Quantizing the maximal spectrum

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Recent Advances and New Directions in the Interplay of Noncommutative Algebra and Geometry

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In honor of S. Paul Smith

Motivating question

Question: What is the “noncommutative space” corresponding to a noncommutative algebra?

Motivation:

- Solution would yield a rich invariant for noncommutative rings, which could control “coarser” invariants
- Help us “see” which rings are “geometrically nice”
- Another perspective on the state space of a quantum system

Modeling all rings via "spaces"?

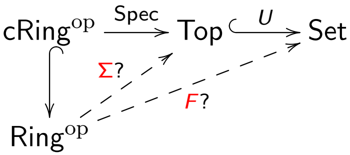
Question: What is the "noncommutative space" corresponding to a noncommutative algebra?

To make this a rigorous problem, we should first set some requirements:

(A) Keep the classical construction if the ring is commutative.
(B) Make it a *functorial* construction.
(To ensure it's truly geometric, and to aid computation.)
(C) Assign a nontrivial spectrum to each nonzero ring.

- Lots of interesting noncommutative spectra satisfy (A) and (C).
- Fewer satisfy both (A) and (B), and they all tend to fail (C).
- Sadly, this is not a coincidence...

Obstructions to Spec functors



Theorem (R., 2012): Any functor $F\colon \mathrm{Ring}^{\mathrm{op}} \to \mathrm{Set}$ whose restriction to the full subcategory $\mathrm{cRing}^{\mathrm{op}}$ is isomorphic to Spec has $F(\mathbb{M}_n(\mathbb{C})) = \varnothing$ for $n \geq 3$. (Same holds for C*-algebras.)

- Still holds when replacing $\mathbb{C}$ by any ring (Ben-Zvi, Ma, R. 2017)
- Proof depends a "no-hidden-variables" theorem from quantum mechanics! (Kochen-Specker, 1967)

Topology without sets

Does the problem lie with points? There are "point-free" approaches to topology. Try a different forgetful functor?

$$\begin{array}{lcl} \mathsf{Top} & \longrightarrow & \mathsf{Set} \\ \downarrow & & \\ \mathsf{Loc} & & \text{("pointless" spaces)} \\ \downarrow & & \\ \mathsf{Topos} & & \text{(categories of sheaves)} \end{array}$$

Theorem (van den Berg, Heunen 2014): The same type of obstruction holds when viewing Spec as a functor to Loc or Topos.

Interpretation: Sets and topologies are "too commutative" to allow for a NC spectrum functor

A deeper question

Q: What objects should be viewed as *noncommutative sets* (i.e., NC *discrete* spaces) in NCG?

Ideal solution: a fully faithful embedding $\mathsf{Set} \hookrightarrow \mathfrak{S}$ into a category of NC sets, and a NC spectrum functor $\Sigma \colon \mathsf{Ring}^{\mathrm{op}} \to \mathfrak{S}$ with commuting diagram

$$\begin{array}{ccccc} \mathsf{cRing}^{\mathrm{op}} & \xrightarrow{\mathsf{Spec}} & \mathsf{Top} & \xrightarrow{U} & \mathsf{Set} \\ \big\downarrow & & & & \big\downarrow \\ \mathsf{Ring}^{\mathrm{op}} & & \xrightarrow{\quad\Sigma\quad} & & \mathfrak{S} \end{array}$$

For now: Replace Spec $\rightsquigarrow$ Max, restrict the class of algebras, and find an approximate solution. . .

