# QUANTUM FIELD THEORY SEMINAR
## (School of Wightman et al.)

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## TOPICS

# QUANTUM FIELD THEORY

The Wightman Axioms   To minimize the technicalities, I'll start with the simplest case.

[Note: We assume throughout that $c=1$ and $\hbar=1$.]

Data:

(1) A separable Hilbert space $\mathcal{H}$ ;

(2) A unitary representation $U$ of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$ ;

(3) A $\mathcal{P}_+^\uparrow$ - invariant dense linear subspace $\mathcal{D} \subset \mathcal{H}$ ;

(4) A $\mathcal{P}_+^\uparrow$ - equivariant linear map $\underset{\sim}{\varphi} : \mathcal{S}(\underset{\sim}{R}^4) \to \underline{\text{End}}\ \mathcal{D}$ ;

(5) A $\mathcal{P}_+^\uparrow$ - invariant unit vector $\Omega_0 \in \mathcal{D}$ .

Remark: The action of $\mathcal{P}_+^\uparrow$ on $\underline{\text{End}}\,\mathcal{D}$ is by conjugation, so the equivariance requirement on $\underset{\sim}{\varphi}$ is that $\forall\, f \in \mathcal{S}(\underset{\sim}{R}^4)$,

$$U(\Lambda,a)\ \underset{\sim}{\varphi}(f)U(\Lambda,a)^{-1} = \underset{\sim}{\varphi}(f_{\Lambda,a}),$$

where

$$f_{\Lambda,a}(x) = f(\Lambda^{-1}(x-a)).$$

[Note: By definition, $((\Lambda,a)\cdot f)(x) = f((\Lambda,a)^{-1}\cdot x) =$ $f((\Lambda^{-1}, -\Lambda^{-1}a)\cdot x) = f(\Lambda^{-1}x - \Lambda^{-1}a) = f(\Lambda^{-1}(x-a))$.]

This data is said to constitute a <u>neutral scalar quantum field theory</u> (<u>QFT</u>) if the following assumptions are satisfied.

<u>W1</u>: The support of the spectral measure $E$ associated with the restriction of $U$ to $\underset{\sim}{R}^4 \approx \{I\} \times \underset{\sim}{R}^4$ is positive, i.e., $\underline{\text{spt}}\ E$ is contained in the closed forward light cone $\overline{V}_+$ ( $= \{(p_0,\underset{\sim}{p}) : p_0 \geq 0,\ p^2 = p_0^2 - \underset{\sim}{p}^2 \geq 0\}$ ).

[Note: By Stone, $\exists$ four commuting selfadjoint operators $P_\mu$ such that $U(I,a) = \underline{\exp}(\sqrt{-1}\,(a_0P_0 - a_1P_1 - a_2P_2 - a_3P_3))\ \forall\, a$, thus

$$\langle x, U(I,a)y\rangle = \int_{\underset{\sim}{R}^4} e^{\sqrt{-1}(a_0 p_0 - \underset{\sim}{a}\cdot\underset{\sim}{p})} \, d\langle x, E_p y\rangle \quad \forall\, x,y \in \mathcal{H}.$$

One calls $P_0$ the <u>hamiltonian</u> and $P=(P_1,P_2,P_3)$ the <u>momentum</u>. It follows from W1 that $P_0$ is a positive operator. The same is true of $M^2 \equiv P_0^2 - (P_1^2 + P_2^2 + P_3^2)$, so it makes sense to form $M = (M^2)^{1/2}$, the <u>mass</u>.]

<u>W2</u>: The space of $\mathcal{P}_+^\uparrow$ - invariants in $\mathcal{H}$ is 1-dimensional.

[Note: Therefore $\Omega_0$ (the <u>vacuum</u>) is unique up to phase. In this connection, observe that W2 is implied by the condition: $\underline{\dim}\,\{x \in \mathcal{H} : U(I,a)x = x \;\; \forall\, a\} = 1$. For suppose that $x \in \mathcal{H}$ has the stated property -- then so does $U(\Lambda,0)x$ (since $U(I,a)\; U(\Lambda,0)x = U(\Lambda,0)\; U(I,\Lambda^{-1}a)x = U(\Lambda,0)x$). But the only one dimensional unitary representation of $\mathcal{L}_+^\uparrow$ is the identity representation, hence $U(\Lambda,a)x = U(I,a)\; U(\Lambda,0)x = x$.]

<u>W3</u>: $\forall\, f \in \mathcal{S}(\underset{\sim}{R}^4)$,

$$\mathcal{D} \subset \underline{\mathrm{Dom}}\; \underset{\sim}{\varphi}(f)^*$$

and

$$\underset{\sim}{\varphi}(f)^* \,|\, \mathcal{D} = \underset{\sim}{\varphi}(\bar{f}).$$

[Note: Therefore $\underset{\sim}{\varphi}$ (the <u>field map</u>) sends the real valued elements of $\mathcal{S}(\underset{\sim}{R}^4)$ to symmetric operators on $\mathcal{H}$. Incidentally, it is tempting to conjecture that for real f, $\underset{\sim}{\varphi}(f)$ is essentially selfadjoint but this turns out to be false.]

<u>W4</u>: $\forall\, x,y \in \mathcal{D}$, the assignment

$$f \longrightarrow \langle x, \underset{\sim}{\varphi}(f)y\rangle$$

is continuous.

[Note: Otherwise said, $\langle x, \underset{\sim}{\varphi}(?)y \rangle$ is a tempered distribution. From this, one can deduce that $\forall\, x \in \mathcal{D}$, the map

$$\begin{cases} \mathcal{S}(\underset{\sim}{R}^4) \longrightarrow \mathcal{H} \\ f \longrightarrow \underset{\sim}{\varphi}(f)x \end{cases}$$

is continuous.]

<u>W5</u>: The set of all finite linear combinations $\underset{\sim}{\varphi}(f_1)\cdots\underset{\sim}{\varphi}(f_n)\Omega_0$ is dense in $\mathcal{H}$.

[Note: Interpret the empty product as the identity so as to include $\Omega_0$. By the way, it can be shown (Reeh-Schlieder) that for any nonempty open set $\mathcal{O} \subset \underset{\sim}{R}^4$, the set of all finite linear combinations $\underset{\sim}{\varphi}(f_1)\cdots\underset{\sim}{\varphi}(f_n)\Omega_0$ is dense in $\mathcal{H}$, where now $\underline{\text{spt}}\ f_j \subset \mathcal{O}\ \forall\, j$.]

<u>W6</u>: If the supports of $f,g \in \mathcal{S}(\underset{\sim}{R}^4)$ are spacelike separated, then

$$[\underset{\sim}{\varphi}(f), \underset{\sim}{\varphi}(g)]=0.$$

[Note: This is a crucial assumption. Roughly speaking, the idea is that measurements of field components at points which are separated by a spacelike interval are independent, i.e., neither can effect the result of the other. To account for a "fundamental length" one could weaken W6 to $[\underset{\sim}{\varphi}(f), \underset{\sim}{\varphi}(g)]=0$ if $(x-y)^2 < -\ell^2$ ($x \in \underline{\text{spt}}\ f$, $y \in \underline{\text{spt}}\ g$). However, nothing is gained in so doing: The condition actually implies W6! So, to essentially weaken W6, the commutator must be allowed to be different from zero at spacelike distances.]

Here is a final point. By definition, the field map has for its domain the Schwartz space on $\underset{\sim}{R}^4$. Can one instead define it directly on $\underset{\sim}{R}^4$? In other words, does $\exists$ a function $\underset{\sim}{\varphi}:\underset{\sim}{R}^4 \longrightarrow \underline{\text{End}}\ \mathcal{D}$ such that $\underset{\sim}{\varphi}(f) = \int_{\underset{\sim}{R}^4} f(x)\underset{\sim}{\varphi}(x)\,dx$? We shall show later that the answer is "no".

Example The free relativistic particle of spin zero and mass $m > 0$ carries the structure of a neutral scalar QFT. To see this, one first has to specify the data, i.e., items (1)-(5).

Recall that $\exists$ an irreducible unitary representation $U^{(m)}$ of $\mathcal{P}_+^\uparrow$ on $L^2(X_m, \mu_m)$, where

$$X_m = \{p \in \underset{\sim}{R}^4 : p_0 > 0,\ p^2 = p_0^2 - \underset{\sim}{p} \cdot \underset{\sim}{p} = m^2\}$$

and

$$\int_{X_m} f \, d\mu_m = \int_{\underset{\sim}{R}^3} \frac{f((m^2 + |\underset{\sim}{p}|^2)^{1/2}, \underset{\sim}{p})}{(m^2 + |\underset{\sim}{p}|^2)^{1/2}} \, dp_1 dp_2 dp_3 .$$

Here

$$(U^{(m)}(\Lambda, a) f)(p) = e^{\sqrt{-1}\,(a_0 p_0 - \underset{\sim}{a} \cdot \underset{\sim}{p})} f(\Lambda^{-1} p).$$

Moreover

$$U^{(m)}(I, a) = \underline{\exp}(\sqrt{-1}\,(a_0 P_0 - a_1 P_1 - a_2 P_2 - a_3 P_3)),$$

so

$$(P_0 f)(p) = \sqrt{m^2 + |\underset{\sim}{p}|^2}\, f(p) \ \ \& \ \ (P_j f)(p) = p_j \, f(p).$$

Notation: Given $f \in \mathcal{S}(\underset{\sim}{R}^4)$, put

$$\hat{f}(p) = \frac{1}{(2\pi)^2} \int_{\underset{\sim}{R}^4} e^{\sqrt{-1}\,(p_0 x_0 - \underset{\sim}{p} \cdot \underset{\sim}{x})} f(x) dx$$

and define a map $E_m : \mathcal{S}(\underset{\sim}{R}^4) \longrightarrow L^2(X_m, \mu_m)$ by

$$E_m f = \sqrt{2\pi}\, \hat{f} \,|X_m .$$

Ad(1): Take for $\mathcal{H}$ the symmetric Fock space over $L^2(X_m, \mu_m)$, i.e., let

$$\mathcal{H} = \mathcal{F}_s\ (L^2(X_m, \mu_m)).$$

Ad(2): To specify U, it suffices to note that $\forall\,(\Lambda, a) \in \mathcal{P}_+^\uparrow$,

$$U_n^{(m)}(\Lambda, a) \equiv U^{(m)}(\Lambda, a) \otimes \cdots \otimes U^{(m)}(\Lambda, a) \quad \text{(n factors)}$$

is a unitary operator on $L^2(X_m, \mu_m)_{n,s}$ (conventionally the identity if n=0). The elements of $L^2(X_m, \mu_m)_{n,s}$ are, of course, the square integrable functions on $X_m \times \cdots \times X_m$ (n factors) which are symmetric under permutation of their arguments, the inner product being

$$\langle f, g \rangle = \int_{X_m} \cdots \int_{X_m} \overline{f(x_1, \ldots, x_n)}\ g(x_1, \ldots, x_n) \prod_1^n d\mu_m .$$

And:

$$(U_n^{(m)}(\Lambda, a) f)(x_1, \ldots, x_n)$$

$$= \underline{\exp}(\sqrt{-1} \sum_{j=1}^{n} \langle a, x_j \rangle) f(\Lambda^{-1} x_1, \ldots, \Lambda^{-1} x_n).$$

Ad(3): Take for $\mathcal{D}$ the subspace of algebraic tensors, i.e., let $\mathcal{D} = \mathcal{D}_s$.

Ad(4): Define the field map by

$$\underset{\sim}{\varphi}_m(f) = \Phi_S\ (E_m\ \underline{\mathrm{Re} f}) + \sqrt{-1}\ \Phi_S\ (E_m\ \underline{\mathrm{Im} f}),$$

where $\Phi_S$ is the Segal field operator. It is easy to check $\mathcal{P}_+^\uparrow$-equivariance. Thus suppose that f is real -- then

$$\Pi(U^{(m)}(\Lambda, a))\ \underset{\sim}{\varphi}_m(f)\ \Pi(U^{(m)}(\Lambda, a))^{-1}$$

$$= \Pi(U^{(m)}(\Lambda, a))\ \Phi_S(E_m f)\ \Pi(U^{(m)}(\Lambda, a))^{-1}$$

$$= \Phi_S\ (U^{(m)}(\Lambda,a)E_m f)$$

$$= \Phi_S\ (E_m\ f_{\Lambda,a})$$

$$= \underset{\sim}{\varphi}_m(f_{\Lambda,a}).$$

Ad(5): Take for $\Omega_0$ the vacuum $\{1,0,\ldots\}$.

We now have to verify assumptions W1-W6.

Ad W1: From the definitions,

$$\langle f,\ U^{(m)}(I,a)f\rangle = \int_{X_m} e^{\sqrt{-1}(a_0 p_0\ -\ \underset{\sim}{a}\cdot\underset{\sim}{p})}\,|f(p)|^2\ d\mu_m(p).$$

But $\mu_m$ is a $\mathcal{O}_+^{\uparrow}$-invariant measure on $\underset{\sim}{R}^4$ with support in $X_m \subset \bar{V}_+$, hence

$$d\langle f,E_p f\rangle = |f(p)|^2\ d\mu_m(p).$$

This settles the case n=1. The general case is similar but computationally more involved.

Ad W2: Trivially, $\underset{\sim}{C}\Omega_0$ is $\mathcal{O}_+^{\uparrow}$-invariant. On the other hand, no nonzero vector in $L^2(X_m,\mu_m)$ is $\mathcal{O}_+^{\uparrow}$-invariant ($U^{(m)}$ is irreducible). In addition, no nonzero vector in $L^2(X_m,\mu_m)_{n,s}$ is $\mathcal{O}_+^{\uparrow}$-invariant. To see this, let us assume that

$$U_n^{(m)}(I,a)f=f\ \ \forall a.$$

Given $(x_1,\ldots,x_n)\in X_m\times\cdots\times X_m$ (n factors), $\exists$ a: $\exp(\sqrt{-1}\sum_{j=1}^{n}\langle a,y_j\rangle)-1$ is not zero for all $(y_1,\ldots,y_n)$ in some neighborhood of $(x_1,\ldots,x_n)$, hence f=0 a.e. on this neighborhood. Therefore f is locally 0 a.e., so f is 0 a.e.

Ad W3: Suppose first that f is real. Since $\Phi_S$ of anything is symmetric,

$$\mathcal{D} \subset \mathrm{Dom}\,\underset{\sim}{\varphi}_m(f)^*$$

and

$$\underset{\sim}{\varphi}_m(f)^* \mid \mathcal{D} = \underset{\sim}{\varphi}_m(f).$$

Next, write $f = \underline{\mathrm{Re}}\, f + \sqrt{-1}\, \underline{\mathrm{Im}}\, f$ -- then

$$\underset{\sim}{\varphi}_m(f)^* = (\Phi_S(E_m \underline{\mathrm{Re}}\, f) + \sqrt{-1}\, \Phi_S(E_m \underline{\mathrm{Im}}\, f))^*$$

and $\mathcal{D}$ is contained in the domain of the operator on the right. Finally,

$$\begin{aligned}
&\underset{\sim}{\varphi}_m(f)^* \mid \mathcal{D} \\
&\quad = \Phi_S(E_m \underline{\mathrm{Re}}\, f)^* \mid \mathcal{D} - \sqrt{-1}\, \Phi_S(E_m \underline{\mathrm{Im}}\, f)^* \mid \mathcal{D} \\
&\quad = \Phi_S(E_m \underline{\mathrm{Re}}\, f) + \sqrt{-1}\, \Phi_S(E_m(-\underline{\mathrm{Im}}\, f)) \\
&\quad = \underset{\sim}{\varphi}_m(\bar{f}).
\end{aligned}$$

<u>Ad W4</u>: Suppose that $f_k \rightarrow f$ in $\mathcal{S}(\underset{\sim}{R}^4)$, with $f_k$ real $\forall\, k$ -- then $E_m \widehat{\ } f_k \rightarrow E_m \widehat{\ } f$ in $L^2(X_m, \mu_m)$, hence $\forall\, \psi \in \mathcal{D}$,

$$\Phi_S(E_m f_k)\psi \rightarrow \Phi_S(E_m f)\psi,$$

so $\forall\, \phi \in \mathcal{D}$,

$$\langle \phi, \underset{\sim}{\varphi}_m(f_k)\psi \rangle \rightarrow \langle \phi, \underset{\sim}{\varphi}_m(f)\psi \rangle .$$

<u>Ad W5</u>: Modulo the fact that the range of $E_m$ is dense in $L^2(X_m, \mu_m)$, this is just a restatement of a property of $\Phi_S$.

