TWISTS OF ALGEBRAS AND 2-COCYCLE TWISTS OF CERTAIN HOPF ALGEBRAS

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arXiv: 2109.11585

TENNESSEE TECH UNIVERSITY
(~ 1 HOUR FROM NASHVILLE)

BIG PICTURE

A graded algebra, ϕ graded automorphism of A

A^{ϕ} = twist of A by ϕ

$r *_{\phi} s = r\phi^{|r|}(s)$

Graded module categories of A and A^{ϕ} are equivalent

B = bialgebra satisfying twisting conditions, ϕ = graded bialgebra automorphism of B

B^{ϕ} = Zhang twist of B

2-cocycle twist of Hopf algebra

Twisting pair

OUR SETUP

- $\Bbbk$ = base field
- $(B, m, u, \Delta, \varepsilon)$ = bialgebra
- $\mathcal{H}$: Hopf envelope of B (Takeuchi's construction)
- $(\mathcal{H}, m, u, \Delta, \varepsilon)$ = Hopf algebra

MOTIVATION (ZHANG TWIST)

- [ATV2, § 8] Twist by an automorphism of a graded algebra

- [Zhang] Generalized to a twisting system

- **Right Zhang twist, A^ϕ**

A = graded algebra
ϕ = graded automorphism of A

  - $A^\phi = A$, as graded vector spaces
  - new multiplication given by

$$r *_\phi s = r\,\phi^{|r|}(s) \qquad \forall \text{ homogeneous } r, s \in A$$

- [Zhang] Graded module categories of A and A^ϕ are equivalent

MOTIVATION . . .

- [Drinfeld, 1987] Notion of a Drinfeld twist, J, of H was introduced

defined on $H \otimes H$

- $H^J = H$, as algebras
  H^J has a deformed coalgebra structure

- If the module categories of 2 Hopf algebras are tensor equivalent, then they are Drinfeld twists of each other.

2-cocycle twist

- [Doi, 1993] + [Doi & Takeuchi, 1994] introduced the dual to a Drinfeld twist, called a 2-cocycle twist, σ, of a Hopf algebra
  - $H^\sigma = H$, as a coalgebra
  - $H^\sigma \neq H$ as an algebra

- **2-COCYCLE ON H** is a convolution invertible linear map $\sigma : H \otimes H \longrightarrow \Bbbk$ satisfying

$$\boxed{\begin{gathered}\sum \sigma(x_1, y_1)\sigma(x_2 y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2 z_2) \\ \text{and } \sigma(x,1) = \sigma(1,x) = \varepsilon(x), \ \forall x, y, z \in H\end{gathered}}$$

- H finite dimensional
  $J \in H \otimes H$ is a Drinfeld twist $\Leftrightarrow$ the map
  $\sigma : H^* \otimes H^* \longrightarrow \Bbbk$, $f \otimes g \longmapsto (f \otimes g)(J)$
  is a 2-cocycle.

OUR QUESTIONS

H = Hopf algebra that is $\mathbb{Z}$-graded as an algebra

- When is a Zhang twist of H a 2-cocycle twist?
    - When is a Zhang twist of H a Hopf algebra?
    - When can a Zhang twist be realized as a 2-cocycle twist?

LET $(B, m, u, \Delta, \varepsilon)$ be a bialgebra

DEFINITION (TWISTING CONDITIONS)

$$\left.\begin{array}{l}\text{(T1) } B \text{ is } \mathbb{Z}\text{-graded} \\ \text{(T2) } \Delta(B_n) \subseteq B_n \otimes B_n\end{array}\right\} \Rightarrow \begin{array}{c} S(H_n) \subseteq H_{-n} \\ \text{for any } n \in \mathbb{Z} \\ \text{(T3)}\end{array}$$

EXAMPLE

For $n \in \mathbb{N}_{\geq 2}$, the polynomial ring $\Bbbk[x_{ij}], 1 \leq i,j \leq n$ with coalgebra structure

$$\Delta(x_{ij}) = \sum_{1 \leq k \leq n} x_{ik} \otimes x_{kj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{ij}$$

$\forall \quad 1 \leq i,j \leq n$

Set $\deg(x_{ij}) = 1$. The polynomial ring $\Bbbk[x_{ij}]$ satisfies the twisting conditions.

PROPOSITION (HUANG - NGUYEN - URE - YASHAW - V - WANG

B = Bialgebra satisfying twisting conditions

1. ϕ = bialgebra automorphism of B
$\Rightarrow B^{\phi}$ = bialgebra satisfying twisting conditions
Zhang twist of B together with coalgebra structure of B

2. B = Hopf algebra with antipode S
ϕ = graded Hopf algebra automorphism
$\Rightarrow B^{\phi}$ = also Hopf algebra satisfying twisting conditions with antipode, $S^{\phi}(r) = \phi^{-|r|} S(r)$ $\forall$ homogeneous r in B^{ϕ}

PROPOSITION (HUANG - NGUYEN - URE - YASHAW - V - WANG

category of graded left H-modules

Monoidal category $Gr-H$

$\Longleftrightarrow$ Monoidal category $Gr-^{\phi}H$.

