TOWARDS A RELATIVE SUPPORT THEORY

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REPRESENTATIONS OF A FINITE GROUP

<u>Strong Maschke's Theorem:</u> kG is semisimple if and only if the characteristic of k does <u>not</u> divide the order of G.

<u>Artin-Wedderburn Theorem:</u> Semisimple rings are isomorphic to a product of finitely many matrix rings over division rings.

We can measure the failure of semisimplicity using the $\boxed{\text{stable category.}}$

2

THE BALMER SPECTRUM

<u>Commutative algebra:</u>		<u>Tensor triangular geometry:</u>
R ring	algebraic object	$\mathcal{K}$ $\otimes$-Δ-cat
$\rightsquigarrow$	$\rightsquigarrow$	$\rightsquigarrow$
$\operatorname{Spc}(R)$	topological space	$\operatorname{Spc}(\mathcal{K})$

This comes with a universal notion of support that detects thick subcategories.

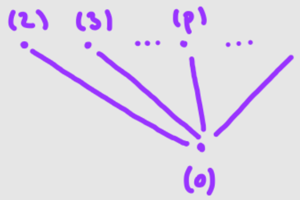
③

THIS IS USEFUL!

Example: R commutative Noetherian: $\mathrm{Spec}(R) \simeq \mathrm{Spc}(D^{perf}(R)) \simeq \mathrm{Spc}(K^b(\mathrm{proj}\, R))$.

$R = \mathbb{Z}$:

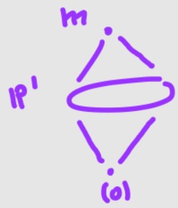
$\mathrm{Spec}(\mathbb{Z}) \simeq \mathrm{Spc}(K^b(\mathrm{proj}\, \mathbb{Z}))$



$G = \mathbb{Z}_2 \times \mathbb{Z}_2$:

$\mathrm{Spc}(\mathrm{st}(\mathrm{mod}\, kG)) \simeq \mathrm{Spc}\left(\dfrac{D^b(\mathrm{mod}\, kG)}{K^b(\mathrm{proj}\, kG)}\right)$

Lattice:



$\mathrm{Max}(H^\bullet(G,k))$:

$\mathrm{Proj}(H^\bullet(G,k))$:

④

SUPPORT THEORIES

Depending on the object of interest, they specialize in different homologies:

G group $\longrightarrow$ $H^\bullet(G,k)$ group cohomology

A Hopf algebra $\longrightarrow$ $H^\bullet(A,k)$ Hopf cohomology

A unital associative algebra $\longrightarrow$ $H^\bullet(A,A)$ Hochschild cohomology

If there is a natural subalgebra $B \subset A$, these theories ignore it.

⑤

RELATIVE HOCHSCHILD COHOMOLOGY

Can handle natural subalgebras: for $B \subseteq A$ unital algebras:

$HH^i(A) := \mathrm{Ext}^i(A,A)$ and $HH^\bullet(A) := \bigoplus HH^i(A)$

$HH^i_{(A,B)}(A) = Ext^i_{(A^e,B^e)}(A,A)$ and $HH^\bullet_{(A,B)}(A) = \bigoplus_{i\in\mathbb{N}} HH^i_{(A,B)}(A)$.

Theorem: 1. $HH^\bullet_{(A,B)}(A)$ is a graded commutative algebra with a cup product.

2. $HH^{\bullet-1}_{(A,B)}(A)$ is a graded Lie algebra.

3. $HH^\bullet_{(A,B)}(A)$ is a Gerstenhaber algebra.

RELATIVE HOMOLOGICAL ALGEBRA

⑥

Let $B \subseteq A$ unital subring.

(A,B)-exact:

$$\cdots \longrightarrow M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$

(i) $\ker(d_i) = \mathrm{im}(d_{i+1})$ ⟵ A-exact.

(ii) $M_i \cong \ker(d_i) \oplus Q_i$ in mod B.

Equivalently:

$$\begin{tikzcd} \cdots \arrow[r] & M_i \arrow[r, "d_i"] & M_{i-1} \arrow[l, bend left, "s_i"] \arrow[r] & \cdots \end{tikzcd}$$

(1) Over mod B we have:

$d_i d_{i+1} = 0$

$d_{i+1} s_i + s_i d_i = 1_{M_i}$

(2) Over mod B $M_\cdot$ is split exact.