<u>Ad W6</u>: Assuming that the supports of f and g are spacelike separated, it suffices to establish that

$$[\underset{\sim}{\varphi}_m(f), \underset{\sim}{\varphi}_m(g)] = 0.$$

But when f and g are real,

$$\begin{aligned}
&[\underset{\sim}{\varphi}_m(f), \underset{\sim}{\varphi}_m(g)] = [\Phi_S(E_m f), \Phi_S(E_m g)] \\
&= \sqrt{-1}\, \underline{\mathrm{Im}}\, \langle E_m f, E_m g \rangle
\end{aligned}$$

$$= \frac{2\pi}{2} \int_{X_m} (\overline{\hat{f}(p)}\hat{g}(p) - \hat{f}(p)\overline{\hat{g}(p)})\, d\mu_m(p)$$

$$= \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{1}{\sqrt{-1}} \Delta_m(x-y) f(x) g(y) dxdy,$$

where

$$\Delta_m(a) = \frac{\sqrt{-1}}{2(2\pi)^3} \int_{X_m} (e^{-\sqrt{-1}(p_0 a_0 - \underset{\sim}{p}\cdot\underset{\sim}{a})} - e^{\sqrt{-1}(p_0 a_0 - \underset{\sim}{p}\cdot\underset{\sim}{a})}) d\mu_m(p).$$

It will be shown below that

$$\underline{\mathrm{spt}}\, \Delta_m \subset \overline{V}_+ \cup \overline{V}_-,$$

hence

$$\Delta_m(x-y) \neq 0 \Rightarrow (x_0 - y_0)^2 - \sum_\mu (x_\mu - y_\mu)^2 \geq 0.$$

On the other hand,

$$\left\{ \begin{array}{l} x \in \underline{\mathrm{spt}}\ f \\ y \in \underline{\mathrm{spt}}\ g \end{array} \right. \Rightarrow (x_0 - y_0)^2 - \sum_\mu (x_\mu - y_\mu)^2 < 0,$$

so

$$\int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \Delta_m(x-y)\ f(x) g(y) dxdy = 0.$$

Observation: $\forall\, f \in \mathcal{S}(\mathbb{R}^4)$,

$$\underset{\sim}{\varphi}_m((\Box^2 + m^2) f) = 0 \quad (\Box^2 = \frac{\partial^2}{\partial t^2} - \nabla^2).$$

[In fact,

$$\widehat{(\Box^2 + m^2)f}(p) = -(p_0^2 - \underset{\sim}{p}\cdot\underset{\sim}{p} - m^2)\hat{f}(p)$$

$$\Rightarrow E_m((\Box^2 + m^2)f) = 0.]$$

Remark: $\Delta_m$ is the unique distribution satisfying the equation

$$(\Box^2 + m^2)\,\Delta_m = 0$$

with Cauchy data

$$\Delta_m(0,\underset{\sim}{x})=0,\ \frac{\partial}{\partial t}\Delta_m(t,\underset{\sim}{x})\Big|_{t=0} = \delta(\underset{\sim}{x}).$$

It can be expressed in terms of Bessel functions. One has

$$\Delta_m(-x) = -\Delta_m(x)$$

and $\forall\ \Lambda \in \mathcal{L}_+^{\uparrow}$,

$$\Delta_m(\Lambda x) = \Delta_m(x).$$

For a spacelike x, $\exists\ \Lambda \in \mathcal{L}_+^{\uparrow}$: $\Lambda x = -x$, thus

$$\Delta_m(x) = \Delta_m(\Lambda x) = \Delta_m(-x) = -\Delta_m(x),$$

which implies that

$$\underline{\text{spt}}\ \Delta_m \subset \bar{V}_+ \cup \bar{V}_- .$$

In what follows, the neutral scalar QFT constructed above will be referred to as the <u>free QFT of mass $m > 0$</u>.

Let us now consider this setup from the traditional point of view.

Working in $\mathcal{F}_s(L^2(\underset{\sim}{R}^3))$, let $\mathcal{D}_\mathcal{S}$ be the dense linear subspace consisting of those strings $\Psi = \{\Psi_0, \Psi_1, \ldots\}$ such that $\Psi_n \in \mathcal{S}(\underset{\sim}{R}^{3n})$ $\forall n$ and $\Psi_n = 0$ $(n >> 0)$.

Given $\underset{\sim}{p} \in \underset{\sim}{R}^3$, define operators $\begin{cases} \underset{\sim}{a}(\underset{\sim}{p}) \\ \underset{\sim}{c}(\underset{\sim}{p}) \end{cases}$ on $\mathcal{D}_{\mathcal{S}}$ by

$$\begin{cases} (\underset{\sim}{a}(\underset{\sim}{p})\Psi)_n (x_1,\ldots,x_n) = \sqrt{n+1}\; \Psi_{n+1}(\underset{\sim}{p},x_1,\ldots,x_n) \\ (\underset{\sim}{c}(\underset{\sim}{p})\Psi)_n (x_1,\ldots,x_n) = \dfrac{1}{\sqrt{n}} \displaystyle\sum_{j=1}^{n} \delta(\underset{\sim}{p}-x_j)\; \Psi_{n-1}(x_1,\ldots,\hat{x}_j,\ldots,x_n). \end{cases}$$

[Note: The physical interpretation is that $\underset{\sim}{a}(\underset{\sim}{p})$ and $\underset{\sim}{c}(\underset{\sim}{p})$ annihilate and create particles of rest mass m, momentum $\underset{\sim}{p}$, and energy $E_{\underset{\sim}{p}} = \sqrt{m^2 + |\underset{\sim}{p}|^2}$.]

Properties:

(1) $\underset{\sim}{a}(\underset{\sim}{p})^* = \underset{\sim}{c}(\underset{\sim}{p})$ & $\underset{\sim}{c}(\underset{\sim}{p})^* = \underset{\sim}{a}(\underset{\sim}{p})$;

(2) $[\underset{\sim}{a}(\underset{\sim}{p}), \underset{\sim}{a}(\underset{\sim}{q})] = 0$ & $[\underset{\sim}{c}(\underset{\sim}{p}), \underset{\sim}{c}(\underset{\sim}{q})] = 0$;

(3) $[\underset{\sim}{a}(\underset{\sim}{p}), \underset{\sim}{c}(\underset{\sim}{q})] = \delta(\underset{\sim}{p}-\underset{\sim}{q}) \cdot I$.

Remark: The number operator N is given by the expression

$$\int_{\underset{\sim}{R}^3} \underset{\sim}{c}(\underset{\sim}{p})\underset{\sim}{a}(\underset{\sim}{p})\, d^3p.$$

E.g.: Suppose that $\Psi(x_1,x_2) = \Psi(x_2,x_1)$ -- then $N\Psi = 2\Psi$. On the other hand,

$$\int_{\underset{\sim}{R}^3} \underset{\sim}{c}(\underset{\sim}{p})(\underset{\sim}{a}(\underset{\sim}{p})\Psi)(x_1,x_2)\, d^3p$$

$$= \int_{\underset{\sim}{R}^3} \left\{ \frac{1}{\sqrt{2}}\, \delta(\underset{\sim}{p}-x_1)(\underset{\sim}{a}(\underset{\sim}{p})\Psi)(x_2) + \frac{1}{\sqrt{2}}\, \delta(\underset{\sim}{p}-x_2)(\underset{\sim}{a}(\underset{\sim}{p})\Psi)(x_1) \right\} d^3p$$

$$= \int_{\underset{\sim}{R}^3} \frac{1}{\sqrt{2}}\, \delta(\underset{\sim}{p}-x_1)\, [\sqrt{2}\, \Psi(\underset{\sim}{p},x_2)]\, d^3p$$

$$+ \int_{\mathbf{R}^3} \frac{1}{\sqrt{2}}\, \delta(\mathbf{p}-x_2)\; [\sqrt{2}\; \Psi(\mathbf{p},x_1)]\, d^3p$$

$$= \Psi(x_1,x_2) + \Psi(x_2,x_1)$$

$$= 2\, \Psi(x_1,x_2).$$

Given $f \in \mathcal{S}(\mathbf{R}^3)$, define operators $\left\{\begin{array}{l}\mathbf{a}(f)\\ \mathbf{c}(f)\end{array}\right.$ on $\mathcal{D}_{\mathcal{S}}$ by

$$\left\{\begin{array}{l} \mathbf{a}(f) = \displaystyle\int_{\mathbf{R}^3} \mathbf{a}(\mathbf{p})\; \overline{f(\mathbf{p})}\; d^3p \\[2ex] \mathbf{c}(f) = \displaystyle\int_{\mathbf{R}^3} \mathbf{c}(\mathbf{p})\; f(\mathbf{p})\; d^3p. \end{array}\right.$$

Remark: These definitions are formally consistent with the usual agreements. Thus

$$(\mathbf{a}(f)\,\Psi)_n\;(x_1,\dots,x_n) = \cancel{\sqrt{n+1}\int (\mathbf{a}(\mathbf{p})\Psi)_n(x_1,\dots,x_n)\,\overline{f(\mathbf{p})}\,d^3p}$$

$$= \sqrt{n+1} \int_{\mathbf{R}^3} \Psi_{n+1}(\mathbf{p},x_1,\dots,x_n)\; \overline{f(\mathbf{p})}\, d^3p$$

and

$$(\mathbf{c}(f)\,\Psi)_n\;(x_1,\dots,x_n)$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[\int_{\mathbf{R}^3} \delta(\mathbf{p}-x_j)\, f(\mathbf{p})\, d^3p\right] \Psi_{n-1}(x_1,\dots,\hat{x}_j,\dots,x_n)$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j)\; \Psi_{n-1}(x_1,\dots,\hat{x}_j,\dots,x_n).$$

Note too that

$$[\underset{\sim}{a}(f),\underset{\sim}{c}(g)] = \int_{\underset{\sim}{R}^3}\int_{\underset{\sim}{R}^3} [\underset{\sim}{a}(\underset{\sim}{p}),\underset{\sim}{c}(\underset{\sim}{q})]\overline{f(\underset{\sim}{p})}g(\underset{\sim}{q})d^3pd^3q$$

$$= \int_{\underset{\sim}{R}^3}\int_{\underset{\sim}{R}^3} \delta(\underset{\sim}{p}-\underset{\sim}{q})\overline{f(\underset{\sim}{p})}g(\underset{\sim}{q})d^3pd^3q$$

$$= \langle f,g\rangle\cdot I.$$

The <u>free Klein-Gordon field of mass $m > 0$ at time $t=0$</u> is the operator

$$\underset{\sim}{\Phi}_m(\underset{\sim}{x}) = \frac{1}{(2\pi)^{3/2}}\int_{\underset{\sim}{R}^3}\left(e^{\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}}\underset{\sim}{a}(\underset{\sim}{p}) + e^{-\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}}\underset{\sim}{c}(\underset{\sim}{p})\right)\frac{d^3p}{\sqrt{2\mu(\underset{\sim}{p})}},$$

where

$$\mu(\underset{\sim}{p}) = \sqrt{m^2 + |\underset{\sim}{p}|^2}.$$

Its conjugate is the operator

$$\underset{\sim}{\Pi}_m(\underset{\sim}{x}) = \frac{\sqrt{-1}}{(2\pi)^{3/2}}\int_{\underset{\sim}{R}^3}\left(-e^{\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}}\underset{\sim}{a}(\underset{\sim}{p}) + e^{-\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}}\underset{\sim}{c}(\underset{\sim}{p})\right)\sqrt{\frac{\mu(\underset{\sim}{p})}{2}}\,d^3p.$$

Properties:

(1) $\underset{\sim}{\Phi}_m(\underset{\sim}{x})^* = \underset{\sim}{\Phi}_m(\underset{\sim}{x})$ & $\underset{\sim}{\Pi}_m(\underset{\sim}{x})^* = \underset{\sim}{\Pi}_m(\underset{\sim}{x})$;

(2) $[\underset{\sim}{\Phi}_m(\underset{\sim}{x}), \underset{\sim}{\Phi}_m(\underset{\sim}{y})] = 0$ & $[\underset{\sim}{\Pi}_m(\underset{\sim}{x}), \underset{\sim}{\Pi}_m(\underset{\sim}{y})] = 0$;

(3) $[\underset{\sim}{\Phi}_m(\underset{\sim}{x}), \underset{\sim}{\Pi}_m(\underset{\sim}{y})] = \sqrt{-1}\,\delta(\underset{\sim}{x}-\underset{\sim}{y})\cdot I.$

[Note: To check the third property, it is necessary to examine

$$\frac{\sqrt{-1}}{(2\pi)^3}\int_{\underset{\sim}{R}^3}\int_{\underset{\sim}{R}^3}\frac{1}{2}\,[e^{\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}}\,a(\underset{\sim}{p}) + e^{-\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}}\,c(\underset{\sim}{p}),$$

$$-e^{\sqrt{-1}\,\underset{\sim}{y}\cdot\underset{\sim}{q}}\,a(\underset{\sim}{q}) + e^{-\sqrt{-1}\,\underset{\sim}{y}\cdot\underset{\sim}{q}}\,c(\underset{\sim}{q})]\sqrt{\frac{\mu(\underset{\sim}{q})}{\mu(\underset{\sim}{p})}}\,d^3p\,d^3q$$

or still,

$$\frac{\sqrt{-1}}{(2\pi)^3}\cdot\frac{1}{2}\int_{\underset{\sim}{R}^3}\int_{\underset{\sim}{R}^3}\Big\{e^{\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}-\sqrt{-1}\,\underset{\sim}{y}\cdot\underset{\sim}{q}}\,[a(\underset{\sim}{p}),\,c(\underset{\sim}{q})]$$

$$-e^{-\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}+\sqrt{-1}\,\underset{\sim}{y}\cdot\underset{\sim}{q}}\,[c(\underset{\sim}{p}),a(\underset{\sim}{q})]\,\sqrt{\frac{\mu(\underset{\sim}{q})}{\mu(\underset{\sim}{p})}}\,d^3p\,d^3q$$

or still,

$$\frac{\sqrt{-1}}{(2\pi)^3}\cdot\frac{1}{2}\Big[\int_{\underset{\sim}{R}^3}\int_{\underset{\sim}{R}^3}e^{\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}-\sqrt{-1}\,\underset{\sim}{y}\cdot\underset{\sim}{q}}\,\delta(\underset{\sim}{p}-\underset{\sim}{q})\,\sqrt{\frac{\mu(\underset{\sim}{q})}{\mu(\underset{\sim}{p})}}\,d^3p\,d^3q$$

$$+\int_{\underset{\sim}{R}^3}\int_{\underset{\sim}{R}^3}e^{-\sqrt{-1}\,\underset{\sim}{x}\cdot\underset{\sim}{p}+\sqrt{-1}\,\underset{\sim}{y}\cdot\underset{\sim}{q}}\,\delta(\underset{\sim}{p}-\underset{\sim}{q})\,\sqrt{\frac{\mu(\underset{\sim}{q})}{\mu(\underset{\sim}{p})}}\,d^3p\,d^3q\Big]$$

or still,

$$\frac{\sqrt{-1}}{(2\pi)^3}\cdot\frac{1}{2}\Big[\int_{\underset{\sim}{R}^3}e^{\sqrt{-1}\,(\underset{\sim}{x}-\underset{\sim}{y})\cdot\underset{\sim}{q}}\,d^3q+\int_{\underset{\sim}{R}^3}e^{\sqrt{-1}\,(\underset{\sim}{y}-\underset{\sim}{x})\cdot\underset{\sim}{q}}\,d^3q\Big]$$

or still,

$$\sqrt{-1}\cdot\frac{1}{(2\pi)^3}\int_{\underset{\sim}{R}^3}e^{\sqrt{-1}\,(\underset{\sim}{x}-\underset{\sim}{y})\cdot\underset{\sim}{q}}\,d^3q$$

or still,

$$\sqrt{-1}\,\delta(\underset{\sim}{x}-\underset{\sim}{y}),$$

as claimed.]

In general, the <u>free Klein-Gordon field of mass $m > 0$ at time $t$</u> is the operator

$$\mathbf{\Phi}_m(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{R}^3} [\, e^{-\sqrt{-1}\,(\mu(\mathbf{p})t - \mathbf{x}\cdot\mathbf{p})}\, \mathbf{a}(\mathbf{p}) + e^{\sqrt{-1}\,(\mu(\mathbf{p})t - \mathbf{x}\cdot\mathbf{p})}\, \mathbf{c}(\mathbf{p})\,] \frac{d^3p}{\sqrt{2\mu(\mathbf{p})}} \,.$$

[Note: The exponentials

$$e^{\pm\sqrt{-1}\,(\mu(\mathbf{p})t - \mathbf{x}\cdot\mathbf{p})}$$

are solutions to the Klein-Gordon equation, i.e.,

$$(\Box^2 + m^2)\, e^{\pm\sqrt{-1}\,\cdots} = 0.]$$

To calculate

$$[\,\mathbf{\Phi}_m(t,\mathbf{x}),\ \mathbf{\Phi}_m(t',\mathbf{x}')\,],$$

it is convenient to introduce

$$\Delta_{\pm}(t,\mathbf{x};m^2) = \pm\frac{\sqrt{-1}}{2(2\pi)^3} \int_{\mathbf{R}^3} e^{\pm\sqrt{-1}\,\mathbf{x}\cdot\mathbf{p}}\, \frac{e^{\mp\sqrt{-1}\,\mu(\mathbf{p})t}}{\mu(\mathbf{p})}\, d^3p,$$

the integrals being Fourier transforms of tempered distributions. Since

$$\Delta_m(t,\mathbf{x}) = \frac{\sqrt{-1}}{2(2\pi)^3} \int_{X_m} (e^{-\sqrt{-1}\,(tp_0 - \mathbf{x}\cdot\mathbf{p})} - e^{\sqrt{-1}\,(tp_0 - \mathbf{x}\cdot\mathbf{p})})\, d\mu_m(p)$$

$$= \frac{\sqrt{-1}}{2(2\pi)^3} \int_{\mathbb{R}^3} \left(e^{-\sqrt{-1}\,(\mu(\mathbf{p})t - \mathbf{x}\cdot\mathbf{p})} - e^{\sqrt{-1}\,(\mu(\mathbf{p})t - \mathbf{x}\cdot\mathbf{p})}\right) \frac{d^3p}{\mu(\mathbf{p})},$$

it follows that

$$\Delta_m(t,\mathbf{x}) = \Delta_+(t,\mathbf{x};m^2) + \Delta_-(t,\mathbf{x};m^2).$$

Of course,

$$\Delta_-(t,\mathbf{x};m^2) = -\Delta_+(-t,-\mathbf{x};m^2).$$

---

LEMMA We have

$$[\mathbf{\Phi}_m(t,\mathbf{x}), \mathbf{\Phi}_m(t',\mathbf{x}')] = \frac{1}{\sqrt{-1}}\,\Delta_m(t-t',\mathbf{x}-\mathbf{x}').$$

[The verification is straightforward symbol pushing.]