DEFINITION (TWISTING PAIR)

A pair (ϕ_1, ϕ_2) of algebra automorphisms of B is a twisting pair if the following conditions hold:

(P1) $\Delta \circ \phi_1 = (\mathrm{id} \otimes \phi_1) \circ \Delta$ and $\Delta \circ \phi_2 = (\phi_2 \otimes \mathrm{id}) \circ \Delta$

(P2) $\varepsilon \circ (\phi_1 \circ \phi_2) = \varepsilon$.

THEOREM CHUANG – NGUYEN – URE – VASHAW – V – WANG

B = bialgebra satisfying twisting conditions.

For any twisting pair (ϕ_1, ϕ_2) of B, there is a unique twisting pair $(\mathcal{H}(\phi_1), \mathcal{H}(\phi_2))$ of the Hopf envelope $\mathcal{H}(B)$ extending (ϕ_1, ϕ_2).

Moreover, the 2-cocycle twist $\mathcal{H}(B)^{\sigma}$, with the 2-cocycle $\sigma : \mathcal{H}(B) \otimes \mathcal{H}(B) \longrightarrow \mathbb{k}$ given by

$$\sigma(x,y) = \varepsilon(x)\,\varepsilon(\mathcal{H}(\phi_2)^{|x|}(y)),$$ for homogeneous elements $x, y \in \mathcal{H}(B)$

is the right Zhang twist $\mathcal{H}(B)^{\mathcal{H}(\phi_1 \circ \phi_2)}$.

CAN A 2-COCYCLE BE A ZHANG TWIST?

PROPOSITION (HUANG - NGUYEN - URE - VASHAW - V - WANG

H = Hopf algebra satisfying twisting conditions.
For any twisting pair (ϕ_1, ϕ_2) of H, have

① Map $\phi_1 \circ \phi_2$ is a graded Hopf automorphism of H.

② Linear map $\sigma : H \otimes H \longrightarrow \Bbbk$ defined by
$\sigma(x,y) = \varepsilon(x)\varepsilon(\phi_2^{|x|}(y))$, for any homogeneous elements $x, y \in H$,
is a 2-cocycle, whose convolution inverse σ^{-1} is given by $\sigma^{-1}(x,y) = \varepsilon(x)\varepsilon(\phi_1^{|x|}(y))$

③ right Zhang twist $H^{\phi_1 \circ \phi_2} \cong$ 2-cocycle twist H^{σ}

As a consequence, H^{σ} and $H^{\phi_1 \circ \phi_2}$ are Morita-Takeuchi equivalent.

APPLICATION TO MANIN'S UNIVERSAL QUANTUM GROUPS

IDEA

- Examine Zhang twists of Manin's universal quantum groups of quadratic algebras
- Automorphisms coming from the underlying algebras
- Connect to 2-cocycle twists of these universal quantum groups

MANIN'S UNIVERSAL QUANTUM GROUPS OF QUADRATIC ALGEBRAS

- A = quadratic algebra and write as,

$$A = \Bbbk\langle A_1\rangle / \langle R(A)\rangle,$$
$$R(A) \subseteq A_1 \otimes A_1, \dim_{\Bbbk} A_1 < \infty$$

- $\underline{\text{end}}^l(A)$ = universal bialgebra that lefts coacts on A via $\rho_A : A \longrightarrow \underline{\text{end}}^l(A) \otimes A$

- $\underline{\text{aut}}^l(A)$ = universal quantum group that left coacts on A

THEOREM (HUANG - NGUYEN - URE - VASHAW - V - WANG

- Every twisting pair of the universal quantum group $\underline{\text{aut}}^l(A)$ associated to a quadratic algebra A is given by a graded algebra automorphism of A in an explicit way.
- As a consequence, the corepresentation theory of $\underline{\text{aut}}^l(A)$ only depends on the graded module category over A.

