A certain 4-dimensional Artin-Schelter regular algebra from noncommutative invariant theory

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Outline

I. Definition of the algebra A

II. Proofs that A is Artin-Schelter regular

III. Homological/algebraic properties of A

IV. Geometry associated to A

V. Work in progress

Definition of the algebra A

Let $\mathbb{K}$ be an algebraically closed field, $\operatorname{char} \mathbb{K} = 0$.

Let A be the factor of the free algebra $\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle$ by the ideal generated by:

$$\begin{array}{ll} x_1x_2 - x_3^2, & x_2x_1 - x_4^2, \\ x_1x_3 - x_2x_4, & x_3x_1 - x_2x_3, \\ x_1x_4 - x_4x_2, & x_4x_1 - x_3x_2. \end{array}$$

A is graded by a semidihedral group

The **semidihedral group of order** 16 is:

$$G = \langle g, h : g^8 = h^2 = e, hg = g^3 h \rangle.$$

Setting

$$\deg_G(x_1) = h, \quad \deg_G(x_2) = g^6 h, \quad \deg_G(x_3) = g, \quad \deg_G(x_4) = g^7,$$

one checks the relations of A are G-homogeneous, so A is G-graded:

$$A = \bigoplus_{\gamma \in G} A_\gamma.$$

This means: the Hopf algebra $\mathbb{K}^G$ acts homogeneously and inner faithfully on A.

Artin-Schelter regular algebras

Definition

A connected graded $\mathbb{K}$-algebra S is **Artin-Schelter regular** or **regular** if:

(a) $\mathrm{gldim} S = d < \infty$, on the left and on the right,

(b) S is Gorenstein:

$$\mathrm{Ext}^i_S({}_S\Bbbk, {}_S S) = \mathrm{Ext}^i_S(\Bbbk_S, S_S) = \begin{cases} 0, & \text{for } i \neq d, \\ \Bbbk(\ell), & i = d, \text{ some } \ell \in \mathbb{Z}. \end{cases}$$

(c) S has finite Gelfand-Kirillov dimension.

Proof that A is Artin-Schelter regular

Gröbner basis ($x_3 < x_2 < x_1 < x_4$) for the defining ideal of A:

$$x_1x_2 - x_3^2,\ x_4^2 - x_2x_1,\ x_1x_3 - x_2x_4,\ x_4x_1 - x_3x_2,\ x_2x_3 - x_3x_1,\ x_4x_2 - x_1x_4,$$

$$x_4x_3^2 - x_3x_2^2,\ x_4x_3x_2 - x_2x_1^2,\ x_4x_3x_1 - x_1x_4x_3$$

Hilbert series of A:

$$H_A(t) = (1-t)^{-4}$$

A is a Koszul algebra:

$$0 \to A(-4) \xrightarrow{M_4} A(-3)^4 \xrightarrow{M_3} A(-2)^6 \xrightarrow{M_2} A(-1)^4 \xrightarrow{M_1} A \to \mathbb{K}_A \to 0$$

Proof that A is Artin-Schelter regular

A has global dimension 4, and $\text{GKdim} A = 4$

The quadratic dual algebra, $A^!$, is Frobenius.

Theorem (P. Smith)

Let S be a Koszul algebra of finite global dimension. Then S is Gorenstein if and only if $S^!$ is Frobenius.

Therefore A is Artin-Schelter regular, of dimension 4.

SD_{16} is a dual reflection group

The fixed ring of A:

$$A_e = \mathbb{K}[x_1^2, x_2^2, x_3x_4, x_4x_3]$$

is a commutative polynomial ring.

Therefore, the semidihedral group of order 16 is a *dual reflection group*.

Other homological properties of A

The Klein 4-group, V, acts as graded automorphisms of A.

There is a 2-cocycle $\mu : V \times V \to \mathbb{K}$ such that $A^{V,\mu} \cong R$, where R is an iterated Ore extension of $\mathbb{K}[r]$.

By work of Andrew Davies, A:

- is Artin-Shelter regular of dimension 4
- is GK-Cohen-Macaulay,
- is Auslander regular,
- is a strongly noetherian domain,
- satisfies the χ condition.

