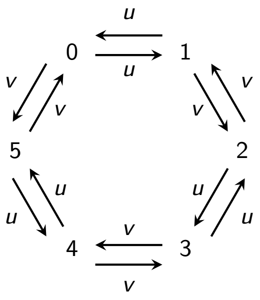

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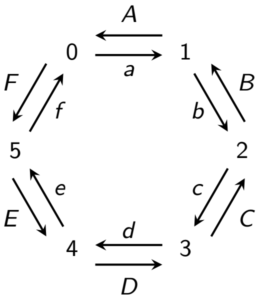

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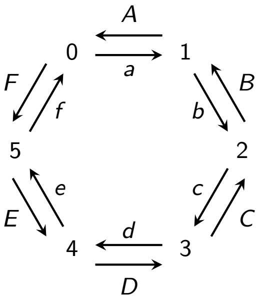

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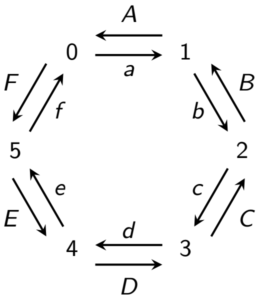

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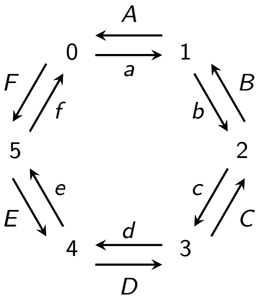

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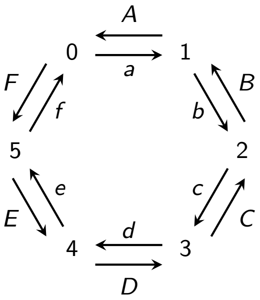

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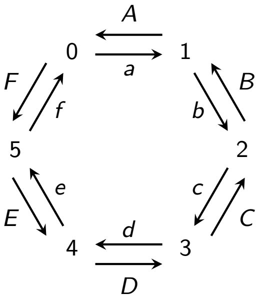

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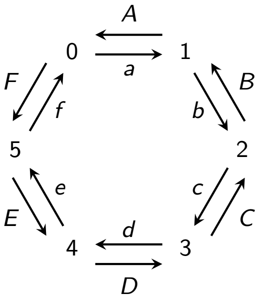

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- Set $x = abcdef, y = FEDCBA, z = aA$. Then $xy = z^6$, and

$$e_0 \Lambda e_0 \cong \frac{\mathbb{k}[x, y, z]}{\langle xy - z^6 \rangle} \cong A^H.$$

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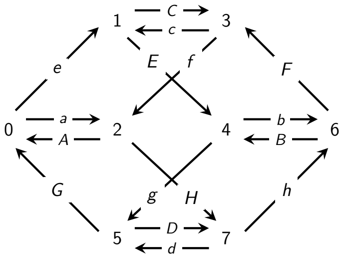

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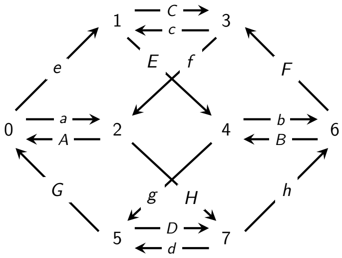

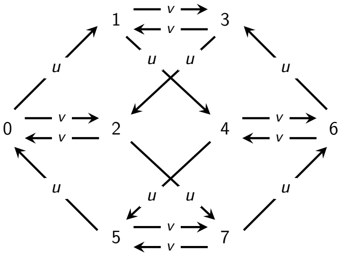

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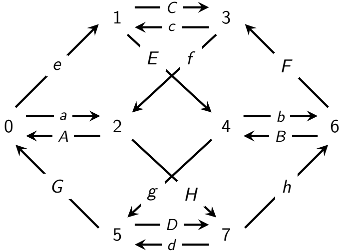