APPLICATION TO $\mathcal{O}_q(M_n(\Bbbk))$ VIA FRT CONSTRUCTION

- Application of $H^\sigma \cong H^{\phi_1 \circ \phi_2}$ with σ constructed using (ϕ_1, ϕ_2)

DEFINITION

$R \in \mathrm{End}_{\Bbbk}(V \otimes V)$ where $V =$ finite dimensional vector space

R is called a solution to the quantum Yang-Baxter equation (QYBE) or an R-matrix if R satisfies

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}.$$

where $R = \sum R_1 \otimes R_2$ and $R^{12} = \sum R_1 \otimes R_2 \otimes \mathrm{id}$, $R^{13} = \sum R_1 \otimes \mathrm{id} \otimes R_2$ and $R^{23} = \sum \mathrm{id} \otimes R_1 \otimes R_2$

- Start with $V = \mathrm{span}(x_1, \ldots, x_n)$ and $R \in M_{n^2 \times n^2}(\Bbbk)$ with entries $R^{ij}_{k\ell}$ such that

$$R(x_k \otimes x_\ell) = \sum_{1 \leq i,j \leq n} R^{ij}_{k\ell} x_i \otimes x_j, \quad \text{for } 1 \leq k, \ell \leq n.$$

- Apply Fadeev-Reshetikin-Takhtajan (FRT) construction to obtain a coquasitriangular bialgebra $(A(R), \theta)$:
  - As an algebra, generated by n^2 generators $\{t^j_i\}_{i,j=1}^n$ such that

n^4 relations

$$\sum_{1 \leq k, \ell \leq n} R^{k\ell}_{ij} t^j_u t^i_v = \sum_{1 \leq k, \ell \leq n} R^{ij}_{vu} t^k_i t^\ell_j$$

$\forall \ 1 \leq u, v, i, j \leq n$

  - As a coalgebra, $\Delta(t^j_i) = \sum_{1 \leq k \leq n} t^k_j \otimes t^k_i$, $\varepsilon(t^j_i) = \delta^j_i$
  - With coquasitriangular structure $\theta : A(R) \otimes A(R) \to \Bbbk$ satisfying $\theta(t^i_v, t^j_u) = R^{ij}_{vu}$ $\forall \ 1 \leq i, j, v, u \leq n$.

APPLICATION TO $\mathcal{O}_q(M_n(\Bbbk))$ VIA FRT CONSTRUCTION

- Can classify the twisting pairs (ϕ_1, ϕ_2) of $A(R)$
  - Suppose $\{t_i^j\}_{1 \leq i,j \leq n}$ are generators of $A(R)$.
  - (ϕ_1, ϕ_2) determined by values on generators

$$\phi_1(t_i^j) = \sum_{1 \leq u \leq n} \alpha_i^u t_u^j \quad \text{and}$$

$$\phi_2(t_i^j) = \sum_{1 \leq u \leq n} \beta_u^j t_i^u$$

and, $$\sum_{1 \leq k, \ell \leq n} R_{ij}^{k\ell} \alpha_u^j \alpha_v^i = \sum_{1 \leq k, \ell \leq n} R_{vu}^{ij} \alpha_i^k \alpha_j^\ell$$

hold $\forall\ 1 \leq u, v, i, j \leq n$, where (α_i^j) and (β_u^v) are inverses of each other.

APPLICATION TO $\mathcal{O}_q(M_n(\mathbb{k}))$ VIA FRT CONSTRUCTION

- Consider $\mathcal{O}_q(M_n(\mathbb{k}))$, quantized coordinate ring of $n\times n$ matrices.

  - As an algebra, $\mathcal{O}_q(M_n(\mathbb{k}))$ is generated by $\{x_{ij}\}_{1\leq i,j\leq n}$ with defining relations

$$q x_{ks} x_{us} = x_{us} x_{ks} \qquad \text{if } k<u$$

$$q x_{ks} x_{kv} = x_{kv} x_{ks} \qquad \text{if } s<v$$

$$x_{us} x_{kv} = x_{kv} x_{us} \qquad \text{if } s<v,\ k<u$$

$$x_{us} x_{kv} = x_{kv} x_{us} + (q-q^{-1}) x_{ks} x_{uv} \qquad \text{if } s<u,\ u<k$$

Here $n\geq 2$, $q\in\mathbb{k}^\times$

  - As a coalgebra, $\mathcal{O}_q(M_n(\mathbb{k}))$ is defined by $\Delta(x_{ij}) = \sum_{1\leq k\leq n} x_{ik}\otimes x_{kj}$ and

$$\varepsilon(x_{ij}) = \delta_{ij} \qquad \forall\ 1\leq i,j\leq n$$

$\mathcal{O}_q(M_n(\mathbb{k}))$ satisfies our twisting conditions

  - There is a central group-like element g of $\mathcal{O}_q(M_n(\mathbb{k}))$ corresponding to the q-determinant of $\mathcal{O}_q(M_n(\mathbb{k}))$