---

The assignment $f \longrightarrow Jf$, where

$$(Jf)(\mathbf{p}) = \frac{f(\mu(\mathbf{p}),\mathbf{p})}{\sqrt{\mu(\mathbf{p})}},$$

defines an isometric isomorphism

$$L^2(X_m,\mu_m) \longrightarrow L^2(\mathbb{R}^3)$$

which by functoriality extends to an isometric isomorphism

$$\mathfrak{F}_s(L^2(X_m,\mu_m)) \longrightarrow \mathfrak{F}_s(L^2(\mathbb{R}^3)).$$

For short, call it J (rather than $\Pi J$).

Everything can then be transferred over to $\underset{\sim}{R}^3$. In particular:

$$\begin{cases} \underset{\sim}{a}(Jf) = J\underset{\sim}{a}(f)J^{-1} \\ \underset{\sim}{c}(Jf) = J\underset{\sim}{c}(f)J^{-1} \end{cases} \qquad (f \in L^2(X_m, \mu_m)).$$

Now write

$$\underset{\sim}{\Phi}_m(x) = \frac{1}{(2\pi)^{3/2}} \int_{\underset{\sim}{R}^3} [e^{-\sqrt{-1}(\underset{\sim}{\mu}(p)x_0 - \underset{\sim}{x}\cdot\underset{\sim}{p})}\underset{\sim}{a}(\underset{\sim}{p}) + e^{\sqrt{-1}(\underset{\sim}{\mu}(p)x_0 - \underset{\sim}{x}\cdot\underset{\sim}{p})}\underset{\sim}{c}(\underset{\sim}{p})] \frac{d^3p}{\sqrt{2\underset{\sim}{\mu}(p)}}.$$

Claim: $\forall\ f \in \mathcal{S}(\underset{\sim}{R}^4)$,

$$\underset{\sim}{\varphi}_m(f) = \int_{\underset{\sim}{R}^4} f(x)\underset{\sim}{\varphi}_m(x)dx,$$

where

$$\underset{\sim}{\varphi}_m(x) = J^{-1}\underset{\sim}{\Phi}_m(x)J.$$

Thus take f real -- then

$$\int_{\underset{\sim}{R}^4} f(x)\underset{\sim}{\varphi}_m(x)dx$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} J^{-1}\Big(\int_{\underset{\sim}{R}^3} \underset{\sim}{a}(\underset{\sim}{p})\Big(\frac{1}{(2\pi)^2}\int_{\underset{\sim}{R}^4} e^{-\sqrt{-1}(\underset{\sim}{\mu}(p)x_0 - \underset{\sim}{x}\cdot\underset{\sim}{p})} f(x)dx\Big)\frac{d^3p}{\sqrt{\underset{\sim}{\mu}(p)}}\Big)J$$

$$+ \frac{\sqrt{2\pi}}{\sqrt{2}} J^{-1}\Big(\int_{\underset{\sim}{R}^3} \underset{\sim}{c}(\underset{\sim}{p})\Big(\frac{1}{(2\pi)^2}\int_{\underset{\sim}{R}^4} e^{\sqrt{-1}(\underset{\sim}{\mu}(p)x_0 - \underset{\sim}{x}\cdot\underset{\sim}{p})} f(x)dx\Big)\frac{d^3p}{\sqrt{\underset{\sim}{\mu}(p)}}\Big)J$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} J^{-1} \left( \int_{\underset{\sim}{R}^3} \underset{\sim}{a}(\underset{\sim}{p}) \overline{\frac{\hat{f}(\mu(\underset{\sim}{p}),\underset{\sim}{p})}{\sqrt{\mu(\underset{\sim}{p})}}} d^3p \right) J$$

$$+ \frac{\sqrt{2\pi}}{\sqrt{2}} J^{-1} \left( \int_{\underset{\sim}{R}^3} \underset{\sim}{c}(\underset{\sim}{p}) \frac{\hat{f}(\mu(\underset{\sim}{p}),\underset{\sim}{p})}{\sqrt{\mu(\underset{\sim}{p})}} d^3p \right) J$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} J^{-1} \left( \int_{\underset{\sim}{R}^3} \underset{\sim}{a}(\underset{\sim}{p}) \overline{J\hat{f}(\underset{\sim}{p})} d^3p \right) J$$

$$+ \frac{\sqrt{2\pi}}{\sqrt{2}} J^{-1} \left( \int_{\underset{\sim}{R}^3} \underset{\sim}{c}(\underset{\sim}{p}) J\hat{f}(\underset{\sim}{p}) d^3p \right) J$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} J^{-1} \underset{\sim}{a}(J\hat{f} \mid X_m) J + \frac{\sqrt{2\pi}}{\sqrt{2}} J^{-1} \underset{\sim}{c}(J\hat{f} \mid X_m) J$$

$$= \frac{1}{\sqrt{2}} [J^{-1} \underset{\sim}{a}(JE_m f) J + J^{-1} \underset{\sim}{c}(JE_m f) J]$$

$$= \frac{1}{\sqrt{2}} [\underset{\sim}{a}(E_m f) + \underset{\sim}{c}(E_m f)]$$

$$= \Phi_S (E_m f)$$

$$= \underset{\sim}{\varphi}_m(f).$$

Of course, we could have worked from the beginning on $X_m$. Indeed,

$$\underset{\sim}{\varphi}_m(x) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int_{X_m} [ e^{-\sqrt{-1}\langle p,x\rangle} \underset{\sim}{a}(p) + e^{\sqrt{-1}\langle p,x\rangle} \underset{\sim}{c}(p)] \, d\mu_m(p),$$

where now $\underset{\sim}{a}(p)$ & $\underset{\sim}{c}(p)$ are defined directly on $X_m$. To confirm this, take f real -- then

$$\int_{\underset{\sim}{R}^4} f(x)\underset{\sim}{\varphi}_m(x)\,dx$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} \left( \int_{X_m} \underset{\sim}{a}(p) \frac{1}{(2\pi)^2} \left( \int_{\underset{\sim}{R}^4} e^{-\sqrt{-1}\langle p,x\rangle} f(x)dx\right) d\mu_m(p) \right.$$

$$\left. + \int_{X_m} \underset{\sim}{c}(p) \frac{1}{(2\pi)^2} \left( \int_{\underset{\sim}{R}^4} e^{\sqrt{-1}\langle p,x\rangle} f(x)dx\right) d\mu_m(p) \right)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} \left( \int_{X_m} \underset{\sim}{a}(p)\, \overline{\hat{f}(p)}\, d\mu_m(p) \right.$$

$$\left. + \int_{X_m} \underset{\sim}{c}(p)\, \hat{f}(p)\, d\mu_m(p) \right)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} \left( \underset{\sim}{a}(\hat{f}\,|\,X_m) + \underset{\sim}{c}(\hat{f}\,|\,X_m)\right)$$

$$= \frac{1}{\sqrt{2}} \left( \underset{\sim}{a}(E_m f) + \underset{\sim}{c}(E_m f)\right)$$

$$= \Phi_S(E_m f)$$

$$= \underset{\sim}{\varphi}_m(f).$$

Remark: The definition of the pointwise creation operator utilizes the $\delta$-function on $X_m$:

$$\delta(p-q) = p_0\,\delta(\underset{\sim}{p}-\underset{\sim}{q}).$$

To check this, simply note that

$$\int_{X_m} f(p)\,\delta\,(p-q)\,d\mu_m(p)$$

$$= \int_{\underset{\sim}{R}^3} f(\mu(\underset{\sim}{p}),\underset{\sim}{p})\;\mu(\underset{\sim}{p})\;\delta\,(\underset{\sim}{p}-\underset{\sim}{q})\;\frac{d^3p}{\mu(\underset{\sim}{p})}$$

$$= f(\mu(\underset{\sim}{q}),\underset{\sim}{q})$$

$$= f(q).$$

<u>Correlation Functions</u> We shall continue to work with a fixed neutral scalar QFT.

Rappel (Schwartz Kernel Theorem): Let $B: \mathcal{S}(\underset{\sim}{R}^n) \times \mathcal{S}(\underset{\sim}{R}^m) \longrightarrow \underset{\sim}{C}$ be a separately continuous bilinear functional -- then there is a unique tempered distribution T on $\underset{\sim}{R}^{n+m}$ such that

$$B(f,g) = \langle T, f \times g \rangle$$

for all $f \in \mathcal{S}(\underset{\sim}{R}^n)$, $g \in \mathcal{S}(\underset{\sim}{R}^m)$.

[Note: Here

$$(f \times g)(x_1,\ldots,x_{n+m})$$
$$= f(x_1,\ldots,x_n)g(x_{n+1},\ldots,x_{n+m}).]$$

Therefore, $\forall\, n \geq 1$, $\exists!$ tempered distribution $\mathcal{W}_n$ on $\underset{\sim}{R}^{4n}$ such that

$$\mathcal{W}_n(f_1 \times \cdots \times f_n) = \langle \Omega_0, \underset{\sim}{\varphi}(f_1)\cdots\underset{\sim}{\varphi}(f_n)\Omega_0 \rangle.$$

One calls $\mathcal{W}_n$ the $n^{\underline{th}}$ <u>correlation function</u> of the theory.

[Note: Another name for $\mathcal{W}_n$ is the <u>n-point function</u>. Conventionally, $\mathcal{W}_0=1$.]

Since

$$\begin{aligned}
&\langle \Omega_0, \underset{\sim}{\varphi}(f_{\Lambda,a})\Omega_0 \rangle \\
&= \langle \Omega_0, U(\Lambda,a)\underset{\sim}{\varphi}(f)U(\Lambda,a)^{-1}\Omega_0 \rangle \\
&= \langle U(\Lambda,a)^{-1}\Omega_0, \underset{\sim}{\varphi}(f)U(\Lambda,a)^{-1}\Omega_0 \rangle \\
&= \langle \Omega_0, \underset{\sim}{\varphi}(f)\Omega_0 \rangle,
\end{aligned}$$

it follows that $\mathcal{W}_n$ is $\mathcal{P}_+^\uparrow$-invariant:

$$(\Lambda,a)\cdot\mathcal{W}_n = \mathcal{W}_n \quad \forall\,(\Lambda,a).$$

In particular, using symbolic notation, $\forall$ a,

$$\mathcal{W}_n(x_1+a,\ldots,x_n+a) = \mathcal{W}_n(x_1,\ldots,x_n).$$

This means that $\exists$ a tempered distribution $W_n$ on $\underset{\sim}{R}^{4n-4}$ such that

$$\mathcal{W}_n(x_1,\ldots,x_n) = W_n(x_1-x_2,\ldots,x_{n-1}-x_n).$$

I.e.: $\forall\ f \in \mathcal{S}(\underset{\sim}{R}^{4n})$,

$$\langle \mathcal{W}_n, f \rangle = \int_{\underset{\sim}{R}^4} \langle W_n, f_{(x)} \rangle dx,$$

where

$$f_{(x)}(\xi_1,\ldots,\xi_{n-1})$$
$$= f(x, x-\xi_1, x-\xi_1-\xi_2,\ldots, x-\xi_1-\cdots-\xi_{n-1}).$$

[Note: $W_n$ is $\mathcal{L}_+^{\uparrow}$ -invariant: $\forall\ \Lambda \in \mathcal{L}_+^{\uparrow}$,

$$W_n(\Lambda\xi_1,\ldots,\Lambda\xi_{n-1}) = W_n(\xi_1,\ldots,\xi_{n-1}).]$$

---

<u>LEMMA</u> The support of $\hat{W}_n$ is contained in $\bar{V}_+ \times \cdots \times \bar{V}_+$ (n-1 factors).

[Note: I have chosen the plus sign in the definition of Fourier transform. It is not difficult to see that

$$\hat{\mathcal{W}}_n(p_1,\ldots,p_n)$$

$$=(2\pi)^2\ \delta\left(\sum_{j=1}^{n} p_j\right)\hat{W}_n(p_1, p_1+p_2,\ldots, p_1+p_2+\cdots+p_{n-1}).]$$

---

Remark: In general, if T is a tempered distribution on $\underset{\sim}{R}^{4n-4}$ with <u>spt</u> $\hat{T} \subset \bar{V}_+ \times \cdots \times \bar{V}_+$ (n-1 factors), then $T \not\equiv 0 \Rightarrow$ <u>spt</u> $T = \underset{\sim}{R}^{4n-4}$.

Taking $T=W_n$, it follows that there are just two possibilities:

(1) $W_n \equiv 0$; (2) $\underline{\text{spt}}\ W_n = \mathbb{R}^{4n-4}$.

Much is known about the correlation functions but I shall omit the specifics here (it is a chapter in the theory of several complex variables). One consequence of these investigations is the following central result.

---

THEOREM For all $(x_1,\ldots,x_n) \in \mathbb{R}^{4n}$,

$$\mathcal{W}_n(x_1,\ldots,x_n) = \mathcal{W}_n(-x_n,\ldots,-x_1).$$

---

Notation: Given $f \in \mathcal{S}(\mathbb{R}^n)$, let

$$\Pi f(x) = f(-x) \quad (x \in \mathbb{R}^n).$$

Write

$$\begin{cases} A = \varphi(f_1)\cdots\varphi(f_n) \\ B = \varphi(g_1)\cdots\varphi(g_m) \end{cases}$$

and

$$\begin{cases} A' = \varphi(\Pi\bar{f}_1)\cdots\varphi(\Pi\bar{f}_n) \\ B' = \varphi(\Pi\bar{g}_1)\cdots\varphi(\Pi\bar{g}_m). \end{cases}$$

---

LEMMA We have

$$\langle B\Omega_0, A\Omega_0\rangle = \langle A'\Omega_0, B'\Omega_0\rangle .$$

[This is a formal consequence of the theorem. E.g.:

$$\langle \underset{\sim}{\varphi}(g)\Omega_0, \ \underset{\sim}{\varphi}(f)\Omega_0\rangle$$

$$= \langle \Omega_0, \ \underset{\sim}{\varphi}(g)^* \underset{\sim}{\varphi}(f)\Omega_0\rangle$$

$$= \langle \Omega_0, \ \underset{\sim}{\varphi}(\bar{g}) \ \underset{\sim}{\varphi}(f)\Omega_0\rangle$$

$$= \int \mathcal{W}_2(y,x)g(y)f(x)dydx.$$

On the other hand,

$$\langle \underset{\sim}{\varphi}(\Pi\bar{f})\Omega_0, \underset{\sim}{\varphi}(\Pi\bar{g})\Omega_0\rangle$$

$$= \int \mathcal{W}_2(x,y)f(-x)\overline{g(-y)} \ dxdy$$

$$= \int \mathcal{W}_2(-x,-y) \ f(x)\overline{g(y)} \ dxdy$$

$$= \int \mathcal{W}_2(y,x) \ f(x)\overline{g(y)} \ dxdy.]$$

Define now an operator $\Theta$ by the prescription

$$\Theta \ \underset{\sim}{\varphi}(f_1)\cdots\underset{\sim}{\varphi}(f_n)\Omega_0 = \underset{\sim}{\varphi}(\Pi\bar{f}_1)\cdots\underset{\sim}{\varphi}(\Pi\bar{f}_n)\Omega_0 \ .$$

Then $\Theta$ is well defined. To see this, suppose that $A\,\Omega_0 = B\,\Omega_0$, the claim being that $A'\Omega_0 = B'\Omega_0$. But

$$\langle A'\,\Omega_0 - B'\,\Omega_0, \ A'\,\Omega_0 - B'\,\Omega_0\rangle$$

$$= \langle A'\,\Omega_0, \ A'\,\Omega_0\rangle - \langle A'\,\Omega_0\,, B'\,\Omega_0\rangle$$

$$- \langle B'\,\Omega_0, A'\,\Omega_0\rangle + \langle B'\,\Omega_0, \ B'\,\Omega_0\rangle$$

$$= \langle A\Omega_0, A\Omega_0 - \langle B\Omega_0, A\Omega_0\rangle$$

$$- \langle A\Omega_0, B\Omega_0\rangle + \langle B\Omega_0, B\Omega_0\rangle$$

$$= 0$$

$$\Longrightarrow$$

$$A'\Omega_0 = B'\Omega_0.$$

The upshot is that $\Theta$ extends to a norm preserving map $\mathcal{H} \rightarrow \mathcal{H}$ which leaves the vacuum invariant with

$$\begin{cases} \Theta(x+y) = \Theta x + \Theta y \\ \Theta(cx) = \bar{c}\ \Theta x \end{cases} \quad \& \quad \Theta^2 = I.$$

And: $\forall\ f \in \mathcal{S}(\underset{\sim}{R}^4)$,

$$\Theta\ \underset{\sim}{\varphi}(f)\ \Theta^{-1} = \underset{\sim}{\varphi}(\Pi \bar{f}).$$

This conclusion is the PCT theorem for a neutral scalar QFT.