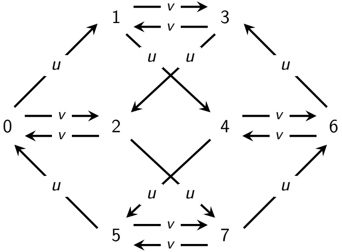

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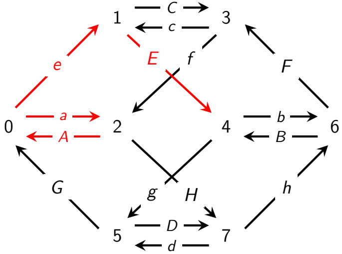

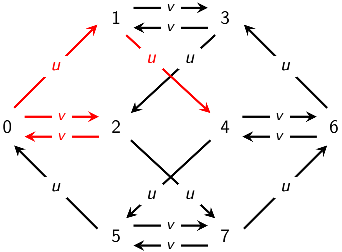

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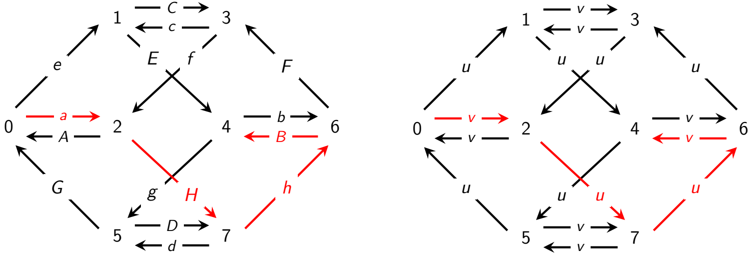

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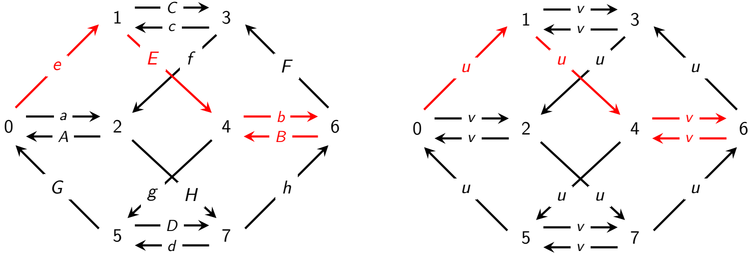

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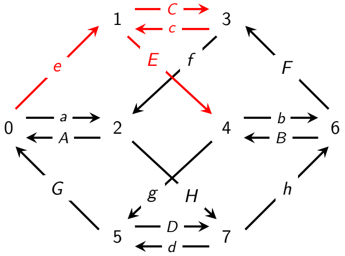

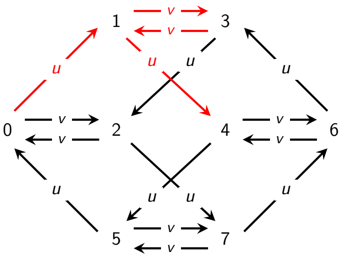

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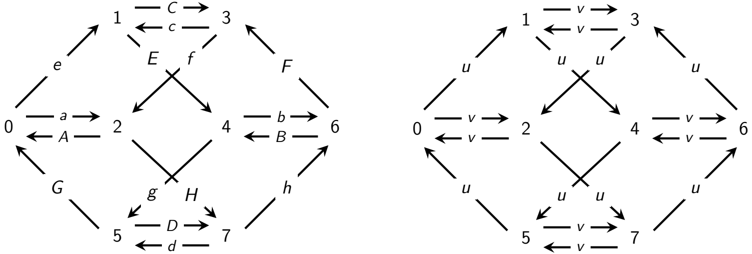

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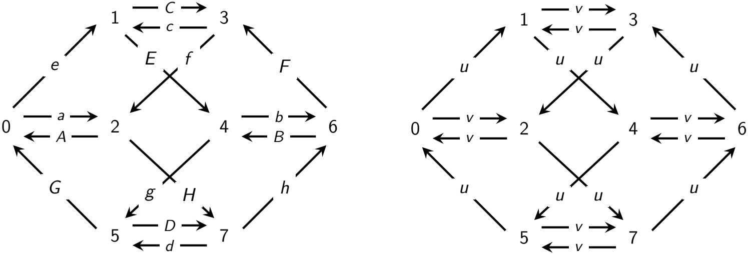

