Noetherian rings with Auslander dualizing complex are bounded factorization

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joint work with Jason Bell, Ken Brown, and Daniel Smertnig

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Setting

Let R be a ring (unital, usually noncommutative), with $R^\bullet$ its submonoid of cancellative (regular) elements, and $R^\times$ its group of units.

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Remark

During this talk, you may suppose that R is a domain and then $R^\bullet = R \setminus \{0\}$ and you are not going to miss something ...

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- A **factorization** of a non-unit $a \in R^\bullet$ is a representation

$$a = u_1 u_2 \cdots u_k \qquad \text{with } u_1, \ldots, u_k \text{ atoms},$$

Lengths and Bounded Factorization (BF)

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The commutative world

Theorem

Every **commutative** noetherian domain has BF.

Noncommutative setting

Question

Do noncommutative noetherian domains/prime rings have BF?

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? Group ring kG, that G is polycyclic-by -finite group.

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We say j is **finitely partitive on** α (α an ordinal) if for every $M \in \text{Mod}_f(R)$ with $j(M) = \alpha$, there is a finite bound on the length of chains of submodules of M of the form

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

with $j(M_{i+1}/M_i) = \alpha$ for all $0 \leq i < n$.

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Lemma

Let R be a noetherian ring having a map j so that, for some fixed ordinal α, the map j is finitely partitive on α and $j(R/aR) = \alpha$ for all non-units $a \in R^{\bullet}$. Then R has BF.

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1. Let M be a left [right] R-module. The **(homological) grade** of M is

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2. A noetherian ring R is **Auslander-Gorenstein** if
   - R has finite (and equal) right and left injective dimension, and
   - R satisfies the Auslander condition, i.e., for every left [right] R-module M and all $i \geq 0$, $j(N) \geq i$ for every submodule N of $\operatorname{Ext}^i_R(M, R)$.

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For an Auslander-Gorenstein ring R, the grade j is exact and finitely partitive ($-j$ is an exact finite partitive dimension function).

Theorem

Every Auslander-Gorenstein ring is a BF-ring.

Many rings are Auslander-Gorenstein, e.g.:

- All known noetherian Hopf algebras (group algebras of polycyclic-by-finite groups, enveloping algebras of finite-dimensional Lie algebras, quantised enveloping algebras, connected Hopf-algebras of finite Gelfand-Kirillov dimension, quantised coordinate rings of semisimple groups).
- Rings with locally finite $\mathbb{N}$-filtration whose associated graded ring is commutative Gorenstein (e.g., Weyl algebras $A_n(k)$).
- Local FBN rings of finite global dimension.
- Sklyanin algebras.

Auslander-dualizing complexes

- Auslander-Gorenstein rings always have finite injective dimension.
- For noetherian algebras over a field there is the more general notion of an Auslander-dualizing complex, developed by Yekutieli and Zhang.
- Again: define a grade j, with $-j$ being finitely partitive and an exact dimension function ($-j$ is the **canonical dimension function**); extend to non-f.g. modules.
- Harder to establish $j(R/xR) = j(R) + 1$ (uses **Gabber's maximality principle**)

Theorem (Bell, Brown, N., Smertnig)

Let R be a noetherian K-algebra with an Auslander dualizing complex. If P is a prime ideal of R, then R/P is a BF-ring.

Still Open:

Still open: Does every noetherian prime ring [noetherian domain] have bounded factorizations?

More specifically, what about the following settings:

- FBN rings — does the variant of the principal ideal theorem hold without the condition that every nonzero ideal contains a nonzero central element?
- Noetherian semigroup algebras of the form $K[S]$ with S a submonoid of a polycyclic-by-finite group satisfying ACC on left [right] ideals.
- Affine noetherian algebras of finite Gelfand-Kirillov dimension.