Remark: As regards the relation

$$\Theta\ \underset{\sim}{\varphi}(f)\ \Theta^{-1} = \underset{\sim}{\varphi}(\Pi \bar{f}),$$

I feel it necessary to inject a proviso. While formally true on a dense subspace of $\mathcal{D}$, why does it hold on all of $\mathcal{D}$? The experts pass in silence on this issue. Have they forgotten that the field operators are unbounded, hence discontinuous?

Cluster Property  The assignment

$$\begin{cases} \mathcal{S}(\underset{\sim}{R}^4) \times \cdots \times \mathcal{S}(\underset{\sim}{R}^4) \longrightarrow \mathcal{H} \\ (f_1, \ldots, f_n) \longrightarrow \underset{\sim}{\varphi}(f_1) \cdots \underset{\sim}{\varphi}(f_n) \Omega_0 \end{cases}$$

is a separately continuous multilinear $\mathcal{H}$-valued function, thus $\exists$ a continuous map $\Phi_n : \mathcal{S}(\underset{\sim}{R}^{4n}) \longrightarrow \mathcal{H}$ such that

$$\langle \Phi_n, f_1 \times \cdots \times f_n \rangle = \underset{\sim}{\varphi}(f_1) \cdots \underset{\sim}{\varphi}(f_n) \Omega_0$$

$$\Longrightarrow$$

$$\mathcal{W}_n = \langle \Omega_0, \Phi_n \rangle .$$

---

SUBLEMMA  Suppose that a is spacelike -- then in the weak operator topology,

$$\lim_{\lambda \to +\infty} U(I, \lambda a) = P_{\Omega_0} ,$$

where $P_{\Omega_0}$ is the orthogonal projection of $\mathcal{H}$ onto $\underset{\sim}{C}\, \Omega_0$.

[Note: This is a nontrivial assertion. It amounts to saying that $\forall x, y \in \mathcal{H}$,

$$\langle x, U(I, \lambda a) y \rangle \longrightarrow \langle x, \Omega_0 \rangle \langle \Omega_0, y \rangle \text{ as } \lambda \to +\infty$$

which, of course, is obvious only if either x or y is equal to $\Omega_0$. However, the verification is straightforward in the special case of a free QFT of mass $m > 0$. Thus take $a = (0,0,0,1)$ (there is no essential loss of generality ~~XXX~~ in so doing) -- then from the definitions,

$$\langle f, U^{(m)}(I, \lambda a) g \rangle = \int_{X_m} e^{-\sqrt{-1}\,\lambda p_3}\, \overline{f(p)}\, g(p)\, d\mu_m(p)$$

$$= \int_{\underset{\sim}{R}^3} e^{-\sqrt{-1}\,\lambda p_3}\, \overline{Jf(p)}\; Jg(p)\; dp_1 dp_2 dp_3$$

which tends to 0 as $\lambda \rightarrow +\infty$ (Riemann-Lebesgue lemma). The extension to functions on $X_m \times \cdots \times X_m$ is clear. One way to handle the general case is to use the fact that $d\langle x, (1-P_{\Omega_0}) E_p y \rangle$ is absolutely continuous w.r.t. Lebesgue measure, hence with a=(0,0,0,1),

$$\langle x, U(I, \lambda a) y \rangle = \int_{\underset{\sim}{R}^4} e^{-\sqrt{-1}\,\lambda p_3}\, d\langle x, E_p y \rangle$$

$$= \langle x, P_{\Omega_0} y \rangle + \int_{\underset{\sim}{R}^4} e^{-\sqrt{-1}\,\lambda p_3}\, \frac{d\langle x, (1-P_{\Omega_0}) E_p y \rangle}{d^4 p}\, d^4 p$$

$$\rightarrow \langle x, P_{\Omega_0} y \rangle = \langle x, \Omega_0 \rangle \langle \Omega_0, y \rangle \text{ as } \lambda \rightarrow +\infty.]$$

---

LEMMA Suppose that a is spacelike -- then

$$\lim_{\lambda \rightarrow +\infty} \mathcal{W}_n(x_1, \ldots, x_j, x_{j+1} + \lambda a, \ldots, x_n + \lambda a)$$

$$= \mathcal{W}_j(x_1, \ldots, x_j)\, \mathcal{W}_{n-j}(x_{j+1}, \ldots, x_n).$$

[Unraveled, the assertion is that

$$\lim_{\lambda \rightarrow +\infty} \langle \Omega_0, \Phi_j(f)\, U(I, \lambda a)\, \Phi_{n-j}(g)\, \Omega_0 \rangle$$

$$= \langle \mathcal{W}_j, f \rangle \langle \mathcal{W}_{n-j}, g \rangle .$$

Reeh-Schlieder This is the assertion that for any nonempty open set $\mathcal{O} \subset \underset{\sim}{R}^4$, the set $\mathcal{D}_{\mathcal{O}}$ of all finite linear combinations $\underset{\sim}{\varphi}(f_1) \cdots \underset{\sim}{\varphi}(f_n)\Omega_0$ is dense in $\mathcal{H}$, where now $\underline{\text{spt}}\ f_j \subset \mathcal{O}\ \forall\ j$.

Thanks to the cyclicity of the vacuum, it suffices to prove that

$$x \in \mathcal{D}_{\mathcal{O}}^{\perp} \Rightarrow x \in \mathcal{D}_{\underset{\sim}{R}^4}^{\perp} \quad (= \{0\}).$$

Fix $n \geq 1$ -- then, as has been noted earlier, $\exists$ a continuous map $\Phi_n : \mathcal{S}(\underset{\sim}{R}^{4n}) \rightarrow \mathcal{H}$ such that

$$\langle \Phi_n, f_1 \times \cdots \times f_n \rangle = \underset{\sim}{\varphi}(f_1) \cdots \underset{\sim}{\varphi}(f_n)\, \Omega_0,$$

thus $\exists$! tempered distribution $\mathcal{W}_{n,x}$ on $\underset{\sim}{R}^{4n}$ such that

$$\mathcal{W}_{n,x}\,(f_1 \times \cdots \times f_n) = \langle x,\ \Phi_n(f_1 \times \cdots \times f_n) \rangle\ .$$

Using the spectrum condition and the identity

$$\langle\ x,\ \underset{\sim}{\varphi}(x_1) \cdots \underset{\sim}{\varphi}(x_{k-1})\ \underset{\sim}{\varphi}(x_k + a) \cdots \underset{\sim}{\varphi}(x_n + a)\ \Omega_0 \rangle$$

$$= \langle x,\ \underset{\sim}{\varphi}(x_1) \cdots \underset{\sim}{\varphi}(x_{k-1}) U(I,a)\ \underset{\sim}{\varphi}(x_k) \cdots \underset{\sim}{\varphi}(x_n)\, \Omega_0 \rangle,$$

one can check that the support of the Fourier transform

$$\hat{\mathcal{W}}_{n,x}\,(p_1, \ldots, p_n)$$

$$= \frac{1}{(2\pi)^{2n}} \int_{\underset{\sim}{R}^{4n}} \underline{\exp}\Big(\sqrt{-1} \sum_{j=1}^{n} \langle p_j, x_j \rangle\Big) \langle x,\ \underset{\sim}{\varphi}(x_1) \cdots \underset{\sim}{\varphi}(x_n)\Omega_0 \rangle\, dx_1 \cdots dx_n$$

is contained in the intersection of the sets

$$\Big\{ p \in \underset{\sim}{R}^{4n} :\ p_k + p_{k+1} + \cdots + p_n \in \overline{V}_- \Big\}\quad (k = 1, \ldots, n).$$

On the other hand, the map

$$x_1 \to -x_1$$
$$x_2 \to x_1 - x_2$$
$$\vdots$$
$$x_n \to x_{n-1} - x_n$$

is a diffeomorphism $\underset{\sim}{R}^{4n} \xrightarrow{\phi} \underset{\sim}{R}^{4n}$, so $\exists\,!$ tempered distribution $W_{n,x}$ on $\underset{\sim}{R}^{4n}$ such that

$$W_{n,x} \circ \phi = \mathcal{W}_{n,x} ,$$

i.e.,

$$W_{n,x}\,(-x_1, x_1 - x_2, \cdots, x_{n-1} - x_n)$$
$$= \mathcal{W}_{n,x}\,(x_1, x_2, \ldots, x_n).$$

And:

$$\underline{\mathrm{spt}}\ \hat{W}_{n,x} \subset \overline{V}_+ \times \cdots \times \overline{V}_+ \ \text{(n factors)}.$$

E.g.: Take n=3 -- then

$$\hat{\mathcal{W}}_{n,x}\,(p_1, p_2, p_3)$$

$$= \frac{1}{(2\pi)^6} \int_{\underset{\sim}{R}^{12}} e^{\sqrt{-1}(p_1x_1+p_2x_2+p_3x_3)}\,\mathcal{W}_{n,x}\,(x_1,x_2,x_3)dx_1dx_2dx_3$$

$$= \frac{1}{(2\pi)^6} \int_{\underset{\sim}{R}^{12}} \underline{\exp}\ \sqrt{-1}\,(-(p_1+p_2+p_3)(-x_1)-(p_2+p_3)(x_1-x_2)$$
$$-p_3(x_2-x_3))W_{n,x}(-x_1,x_1-x_2,x_2-x_3)dx_1dx_2dx_3$$

$$= \hat{W}_{n,x}\,(-(p_1+p_2+p_3),\ -(p_2+p_3),\ -p_3).$$

Assuming that $\hat{W}_{n,x}\ (q_1,q_2,q_3)\neq 0$, put

$$\begin{cases} q_1 = -(p_1+p_2+p_3) \\ q_2 = -(p_2+p_3) \\ q_3 = -p_3. \end{cases}$$

Then

$$\hat{\mathcal{W}}_{n,x}\ (p_1,p_2,p_3)\neq 0$$

$$\Longrightarrow \begin{cases} p_1+p_2+p_3 \in \overline{V}_- \\ p_2+p_3 \in \overline{V}_- \\ p_3 \in \overline{V}_- \end{cases}$$

$$\Longrightarrow \quad q_1,q_2,q_3 \in \overline{V}_+.$$

Proceeding,

$$\underline{\mathrm{spt}}\ W_{n,x} \neq \underset{\sim}{R}^{4n}$$

if $\mathcal{O}$ is proper, as we suppose. But this implies that

$$W_{n,x} \equiv 0,$$

which finishes the proof.

<u>Euclidean QFT</u> Starting from the correlation functions $\mathcal{W}_n$ of a neutral scalar QFT, one can use analytic continuation and Laplace transform techniques to produce a certain collection of real analytic functions $\mathfrak{S}_n$ on $\underset{\sim}{R}^{4n}_{\neq}$, the <u>Schwinger functions</u>.

Notation: (1) Let

$$\underset{\sim}{R}^{4n}_{\neq} = \left\{ (x_1,\ldots,x_n) \in \underset{\sim}{R}^{4n} : x_i \neq x_j \ (i \neq j) \right\}$$

and denote by $\mathcal{S}(\underset{\sim}{R}^{4n}_{\neq})$ the subspace of $\mathcal{S}(\underset{\sim}{R}^{4n})$ consisting of those f which, together with their derivatives, vanish on each hyperplane $x_i - x_j = 0$ $(i \neq j)$.

(2) Let

$$\underset{\sim}{R}^{4n}_{<} = \left\{ (x_1,\ldots,x_n) \in \underset{\sim}{R}^{4n} : 0 < x_1^0 < \cdots < x_n^0 \right\}$$

and denote by $\mathcal{S}(\underset{\sim}{R}^{4n}_{<})$ the subspace of $\mathcal{S}(\underset{\sim}{R}^{4n})$ consisting of those f whose support is contained in $\underset{\sim}{R}^{4n}_{<}$.

A given Schwinger function $\mathfrak{S}_n$ defines an element of $\mathcal{S}'(\underset{\sim}{R}^{4n}_{\neq})$ in the sense that the functional defined by the integral

$$\int \mathfrak{S}_n(x_1,\ldots,x_n)\ f(x_1,\ldots,x_n)\ d^{4n}x$$

is absolutely convergent for all $f \in \mathcal{S}(\underset{\sim}{R}^{4n}_{\neq})$ and the assignment

$$f \longrightarrow \int \mathfrak{S}_n(x_1,\ldots,x_n)\ f(x_1,\ldots,x_n)\ d^{4n}x$$

is continuous.

**OS1** $\forall\, f \in \mathcal{S}(\underset{\sim}{R}^{4n}_{\neq})$,

$$\overline{\langle \mathfrak{S}_n, f \rangle} = \langle \mathfrak{S}_n, \overline{\Theta f} \rangle,$$

where

$$\Theta f((x_1^0, \underset{\sim}{x}_1), \ldots, (x_n^0, \underset{\sim}{x}_n)) = f((-x_1^0, \underset{\sim}{x}_1), \ldots, (-x_n^0, \underset{\sim}{x}_n)).$$

**OS2** $\forall\, f \in \mathcal{S}(\underset{\sim}{R}^{4n}_{\neq})$,

$$\langle \mathfrak{S}_n, f \rangle = \langle \mathfrak{S}_n, f_{\Lambda, a} \rangle$$

where $(\Lambda, a) \in \underset{\sim}{SO}(4) \rtimes \underset{\sim}{R}^4$ and

$$f_{\Lambda, a}(x_1, \ldots, x_n) = f(\Lambda^{-1}(x_1 - a), \ldots, \Lambda^{-1}(x_n - a)).$$

**OS3** Let $f_0 \in \underset{\sim}{C}$, $f_1 \in \mathcal{S}(\underset{\sim}{R}^4_>), \ldots, f_n \in \mathcal{S}(\underset{\sim}{R}^{4n}_>)$ -- then

$$\sum_{n,m} \langle \mathfrak{S}_{n+m}, \overline{\Theta f_n} \times f_m \rangle \geq 0.$$

**OS4** $\forall\, f \in \mathcal{S}(\underset{\sim}{R}^{4n}_{\neq})$,

$$\langle \mathfrak{S}_n, f \rangle = \langle \mathfrak{S}_n, f \circ \sigma \rangle \quad (\sigma \in S_n).$$

**OS5** $\forall\, f \in \mathcal{S}(\underset{\sim}{R}^{4n}_{\neq}) \cap C_c^\infty(\underset{\sim}{R}^{4n})$ and $\forall\, g \in \mathcal{S}(\underset{\sim}{R}^{4m}_{\neq}) \cap C_c^\infty(\underset{\sim}{R}^{4m})$,

$$\lim_{t \to +\infty} \langle \mathfrak{S}_{n+m}, f \times T_t g \rangle = \langle \mathfrak{S}_n, f \rangle \langle \mathfrak{S}_m, g \rangle,$$

where

$$T_t g((x_1^0, \underset{\sim}{x}_1), \ldots, (x_m^0, \underset{\sim}{x}_m)) = g((x_1^0 - t, \underset{\sim}{x}_1), \ldots, (x_m^0 - t, \underset{\sim}{x}_m)).$$

Remark: OS stands for Osterwalder-Schrader. Their reconstruction theorem reverses the procedure, viz. they show that if you start with distributions $\mathfrak{S}_n$ satisfying OS1-OS5, then these distributions are in fact real analytic functions on $\underset{\sim}{R}^{4n}_{\neq}$ and are the Schwinger functions associated with an essentially unique neutral scalar QFT.

[Note: This is not quite true in that the requirement $\mathfrak{S}_n \in \mathcal{S}'(\underset{\sim}{R}^{4n}_{\neq})$ has to be reinforced. Unfortunately, this auxiliary condition is virtually impossible to check in practice.]