From sets to “quantum sets”

$$\begin{array}{ccc} \{\text{commutative algebras}\} & \xrightarrow{\text{Spec}} & \{\text{sets}\} \\ \big\downarrow & & \big\downarrow \\ \{\text{noncommutative algebras}\} & \longrightarrow & \{???\} \end{array}$$

Inspired by quantum mechanics: If X is our set of “states,” we should also allow linear combinations of states: $X \rightsquigarrow kX = \mathrm{Span}(X)$

This vector space $Q = kX$ carries the structure of a coalgebra:

- Comultiplication $\Delta \colon Q \to Q \otimes Q$ given by $x \mapsto x \otimes x$
- Counit $\eta \colon Q \to k$ given by $x \mapsto 1$

Coalgebra maps correspond to set maps: $\mathrm{Set}(X, Y) \cong \mathrm{Coalg}(kX, kY)$.
Gives a full and faithful embedding $\mathrm{Set} \hookrightarrow \mathrm{Coalg}$.

"Quantum sets" for algebras over a field

Therefore: We view a coalgebra (Q, Δ, η) as a "quantum set" (over k). Its *algebra of observables* is the dual algebra $\mathrm{Obs}(Q) = Q^*$.

History: Coalgebras were considered as "discrete objects" by Takeuchi (1974), and in the noncommutative context by Kontsevich-Soibelman (*noncommutative thin schemes*) and Le Bruyn.

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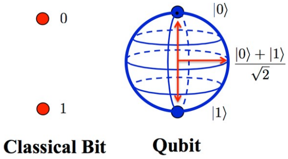
Ex: $\mathbb{M}^d = \mathbb{M}_d(k)^*$ has "algebra of observables" $(\mathbb{M}^d)^* \cong \mathbb{M}_d(k)$. Thus we may view it as a quantum d-level system or qu*d*it.



http://qoqms.phys.strath.ac.uk/research_qc.html

"Underlying coalgebras" of k-schemes

Motivating fact: The underlying set $|X|$ of a Hausdorff space X is the directed limit of its finite discrete subspaces.

Note: A scheme S finite over k is of the form $S \cong \operatorname{Spec}(B)$ for f.d. algebra B. The functor $S \mapsto \Gamma(S, \mathcal{O}_S)^* \cong B^*$ is an equivalence

$$\{\text{finite schemes over } k\} \xrightarrow{\sim} \{\text{f.d. cocomm. coalg's}\}$$

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Def: For a k-scheme X, the coalgebra of distributions is

$$\mathrm{Dist}(X) = \varinjlim \Gamma(S, \mathcal{O}_S)^*,$$

where S ranges over the closed subschemes of X that are finite over k. This gives a functor $\mathrm{Dist}\colon \mathsf{Sch}_k \to \mathsf{Coalg}$.

Local nature of distributions

Affine case: $X = \operatorname{Spec}(A)$ with A an affine commutative k-algebra. Distributions given by the finite dual coalgebra

$$\operatorname{Dist}(X) \cong A^\circ := \varinjlim (A/I)^*$$

where I ranges over all ideals of finite codimension.

Theorem: Suppose X is of finite type over k, and let X_0 be its set of closed points.

1. There is an isomorphism of coalgebras $\operatorname{Dist}(X) \cong \bigoplus_{x \in X_0} (\mathcal{O}_{X,x})^\circ$
2. If $k = \overline{k}$, then $\operatorname{Dist}(X)$ has a subcoalgebra isomorphic to kX_0.

Moral: $\operatorname{Dist}(X)$ linearizes the set of closed points, and includes the formal neighborhood of each point.

The quantized maximal spectrum

Thus for commutative affine k-algebras A,

$$A^\circ \cong \mathrm{Dist}(\mathrm{Spec}(A)) \cong \bigoplus_{\mathfrak{m} \in \mathrm{Max}(A)} (A_{\mathfrak{m}})^\circ$$

Thesis: For affine k-algebras with “many” f.d. rep’s, the functor $A \mapsto A^\circ$ is a suitable candidate for a quantized maximal spectrum.

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We end up with a diagram similar to our goal:

$$\begin{array}{ccccc} \mathsf{cAff}^{\mathrm{op}} & \xrightarrow{\mathsf{Max}} & \mathsf{Top} & \xrightarrow{U} & \mathsf{Set} \\ \big\downarrow & & & & \big\downarrow \\ \mathsf{Alg}^{\mathrm{op}} & & \xrightarrow{(-)^\circ} & & \mathsf{Coalg} \end{array}$$

But it does not commute! ("Best approximation" in a sense...)