APPLICATION TO $\mathcal{O}_q(M_n(\Bbbk))$ VIA FRT CONSTRUCTION

- All twisting pairs of $\mathcal{O}_q(GL_n(\Bbbk))$ are of the form (ϕ_1, ϕ_2) where ϕ_1 and ϕ_2 are

$$\begin{pmatrix} \phi_1(x_{11}) & \cdots & \phi_1(x_{1n}) \\ \vdots & \ddots & \vdots \\ \phi_1(x_{n1}) & \cdots & \phi_1(x_{nn}) \end{pmatrix} = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}$$

$$\begin{pmatrix} \phi_2(x_{11}) & \cdots & \phi_2(x_{1n}) \\ \vdots & \ddots & \vdots \\ \phi_2(x_{n1}) & \cdots & \phi_2(x_{nn}) \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}^{-1} \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

  1. If $q = 1$, then for any $(\alpha_{ij}) \in GL_n(\Bbbk)$, (ϕ_1, ϕ_2), as defined above form a twisting pair
  2. If $\mathrm{char}(\Bbbk) \neq 2$ and $q = -1$, then (α_{ij}) defines a twisting pair $\Leftrightarrow$ it is a generalized permutation matrix
  3. If $q \neq \pm 1$, then (α_{ij}) defines a twisting pair $\Leftrightarrow$ it is diagonal.

- right Zhang twist of $\mathcal{O}_q(GL_n(\Bbbk))$ by $\phi_1 \circ \phi_2$

  $\cong$ 2-cocycle twist of $\mathcal{O}_q(GL_n(\Bbbk))$ by the

  2-cocycle σ (σ can be given explicitly).

THANK-YOU !

HAPPY BIRTHDAY PAUL!

References

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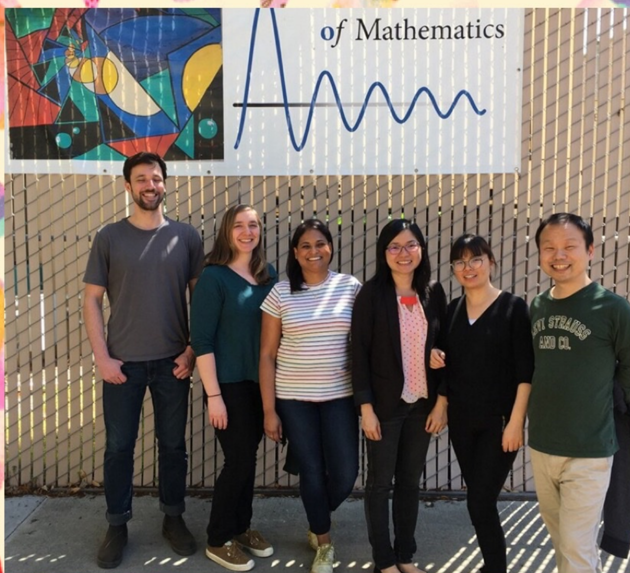
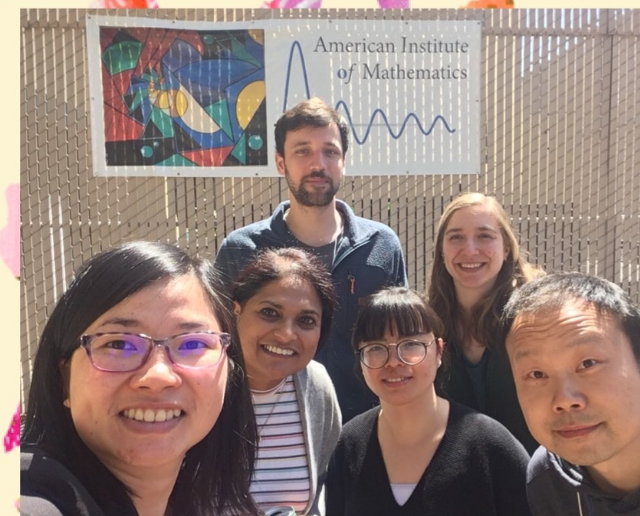
③ Y. I. Manin. Quantum groups and noncommutative geometry. CRM Short Courses. Centre de Recherches Mathématiques, [Montreal], QC; Springer, Cham, 2nd ed., 2018. With a contribution by T. Raedschelders and M. Van den Bergh.

④ M. Takeuchi. Free Hopf algebras generated by coalgebras. J. Math. Soc. Japan, 23: 561–582, 1971.

⑤ J. J. Zhang. Twisted graded algebras and equivalences of graded categories. Proc. London Math Soc. (3), 72(2): 281–311, 1996.