Example: Consider the free QFT of mass $m > 0$ -- then

$$\mathfrak{S}_2(x,y) = \frac{1}{(2\pi)^2} \int_{\underset{\sim}{R}^4} e^{\sqrt{-1}\langle p, x-y\rangle} \frac{1}{p^2+m^2} d^4p$$

$$= \frac{m}{|x-y|} K_1(m|x-y|).$$

<u>The Källen-Lehmann Representation</u> A distribution T on $\underset{\sim}{R}^n$ is said to be of <u>positive type</u> if

$$\langle T*f,\bar{f}\rangle = \langle T, \Pi f*\bar{f}\rangle \geq 0 \quad \forall f \in C_c^\infty(\underset{\sim}{R}^n).$$

[Note: Symbolically,

$$\langle T, \Pi f*\bar{f}\rangle$$

$$= \int_{\underset{\sim}{R}^n} (\Pi f*\bar{f})(x)\ T(x)dx$$

$$= \int_{\underset{\sim}{R}^n} \Big( \int_{\underset{\sim}{R}^n} \overline{f(y)}\ \Pi f(x-y)dy\Big)\ T(x)dx$$

$$= \int_{\underset{\sim}{R}^n} \Big( \int_{\underset{\sim}{R}^n} \overline{f(y)} f(y-x)dy\Big)\ T(x)dx$$

$$= \int_{\underset{\sim}{R}^n} \int_{\underset{\sim}{R}^n} \overline{f(x)}\ f(y)\ T(x-y)dxdy.]$$

---

THEOREM (Bochner-Schwartz) Suppose that T is of positive type-- then $T=\hat{\mu}$, where $\mu$ is a tempered positive measure.

---

LEMMA Let $\mu$ be a tempered $\mathcal{L}_+^\uparrow$-invariant measure on $\underset{\sim}{R}^4$ with support in $\bar{V}_+$ -- then $\exists$ a tempered measure $\rho$ on $[0,+\infty[$ such that $\forall\ f \in \mathcal{S}(\underset{\sim}{R}^4)$,

$$\int_{\underset{\sim}{R}^4} f\,d\mu = \mu(\{0\})f(0) + \int_0^\infty \Big( \int_{X_m} f d\mu_m\Big) d\rho(m).$$

---

Returning to our neutral scalar QFT, consider $W_2$. Thus from the definitions,

$$\int_{\underset{\sim}{R}^4} \int_{\underset{\sim}{R}^4} \overline{f(x)}\, f(y)\, W_2(x-y)\,dxdy$$

$$= \langle \mathcal{W}_2, \bar{f} \times f \rangle .$$

And:

$$\langle \mathcal{W}_2, \bar{f} \times f \rangle$$

$$= \langle \Omega_0, \underset{\sim}{\varphi}(\bar{f})\, \underset{\sim}{\varphi}(f) \Omega_0 \rangle$$

$$= \langle \underset{\sim}{\varphi}(\bar{f})^* \Omega_0, \underset{\sim}{\varphi}(f) \Omega_0 \rangle$$

$$= \langle \underset{\sim}{\varphi}(f)\, \Omega_0, \underset{\sim}{\varphi}(f) \Omega_0 \rangle$$

$$= || \underset{\sim}{\varphi}(f) \Omega_0 ||^2 \geq 0.$$

This shows that $W_2$ is of positive type, hence by Bochner-Schwartz, $\exists$ a tempered positive measure $\mu$: $W_2 = \check{\mu} \Rightarrow \hat{W}_2 = \hat{\check{\mu}} = \mu$. But the support of $\hat{W}_2$ is contained in $\overline{V}_+$, so an application of the lemma gives

$$\langle \hat{W}_2, f \rangle = \hat{W}_2(\{0\}) f(0) + \int_0^\infty \left( \int_{X_m} f d\mu_m \right) d\rho(m)$$

or still,

$$\langle W_2, f \rangle = \langle W_2, \hat{\check{f}} \rangle$$

$$= \langle \hat{W}_2, \check{f} \rangle$$

$$= \hat{W}_2(\{0\}) \cdot \frac{1}{(2\pi)^2} \int_{\underset{\sim}{R}^4} f(x)\,dx$$

$$+ \int_0^{\infty} \left( \int_{X_m} \check{f} d\mu_m \right) d\rho(m),$$

which is the Källen-Lehman representation for $W_2$. Here

$$\check{f}(p) = \frac{1}{(2\pi)^2} \int_{\underset{\sim}{R}^4} e^{-\sqrt{-1}\,(p_0 x_0 - \underset{\sim}{p} \cdot \underset{\sim}{x})} f(x)dx,$$

so by formal manipulation,

$$\langle W_2, f \rangle = \hat{W}_2(\{0\}) \cdot \frac{1}{(2\pi)^2} \int_{\underset{\sim}{R}^4} f(x)dx$$

$$+ 4\pi \int_{\underset{\sim}{R}^4} f(x)\; W_2(x)dx,$$

where

$$W_2(x) = \int_0^{\infty} \frac{1}{\sqrt{-1}} \Delta_+(x;m^2) d\rho(m)$$

and

$$\Delta_+(x;m^2) = \frac{\sqrt{-1}}{2(2\pi)^3} \int_{X_m} e^{-\sqrt{-1}\,(p_0 x_0 - \underset{\sim}{p} \cdot \underset{\sim}{x})} d\mu_m(p).$$

It remains to explicate $\hat{W}_2(\{0\})$. For this, note first that $\mathcal{W}_1$ is translation invariant, hence $\exists$ a constant K:

$$\langle \mathcal{W}_1, f \rangle = K \int_{\underset{\sim}{R}^4} f(x)dx.$$

4.

---

<u>LEMMA</u> We have

$$\hat{W}_2(\{0\}) = (2\pi)^2 \cdot |K|^2.$$

[Given $f \in \mathcal{S}(\mathbb{R}^4)$, let

$$F(a) = \langle \boldsymbol{\varphi}(f)\Omega_0, U(I,-a)\, \boldsymbol{\varphi}(f)\Omega_0 \rangle .$$

Then

$$F(a) = (W_2 * f * \Pi \bar{f})(a),$$

hence

$$\hat{F} = (2\pi)^4 \, |\hat{f}|^2 \, \hat{W}_2.$$

On the other hand,

$$F(a) = \int_{\mathbb{R}^4} e^{-\sqrt{-1}\,\langle a,\lambda\rangle} \, d\langle \boldsymbol{\varphi}(f)\Omega_0, E_\lambda \boldsymbol{\varphi}(f)\Omega_0 \rangle ,$$

so

$$F = (2\pi)^2 \times \big(d\langle \boldsymbol{\varphi}(f)\Omega_0, E_\lambda \boldsymbol{\varphi}(f)\Omega_0 \rangle\big)^{\vee}$$

$\Longrightarrow$

$$\hat{F} = (2\pi)^2 \times d\langle \boldsymbol{\varphi}(f)\Omega_0, E_\lambda \boldsymbol{\varphi}(f)\Omega_0 \rangle .$$

The mass of $d\langle \boldsymbol{\varphi}(f)\Omega_0, E_\lambda \boldsymbol{\varphi}(f)\Omega_0 \rangle$ at the origin is

$$\langle \boldsymbol{\varphi}(f)\Omega_0, E_0 \boldsymbol{\varphi}(f)\Omega_0 \rangle .$$

But since the vacuum is unique,

$$\begin{aligned} E_0 \boldsymbol{\varphi}(f)\Omega_0 &= \langle \Omega_0, E_0 \boldsymbol{\varphi}(f)\Omega_0 \rangle \Omega_0 \\ &= \langle E_0 \Omega_0, \boldsymbol{\varphi}(f)\Omega_0 \rangle \Omega_0 \\ &= \langle \Omega_0, \boldsymbol{\varphi}(f)\Omega_0 \rangle \Omega_0 \end{aligned}$$

$\Longrightarrow$

$$\langle \underset{\sim}{\varphi}(f)\Omega_0, E_0\underset{\sim}{\varphi}(f)\Omega_0\rangle$$

$$= \langle \underset{\sim}{\varphi}(f)\Omega_0,\Omega_0\rangle \cdot \langle \Omega_0, \underset{\sim}{\varphi}(f)\Omega_0\rangle$$

$$= |\langle \Omega_0, \underset{\sim}{\varphi}(f)\Omega_0\rangle|^2$$

$$= |\langle \mathcal{W}_1, f\rangle|^2$$

$$= |K|^2 \cdot \int_{\underset{\sim}{R}^4} f(x)dx \cdot \int_{\underset{\sim}{R}^4} \overline{f(x)}dx$$

$$= (2\pi)^4 \cdot |\hat{f}(0)|^2 \cdot |K|^2.$$

One can then eliminate f and conclude that $\hat{W}_2(\{0\}) = (2\pi)^2 \cdot |K|^2$.]

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Definition: The <u>mass spectrum</u> of a neutral scalar QFT is the support of $\rho$, the <u>mass</u> then being the infimum of its mass spectrum with 0 removed.

Example: Consider the free QFT of mass $m > 0$ -- then $\rho = \delta_m$.

[Note: Take f,g real. Using definitions only, one finds that in this case

$$\mathcal{W}_2(f \times g) = \int_{\underset{\sim}{R}^4} \int_{\underset{\sim}{R}^4} \frac{1}{\sqrt{-1}} \Delta_+(x-y;m^2) f(x) g(y) dxdy.]$$

We have mentioned earlier that it is impossible to define the field map directly on $\underset{\sim}{R}^4$ (subject, of course, to the assumptions). In brief, here is why.

(1) We have

$$\mathcal{W}_2(x,y) = \langle \Omega_0, \underset{\sim}{\varphi}(x)\, \underset{\sim}{\varphi}(y)\, \Omega_0 \rangle = W_2(x-y)$$

$\Longrightarrow$

$$W_2(x) = \langle \Omega_0, \underset{\sim}{\varphi}(x)\, \underset{\sim}{\varphi}(0)\, \Omega_0 \rangle .$$

(2) We have

$$\underset{\sim}{\varphi}(x) = U(I,x)\, \underset{\sim}{\varphi}(0) U(I,x)^{-1}$$

$\Longrightarrow$

$$W_2(x) = \langle \Omega_0, \underset{\sim}{\varphi}(0) U(I,-x)\, \underset{\sim}{\varphi}(0)\, \Omega_0 \rangle .$$

(3) We have

$$W_2(x) = \int_{\underset{\sim}{R}^4} e^{-\sqrt{-1}\langle x,p \rangle}\, d\mu(p),$$

where $\mu$ is a tempered $\mathcal{L}_+^{\uparrow}$ -invariant measure.

(4) We have

$$W_2(0) = \mu(\underset{\sim}{R}^4) < +\infty$$

$\Longrightarrow$

$$\mu = \mu(\{0\})\ \delta_0$$

$\Longrightarrow$

$$W_2(x) = W_2(0) = \mu(\{0\}).$$

(5) We have

$$\underset{\sim}{\varphi}(x)\, \Omega_0 = \underset{\sim}{\varphi}(0)\, \Omega_0 .$$

(6) We have

$$U(\Lambda,a)(\underset{\sim}{\varphi}(0)\, \Omega_0) = U(\Lambda,a)\, \underset{\sim}{\varphi}(0) U(\Lambda,a)^{-1} \Omega_0$$

$$= \underset{\sim}{\varphi}(a)\, \Omega_0$$

$$= \underset{\sim}{\varphi}(0)\,\Omega_0 \qquad \forall\ (\Lambda, a)$$

$\Longrightarrow$

$$\underset{\sim}{\varphi}(0)\,\Omega_0 = c\,\Omega_0 \qquad (\exists\ c)$$

$\Longrightarrow$

$$\underset{\sim}{\varphi}(x)\,\Omega_0 = c\,\Omega_0.$$

But this can't happen if the vacuum is to be cyclic.

<u>Spin and Statistics</u> Recall the statement of W6: If the supports of $f,g \in \mathcal{S}(\underset{\sim}{R}^4)$ are spacelike separated, then

$$\underset{\sim}{\varphi}(f)\,\underset{\sim}{\varphi}(g) - \underset{\sim}{\varphi}(g)\,\underset{\sim}{\varphi}(f) = 0.$$

This is the way it has to be: It is impossible to have

$$\underset{\sim}{\varphi}(f)\,\underset{\sim}{\varphi}(g) + \underset{\sim}{\varphi}(g)\,\underset{\sim}{\varphi}(f) = 0$$

for all $f,g \in \mathcal{S}(\underset{\sim}{R}^4)$ with spacelike separated supports.

The proof goes as follows. Agreeing to use symbolic notation, $\exists\ \mathcal{L}_+^{\uparrow}$-invariant tempered distributions F and G on $\underset{\sim}{R}^4$ such that

$$\begin{cases} F(x-y) = \langle \Omega_0, \underset{\sim}{\varphi}(x)\,\underset{\sim}{\varphi}(y)\,\Omega_0 \rangle - \langle \Omega_0, \underset{\sim}{\varphi}(y)\,\underset{\sim}{\varphi}(x)\,\Omega_0 \rangle \\ G(x-y) = \langle \Omega_0, \underset{\sim}{\varphi}(x)\,\underset{\sim}{\varphi}(y)\,\Omega_0 \rangle + \langle \Omega_0, \underset{\sim}{\varphi}(y)\,\underset{\sim}{\varphi}(x)\,\Omega_0 \rangle . \end{cases}$$

Because F is odd,

$$\xi^2 < 0 \Rightarrow F(\xi) = 0.$$

Here the argument is exactly the same as that employed earlier for $\Delta_m$. On the other hand,

$$\begin{aligned} W_2(x-y) &= \langle \Omega_0, \underset{\sim}{\varphi}(x)\,\underset{\sim}{\varphi}(y)\,\Omega_0 \rangle \\ &= \frac{1}{2}\,(F(x-y) + G(x-y)). \end{aligned}$$

But anticommutativity means

$$(x-y)^2 < 0 \Rightarrow G(x-y) = 0,$$

hence

$$\xi^2 < 0 \Rightarrow W_2(\xi) = 0.$$

In particular: $\underline{\text{spt}}\ W_2 \neq \mathbb{R}^4 \Rightarrow W_2 \equiv 0$

$$\Rightarrow$$

$$0 = \langle \mathcal{W}_2, \bar{f} \times f \rangle = \| \varphi(f)\Omega_0 \|^2 \quad \forall f \in \mathcal{S}(\mathbb{R}^4),$$

which contradicts the cyclicity of $\Omega_0$.

<u>Irreducibility of the Field Operators</u> Suppose given a neutral scalar QFT with the property that the space of $\underset{\sim}{R}^4$-invariants is one dimensional (recall that this condition implies W2) -- then the $\underset{\sim}{\varphi}(f)$ $(f \in \mathcal{S}(\underset{\sim}{R}^4))$ are an irreducible set. By this we mean that every bounded linear operator A for which

$$\langle x, A\underset{\sim}{\varphi}(f)y\rangle = \langle \underset{\sim}{\varphi}(f)^* x, Ay\rangle \ \forall\, x,y \in \mathcal{D} \ \& \ \forall f \in \mathcal{S}(\underset{\sim}{R}^4)$$

is necessarily a constant multiple of the identity.

As a preliminary, note that the function

$$a \longrightarrow \langle x, U(I,a)y\rangle = \int_{\underset{\sim}{R}^4} e^{\sqrt{-1}(a_0 p_0 - \underset{\sim}{a}\cdot\underset{\sim}{p})}\, d\langle x, E_p y\rangle$$

is a bounded continuous function of a, thus defines a tempered distribution $T_{x,y}$.

---

<u>LEMMA</u> The support of $\hat{T}_{x,y}$ is contained in $\overline{V}_+$.

[Take any f whose support is contained in the complement of $\overline{V}_+$ -- then

$$\langle \hat{T}_{x,y}, f\rangle = \langle T_{x,y}, \hat{f}\rangle$$

$$= \int_{\underset{\sim}{R}^4} \langle x, U(I,a)y\rangle \hat{f}(a)da$$

$$= \int_{\underset{\sim}{R}^4} \left( \int_{\underset{\sim}{R}^4} e^{\sqrt{-1}(a_0 p_0 - \underset{\sim}{a}\cdot\underset{\sim}{p})} \hat{f}(a)da\right) d\langle x, E_p y\rangle$$

$$= (2\pi)^2 \int_{\underset{\sim}{R}^4} f(p)\,d\langle x, E_p y\rangle = 0$$

$$\Longrightarrow \quad \underline{\text{spt}}\ \hat{T}_{x,y} \subset \overline{V}_+ .$$

[Note: Here, of course, we have used the fact that the support of $d\langle x, E_p y\rangle$ is contained in $\overline{V}_+$.]

---

Suppose now that A is a bounded linear operator with the stated property.

