Gaps and approximations in the space of growth functions

From pathological examples to natural constructions

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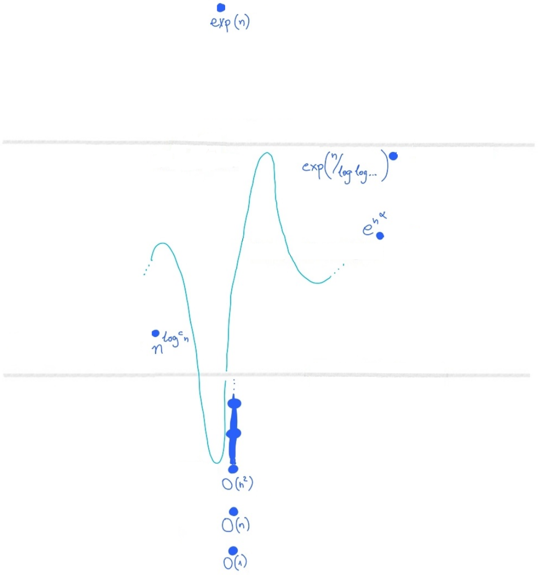
Recent Advances and New Directions in the Interplay of Noncommutative Algebra and Geometry
In Honor of S. Paul Smith on the occasion of his 65th Birthday

The space of growth functions



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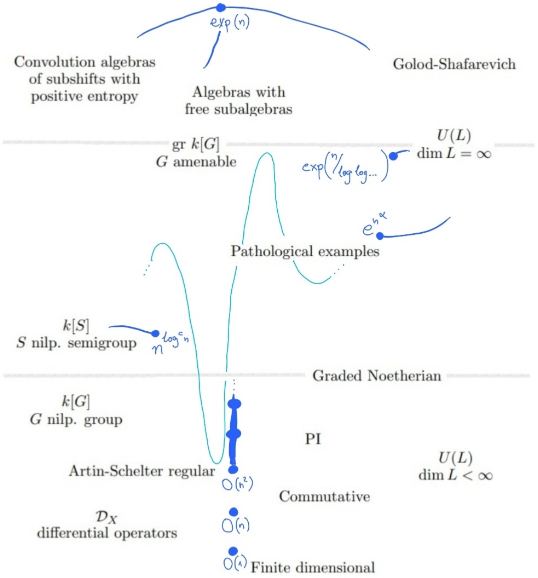

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Question (Zelmanov, '17; Alahmadi-Alsulami-Jain-Zelmanov, '17)

Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing and submultiplicative function s.t. $f(n) \succeq n^2$ (or: sufficiently rapid). Is $f \sim$ growth function of an algebra?

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But partial affirmative answers...

- $\exists$ algebra A: $f(n) \preceq \gamma_A(n) \preceq n^3 f(n)$ (Smoktunowicz-Bartholdi, '14)
- $f(n) \preceq \gamma_A(n) \preceq nf(n)$ (follows from Bell-Zelmanov '20)

Pathological examples vs. Algebras ‘from nature’

Examples are generically pathological (nilpotent radicals, non-prolongable)

Monomial algebras: Prolongable algebras $\leftrightarrow$ Subshifts; Prime $\leftrightarrow$ Transitive w/o isolated pts.; Just-infinite $\leftrightarrow$ Minimal subshifts

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Conjecture I (Zelmanov, '17; Alahmadi-Alsulami-Jain-Zelmanov, '17)

$$\left\{\begin{matrix}\textit{Growth functions*} \\ \textit{of algebras}\end{matrix}\right\} = \left\{\begin{matrix}\textit{Growth functions} \\ \textit{of primitive algebras}\end{matrix}\right\}$$

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Theorem (G., '22)

- *For any growth function f there exists a primitive algebra s.t.* $f(n) \preceq \gamma_A(n) \preceq n^2 f(n)$
- *There exist graded algebras whose growth functions encode graded nilpotent ideals, i.e. the 'Graded Conjecture I' is false*

(Convolution algebras of étale groupoids of Cantor systems)

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Theorem (G.-Zelmanov, '22)

For any increasing, submultiplicative function f and an arbitrarily slow 'distortion' $\omega(n) \to \infty$ there exists a nil algebra/Lie algebra A such that:

$$f(n/\omega(n)) \preceq \gamma_A(n) \preceq poly(n) \cdot f(n)$$

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Are there such example among semiprime algebras?

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Theorem (G.-Zelmanov, '22)

For any $\alpha, \beta \geq 2$ *and* $\gamma \in [\min\{\alpha, \beta\} + 2, \alpha + \beta]$ *there exist primitive monomial algebras* $\mathrm{GKdim}(A) = \alpha, \mathrm{GKdim}(B) = \beta, \mathrm{GKdim}(A \otimes_F B) = \gamma$.

Thank you!

Questions?