HEXAGON GROUP @ AIM





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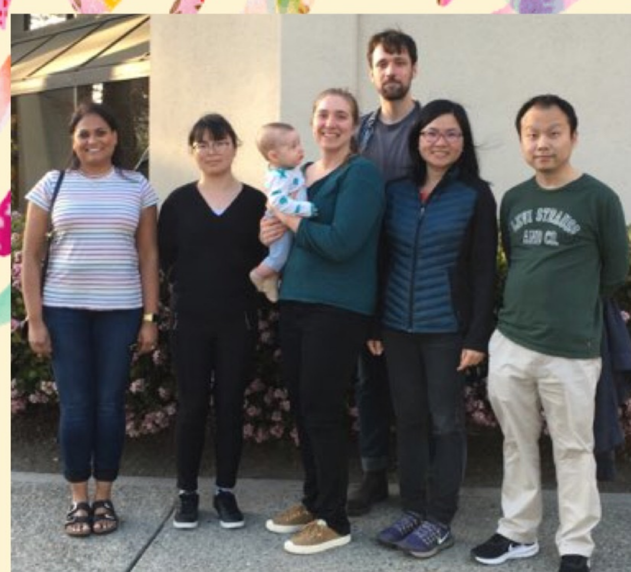

1. 1^{st} page – lit. review
2. our paper – bridge bet. 2
3. Twisting conditions + twisting pair
4. Main result
5. + Corollary module categories
6. Link to Tim Hodges FRT construction + Toby Stafford's students
7. Examples of bialg. satisfying twisting conditions (maybe grp algebra)

PROPOSITION B = bialgebra satisfying twisting conditions

① ϕ = bialgebra automorphism of B
$\Rightarrow B^{\phi}$ = bialgebra satisfying twisting conditions
Zhang twist of B together with coalgebra structure of B

② B = Hopf algebra with antipode S
ϕ = graded Hopf algebra automorphism
$\Rightarrow B^{\phi}$ = also Hopf algebra satisfying twisting conditions with antipode, $S^{\phi}(r) = \phi^{-|r|} S(r)$ $\forall$ homogeneous r in B^{ϕ}

Sketch of Proof ① Show that Δ, ε are algebra maps w.r.t new multiplication
- B^{ϕ} is a graded algebra with grading $B^{\phi} = \bigoplus_{i \in \mathbb{Z}} B_i$

- Endow B^{ϕ} with same coalgebra structure as B
- Know ϕ is degree-preserving bialgebra automorphism
$\Rightarrow (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$ and $\varepsilon \circ \phi = \varepsilon$
$\Rightarrow \Delta(r * s) = \Delta(r \phi^{|r|}(s)) = \Delta(r) \Delta(\phi^{|r|}(s)) =$
$= \sum r_1 \phi^{|r|}(s)_1 \otimes \sum r_2 \phi^{|r|}(s)_2$
$= \sum r_1 \phi^{|r_1|}(s_1) \otimes \sum r_2 \phi^{|r_2|}(s_2)$
$= \sum r_1 * s_1 \otimes r_2 * s_2$

- $\varepsilon(r * s) = \varepsilon(r \phi^{|r|}(s)) = \varepsilon(r) \varepsilon(\phi^{|r|}(s))$
$= \varepsilon(r) * \varepsilon(s)$

② B = Hopf algebra satisfying twisting conditions

- ① ⇒ B^ϕ is a bialgebra

Question: Does B^ϕ have an antipode?

- Define map $S^\phi : B^\phi \longrightarrow B^\phi$ by $S^\phi(r) = \phi^{-|r|}(S(r))$

original antipode of B

$$
\begin{aligned}
- \sum S^\phi(r_1) * r_2 &= \sum \phi^{-|r_1|}(S(r_1)) * r_2 \\
&= \sum \phi^{-|r_1|}(S(r_1))\, \phi^{|S(r_1)|}(r_2) \\
&= \sum \phi^{|S(r_1)|}\left(\phi^{-|S(r_1)|} \circ \phi^{-|r_1|}(S(r_1))\, r_2\right) \\
&\overset{(T3)}{=} \sum \phi^{|S(r_1)|}(\varepsilon(r)) \\
&= \varepsilon(r)
\end{aligned}
$$

- Similarly, for $\sum r_1 * S^\phi(r_2)$

$\therefore$ $(B^\phi, *, \Delta, S^\phi)$ is a Hopf algebra. ■

category of graded left H-modules

PROPOSITION Monoidal category Gr-H

$\Longleftrightarrow$ Monoidal category $Gr\text{-}^{\phi}H$. ■

THEOREM $\mathcal{H}(B^\phi)$ = Hopf envelope of B^ϕ satisfies twisting conditions. ■

- Construction of $\mathcal{H}(B^\phi)$ uses [Takeuchi, 1971]

2-COCYCLE TWISTS OF HOPF ALGEBRAS

BACKGROUND H = Hopf algebra

DEFINITION(S)

- **Left H-Galois object** is left H-comodule algebra $A \neq 0$ such that if $\alpha : A \longrightarrow H \otimes A$ is the coaction of H on A, the linear map defined by the following composition

$$\boxed{A \otimes A \xrightarrow{\alpha \otimes id} H \otimes A \otimes A \xrightarrow{id \otimes m} H \otimes A}$$

is an isomorphism of vector spaces

- **(H, K)-biGalois object** is H-K-bicomodule algebra which is both a left H-Galois object and a right K-Galois object.