Claim: $\exists\ \lambda : A\Omega_0 = \lambda\Omega_0$. To see this, observe first that

$$\langle \Omega_0, A\,\underset{\sim}{\varphi}((I,a)\cdot f_1)\cdots\underset{\sim}{\varphi}((I,a)\cdot f_n)\Omega_0\rangle$$

$$= \langle \underset{\sim}{\varphi}((I,a)\cdot f_n)^*\cdots\underset{\sim}{\varphi}((I,a)\cdot f_1)^*\Omega_0, A\Omega_0\rangle$$

$\Longrightarrow$

$$\langle A^*\Omega_0, U(I,a)\,\underset{\sim}{\varphi}(f_1)\cdots\underset{\sim}{\varphi}(f_n)\Omega_0\rangle$$

$$= \langle \underset{\sim}{\varphi}(f_n)^*\cdots\underset{\sim}{\varphi}(f_1)^*\Omega_0, U(I,-a)A\Omega_0\rangle .$$

According to the lemma, the support of the Fourier transform of the LHS is $\overline{V}_+$, while the support of the Fourier transform of the RHS is $\overline{V}_-$. Therefore the support of the Fourier transform of either side is the origin, hence is a finite linear combination of derivatives of the Dirac delta. But then, by Fourier inversion, our function must be a polynomial in a, thus is a constant in a (being bounded). So: $\forall$ a,

$$\langle \underset{\sim}{\varphi}(\overline{f}_n)\cdots\underset{\sim}{\varphi}(\overline{f}_1)\Omega_0, U(I,-a)A\Omega_0\rangle$$

$$= \langle \underset{\sim}{\varphi}(\overline{f}_n)\cdots\underset{\sim}{\varphi}(\overline{f}_1)\Omega_0, A\Omega_0\rangle$$

$\Longrightarrow$

$$U(I,a)A\,\Omega_0 = A\,\Omega_0 \quad \forall a$$

$$\Longrightarrow \quad A\,\Omega_0 = \lambda\Omega_0 \ (\exists\, \lambda).$$

Accordingly, $\forall\, x \in \mathcal{D}$,

$$\langle x, A\,\underset{\sim}{\varphi}(f_1) \cdots \underset{\sim}{\varphi}(f_n)\,\Omega_0 \rangle$$

$$= \langle \underset{\sim}{\varphi}(f_n)^* \cdots \underset{\sim}{\varphi}(f_1)^* x, A\,\Omega_0 \rangle$$

$$= \langle \underset{\sim}{\varphi}(f_n)^* \cdots \underset{\sim}{\varphi}(f_1)^* x, \lambda\Omega_0 \rangle$$

$\Longrightarrow$

$$(A-\lambda)\,\underset{\sim}{\varphi}(f_1) \cdots \underset{\sim}{\varphi}(f_n)\,\Omega_0 = 0$$

$$\Longrightarrow \quad A = \lambda I.$$

<u>The Borchers Algebra</u> Set theoretically, this is

$$\mathcal{S} = \underset{\sim}{C} \oplus \mathcal{S}(\underset{\sim}{R}^4) \oplus \mathcal{S}(\underset{\sim}{R}^8) \oplus \cdots \quad .$$

Equip $\mathcal{S}$ with the direct sum topology per the injections $\mathcal{S}(\underset{\sim}{R}^{4n}) \longrightarrow \mathcal{S}$ -- then $\mathcal{S}$ becomes a separable LCTVS.

(1) $\mathcal{S}$ admits a continuous involution *.

[Let $f = \{f_n\} \in \mathcal{S}$ and put

$$f^* = \{f_n^*\},$$

where

$$f_n^*(x_1, \ldots, x_n) = \overline{f_n(x_n, \ldots, x_1)}.$$

(2) $\mathcal{S}$ admits a continuous multiplication $\times$ .

[Let $f = \{f_n\}$ & $g = \{g_n\} \in \mathcal{S}$ and define $f \times g$ by

$$(f \times g)_n = \sum_{k=0}^{n} f_k(x_1, \ldots, x_k)\, g_{n-k}(x_{k+1}, \ldots, x_n).]$$

[Note: Multiplication is continuous in each variable separately and is jointly continuous as a bilinear map

$$\bigoplus_{n=0}^{N} \mathcal{S}(\underset{\sim}{R}^{4n}) \times \bigoplus_{n=0}^{N} \mathcal{S}(\underset{\sim}{R}^{4n}) \longrightarrow \bigoplus_{n=0}^{2N} \mathcal{S}(\underset{\sim}{R}^{4n})$$

for $N < +\infty$ but not for $N = +\infty$.]

So: $\mathcal{S}$ is a graded topological *-algebra with unit $I = (1, 0, \ldots)$ on which $\mathcal{G}_+^\uparrow$ operates in the evident way.

[Note: Explicitly,

$$\begin{cases} (\Lambda, a)\cdot(\alpha f + \beta g) = \alpha((\Lambda, a)\cdot f) + \beta((\Lambda, a)\cdot g) \\ (\Lambda, a)\cdot(f \times g) = (\Lambda, a)\cdot f \times (\Lambda, a)\cdot g \\ ((\Lambda, a)\cdot f)^* = (\Lambda, a)\cdot f^*, \ (\Lambda, a)\cdot I = I.] \end{cases}$$

Let $\mathcal{W} : \mathcal{S} \to \underset{\sim}{C}$ be a linear functional -- then

$$\left\{\begin{array}{l} \mathcal{W} \text{ is } \underline{\text{positive}} \text{ if } \mathcal{W}(f^* \times f) \geq 0 \\ \mathcal{W} \text{ is } \underline{\text{hermitian}} \text{ if } \mathcal{W}(f^*) = \overline{\mathcal{W}(f)} \end{array}\right. \quad \forall f \in \mathcal{S}.$$

---

LEMMA $\mathcal{W}$ positive $\Rightarrow$ $\mathcal{W}$ hermitian.

---

LEMMA $\mathcal{W}$ positive $\Rightarrow$

$$|\mathcal{W}(f^* \times g)|^2 \leq \mathcal{W}(f^* \times f)\, \mathcal{W}(g^* \times g).$$

---

Definition: A state on $\mathcal{S}$ is a positive linear functional $\mathcal{W}$ such that $\mathcal{W}(I)=1$.

Example: Fix a neutral scalar QFT -- then the correlation functions $\mathcal{W}_n$ of the theory determine a state $\mathcal{W}$ on $\mathcal{S}$.

---

THEOREM Every positive linear functional on $\mathcal{S}$ is continuous.

[Note: It is therefore automatic that the correlation functions $\mathcal{W}_n$ of a neutral scalar QFT are tempered.]

---

Given a $\mathcal{P}_+^\uparrow$-invariant state $\mathcal{W}$, put

$$N_{\mathcal{W}} = \{f: \mathcal{W}(f^* \times f)=0\}.$$

Then $N_{\mathcal{W}}$ is a $\mathcal{P}_+^\uparrow$-invariant left ideal in $\mathcal{S}$. As such, it is a closed subspace ($\mathcal{W}$ being continuous).

[Note: Observe that

$$\left\{\begin{array}{l} f \in N_{\mathcal{W}} \\ g \in \mathcal{S} \end{array}\right. \Rightarrow \mathcal{W}(g \times f)=0.$$

Indeed,

$$0 \leq |\mathcal{W}(g \times f)| = |\mathcal{W}(g^{**} \times f)|$$
$$\leq \sqrt{\mathcal{W}(g \times g^*)}\sqrt{\mathcal{W}(f^* \times f)} = 0.]$$

The prescription

$$\langle [f],[g] \rangle = \mathcal{W}(f^* \times g)$$

is an inner product on the quotient $\mathcal{D} = \mathcal{S}/N_{\mathcal{W}}$. Let $\mathcal{H}$ be the corresponding completion -- then $\mathcal{D}$ is a dense linear subspace of $\mathcal{H}$.

The following properties obtain.

(1) $\mathcal{H}$ is a separable Hilbert space.

(2) The action of $\mathcal{P}_+^\uparrow$ on $\mathcal{S}$ passes to the quotient to define an action of $\mathcal{P}_+^\uparrow$ on $\mathcal{S}/N_{\mathcal{W}}$ : $(\Lambda,a)\cdot[f]=[(\Lambda,a)\cdot f]$. Since

$$\begin{aligned}\langle (\Lambda,a)\cdot[f], \ (\Lambda,a)\cdot[g] \rangle &= \mathcal{W}(((\Lambda,a)\cdot f)^* \times (\Lambda,a)\cdot g) \\ &= \mathcal{W}((\Lambda,a)\cdot f^* \times (\Lambda,a)\cdot g) \\ &= \mathcal{W}((\Lambda,a)\cdot(f^* \times g)) \\ &= \mathcal{W}(f^* \times g) \\ &= \langle [f],[g] \rangle,\end{aligned}$$

the action extends by continuity to a unitary representation U of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$.

(3) From its very construction, $\mathcal{D} \subset \mathcal{H}$ is dense and $\mathcal{P}_+^\uparrow$-invariant.

(4) Define a linear map $\underset{\sim}{\varphi} : \mathcal{S} \longrightarrow \underline{\text{End}}\,\mathcal{D}$ by $\underset{\sim}{\varphi}(f)[g]=[f \times g]$ -- then

$$\begin{aligned}\underset{\sim}{\varphi}((\Lambda,a)\cdot f)[g] &= [(\Lambda,a)\cdot f \times g] \\ &= [(\Lambda,a)\cdot(f \times (\Lambda,a)^{-1}g)]\end{aligned}$$

$$= U(\Lambda,a)[f \times (\Lambda,a)^{-1}g]$$

$$= U(\Lambda,a)\, \underset{\sim}{\varphi}(f) U(\Lambda,a)^{-1}[g],$$

thus $\underset{\sim}{\varphi}$ is $\mathcal{P}_+^\uparrow$-equivariant.

(5) Put $\Omega_0=[1]$ -- then $\Omega_0 \in \mathcal{D}$ is a $\mathcal{P}_+^\uparrow$-invariant unit vector.

Conclusion: All the data lying behind a neutral scalar QFT is in place but there is no guarantee that assumptions W1-W6 are satisfied.

[Note: To get the field map, restrict to $\mathcal{S}(\underset{\sim}{R}^4) \hookrightarrow \mathcal{S}$. Obviously, on the basis of the construction itself, we have

$$\mathcal{W}_n(f_1 \times \cdots \times f_n) = \langle \Omega_0, \underset{\sim}{\varphi}(f_1) \cdots \underset{\sim}{\varphi}(f_n)\Omega_0\rangle.]$$

Claim: W3 holds. To see this, take $f,g,h \in \mathcal{S}$ and compute:

$$\langle \underset{\sim}{\varphi}(f)[g],[h]\rangle = \langle [f \times g],[h]\rangle$$

$$= \mathcal{W}((f \times g)^* \times h)$$

$$= \mathcal{W}(g^* \times f^* \times h)$$

$$= \langle [g],[f^* \times h]\rangle$$

$$= \langle [g], \underset{\sim}{\varphi}(f^*)[h]\rangle .$$

Therefore

$$\mathcal{D} \subset \underline{\mathrm{Dom}}\ \underset{\sim}{\varphi}(f)^*$$

and

$$\underset{\sim}{\varphi}(f)^* \mid \mathcal{D} = \underset{\sim}{\varphi}(f^*).$$

It remains only to specialize to $f \in \mathcal{S}(\underset{\sim}{R}^4)$, observing that in this case $f^*=\bar{f}$.

Claim: W4 holds. This boils down to two things: (i) $\mathcal{W}$ is a state, hence is positive, hence is continuous; (ii) $\times : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is separately continuous. Therefore the assignment

$$\begin{aligned} f \to & \langle [g], \underset{\sim}{\varphi}(f)[h] \rangle \\ & = \langle [g], [f \times h] \rangle \\ & = \mathcal{W}(g^* \times f \times h) \end{aligned}$$

is continuous.

Claim: W5 holds. From the definitions, $\mathcal{D} = \{\underset{\sim}{\varphi}(f)\Omega_0 : f \in \mathcal{S}\}$ (since $\underset{\sim}{\varphi}(f)\Omega_0 = [f \times I] = [f]$). On the other hand, if $f_1, \ldots, f_n \in \mathcal{S}(\underset{\sim}{R}^4)$, then $\underset{\sim}{\varphi}(f_1) \cdots \underset{\sim}{\varphi}(f_n)\Omega_0 = [f_1 \times \cdots \times f_n]$ and $\prod_1^n \mathcal{S}(\underset{\sim}{R}^4)$ is dense in $\mathcal{S}(\underset{\sim}{R}^{4n})$.

Suppose that $\mathcal{W}_1, \mathcal{W}_2$ are $\mathcal{P}_+^\uparrow$-invariant states -- then the same is true of

$$\lambda_1 \mathcal{W}_1 + \lambda_2 \mathcal{W}_2 \quad \left(\lambda_1 + \lambda_2 = 1, \begin{cases} \lambda_1 > 0 \\ \lambda_2 > 0 \end{cases}\right),$$

thus the set of $\mathcal{P}_+^\uparrow$-invariant states is convex. Its extreme points are the <u>pure</u> $\mathcal{P}_+^\uparrow$-invariant states.

---

THEOREM $\mathcal{W}$ is pure iff the space of $\mathcal{P}_+^\uparrow$-invariants in $\mathcal{H}$ is 1-dimensional, i.e., iff W2 holds.

---

Fix now a pure $\mathcal{P}_+^\uparrow$-invariant state $\mathcal{W}$ -- then the issue which has yet to be resolved is: When do W1 and W6 hold?

($I_{sp}$) The <u>spectrum ideal</u> $I_{sp}$ is the left ideal consisting of those $f \in \mathcal{S}$ such that $f_0=0$ and $\forall\, n \geq 1$,

$$\hat{f}_n(p_1,\ldots,p_n)=0$$

if $\sum_{j=k}^{n} p_j \in \overline{V}_+ \ \forall\, k,\ 1 \leq k \leq n.$

($I_{loc}$) The locality ideal $I_{loc}$ is the ideal generated by elements of the form

$$f_n(x_1,\ldots,x_j,x_{j+1},\ldots,x_n)$$

$$=g(x_1,\ldots,x_j,x_{j+1},\ldots,x_n) - g(x_1,\ldots,x_{j+1},x_j,\ldots,x_n),$$

where $g(x_1,\ldots,x_n)=0$ if $(x_j-x_{j+1})^2 \geq 0.$

Definition: A pure $\mathcal{P}_+^\uparrow$-invariant state $\mathcal{W}$ is said to satisfy the Wightman condition if

$$I_{sp},\ I_{loc} \subset N_{\mathcal{W}} \ .$$

Example: The correlation functions $\mathcal{W}_n$ of a neutral scalar QFT determine a pure $\mathcal{P}_+^\uparrow$-invariant state $\mathcal{W}$ on $\mathcal{S}$ which satisfies the Wightman condition.

---

LEMMA Suppose that $\mathcal{W}$ satisfies the Wightman condition -- then W1 and W6 hold.

[The verifications are straightforward. To illustrate, let's check W6. Thus suppose that the supports of $f,g \in \mathcal{S}(\underset{\sim}{R}^4)$ are spacelike separated -- then

$$\|\, [\underset{\sim}{\varphi}(f),\, \underset{\sim}{\varphi}(g)]\,[h]\,\|^2$$

$$= \mathcal{W}((f\times g\times h-g\times f\times h)^* \times (f\times g\times h-g\times f\times h))$$

$= \mathcal{W}(((f \times g - g \times f) \times h)^* \times ((f \times g - g \times f) \times h))$

$= 0,$

since $f \times g - g \times f \in I_{loc}$.]

---

Consequently, to every pure $\mathcal{P}_+^\uparrow$-invariant state $\mathcal{W}$ satisfying the Wightman condition there is associated a neutral scalar QFT. Moreover, any other neutral scalar QFT with these correlation functions is unitarily equivalent to this one.

Example (Greenberg): Let $\mathcal{W}_0 = 1$, $\mathcal{W}_1 = 0$ and define $W_2$ by

$$\langle W_2, f \times g \rangle = \int_{\underset{\sim}{R}^4} \int_{\underset{\sim}{R}^4} f(x) g(y) W_2(x-y) \, dx dy,$$

where

$$W_2(x) = \int_0^\infty \frac{1}{\sqrt{-1}} \Delta_+(x; m^2) \, d\rho(m),$$

and $\rho$ is a tempered measure on $[0, +\infty[$. For $n > 2$, let $\mathcal{W}_n = 0$ if n is odd but if n is even, let

$$\langle \mathcal{W}_n, f_1 \times \cdots \times f_n \rangle$$
$$= \sum \langle \mathcal{W}_2, f_{i_1} \times f_{i_2} \rangle \cdots \langle \mathcal{W}_2, f_{i_{n-1}} \times f_{i_n} \rangle ,$$

the sum being over all partitions of n into n/2 disjoint pairs with $i_{2k-1} < i_{2k}$ $(k=1,\ldots,n)$.

E.g.:

$$\langle \mathcal{W}_4, f_1 \times f_2 \times f_3 \times f_4 \rangle$$
$$= \langle \mathcal{W}_2, f_1 \times f_2 \rangle \langle \mathcal{W}_2, f_3 \times f_4 \rangle$$
$$+ \langle \mathcal{W}_2, f_1 \times f_3 \rangle \langle \mathcal{W}_2, f_2 \times f_4 \rangle$$

$$+ \langle \mathcal{W}_2, f_1 \times f_4 \rangle \langle \mathcal{W}_2, f_2 \times f_3 \rangle .$$

It can then be shown that $\mathcal{W} = \{\mathcal{W}_n\}$ is a pure $\mathcal{P}_+^\uparrow$-invariant state which satisfies the Wightman condition.

Example (Haag): Suppose given a neutral scalar QFT such that $\mathcal{W}_0=1$, $\mathcal{W}_1=0$, and

$$\langle \mathcal{W}_2, f \times g \rangle = \int_{\underset{\sim}{R}^4} \int_{\underset{\sim}{R}^4} f(x)g(y)W_2(x-y)\,dxdy,$$

where

$$W_2(x-y) = \frac{1}{\sqrt{-1}} \Delta_+(x-y;m^2) \quad (m > 0).$$

Then this neutral scalar QFT "is" the free QFT of mass $m > 0$.

