Invariant Mixed Forms of Modular Reflection Groups

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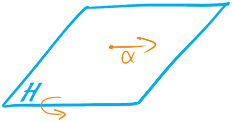

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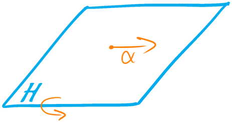

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When $\mathbb{F} = \mathbb{C}$ and $S^G = \mathbb{F}[f_1, \dots, f_n]$, yes. (Hochster, Eagon 1971).

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Theorem (D.H.)

Let G be a finite group acting linearly on $V = \mathbb{F}^n$. Any homogeneous subset $\mathcal{B}$ of $n\binom{n}{k}$ elements in $(S \otimes \wedge^k V^* \otimes V)^G$ is an S^G-basis if and only if

$$\det \mathrm{Coef}(\mathcal{B}) \neq 0 \textit{ and } \sum_{\eta \in \mathcal{B}} \deg \eta = \Delta_k \,,$$

where $\Delta_k = \sum_{H \in \mathcal{A}} \binom{n-1}{k} + (e_H - 1)(n-1)\binom{n-1}{k-1} + e_H a_{H,k}$,
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If G has no transvections, $a_{H,k} = 0$; coincides with Reiner, Shepler 2019.

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Corollary

With $R = \bigcurlywedge_{S^G} \{\omega_1, \ldots, \omega_{n-1}\}$*, we have a direct sum of* R*-submodules:*

$$(S \otimes \wedge V^* \otimes V)^G = \bigoplus_{j=1}^{n} R\,\theta_j \oplus \bigoplus_{j=1}^{n-1} R \curlywedge \omega_n\,\theta_j \oplus R\,d\theta_E .$$

Example

Let $G = \mathrm{SL}_2(\mathbb{F}_3)$. Given these basic 1-forms and their dual derivations:

$$\omega_1 = x_2^3 \otimes x_1 \otimes 1 - x_1^3 \otimes x_2 \otimes 1 \qquad \theta_1 = x_1 \otimes 1 \otimes v_1 + x_2 \otimes 1 \otimes v_2$$
$$\omega_2 = x_2 \otimes x_1 \otimes 1 - x_1 \otimes x_2 \otimes 1 \qquad \theta_2 = x_1^3 \otimes 1 \otimes v_1 + x_2^3 \otimes 1 \otimes v_2$$

with $(\omega_1 \curlywedge \omega_2) = 1 \otimes x_1 \wedge x_2 \otimes 1$ and $d\theta_E = 1 \otimes x_1 \otimes v_1 + 1 \otimes x_2 \otimes v_2$, the following is a basis of $(S \otimes \wedge V^* \otimes V)^G$ as a free S^G-module:

$$\theta_1,\ \theta_2 \quad \omega_1\theta_1\, \omega_1\theta_2\, \omega_2\theta_1\ d\theta_E, \quad (\omega_1 \curlywedge \omega_2)\theta_1,\ (\omega_1 \curlywedge \omega_2)\theta_2$$

Thank you!