SPECIAL MODULES

⑦

(A,B)-free: $A \otimes_B X$, X in mod B.

$$\begin{tikzcd} \mathrm{mod}\,A \arrow[r, phantom, "\perp"] & \mathrm{mod}\,B \arrow[l, bend right, "A\otimes_B -"'] \end{tikzcd}$$

(A,B)-projective:

$$\begin{tikzcd} & P \arrow[dl, dashed, "h'_A"'] \arrow[d, "h_A"] & \\ M \arrow[r, "g_A"] & N \arrow[l, bend left, "s_B"] \arrow[r] & 0 \end{tikzcd}$$

Bottom row is (A,B)-exact.

✱ **(A,B)-flat:** For every (A,B)-exact $0 \to L \to M \to N \to 0$ then:

$$0 \longrightarrow L \otimes_A F \longrightarrow M \otimes_A F \longrightarrow N \otimes_A F \longrightarrow 0 \quad \text{is } (\mathbb{Z},\mathbb{Z})\text{-exact.}$$

RELATIVE LONG EXACT SEQUENCE: TOR

⑧

Theorem: Let $0 \to K \to L \to M \to 0$ be an (A,B)-exact sequence of right A-modules. Then for every left A-module N:

$$\cdots \rightleftarrows \mathrm{Tor}_{i+1}^{(A,B)}(M,N) \to \mathrm{Tor}_i^{(A,B)}(K,N) \rightleftarrows \mathrm{Tor}_i^{(A,B)}(L,N) \rightleftarrows \mathrm{Tor}_i^{(A,B)}(M,N) \to \cdots$$

is split exact in 2-out-of-3 terms.

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Application

Theorem: (Relative Künneth Theorem) Let $(M_\cdot, m_\cdot)$ be a complex of right A-modules in the relative setting. Let $(N_\cdot, n_\cdot)$ be a complex of left A-modules in the relative setting. Then:

$$\bigoplus_{r+s=i} H_r(M_\cdot) \otimes_A H_s(N_\cdot) \rightleftarrows H_i(M_\cdot \otimes_A N_\cdot) \rightleftarrows \bigoplus_{r+s=i-1} \mathrm{Tor}_1^{(A,B)}(H_r(M_\cdot), H_s(N_\cdot))$$

are split short exact sequences of $\mathbb{Z}$-modules.

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Thank you!

(A,B)-Flat

⊛ not the usual definition

For every (A,B)-exact $0 \to L \to M \to N \to 0$ then:

$$0 \to L \otimes_A F \longrightarrow M \otimes_A F \longrightarrow N \otimes_A F \longrightarrow 0 \quad \text{is } (\varkappa,\varkappa)\text{-exact.}$$

Remark:

(A,B)-flat modules preserve (A,B)-exact sequences:

$(M_\bullet, d_\bullet)$ right (A,B)-exact then

$(M_\bullet \otimes_A F, d_\bullet \otimes 1_F)$ is $(\varkappa,\varkappa)$-exact.

Theorem: The following are equivalent:

(1) F is (A,B)-flat.

(2) $\mathrm{Tor}_i^{(A,B)}(M,F) = 0$ for all M and i.

(3) $\mathrm{Tor}_1^{(A,B)}(M,F) = 0$ for all M.

Application

⊛ (A,B)-flat is unusual

Given $0 \to L \rightleftarrows M \rightleftarrows N \to 0$ (A,B)-exact:

F (A,B)-flat:

$$0 \to L \otimes_A F \rightleftarrows M \otimes_A F \rightleftarrows N \otimes_A F \longrightarrow 0 \quad \text{is } (\varkappa,\varkappa)\text{-exact.}$$

F "relatively flat": Weibel

$$0 \to L \otimes_A F \longrightarrow M \otimes_A F \longrightarrow N \otimes_A F \longrightarrow 0 \quad \text{is exact.}$$

Proposition: F is (A,B)-flat $\Leftrightarrow$ F is relatively flat.

Application

Proposition: F is (A,B)-flat $\Leftrightarrow$ F is relatively flat.

Proof: Given $0 \to L \rightleftarrows M \rightleftarrows N \to 0$ (A,B)-exact:

$\Rightarrow)$ Easy.

$\Leftarrow)$ Tor:
$$\cdots \rightleftarrows \mathrm{Tor}_1^{(A,B)}(N,F) \to L \otimes_A F \underset{r}{\overset{f\otimes 1}{\rightleftarrows}} M \otimes_A F \underset{s}{\overset{g\otimes 1}{\rightleftarrows}} N \otimes_A F \longrightarrow 0$$

Relatively flat:
$$0 \to L \otimes_A F \xrightarrow{f\otimes 1} M \otimes_A F \xrightarrow{g\otimes 1} N \otimes_A F \longrightarrow 0$$

$(f\otimes 1)r(f\otimes 1) = f\otimes 1$, $(g\otimes 1)s(g\otimes 1) = g\otimes 1$ $\square$.