- **(Cleft objects - class of Hopf-Galois objects)**
Right H-cleft object is a right H-comodule algebra which admits an H-comodule isomorphism $\phi : H \xrightarrow{\sim} A$ that is also invertible with respect to the convolution product. If ϕ preserves the unit, it is called a **section**.

- **(K, H)-bicleft object** analogously defined

2-COCYCLE TWISTS OF HOPF ALGEBRAS

BACKGROUND H = Hopf algebra

DEFINITION(S)

- **LEFT H-GALOIS OBJECT** is a left H-comodule algebra $A \neq 0$ such that if $\alpha : A \longrightarrow H \otimes A$ is the coaction of H on A, the linear map defined by the following composition

$$\boxed{A \otimes A \xrightarrow{\alpha \otimes id} H \otimes A \otimes A \xrightarrow{id \otimes m} H \otimes A}$$

is an isomorphism of vector spaces.

- **(H-K)-BIGALOIS OBJECT** is an H-K-bicomodule algebra which is both a left H-Galois object and a right K-Galois object.

- **CLEFT OBJECTS (CLASS OF HOPF-GALOIS OBJECTS)**
A right H-cleft object is a right H-comodule algebra which admits an H-comodule isomorphism $\phi : H \xrightarrow{\sim} A$ that is also invertible with respect to the convolution product. If ϕ preserves the unit, it is called a **SECTION**.

- **(K-H)-BICLEFT OBJECT** analogously defined.

- **2-COCYCLE ON H** is a convolution invertible linear map $\sigma : H \otimes H \longrightarrow \mathbb{k}$ satisfying

• **2-COCYCLE ON H** is a convolution invertible linear map $\sigma : H \otimes H \longrightarrow \Bbbk$ satisfying

$$\boxed{\begin{array}{l} \sum \sigma(x_1, y_1)\sigma(x_2 y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2 z_2) \\ \text{and } \sigma(x, 1) = \sigma(1, x) = \varepsilon(x), \ \forall x, y, z \in H \end{array}}$$

Convolution inverse of σ, σ^{-1} satisfies

$$\sum \sigma^{-1}(x_1 y_1, z)\,\sigma^{-1}(x_2, y_2) = \sum \sigma^{-1}(x, y_1 z_1)\sigma^{-1}(y_2, z_2)$$

and

$$\sigma^{-1}(x, 1) = \sigma^{-1}(1, x) = \varepsilon(x), \ \forall x, y, z \in H$$

$\boxed{\text{Doi \& Takeuchi, 1986}}$

Given pair (A, ϕ)

A: right H-cleft object

ϕ: section

SKIP

linear map σ defined on $H \otimes H$ by

$$\sigma(x,y) := \sum \phi(x_1)\,\phi(y_1)\,\bar{\phi}(x_2 y_2)$$

$\bar{\phi}$: convolution inverse

is a 2-cocycle on H

- σ takes on values in $\Bbbk$
- Let ${}_\sigma H$ = right H-comodule algebra H endowed with the original unit and deformed product

$$x \cdot_\sigma y = \sum \sigma(x_1, y_1)\, x_2 y_2, \quad \forall x, y \in {}_\sigma H$$

- Have an isomorphism ${}_\sigma H \xrightarrow{\sim} A$ via $y \mapsto \phi(y)$ as H-comodule algebras

$\boxed{\text{Masuoka, 1994}}$ Every right H-cleft object arises in this way.

Given 2-cocycle $\sigma : H \otimes H \to \Bbbk$, let

H^σ = coalgebra H endowed with the original unit and deformed product

$$x *_\sigma y := \sum (x_1, y_1)\, x_2 y_2\, \sigma^{-1}(x_3, y_3)$$

$\boxed{\text{Doi, 1993}}$ H^σ is a Hopf algebra with deformed antipode S^σ

H^σ = 2-cocycle twist of H by σ

PROPOSITION Let H = bialgebra satisfying twisting conditions. Let H^ϕ = right Zhang twist of H, ϕ = graded algebra automorphism.

① Suppose $\Delta \circ \phi = (\phi \otimes id) \circ \Delta$.

