Universal Quantum Semigroupoids

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Hopf algebras vs weak Hopf algebras

- k is an arbitrary field.
- A group algebra kG is a Hopf algebra.
- Is $kG \oplus kG$ still a Hopf algebra?

Hopf algebras vs weak Hopf algebras

- k is an arbitrary field.
- A group algebra kG is a Hopf algebra.
- Is $kG \oplus kG$ still a Hopf algebra? No!! But $kG \oplus kG$ is a weak Hopf algebra.

What are weak Hopf algebras?

A weak Hopf algebra is a generalization of a Hopf algebra.

Definition

1. A bialgebra H is an algebra and a coalgebra such that Δ and ε are multiplicative.
2. A *weak bialgebra* H is an algebra and a coalgebra such that Δ and ε satisfies some compatibilities (but $\Delta(1) \neq 1 \otimes 1$ and ε is not multiplicative). If H has an antipode S, then H is a *weak Hopf algebra*.

$kG \oplus kG$ is a weak Hopf algebra

$H = kG \oplus kG$ is a weak Hopf algebra. The coalgebra structure and the antipode are given by:

- $\Delta(g+h) = g \otimes g + h \otimes h$
- $\varepsilon(g+h) = \varepsilon(g) + \varepsilon(h)$
- $1_H = 1_{kG} + 1_{kG}$ (H has a "complicated" identity.)
- $S(g+h) = g^{-1} + h^{-1}$

Look:

$\Delta(1_H) = 1_{kG} \otimes 1_{kG} + 1_{kG} \otimes 1_{kG} \neq 1_H \otimes 1_H = (1_{kG} + 1_{kG}) \otimes (1_{kG} + 1_{kG})$

$$\varepsilon(gh) = \varepsilon(g1_{kG})\varepsilon(1_{kG}h) + \varepsilon(g1_{kG})\varepsilon(1_{kG}h) = 2\varepsilon(g)\varepsilon(h) \neq \varepsilon(g)\varepsilon(h),$$

for $g, h \in G$.

Hopf algebras vs weak Hopf algebras

$$\Delta(1_H) = 1_H \otimes 1_H \quad \xrightarrow{\text{weaker}} \quad \Delta(1_H) = 1_1 \otimes 1_2 \neq 1_H \otimes 1_H$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \quad \xrightarrow{\text{weaker}} \quad \varepsilon(ab) = \varepsilon(a1_1)\varepsilon(1_2 b)$$

Write $\Delta(1_H) = 1_1 \otimes 1_2$ using the sumless Sweedler notation.

$$H_s := \{\sum 1_1 \varepsilon(h1_2) \mid h \in H\} \leq H \qquad H_t := \{\sum \varepsilon(1_1 h) 1_2 \mid h \in H\}$$

A weak Hopf algebra H is a Hopf algebra iff $H_t = H_s = k1_H$.

Examples

1. Given a quiver Q, the path algebra kQ is a weak bialgebra.

2. Given a groupoid (a small category with every morphism has inverse), its groupoid algebra is a weak Hopf algebra.

Symmetries

1. If A is an $\mathbb{N}$-graded ($A = \oplus_{i \in \mathbb{N}} A_i$) and connected ($A_0 = k$) algebra, then its classical (quantum) symmetries are semigroups (bialgebras).

2. Q: What if A is not connected? i.e., A_0 has dimension bigger than 1. For example, $A = kQ$ the path alegbra.

Universal quantum semigroupoids

Set up: $A = A_0 \oplus A_1 \oplus \cdots$ with $\dim(A_0) > 1$ and $\dim(A_i) < \infty$ for $i \in \mathbb{N} \cup \{0\}$.

Definition

A left *universal quantum semigroupoid* (UQSGd) of A is a weak bialgebra $\mathcal{O} = \mathcal{O}^{\text{left}}(A)$ that left coacts on A lineraly with $A_0 \cong \mathcal{O}_t$ as left H-comodule algebras, so that for any weak bialgebra H that left coacts on A linearly with $A_0 \cong H_t$ as left H-comodule algebras, $\exists!$ a weak bialgebra map $\pi : \mathcal{O} \longrightarrow H$ that makes the diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\lambda^{\mathcal{O}}} & \mathcal{O} \otimes A \\ & \searrow^{\lambda^{H}} & \downarrow{\scriptstyle \pi \otimes id} \\ & & H \otimes A \end{array}.$$

Universal quantum semigroupoids

If $A = kQ$ for a quiver Q, then surprisingly, we have the following:

Theorem (HWWW)

The universal quantum semigroupoids $\mathcal{O}^{\mathrm{left}}(kQ)$ and $\mathcal{O}^{\mathrm{right}}(kQ)$ are each isomorphic to $\mathfrak{H}(Q)$–the Hayashi face algebra attached to the quiver Q.

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Proposition (HWWW)

Let kQ be a path algebra and let $I \subseteq kQ$ be a graded ideal which is generated in degree 2 or greater. If $\mathcal{O}^*(kQ/I)$ exists (where $*$ means 'left', or 'right'), we have

$$\mathcal{O}^*(kQ/I) \cong \mathfrak{H}(Q)/\mathcal{I},$$

for some biideal $\mathcal{I}$ of $\mathfrak{H}(Q)$.

Thank You!

Happy Birthday!

Hayashi's face algebra attached to a quiver

For a finite quiver $Q = (Q_0, Q_1)$, $\mathfrak{H}(Q)$ is a weak bialgebra. As a k-algebra,

$$\mathfrak{H}(Q) = \frac{k\langle x_{i,j}, x_{p,q} \mid i,j \in Q_0,\ p,q \in Q_1\rangle}{(R)},$$

for indeterminates $x_{i,j}$ and $x_{p,q}$ with relations R, given by:

$$\begin{cases} x_{i,j}\ x_{k,\ell} = \delta_{i,k}\delta_{j,\ell}\ x_{i,j} \\ x_{s(p),s(q)}\ x_{p,q} = x_{p,q} = x_{p,q}\ x_{t(p),t(q)} \\ x_{p,q}\ x_{p',q'} = \delta_{t(p),s(p')}\delta_{t(q),s(q')} x_{p,q}\ x_{p',q'} \end{cases}$$

for all $p, p', q, q' \in Q_1$, and $i, j, k, \ell \in Q_0$.

$$1_{\mathfrak{H}(Q)} = \textstyle\sum_{i,j \in Q_0} x_{i,j}.$$

For $a, b \in Q_\ell$, the coalgebra structure is given by

$$\Delta(x_{a,b}) = \sum_{c \in Q_\ell} x_{a,c} \otimes x_{c,b} \quad \text{and} \quad \varepsilon(x_{a,b}) = \delta_{a,b}.$$