Fully RFD algebras

What does it mean for A to have "many" f.d. representations?

Def: A is left fully residually finite-dimensional if every f.g. left A-module is a subdirect product of f.d. left modules.

Examples:

- Affine noetherian PI algebras
- In particular, lots of "quantum algebras" at roots of unity

Thm: A is left fully RFD $\iff$ A° is a cogenerator in A-Mod.

We are still left with:

Problem: If A° is an important invariant, how can we compute it?

Twisted tensor products

Many concrete examples of NC algebras arise as twisted tensor products of algebras A and B: Given a suitably "nice" linear map

$$\rho\colon B \otimes A \to A \otimes B,$$

$A\#_\rho B$ is the vector space $A \otimes B$ with multiplication

$$m_\rho := (m_A \otimes m_B) \circ (\mathrm{id}_A \otimes \rho \otimes \mathrm{id}_B)$$

Familiar examples:

- Tensor product $A \otimes B$ (if ρ is just the "tensor swap")
- Ore extensions $A[t; \sigma, \delta] \cong A\#_\rho\, k[t]$
- Smash products $A\#H$, where $H \curvearrowright A$

Crossed product coalgebras

Q: Can we describe $(A\#_\rho B)^\circ$ in terms of A° and B°?

For coalgebras C and D and a suitably "nice" linear map

$$\phi\colon C \otimes D \to D \otimes C$$

we can form a crossed product coalgebra $C\#^\phi D$ in a formally dual manner (Caenepeel, Ion, Militaru, Zhu 2000).

Naive guess: Is $(A\#_\rho B)^\circ \cong A^\circ\#^\phi B^\circ$ for some map ϕ?

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Not always! For $A = k[x]$ and $B = k[y]$, choose ρ so that $A\#_\rho B$ has no f.d. modules. Then $(A\#B)^\circ = 0 \neq A^\circ \#^\phi B^\circ$ for any ϕ!

Topology to the rescue

Better guess: $(A\#_\rho B)^\circ \cong A^\circ \#^\phi B^\circ$ whenever $\phi = \rho^\circ$ makes sense.

This hints that we should interpret $(-)^\circ$ as a functor beyond just algebras...

Denote:

- Top_k the category of linearly topologized k-vector spaces
- CF_k full subcategory of spaces with *cofinite* topology (all open subspaces have finite codimension)
- Continuous dual $(-)^\circ := \mathsf{Top}_k(-, k)\colon \mathsf{Top}_k^{\mathrm{op}} \to \mathsf{Vect}_k$

What justifies the notation?

The finite dual is a continuous dual

Monoidal structure: $E \otimes^! F$ is the tensor product topologized by the subspaces $E \otimes F_0 + E_0 \otimes F$ for open $E_0 \subseteq E$, $F_0 \subseteq F$.

Theorem: The continuous dual forms a lax monoidal functor

$$(-)^\circ \colon (\mathsf{Top}_k, \otimes^!, k)^{\mathrm{op}} \to (\mathsf{Vect}_k, \otimes, k)$$

which restricts to a strong monoidal functor on $(\mathsf{CF}_k, \otimes^!, k)$.

(Lax vs. strong: $E^\circ \otimes F^\circ \hookrightarrow (E \otimes F)^\circ$ vs. isomorphism...)

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Corollary: Let A be an algebra. Equip it with the linear topology whose open ideals are the ideals of finite codimension. Then the continuous dual of A is the finite dual A° and

$$\Delta_{A^\circ} = m_A^\circ \colon A^\circ \to (A \otimes^! A)^\circ \xrightarrow{\sim} A^\circ \otimes A^\circ.$$

Duals of twisted tensor products

Theorem: Let A and B be algebras with a twisting map ρ. Suppose that ρ is continuous when viewed as a map

$$\rho\colon B \otimes^! A \to A \otimes^! B.$$

Then $(A \#_\rho B)^\circ \cong A^\circ \#^{\rho^\circ} B^\circ$.