$H^\phi \cong {}_\sigma H$ is right H-cleft with a 2-cocycle $\sigma : H \otimes H \longrightarrow \Bbbk$ given by $\sigma(x,y) = \varepsilon(x)\varepsilon(\phi^{|x|}(y))$, $\forall$ homogeneous elements $x, y \in H$.

② Suppose $\Delta \circ \phi = (id \otimes \phi) \circ \Delta$.

$H^\phi \cong H_{\sigma^{-1}}$ is left H-cleft with a 2-cocycle convolution inverse $\sigma^{-1} : H \otimes H \to \Bbbk$ given by $\sigma^{-1}(x,y) = \varepsilon(x)\varepsilon(\phi^{|x|}(y))$, for homogeneous $x, y \in H$.

Sketch of proof ① Is H^ϕ a right Galois object?

- $H^\phi \cong H$, as graded vector spaces
- $\Delta : H \longrightarrow H \otimes H$ gives H^ϕ a right H-comodule structure via $\Delta^\phi : H^\phi \longrightarrow H^\phi \otimes H$ SKIP
- If $\Delta \circ \phi = (\phi \otimes id) \circ \Delta$, then

$$\begin{aligned} \Delta(\underset{\text{Zhang twist}}{x * y}) &= \Delta(x\,\phi^{|x|}(y)) = \Delta(x)\,\Delta(\phi^{|x|}(y)) \\ &= \Delta(x)(\phi^{|x|} \otimes id)\Delta(y) \\ &= \textstyle\sum x_1 \phi^{|x_1|}(y_1) \otimes x_2 y_2, \end{aligned}$$

$\forall x, y \in H^\phi \Rightarrow H^\phi$ is a right H-comodule algebra

- Can check that ϕ is invertible $\Rightarrow$ H^ϕ = right Galois object

- View identity map $H = H^\phi$ as an isomorphism of right H-modules $\Rightarrow$ H^ϕ is cleft

Doi & Takeuchi

Sketch of proof (cont'd)

SKIP

- H-comodule isomorphism $\phi : H \longrightarrow H^{\phi}$ is invertible with respect to the convolution product in $\mathrm{Hom}(H, H^{\phi})$ with inverse $\bar{\phi}$ given by
$$\bar{\phi}(x) = \phi^{1-|x|}(S(x)).$$

- Know that 2-cocycle $\sigma : H \otimes H \longrightarrow \Bbbk$ associated with H-cleft object H^{ϕ} is given by
$$\begin{aligned}
\sigma(x,y) &= \sum \phi(x_1) * \phi(y_1) * \bar{\phi}(x_2 y_2) \\
&= \sum (\phi(x_1)\phi^{1+|x|}(y_1)) * \phi^{1-|x_2y_2|}(S(x_2y_2)) \\
&= \sum \phi(x_1)\phi^{1+|x|}(y_1)\phi(S(x_2y_2)) \\
&= \sum \phi(x_1 \phi^{|x|}(y_1) S(x_2) S(y_2)) \\
&= \sum \phi(x_1 \sum (\phi^{|x|}(y)_1 S(\phi^{|x|}(y)_2)) S(x_2)) \\
&= \phi(\sum x_1 S(x_2))\, \varepsilon(\phi^{|x|}(y)) \\
&= \varepsilon(x)\varepsilon(\phi^{|x|}(y)). \quad \blacksquare
\end{aligned}$$

PROPOSITION $H =$ Hopf algebra satisfying twisting conditions. For any twisting pair (ϕ_1, ϕ_2) of H, have

① Map $\phi_1 \circ \phi_2$ is a graded Hopf automorphism of H.

② Linear map $\sigma : H \otimes H \longrightarrow \mathbb{k}$ defined by
$\sigma(x, y) = \varepsilon(x)\varepsilon(\phi_2^{|x|}(y))$, for any homogeneous elements $x, y \in H$,
is a 2-cocycle, whose convolution inverse σ^{-1} is given by $\sigma^{-1}(x,y) = \varepsilon(x)\varepsilon(\phi_1^{|x|}(y))$

③ The 2-cocycle twist $H^{\sigma} \cong H^{\phi_1 \circ \phi_2}$ is a right Zhang twist.
As a consequence, H^{σ} and $H^{\phi_1 \circ \phi_2}$ are Morita-Takeuchi equivalent with bi-cleft object given by H^{ϕ_1}.

Sketch of proof

①: $\phi_1 \circ \phi_2$ is compatible with the coassociativity axiom SKIP
Rest follows from twisting pair conditions.

