# Dual Reflection Groups

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Joint work with Pete Goetz, Ellen Kirkman and Kent Vashaw

Throughout, let $k$ be an algebraically closed field with $\operatorname{char} k = 0$.

Let $G$ be a finite subgroup of $\mathrm{GL}_n(k)$. Then $G$ acts on $A = k[x_1, \ldots, x_n]$ via the linear action on $A_1 = kx_1 + \cdots + kx_n$.

The invariant ring is

$$A^G = \{ f \in A \mid g.f = f \ \ \forall g \in G\}.$$

Then $A^G$ is a Noetherian graded ring, and $A$ is a finitely generated $A^G$-module.

There are many interesting results in this context; in this talk we will focus on the following:

**Theorem (Chevalley-Shephard-Todd)**

*The invariant ring $A^G$ is isomorphic to a polynomial ring if and only if $G$ is generated by pseudoreflections.*

An element $g \in \mathrm{GL}_n(k)$ is a pseudoreflection if it is not the identity, has finite order, and fixes a hyperplane in $k^n$.

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**Question**

What are some extensions of this result to the context of noncommutative algebras?

One often broadens the context as follows:

- Require $A$ to be an AS-regular algebra over $k$. (Recall that a $k$-algebra $A$ is AS-regular if $A$ is connected graded, $AS$-Gorenstein ring of finite global dimension and finite GK dimension.)

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- Replace the $G$-action on $A$ with the action of a (usually semisimple) Hopf algebra $H$, acting linearly. This action is defined via the coproduct of $H$:

$$h.(ab) = \sum (h_{(1)}.a)(h_{(2)}.b)$$

where we have used Sweedler notation: $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$.

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In this context, the invariant ring becomes

$$A^H = \{f \in A \mid h.f = \epsilon(h)f\}.$$

where $\epsilon : H \to k$ is the counit of $H$.

**Definition (Kirkman-Kuzmanovich-Zhang)**

If $A$ is an AS-regular algebra, and $H$ acts linearly and inner faithfully on $A$ such that $A^H$ is also an AS-regular algebra, we say that $H$ is a *reflection Hopf algebra for* $A$.

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To search for examples of reflection Hopf algebras $H$ acting on (quadratically defined, say) AS-regular algebras $A$, one may:

- Fix a Hopf algebra $H$
- Choose an inner-faithful representation $V$ of $H$. This determines a linear action of $H$ on $T(V)$, the tensor algebra of $V$.
- Let $W$ be a $H$-subrepresentation of $V \otimes V$. Let $I = \langle W \rangle$, and let $A = T(V)/I$. Restrict your choices of $W$ such that $A$ is AS-regular. (This is the tricky part!)

In this talk, we will focus on the particular case of $H = (kG)^*$, the linear dual of the group algebra of $G$ over $k$.

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- To choose $W$, one need only choose a set of $G$-homogeneous relations to impose in order to define $A = T(V)/\langle W \rangle$.
- The counit of $(kG)^*$ sends the dual basis element corresponding to nonidentity group elements to zero and that of the identity to $1$. Therefore the invariant subring is the component of $A$ in the identity group grade, denoted $A_e$.

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One calls such a group $G$ a *dual reflection group* for the AS-regular algebra $A$.
Finding such examples was the content of Vashaw's 2016 Master's Thesis.

How can one recognize such behavior? We give some necessary conditions due to Kirkman-Kuzmanovich-Zhang.

## Definition

Let $G$ be a finite group and let $\mathcal{R}$ be a generating set of $G$. For $g \in G$, the length of $g$ with respect to $\mathcal{R}$ is defined to be:

$$\ell_{\mathcal{R}}(g) := \min\{r \mid v_1 \cdots v_r = g \text{ for some } v_i \in \mathcal{R}\}.$$

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The Poincaré polynomial is the same as the Hilbert series of the associated graded ring of $kG$ with respect to the length filtration defined by $\mathcal{R}$. This algebra is called the Hasse algebra by [KKZ] and the nilCoxeter algebra by Fomin-Stanley.

## Theorem (KKZ)

*Let $A$ be a Noetherian AS-regular domain generated in degree 1. Let $G$ coact on $A$ inner-faithfully as a dual reflection group. Let $\mathcal{R}$ be the set of nonidentity group grades present in the representation $A_1$. Then the following hold:*

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1. *There is a set of homogeneous elements $\{f_g \mid g \in G\} \subseteq A$ with $f_e = 1$ such that $A = \bigoplus_{g \in G} A_g$ and $A_g = f_g A_e = A_e f_g$ for all $g \in G$.*

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2. $\deg f_g = \ell_{\mathcal{R}}(g)$ *for all $g \in G$.*
3. *The Poincaré polynomial of $G$ with respect to $\mathcal{R}$ is a product of cyclotomic polynomials (and is hence palindromic).*
4. *The unique element $m$ of $G$ of longest length with respect to $\mathcal{R}$ is called the mass element of $G$ with respect to $\mathcal{R}$. For this element $m$, the element $f_m$ defined above is normal in $A$.*

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5. *The left ideal $J = A(A_e)_{\geq 1}$ is two-sided, and the quotient (called the covariant ring of the action, denoted $A^{\text{cov}} := A/J$) is a skew Hasse algebra of $G$ with respect to $\mathcal{R}$ and is a Frobenius algebra.*

Consider the group that Magma calls $M_{16}$ (`SmallGroup(16,6)`), and presents as:

$$G = \langle a, b, c, d \mid a^2 = c, b^2 = d^2 = e, c^2 = d, ba = abd, c, d \text{ central}\rangle.$$

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Consider the generating set of $G$:

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This generating set has Poincaré polynomial

$$(1+t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4.$$

If we consider $k\langle x_1, x_2, x_3, x_4\rangle$ with grades coming from $\mathcal{R}$, we may impose the following quadratic relations to obtain an algebra $A$:

$$\begin{aligned} x_1^2 &= x_4^2 \text{ (grade } c) & x_2^2 &= x_3^2 \text{ (grade } cd) \\ x_4x_1 &= x_2x_3 \text{ (grade } b) & x_1x_4 &= x_3x_2 \text{ (grade } bd) \\ x_1x_3 &= x_2x_4 \text{ (grade } bc) & x_3x_1 &= x_4x_2 \text{ (grade } bcd) \end{aligned}$$

If we make the change of variable

$$\begin{aligned} Y_1 &= x_1 + x_2 + x_3 + x_4 \\ Y_2 &= -x_1 + x_2 - x_3 + x_4 \\ Z_1 &= -x_1 + x_2 + x_3 - x_4 \\ Z_2 &= -x_1 - x_2 + x_3 + x_4 \end{aligned}$$

we find that the following relations hold:

$$\begin{aligned} Y_2Y_1 = -Y_1Y_2 &\qquad Z_2Z_1 = -Z_1Z_2 \\ Y_1Z_1 = -Z_2Y_2 &\qquad Y_2Z_1 = Z_2Y_1 \\ Y_1Z_2 = Z_1Y_2 &\qquad Y_2Z_2 = -Z_1Y_1 \end{aligned}$$

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These relations show that $A$ is a (left or right) double Ore extension of $k_{-1}[Y_1, Y_2]$ in the sense of Zhang-Zhang. It is therefore AS-regular of dimension four.

The identity component is:

$$A_e = k[x_1x_2, x_2x_1, x_3x_4, x_4x_3].$$

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Pete will talk about a particularly interesting example also found using this method, together with some potential new tools to study such algebras.

Thank you!