- Fact: hdet_A tells us that paths in Φ have distinct start and end points, but are equal mod 4. We have:

$$\begin{aligned}\Phi = {} & aAeE - aHhB + eEbB - eCcE + CcEg - CfHd + EgDd - EbBg + AaHh - AeEb + HhBb - HdDh \\ & + cCfH - cEgD + fHdD - fAaH + bBgG - bFfA + gGaA - gDdG + DdGe - DhFc + GeCc - GaAe \\ & + BbFf - BgGa + FfAa - FcCf + dDhF - dGeC + hFcC - hBbF.\end{aligned}$$

- $A \# H$ is Morita equivalent to $\Lambda = \mathscr{D}(\Phi, 1)$. The relations in Λ are, e.g. $\partial_a \Phi = AeE - HhB$, $\partial_b \Phi = EbB - CcE$, etc.

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$$\gamma : A \# H \to \mathrm{End}_{A^H}(A).$$

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- This condition is (sometimes) easy to check!

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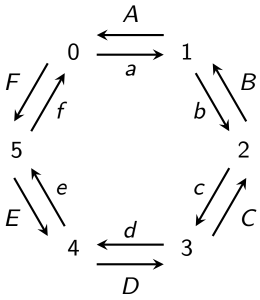

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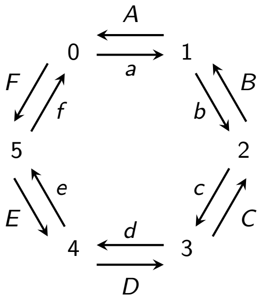

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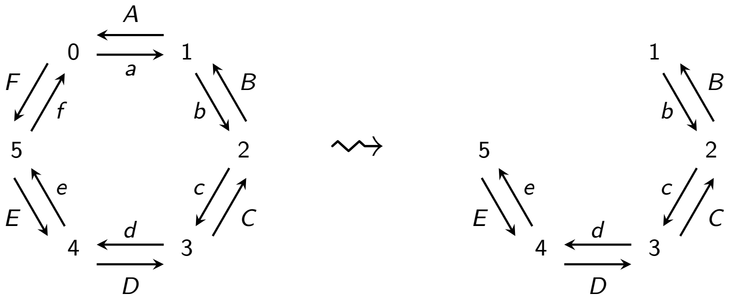

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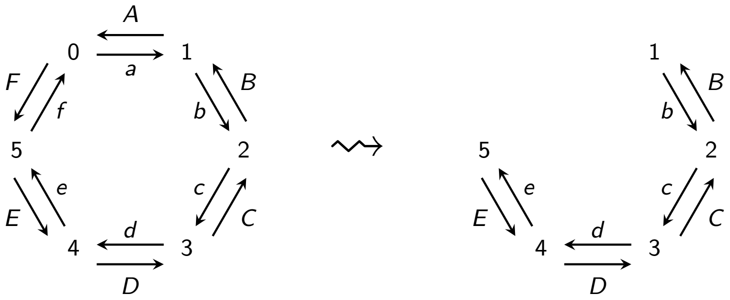

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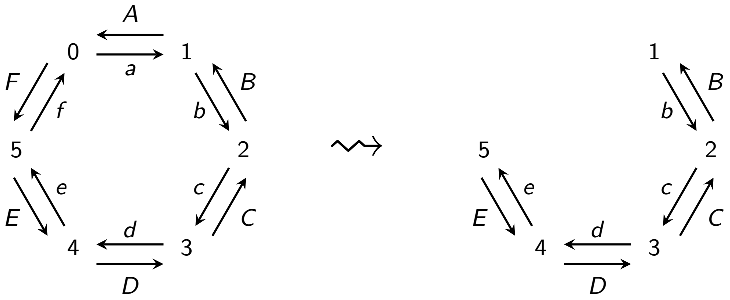

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- A similar argument works for all quantum Kleinian singularities.

Thank you

And congratulations Paul!