②: Show that σ and σ^{-1} are inverses of each other with respect to the convolution product $*$ in $\mathrm{Hom}_{\mathbb{k}}(H \otimes H, \mathbb{k})$

- $(\sigma * \sigma^{-1})(x,y) = \varepsilon(x)\varepsilon(y)$ (similarly for $\sigma^{-1} * \sigma$)
- $\sigma(1, x) = \sigma(x, 1) = \varepsilon(x)$

$\Rightarrow \sum \sigma(x_1, y_1)\sigma(x_2 y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2 z_2)$.
$\forall$ homogeneous $x, y, z \in H$

③: Verify that identity map on H induces an isomorphism of Hopf algebras between the Zhang twist $H^{\phi_1 \circ \phi_2}$ and the 2-cocycle twist H^{σ}. ■

EXAMPLE : APPLICATION TO MANIN'S UNIVERSAL QUANTUM GROUPS

IDEA

- Examine Zhang twists of Manin's universal quantum groups of quadratic algebras
- Automorphisms coming from the underlying algebras
- Connect to 2-cocycle twists of these universal quantum groups

MANIN'S UNIVERSAL QUANTUM GROUPS OF QUADRATIC ALGEBRAS

- A = quadratic algebra and write as,

$$A = \Bbbk\langle A_1\rangle / \langle R(A)\rangle,$$
$$R(A) \subseteq A_1 \otimes A_1,\ \dim_{\Bbbk} A_1 < \infty$$

- $\underline{\mathrm{end}}^l(A)$ = universal bialgebra that lefts coacts on A via $\rho_A : A \longrightarrow \underline{\mathrm{end}}^l(A) \otimes A$

- $\underline{\mathrm{aut}}^l(A)$ = universal quantum group that left coacts on A

- $A \bullet B := \dfrac{\Bbbk\langle A_1 \otimes B_1\rangle}{\langle S_{(23)}(R(A) \otimes R(B))\rangle}$

bullet product of A and B

$S_{(23)}$: flip of middle two tensor factors in the 4-fold tensor product

- $\underline{\mathrm{end}}^r(A) \cong A \bullet A^!$ and $\underline{\mathrm{end}}^l(A) \cong A^! \bullet A$

Link

1st proof ① lit ② own proofs – bridge condition ③

④ Manin itself

⑤ general proof of corollary

⑥ Tim Hodges – FRT construction + Manin's
Toby Stafford links
lawrence about twists

⑦ Examples of bialg satisfying twisting pair condition (group algebra case)

OUR CONSTRUCTION

- $\underline{\text{end}}(A)$ and $\underline{\text{aut}}(A)$ both satisfy our twisting conditions
- $\boxed{\text{GOAL}}$: Construct a twisting pair for $\underline{\text{end}}^l(A)$ (or $\underline{\text{aut}}^l(A)$)

I.e., $\left(\underline{\text{end}}^r((\phi^{-1})^!, \underline{\text{end}}^l(\phi))\right)$ = twisting pair for $\underline{\text{end}}^l(A)$

① Show that $\underline{\text{end}}^l(A) \cong \underline{\text{end}}^r(A^!)$

② Define $\underline{\text{end}}^r(\phi)$ and $\text{end}^l(\phi^!)$ as graded algebra endomorphisms of $\underline{\text{end}}^r(A) \cong A \bullet A^! \cong \underline{\text{end}}^l(A^!)$

③ (Ueyama) $\left({}^{\phi}A\right)^! \cong^{(\phi^{-1})^!} (A^!) \cong (A^!)^{\phi^!}$

④ $\left(\underline{\text{end}}^r((\phi^{-1})^!, \underline{\text{end}}^l(\phi))\right)$ = twisting pair for $\underline{\text{end}}^l(A)$

- $\underline{\text{end}}^l(A^{\phi}) \cong \underline{\text{end}}^l(A)^{\underline{\text{end}}^l(\phi)\circ\underline{\text{end}}^r((\phi^{-1})^!)}$
  bialgebra isomorphism
- Lift $\underline{\text{end}}^l(A^{\phi})$ to $\underline{\text{aut}}^l(A^{\phi})$
  recall $\mathcal{H}(\underline{\text{end}}(A)) \cong \underline{\text{aut}}(A)$
- $\left(\text{aut}^r((\phi^{-1})^!), \text{aut}^l(\phi)\right)$ = twisting pair for $\underline{\text{aut}}^l(A)$

- $\underline{\text{aut}}^l(A^{\phi})$ is a 2-cocycle twist of $\underline{\text{aut}}^l(A)$ with the 2-cocycle $\sigma : \underline{\text{aut}}^l(A) \otimes \underline{\text{aut}}^l(A) \longrightarrow \Bbbk$ given by $\sigma(g,h) = \varepsilon(g)\,\varepsilon(\underline{\text{aut}}^l(\phi)^{|g|}(h))$ for $\forall$ homogeneous elements $g, h \in \underline{\text{aut}}^l(A)$

DEFINITION

Proposition

THEOREM

PROPOSITION

PROPOSITION