Balmer Spectra and Drinfeld Centers

Kent Vashaw

MIT

kentv@mit.edu

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Favorite example: $\mathrm{mod}(H)$, H a finite-dimensional Hopf algebra. (In fact, all finite tensor categories are equivalent to $\mathrm{mod}(A)$ for a finite-dimensional algebra A as abelian categories.)

The stable category

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Straightforward: objects A and B are isomorphic in $\mathrm{st}(\mathbf{C})$ if and only if there exist projective objects P and Q with $A \oplus P \cong B \oplus Q$ in $\mathbf{T}$.

Lemma

The stable category of a finite tensor category is a monoidal triangulated category.

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- Equivalently, these are thick ideals $\mathbf{P}$ such that $A \otimes \mathbf{K} \otimes B \subseteq \mathbf{P}$, then A or $B \in \mathbf{P}$.
- Topology: closed sets generated by $V(A) := \{\mathbf{P} \in \mathrm{Spc}\,\mathbf{K} : A \notin \mathbf{P}\}$.

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- Construction: objects are pairs (A, γ) where $A \in \mathbf{C}$ and $\gamma_B : B \otimes A \xrightarrow{\cong} A \otimes B$ a natural isomorphism called a **half-braiding**, satisfying some natural requirements. Morphisms consist of morphisms from $\mathbf{C}$ which are compatible with the half-braidings.

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Extending the forgetful functor

- Following noncommutative ring theory: for a ring R with center $Z(R)$, if $\mathfrak{p}$ is a prime ideal of R, then $\mathfrak{p} \cap Z(R)$ is a prime ideal in $Z(R)$– this is called prime ideal contraction.

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Lemma (V.)

$f(\mathbf{P})$ is a prime ideal for any prime ideal $\mathbf{P}$, and f is continuous.

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Theorem (V.)

The map $f : \text{Spc}\,\text{stmod}(H_{G,L}) \to \text{Spc}\,\text{stmod}(D(H_{G,L}))$ is a homeomorphism, and the thick ideals of the two categories are in bijection.

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By a result of Nakano-V.-Yakimov, the left hand side is $\mathrm{Proj}(\mathrm{H}^\bullet(G,\Bbbk)^L)$.

Conclusion

Thanks for your time!