[The idea is to show that the correlation functions $\mathcal{W}_n (n > 2)$ are the same as those of the free QFT of mass $m > 0$.]

Extension of the Assumptions The Wightman axioms can be formulated more generally.

Data:

(1) A finite dimensional vector space $\mathcal{V}$ and a separable Hilbert space $\mathcal{H}$;

(2) A representation $\Pi$ of $\widetilde{\mathcal{L}}_+^\uparrow$ on $\mathcal{V}$ and a unitary representation U of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$;

(3) A $\widetilde{\mathcal{P}}_+^\uparrow$-invariant dense linear subspace $\mathcal{D} \subset \mathcal{H}$;

(4) A $\widetilde{\mathcal{P}}_+^\uparrow$-equivariant linear map $\underset{\sim}{\varphi} : \mathcal{S}(\underset{\sim}{R}^4; \mathcal{V}) \longrightarrow \underline{\text{End}}\, \mathcal{D}$;

(5) A $\widetilde{\mathcal{P}}_+^\uparrow$-invariant unit vector $\Omega_0 \in \mathcal{D}$.

Remark: The action of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\underline{\text{End}}\,\mathcal{D}$ is by conjugation, so the equivariance requirement on $\underset{\sim}{\varphi}$ is that $\forall f \in \mathcal{S}(\underset{\sim}{R}^4; \mathcal{V})$,

$$U(\widetilde{\Lambda},a)\underset{\sim}{\varphi}(f)U(\widetilde{\Lambda},a)^{-1} = \underset{\sim}{\varphi}(f_{\widetilde{\Lambda},a}),$$

where

$$f_{\widetilde{\Lambda},a}(x) = \check{\Pi}(\widetilde{\Lambda})f(\widetilde{\Lambda}^{-1}(x-a))$$

and

$$\check{\Pi}(\widetilde{\Lambda}) = \Pi(\widetilde{\Lambda}^{-1})^T.$$

[Note: The rationale for the introduction of the contragredient is this. Let $d = \underline{\dim}\,\mathcal{V}$ -- then $f \longleftrightarrow \{f_1, \ldots, f_d\}$ and $\underset{\sim}{\varphi} \longleftrightarrow \{\underset{\sim}{\varphi}_1, \ldots, \underset{\sim}{\varphi}_d\}$, thus

$$\begin{aligned}\underset{\sim}{\varphi}(f) &= \sum_j \underset{\sim}{\varphi}_j(f_j) \\ &= \sum_j \int_{\underset{\sim}{R}^4} f_j(x)\,\underset{\sim}{\varphi}_j(x)\,dx.\end{aligned}$$

Therefore

$$U(\tilde{\Lambda},a)\,\underset{\sim}{\varphi}(f)\,U(\tilde{\Lambda},a)^{-1} = \underset{\sim}{\varphi}(f_{\tilde{\Lambda},a})$$

$\Longrightarrow$

$$\sum_i \int_{\underset{\sim}{R}^4} f_i(x)\,U(\tilde{\Lambda},a)\,\underset{\sim}{\varphi}_i(x)\,U(\tilde{\Lambda},a)^{-1}dx$$

$$= \sum_j \int_{\underset{\sim}{R}^4} (f_{\tilde{\Lambda},a})_j(x)\,\underset{\sim}{\varphi}_j(x)\,dx$$

$$= \sum_j \int_{\underset{\sim}{R}^4} \left(\sum_i \check{\pi}(\tilde{\Lambda})_{ji}\; f_i(\tilde{\Lambda}^{-1}(x-a))\right)\underset{\sim}{\varphi}_j(x)\,dx$$

$$= \sum_i \int_{\underset{\sim}{R}^4} f_i(x)\left(\sum_j \check{\pi}(\tilde{\Lambda})_{ji}\; \underset{\sim}{\varphi}_j(\tilde{\Lambda}x+a)\right)dx$$

$\Longrightarrow$

$$U(\tilde{\Lambda},a)\,\underset{\sim}{\varphi}_i(x)\,U(\tilde{\Lambda},a)^{-1}$$

$$= \sum_j \pi(\tilde{\Lambda}^{-1})_{ij}\; \underset{\sim}{\varphi}_j(\tilde{\Lambda}x+a),$$

which is the traditional transformation law for $\underset{\sim}{\varphi}(x) \longleftrightarrow \{\underset{\sim}{\varphi}_1(x), \ldots, \underset{\sim}{\varphi}_d(x)\}$.]

Observation:

$$\underset{\sim}{\varphi}(f)^* = \left(\int_{\underset{\sim}{R}^4} f(x)\,\underset{\sim}{\varphi}(x)\,dx\right)^*$$

$$= \int_{\underset{\sim}{R}^4} \overline{f(x)}\;\underset{\sim}{\varphi}^*(x)\,dx$$

$\Longrightarrow$

$$\underset{\sim}{\varphi}(\overline{f})^* = \int_{\underset{\sim}{R}^4} f(x)\,\underset{\sim}{\varphi}^*(x)\,dx.$$

This data is said to constitute a <u>vector quantum field theory (QFT) of type $\pi$</u> if the following assumptions are satisfied.

<u>W1</u>: Same as in the neutral scalar case.

<u>W2</u>: Same as in the neutral scalar case.

<u>W3</u>: Same as in the neutral scalar case except that now no a priori connection between $\underset{\sim}{\varphi}(f)$ and its adjoint $\underset{\sim}{\varphi}(f)^*$ is assumed.

<u>W4</u>: Same as in the neutral scalar case.

<u>W5</u>: The set of all finite linear combinations $A_1 \cdots A_n \Omega_0$, where $A_i = \underset{\sim}{\varphi}(f_i)$ or $\underset{\sim}{\varphi}(g_i)^*$, is dense in $\mathcal{H}$.

The statement of the final assumption, viz. that there is a <u>normal connection between spin and statistics</u>, is a little involved.

Definition: Let D be a finite multiple of some finite dimensional irreducible representation of $\tilde{\mathcal{L}}_+^{\uparrow}$ -- then there are two possibilities:

$$\begin{cases} D(-I) = I & \text{(call D \underline{integral})} \\ D(-I) = -I & \text{(call D \underline{half-integral}).} \end{cases}$$

<u>W6</u>: Decompose $\pi$ as a direct sum $\bigoplus_k D_k$, where $D_k$ is a finite multiple of some finite dimensional irreducible representation of $\tilde{\mathcal{L}}_+^{\uparrow}$, so $\mathcal{V} = \bigoplus_k \mathcal{V}_k$. Given

$$\begin{cases} f = \{f_k\} & (f_k \in \mathcal{S}(\underset{\sim}{R}^4;\ \mathcal{V}_k)) \\ g = \{g_k\} & (g_k \in \mathcal{S}(\underset{\sim}{R}^4;\ \mathcal{V}_k)) \end{cases}$$

whose supports are spacelike separated,

$$\begin{cases} \underset{\sim}{\varphi}(f_k)\,\underset{\sim}{\varphi}(g_\ell) - \underset{\sim}{\varphi}(g_\ell)\,\underset{\sim}{\varphi}(f_k) = 0 \\ \underset{\sim}{\varphi}(f_k)^*\,\underset{\sim}{\varphi}(g_\ell) - \underset{\sim}{\varphi}(g_\ell)\,\underset{\sim}{\varphi}(f_k)^* = 0 \end{cases}$$

if either $D_k$ or $D_\ell$ (or both) are integral but

$$\begin{cases} \underset{\sim}{\varphi}(f_k)\,\underset{\sim}{\varphi}(g_\ell) + \underset{\sim}{\varphi}(g_\ell)\,\underset{\sim}{\varphi}(f_k) = 0 \\ \underset{\sim}{\varphi}(f_k)^*\,\underset{\sim}{\varphi}(g_\ell) + \underset{\sim}{\varphi}(g_\ell)\,\underset{\sim}{\varphi}(f_k)^* = 0 \end{cases}$$

if $D_k$ and $D_\ell$ are half-integral.

Remark: The finite dimensional irreducible representations of $\tilde{\mathcal{L}}_+^\uparrow$ are parameterized by pairs $(u,v)$, where $u,v \in \{0,1/2,1,\dots\}$ : $D^{(u,v)}$ $(D^{(u)}=D^{(u,0)}$, $\bar{D}^{(v)}=D^{(0,v)}$, $D^{(u,v)}=D^{(u)}\otimes \bar{D}^{(v)})$. One has $D^{(u,v)}(-I) = (-1)^{2(u+v)}I$, so $D^{(u,v)}$ is integral iff $u+v \in \underset{\sim}{Z}$. Moreover $D^{(u,v)}$ is real, i.e., equivalent to its complex conjugate, iff $u=v$ (for $\bar{D}^{(u,v)}=D^{(v,u)}$). Finally, $\underline{\dim}\ D^{(u,v)}=(2u+1)(2v+1)$.

Example: Let $\pi$ be the one dimensional identity representation of $\tilde{\mathcal{L}}_+^\uparrow$ (in which case the term "vector" is replaced by "scalar") -- then a neutral scalar QFT is a scalar QFT of type $\pi$ with the additional property that the field map sends the real valued elements of $\mathcal{S}(\underset{\sim}{R}^4)$ to symmetric operators. Consider now two free QFTs of mass $m>0$. Call the field maps $\underset{\sim}{\varphi}_m^\pm$ and realize the data on $\mathcal{H}=\mathcal{H}^+\otimes\mathcal{H}^-$ $(\mathcal{H}^\pm = \mathcal{F}_s(L^2(X_m,\mu_m)^\pm)$. Taking $\mathcal{D}=\mathcal{D}^+\otimes\mathcal{D}^-$, $\Omega_0=\Omega_0^+\otimes\Omega_0^-$, $U = U^+\otimes U^-$, and

$$\underset{\sim}{\varphi}_m(x) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int_{X_m}[e^{-\sqrt{-1}\langle p,x\rangle}\underset{\sim}{a}^+(p)\otimes I^- + e^{\sqrt{-1}\langle p,x\rangle}I^+\otimes\underset{\sim}{c}^-(p)]\,d\mu_m(p)$$

so

$$\underset{\sim}{\varphi}_m(f) = \frac{1}{\sqrt{2}} \left(\underset{\sim}{a}^+(E_m^+ f) \otimes I^- + I^+ \otimes \underset{\sim}{c}^-(E_m^- f)\right)$$

if f is real, one can check that all the requirements for a scalar QFT of type $\Pi$ are satisfied (generally referred to as a charged scalar QFT). Here

$$\left\{\begin{array}{l} [\underset{\sim}{\varphi}_m(f), \underset{\sim}{\varphi}_m(g)] \\ \\ [\underset{\sim}{\varphi}_m(f)^*, \underset{\sim}{\varphi}_m(g)^*] \end{array}\right. = 0$$

and for f,g real,

$$[\underset{\sim}{\varphi}_m(f), \underset{\sim}{\varphi}_m(g)^*]$$

$$= \left( \int_{\underset{\sim}{R}^4} \int_{\underset{\sim}{R}^4} \frac{1}{\sqrt{-1}} \Delta_m(x-y) f(x) g(y) dxdy \right) (I^+ \otimes I^-).$$

[Note: Writing

$$\left\{\begin{array}{l} \mathcal{H}^+ = \bigoplus\limits_{n=0}^{\infty} \mathcal{H}^+_{n,s} \\ \\ \mathcal{H}^- = \bigoplus\limits_{m=0}^{\infty} \mathcal{H}^-_{m,s}, \end{array}\right.$$

we have (in obvious notation)

$$\mathcal{H} = \bigoplus_{n,m=0}^{\infty} \mathcal{H}^{(n,m)} .$$

On $\mathcal{H}^{(n,m)}$, the number operator N is given by

$$N\psi = (n+m)\psi$$

and the charge operator Q is given by

$$Q\psi = (n-m)\psi .$$

Both are selfadjoint and have a purely discrete spectrum. In particular: There is a decomposition

$$\mathcal{H} = \bigoplus_{-\infty}^{\infty} \mathcal{H}_q,$$

where $\mathcal{H}_q$ is the q-charge sector, i.e., the eigenspace of Q corresponding to the eigenvalue q. Physically, $\mathcal{H}^+$ represents the space of particles and $\mathcal{H}^-$ represents the space of antiparticles (of a given type). E.g.: The electrically charged $\pi$ mesons $\pi^+$ and $\pi^-$ or the electrically neutral $K^0$ meson and its electrically neutral antiparticle $K^0$, which carry opposite hypercharge: Y=1 for the $K^0$ and Y=-1 for the $\bar{K}^0$. ]

Remark: The symmetric Fock space associated with

$$L^2(X_m,\mu_m)^+ \oplus L^2(X_m,\mu_m)^-$$

is isomorphic to $\mathcal{H}$ and

$$\underset{\sim}{\varphi}_m^+ \otimes I^- + I^+ \otimes \underset{\sim}{\varphi}_m^- = \underset{\sim}{\varphi}_m + \underset{\sim}{\varphi}_m^* .$$

This setting also carries with it a Borchers algebra. Thus let D be a finite multiple of some finite dimensional irreducible representation of $\tilde{\mathcal{L}}_+^\uparrow$ and put d=dim D. Set

$$\mathcal{S}_D = \underset{\sim}{C} \oplus \mathcal{S}(\underset{\sim}{R}^4;\underset{\sim}{C}^d) \oplus \mathcal{S}(\underset{\sim}{R}^8;\underset{\sim}{C}^{d^2}) \oplus \cdots$$

and equip $\mathcal{S}_D$ with the direct sum topology per the injections $\mathcal{S}(\underset{\sim}{R}^{4n};\underset{\sim}{C}^{d^n}) \rightarrow \mathcal{S}_D$ -- then $\mathcal{S}_D$ becomes a separable LCTVS.

(1) $\mathcal{S}_D$ admits a continuous involution *.

[Let $f=\{f_n\} \in \mathcal{S}_D$ and put

$$f^* = \{f_n^*\},$$

where

$$f_n^*(x_1,\ldots,x_n) = \overleftarrow{r}\ \overline{f_n(x_n,\ldots,x_1)}.$$

Here $\overleftarrow{r}$ acts on $f_n = (f_{i_1,\ldots,i_n} : (i_1,\ldots,i_n) \in \{1,\ldots,d\}^n)$ by

reversing the order of the indexes.

(2) $\mathcal{S}_D$ admits a continuous multiplication $\times$.

[Let $f = \{f_n\}$ & $g = \{g_n\} \in \mathcal{S}_D$ and define $f \times g$ by

$$(f \times g)_n = \sum_{k=0}^{n} f_k \otimes g_{n-k}.]$$

Suppose given a vector QFT of type D -- then the correlation functions $\mathcal{W}_n$ of the theory determine a state $\mathcal{W}$ on $\mathcal{S}_D$ and what has been said earlier in the scalar case extends with but minor changes to the vector case.

Higher Spin The free relativistic particle of spin $s > 0$ and mass $m > 0$ carries the structure of a vector QFT of type $D^{(s)} = D^{(s,0)}$. To see this, one first has to specify the data, i.e., items (1)-(5).

Notation: As usual,

$$\begin{cases} \tilde{\mathcal{L}}_+^\uparrow \rightarrow \mathcal{L}_+^\uparrow & (\tilde{\Lambda} \rightarrow \Lambda) \\ \tilde{\mathcal{P}}_+^\uparrow \rightarrow \mathcal{P}_+^\uparrow & ((\tilde{\Lambda}, a) \rightarrow (\Lambda, a)). \end{cases}$$

(1) Assign to each $p \in X_m$ the boost $\Lambda_p \in \mathcal{L}_+^\uparrow : \Lambda_p(m,0,0,0) = p$. Note that $\tilde{\Lambda}_p \rightarrow \Lambda_p$, where

$$\tilde{\Lambda}_p = \frac{1}{\sqrt{2m(p_0+m)}} \begin{pmatrix} p_0+m+p_3, & p_1 - \sqrt{-1}\, p_2 \\ p_1 + \sqrt{-1}\, p_2, & p_0+m-p_3 \end{pmatrix}$$

is positive.

(2) Assign to each $p \in X_m$ the Wigner rotation $W(\tilde{\Lambda}, p) \in \underset{\sim}{SU}(2)$:

$$W(\tilde{\Lambda}, p) = (\tilde{\Lambda}_p)^{-1} \tilde{\Lambda} \tilde{\Lambda}_{\tilde{\Lambda}^{-1} p}.$$

Thus

$$W(\tilde{\Lambda}, \tilde{\Lambda} p) = (\tilde{\Lambda}_{\tilde{\Lambda} p})^{-1} \tilde{\Lambda} \tilde{\Lambda}_p$$

$$\Rightarrow$$

$$W(\tilde{\Lambda}, \tilde{\Lambda} p)^{-1} = (\tilde{\Lambda}_p)^{-1} \tilde{\Lambda}^{-1} \tilde{\Lambda}_{\tilde{\Lambda} p}.$$

---

LEMMA $\forall\, p \in X_m$,

$$\tilde{\Lambda}_p \tilde{\Lambda}_q = I,$$

where $q=(p_0,-p)$.