Point modules of A

A **point module**, say P, for an $\mathbb{N}$-graded algebra S is a right $\mathbb{N}$-graded S-module, generated in degree 0, such that

$$H_P(t) = (1-t)^{-1} = 1 + t + t^2 + \cdots .$$

For algebras like A, general theory implies there is a closed subscheme $E \subset \mathbb{P}^3$ such that

$$\{\text{closed points of } E\} \longleftrightarrow \{\text{iso. classes of point modules for } A\}$$
$$p \longleftrightarrow P(p)$$

Moreover, general theory implies there exists a scheme automorphism $\sigma : E \to E$.

For A:

- E is reduced and has 20 distinct closed points.
- on closed points, σ has order 4.

Line modules of A

A **line module**, say L for an $\mathbb{N}$-graded algebra S is a right $\mathbb{N}$-graded S-module, generated in degree 0, such that

$$H_L(t) = (1-t)^{-2} = 1 + 2t + 3t^2 + \cdots .$$

For algebras like A, work of Shelton-Vancliff implies there is a closed subscheme $\mathscr{L} \subset \mathbb{P}^5$ such that

$$\begin{aligned}\{\text{closed points of } \mathscr{L}\} &\longleftrightarrow \{\text{iso. classes of line modules for } A\}\\ \ell &\longleftrightarrow L(\ell)\end{aligned}$$

For A:

- $\dim \mathscr{L} = 1$
- $\mathscr{L} = C_1 \cup C_2 \cdots \cup C_{10}$, irreducible components
- each C_i is isomorphic to a smooth conic in $\mathbb{P}^2$
- each C_i can be identified with a certain ruling on a quadric $\mathcal{V}(Q_i) \subset \mathbb{P}^3$.

Incidence relations among points and lines

We say $p \in E$ **lies on** $\ell \in \mathscr{L}$ if there exists a graded surjection:

$$L(\ell) \twoheadrightarrow P(p).$$

For $p \in E$, let

$$\mathscr{L}_p = \{\ell \in \mathscr{L} : p \text{ lies on } \ell\}.$$

Shelton-Vancliff: For algebras like A, if $\mathscr{L}_p$ is finite, then $|\mathscr{L}_p| = 6$, counting multiplicity.

For A and any $p \in E$, $\mathscr{L}_p$ consists of three closed points, each of multiplicity 2.

The line modules that occur correspond exactly to the 30 line modules on the $C_i \cap C_j$.

The center of R, in general

Suppose G is a dual reflection group.

Let $R = \oplus_{g \in G} R_g$ be a noetherian, regular domain, with invariant subring R_e.

Let $Z(R)$ denote the center of R.

Theorem

$Z(R)$ is characterized by the solution set of a certain finite system of equations in the invariant subring R_e.

A is a finite module over its center

The center of A is given by:

$$Z(A) \cong \mathbb{K}[u, v_1, v_2, v_3, t]/(t^2 - v_1 v_2 v_3),$$

$$\deg(u) = 2, \qquad \deg(v_1) = \deg(v_2) = \deg(v_3) = 4, \qquad \deg(t) = 6.$$

A is a finitely generated module over $Z(A)$, so A is PI.

Theorem

PI deg$(A) = 8$.

Work in progress

Generically,

$$\mathfrak{m} \in \text{MaxSpec } Z(A) \rightsquigarrow M,$$

for some 8-dimensional simple A module, M, with $\mathfrak{m} = \text{Ann}_A M \cap Z(A)$.

To do:

- Determine the *Azumaya locus*, $\mathscr{A}$ (dense open subset of MaxSpec $Z(A)$ parametrizing 8-dimensional simple modules).
- **Brown-Yakimov**: compute the top discriminant ideal, I, then $\mathscr{A} = \text{MaxSpec } Z(A) - \mathcal{V}(I)$.
- See if $\mathcal{V}(I)$ is equal to the singular locus of MaxSpec $Z(A)$.
- Determine the fat point modules of A....

Frobenius extensions, in general

$S \subseteq R$, ring extension

$\beta \in \mathrm{Aut}(S)$

R is a *free β-Frobenius extension* of S if:

(i) R is a free right S-module of finite rank, and

(ii) there is an (S, R)-bimodule isomorphism $\varphi : R \to \mathrm{Hom}_S(R_S, (S_\beta)_S)$.

Theorem

Let G be a dual reflection group. Let $R = \bigoplus_{g\in G} R_g$ be a noetherian, regular domain, with invariant subring R_e. Let $m \in G$ denote the "mass element", and let $\mu_m \in \mathrm{Aut}(R_e)$ be the corresponding automorphism. Then R is a free μ_m^{-1}-Frobenius extension of R_e.

The End

Thank you for listening.

Happy birthday, Paul!