Fact: ρ continuous $\iff A \otimes^! B \xrightarrow{\sim} A \#_\rho B$ in Top_k.

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When does this happen? A sufficient condition:

Lem: For $\rho\colon B \otimes A \to A \otimes B$, suppose there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ finite extensions such that ρ restricts to the "tensor swap" on $A_0 \otimes B$ and $A \otimes B_0$. Then ρ is continuous, so $(A\#_\rho B)^\circ \cong A^\circ \#^{\rho^\circ} B^\circ$.

A few applications

Ex: $\theta \in \mathrm{Aut}(A)$ finite order, then

$$A[t;\theta]^\circ \cong A^\circ \#^\phi k[t]^\circ$$

Ex: A a left Hopf H-module algebra. Suppose A is finite over A^H and $H \curvearrowright A$ factors through a finite-dimensional Hopf algebra. Then

$$(A\#H)^\circ \cong A^\circ \#^\phi H^\circ$$

Ex: Fix bialgebras A and B with twisting and cotwisting maps forming a crossed product bialgebra $H = A\#_\rho^\phi B$. If ρ, ϕ are continuous with respect to $A \otimes^! B$ and $B \otimes^! A$, then as bialgebras

$$H^\circ \cong A^\circ \#_{\phi^\circ}^{\rho^\circ} B^\circ$$

"Picturing" the quantum plane

$\mathcal{O}_q(k^2) = k[x]\#_\rho k[y]$ where $\rho = \rho_q \colon y^j \otimes x^i \mapsto q^{ij} x^i \otimes y^j$. This is continuous if q is a root of unity, say of order n. We get

$$\mathcal{O}_q(k^2)^\circ \cong k[x]^\circ \#^{\rho^\circ} k[y]^\circ \cong \mathrm{Dist}(\mathbb{A}^1_k)\#^{\phi_q} \mathrm{Dist}(\mathbb{A}^1_k).$$

How to *visualize* it? Helps to relate it (functorially!) to the center...

Assume $k = \overline{k}$, set $G_q = \langle q \rangle^2 \subseteq k^2 = \mathbb{A}^2_k$, and form the quotient scheme and open subscheme

$$\operatorname{Spec} Z(\mathcal{O}_q(k^2)) \cong \mathbb{A}^2_k / G_q \supseteq D(xy)/G_q =: U_q.$$

Azumaya locus technique then gives

$$\mathcal{O}_q(k^2)^\circ \cong (\mathbb{M}^n \otimes \mathrm{Dist}\, U_q) \oplus \varinjlim_i (\mathcal{O}_q(k^2)/(xy)^i)^\circ$$

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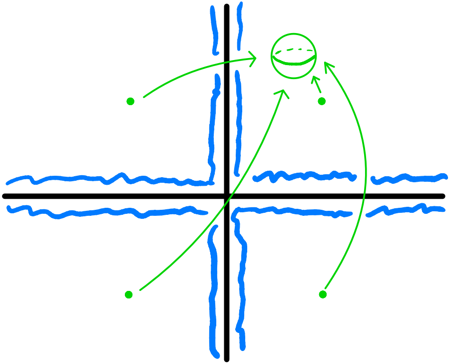
$$\mathcal{O}_q(k^2)^\circ \cong \mathrm{Dist}(\mathbb{A}^1_k)\#^{\phi_q}\mathrm{Dist}(\mathbb{A}^1_k)$$
$$\cong (\mathbb{M}^n \otimes \mathrm{Dist}\, U_q) \oplus \varinjlim(\mathcal{O}_q(k^2)/(xy)^i)^\circ$$

Case $n = 2$: $q = -1$ and $\mathbb{M}^2 = $ qubit



Thank you. . .

And congratulations to Paul!