[Employing the usual notation, let

$$\underset{\sim}{p} = p_0 I + \underset{\sim}{p}\cdot\underset{\sim}{\sigma} .$$

Then

$$\begin{cases} \widetilde{\Lambda}_p = \dfrac{m+\underset{\sim}{p}}{\sqrt{2m(p_0+m)}} \\ \widetilde{\Lambda}_p^2 = \dfrac{1}{m}\,\underset{\sim}{p}. \end{cases}$$

On the other hand,

$$\frac{\underset{\sim}{q}}{m} = \left(\frac{\underset{\sim}{p}}{m}\right)^{-1} .$$

Therefore

$$\widetilde{\Lambda}_p^2\,\widetilde{\Lambda}_q^2 = I.$$

This shows in particular that $\widetilde{\Lambda}_p^2$ and $\widetilde{\Lambda}_q^2$ commute, thus the same is true of their square roots, i.e., $\widetilde{\Lambda}_p$ and $\widetilde{\Lambda}_q$, which in turn implies that the product $\widetilde{\Lambda}_p\widetilde{\Lambda}_q$ is positive. Since

$$\widetilde{\Lambda}_p^2\,\widetilde{\Lambda}_q^2 = (\widetilde{\Lambda}_p\widetilde{\Lambda}_q)^2 ,$$

the assertion follows by taking the square root of I.]

[Note: The disbeliever can multiply out the matrices.]

---

Put

$$L^2(X_m,\mu_m;s) = L^2(X_m,\mu_m;\underset{\sim}{C}^{2s+1}).$$

So, the elements $f\in L^2(X_m,\mu_m;s)$ are strings

$$\left\{ f(p,\sigma) :\ \sigma = s, s-1, \ldots, -s+1, -s \right\}$$

with

$$\int_{X_m} |f(p,\sigma)|^2 \, d\mu_m(p) < +\infty \quad .$$

This said, $\exists$ an irreducible unitary representation $U^{(m,s)}$ of $\tilde{\mathcal{P}}_+^\uparrow$ on $L^2(X_m,\mu_m;s)$, viz.

$$\begin{aligned}(U^{(m,s)}(\tilde{\Lambda},a)f)(p,\sigma) \\ = e^{\sqrt{-1}\langle a,p\rangle} \sum_{\tau=-s}^{s} D^{(s)}_{\sigma\tau}(W(\tilde{\Lambda},p))f(\tilde{\Lambda}^{-1}p,\tau).\end{aligned}$$

Turning now to quantum field theory, one has to start by specifying the Hilbert space, which we shall do by taking for $\mathcal{H}$ the symmetric Fock space over $L^2(X_m,\mu_m;s)$ if s is integral or the antisymmetric Fock space over $L^2(X_m,\mu_m;s)$ if s is half-integral. Modulo the definition of the field map, the other ingredients are obvious. As for the field map, what follows are the preliminaries that lead to its definition. I shall concentrate on the symmetric case, the antisymmetric case being analogous.

An element $\psi \in \mathcal{H}$ is a string $\psi = \{\psi_0, \psi_1, \ldots\}$, where

$$\psi_n = \psi_n(p_1,\sigma_1;\cdots;p_n,\sigma_n)$$

is symmetric w.r.t. permutations of pairs:

$$(p_i,\sigma_i) \longleftrightarrow (p_j,\sigma_j).$$

Given $(p,\sigma)$, define operators $\begin{cases} \underset{\sim}{a}(p,\sigma) \\ \underset{\sim}{c}(p,\sigma) \end{cases}$ by

$$\begin{cases} (\underset{\sim}{a}(p,\sigma)\Psi)_n(p_1,\sigma_1;\cdots;p_n,\sigma_n) \\ \quad = \sqrt{n+1}\,\Psi_{n+1}(p,\sigma;p_1,\sigma_1;\cdots;p_n,\sigma_n) \\ (\underset{\sim}{c}(p,\sigma)\Psi)_n(p_1,\sigma_1;\cdots;p_n,\sigma_n) \\ \quad = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\delta(p-p_j)\,\delta_{\sigma\sigma_j}\,\psi_{n-1}(p_1,\sigma_1;\cdots;\widehat{p_j,\sigma_j};\cdots;p_n,\sigma_n). \end{cases}$$

Properties:

(1) $\underset{\sim}{a}(p,\sigma)^* = \underset{\sim}{c}(p,\sigma)$ & $\underset{\sim}{c}(p,\sigma)^* = \underset{\sim}{a}(p,\sigma)$;

(2) $[\underset{\sim}{a}(p,\sigma),\underset{\sim}{a}(q,\tau)] = 0$ & $[\underset{\sim}{c}(p,\sigma),\underset{\sim}{c}(q,\tau)] = 0$;

(3) $[\underset{\sim}{a}(p,\sigma),\underset{\sim}{c}(q,\tau)] = \delta(q-p)\,\delta_{\tau\sigma}\cdot I$.

[Note: Let us check the third property. Thus

$$(\underset{\sim}{a}(p,\sigma)\underset{\sim}{c}(q,\tau)\Psi)_n(p_1,\sigma_1;\cdots;p_n,\sigma_n)$$

$$= \sqrt{n+1}\,(\underset{\sim}{c}(q,\tau)\Psi)_{n+1}(p,\sigma;p_1,\sigma_1;\cdots;p_n,\sigma_n)$$

$$= \sqrt{n+1}\Bigg(\frac{1}{\sqrt{n+1}}\,\delta(q-p)\,\delta_{\tau\sigma}\,\psi_n(p_1,\sigma_1;\cdots;p_n,\sigma_n)$$

$$+ \frac{1}{\sqrt{n+1}}\sum_{j=1}^{n}\delta(q-p_j)\,\delta_{\tau\sigma_j}\,\psi_n(p,\sigma;\cdots;\widehat{p_j,\sigma_j};\cdots;p_n,\sigma_n)\Bigg).$$

On the other hand,

$$(\underset{\sim}{c}(q,\tau)\underset{\sim}{a}(p,\sigma)\Psi)_n(p_1,\sigma_1;\cdots;p_n,\sigma_n)$$

$$= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\delta(q-p_j)\,\delta_{\tau\sigma_j}(\underset{\sim}{a}(p,\sigma)\Psi)_{n-1}(p_1,\sigma_1;\cdots;\widehat{p_j,\sigma_j};\cdots;p_n,\sigma_n)$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(q-p_j)\, \delta_{\tau\sigma_j} \left( \sqrt{n}\, \psi_n(p,\sigma;\cdots;\widehat{p_j,\sigma_j};\cdots;p_n,\sigma_n) \right).$$

Therefore

$$(\underset{\sim}{a}(p,\sigma)\underset{\sim}{c}(q,\tau)\psi - \underset{\sim}{c}(q,\tau)\underset{\sim}{a}(p,\sigma)\,\psi)_n\,(p_1,\sigma_1;\cdots;p_n,\sigma_n)$$

$$= \delta(q-p)\;\delta_{\tau\sigma}\;\psi_n(p_1,\sigma_1;\cdots;p_n,\sigma_n),$$

from which the claim.]

Given f, define operators $\left\{ \begin{array}{l} \underset{\sim}{a}(f) \\ \underset{\sim}{c}(f) \end{array} \right.$ by

$$\left\{ \begin{array}{l} \underset{\sim}{a}(f) = \displaystyle\sum_{\sigma=-s}^{s} \int_{X_m} \underset{\sim}{a}(p,\sigma)\overline{f(p,\sigma)}\,d\mu_m(p) \\[2ex] \underset{\sim}{c}(f) = \displaystyle\sum_{\sigma=-s}^{s} \int_{X_m} \underset{\sim}{c}(p,\sigma) f(p,\sigma)\,d\mu_m(p). \end{array} \right.$$

Remark: These definitions are formally consistent with the usual agreements. Thus

$$(\underset{\sim}{a}(f)\,\psi)_n\;(p_1,\sigma_1;\cdots;p_n,\sigma_n)$$

$$= \sqrt{n+1} \sum_{\sigma} \int_{X_m} \psi_{n+1}(p,\sigma;p_1,\sigma_1;\cdots;p_n,\sigma_n)\overline{f(p,\sigma)}\,d\mu_m(p)$$

and

$$(\underset{\sim}{c}(f)\,\psi)_n\;(p_1,\sigma_1;\cdots;p_n,\sigma_n)$$

$$= \frac{1}{\sqrt{n}} \sum_{\sigma} \sum_{j=1}^{n} \left[ \int_{X_m} \delta(p-p_j) f(p,\sigma)\;d\mu_m(p) \right] \delta_{\sigma\sigma_j}$$

$$\cdot\; \psi_{n-1}\;(p_1,\sigma_1;\cdots;\widehat{p_j,\sigma_j};\cdots;p_n,\sigma_n)$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{\sigma} f(p_j, \sigma)\ \delta_{\sigma\sigma_j}\ \psi_{n-1}(p_1, \sigma_1; \cdots; \widehat{p_j, \sigma_j}; \cdots; p_n, \sigma_n)$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(p_j, \sigma_j)\ \psi_{n-1}(p_1, \sigma_1; \cdots; \widehat{p_j, \sigma_j}; \cdots; p_n, \sigma_n).$$

Note too that

$$[\underset{\sim}{a}(f), \underset{\sim}{c}(g)]$$

$$= [\sum_{\sigma} \int_{X_m} \underset{\sim}{a}(p,\sigma)\overline{f(p,\sigma)}\,d\mu_m(p), \sum_{\tau} \int_{X_m} \underset{\sim}{c}(q,\tau)g(q,\tau)\,d\mu_m(q)]$$

$$= \sum_{\sigma,\tau} \int_{X_m} \int_{X_m} [\underset{\sim}{a}(p,\sigma), \underset{\sim}{c}(q,\tau)]\overline{f(p,\sigma)}g(q,\tau)\,d\mu_m(p)\,d\mu_m(q)$$

$$= \sum_{\sigma,\tau} \int_{X_m} \int_{X_m} \delta(q-p)\ \delta_{\tau\sigma}\ \overline{f(p,\sigma)}g(q,\tau)\,d\mu_m(p)\,d\mu_m(q)$$

$$= \sum_{\sigma,\tau} \int_{X_m} \overline{f(p,\sigma)}g(p,\tau)\ \delta_{\sigma\tau}\ d\mu_m(p)$$

$$= \sum_{\sigma} \int_{X_m} \overline{f(p,\sigma)}g(p,\sigma)\,d\mu_m(p)$$

$$= \langle f,g \rangle \cdot I.$$

On general grounds, for any unitary operator $U$ on $L^2(X_m, \mu_m; s)$,

$$\begin{cases} (\Pi U)\underset{\sim}{a}(f)(\Pi U)^{-1} = \underset{\sim}{a}(Uf) \\ (\Pi U)\underset{\sim}{c}(f)(\Pi U)^{-1} = \underset{\sim}{c}(Uf). \end{cases}$$

[Note: In the sequel, we shall omit the cap pi from the notation.]

Specialize to

$$U = U^{(m,s)}(\tilde{\Lambda}, 0).$$

Then

$$\underset{\sim}{c}(U^{(m,s)}(\tilde{\Lambda},0)f)$$

$$= \sum_{\sigma} \int_{X_m} \underset{\sim}{c}(p,\sigma)(U^{(m,s)}(\tilde{\Lambda},0)f)(p,\sigma)\,d\mu_m(p)$$

$$= \sum_{\sigma} \int_{X_m} \underset{\sim}{c}(p,\sigma) \sum_{\tau} D^{(s)}_{\sigma\tau}(W(\tilde{\Lambda},p))f(\tilde{\Lambda}^{-1}p,\tau)\,d\mu_m(p)$$

$$= \sum_{\sigma} \int_{X_m} \underset{\sim}{c}(\tilde{\Lambda}p,\sigma) \sum_{\tau} D^{(s)}_{\sigma\tau}(W(\tilde{\Lambda},\tilde{\Lambda}p))f(p,\tau)\,d\mu_m(p)$$

$$= \sum_{\tau} \int_{X_m} \underset{\sim}{c}(\tilde{\Lambda}p,\tau) \sum_{\sigma} D^{(s)}_{\tau\sigma}(W(\tilde{\Lambda},\tilde{\Lambda}p))f(p,\sigma)\,d\mu_m(p)$$

$$= \sum_{\sigma} \int_{X_m} \left( \sum_{\tau} \underset{\sim}{c}(\tilde{\Lambda}p,\tau)D^{(s)}_{\tau\sigma}(W(\tilde{\Lambda},\tilde{\Lambda}p))\right)f(p,\sigma)\,d\mu_m(p).$$

But

$$U^{(m,s)}(\tilde{\Lambda},0)\underset{\sim}{c}(f)U^{(m,s)}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\sigma} \int_{X_m} U^{(m,s)}(\tilde{\Lambda},0)\underset{\sim}{c}(p,\sigma)U^{(m,s)}(\tilde{\Lambda},0)^{-1}f(p,\sigma)\,d\mu_m(p),$$

so by comparison, it follows that

$$U^{(m,s)}(\tilde{\Lambda},0)\underset{\sim}{c}(p,\sigma)U^{(m,s)}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} \underset{\sim}{c}(\tilde{\Lambda}p,\tau)D^{(s)}_{\tau\sigma}(W(\tilde{\Lambda},\tilde{\Lambda}p))$$

or, as is preferable,

$$\sum_{\tau} D^{(s)}_{\tau\sigma}(W(\tilde{\Lambda},\tilde{\Lambda}p))\underset{\sim}{c}(\tilde{\Lambda}p,\tau).$$

Analogously,

$$U^{(m,s)}(\tilde{\Lambda},0)\underset{\sim}{a}(p,\sigma)U^{(m,s)}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} \overline{D^{(s)}_{\tau\sigma}(W(\tilde{\Lambda},\tilde{\Lambda}p))}\, \underset{\sim}{a}(\tilde{\Lambda}p,\tau).$$

These transformation rules can be put into a more convenient form. Thus, since

$$D^{(s)}(W(\tilde{\Lambda},\tilde{\Lambda}p))$$

is unitary, we have

$$\overline{D^{(s)}_{\tau\sigma}(W(\tilde{\Lambda},\tilde{\Lambda}p))} = D^{(s)}_{\sigma\tau}(W(\tilde{\Lambda},\tilde{\Lambda}p)^{-1})$$

$$= D^{(s)}_{\sigma\tau}((\tilde{\Lambda}_p)^{-1}\tilde{\Lambda}^{-1}\tilde{\Lambda}_{\tilde{\Lambda}p}).$$

Therefore

$$U^{(m,s)}(\tilde{\Lambda},0)\underset{\sim}{a}(p,\sigma)U^{(m,s)}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} D^{(s)}_{\sigma\tau}((\tilde{\Lambda}_p)^{-1}\tilde{\Lambda}^{-1}\tilde{\Lambda}_{\tilde{\Lambda}p})\underset{\sim}{a}(\tilde{\Lambda}p,\tau).$$

On general grounds, $\exists$ a unitary 2s+1 by 2s+1 matrix C such that $\forall\ R \in \underset{\sim}{SU}(2)$,

$$\overline{D^{(s)}(R)} = CD^{(s)}(R)C^{-1}.$$

Indeed,

$$C_{\sigma\tau} = (-1)^{s+\sigma}\,\delta_{\tau(-\sigma)}.$$

[Note: Since the entries of C are real, $C^{\top} = C^{-1}$.]

Consequently,

$$\overline{D^{(s)}(R^{-1})} = CD^{(s)}(R^{-1})C^{-1}$$

$$\Longrightarrow$$

$$D^{(s)}(R)^{\top} = CD^{(s)}(R^{-1})C^{-1}.$$

Taking these facts into account then allows us to write

$$U^{(m,s)}(\tilde{\Lambda},0)\,\underset{\sim}{c}(p,\sigma)\,U^{(m,s)}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} \left\{ C D^{(s)}((\tilde{\Lambda}_p)^{-1}\tilde{\Lambda}^{-1}\tilde{\Lambda}_{\tilde{\Lambda}p})C^{-1} \right\}_{\sigma\tau} \underset{\sim}{c}(\tilde{\Lambda}p,\tau),$$

which is structurally similar to its counterpart for $\underset{\sim}{a}(p,\sigma)$.

Bearing in mind that

$$D^{(s)}((\tilde{\Lambda}_p)^{-1}\tilde{\Lambda}^{-1}\tilde{\Lambda}_{\tilde{\Lambda}p})$$

$$= D^{(s)}((\tilde{\Lambda}_p)^{-1})D^{(s)}(\tilde{\Lambda}^{-1})D^{(s)}(\tilde{\Lambda}_{\tilde{\Lambda}p}),$$

introduce now

$$\begin{cases} \underset{\sim}{\alpha}(p,\sigma) = \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}_p)\underset{\sim}{a}(p,\tau) \\ \underset{\sim}{\gamma}(p,\sigma) = \sum_{\tau} \left\{ D^{(s)}(\tilde{\Lambda}_p)C^{-1} \right\}_{\sigma\tau} \underset{\sim}{c}(p,\tau). \end{cases}$$

[Note: It is a fact that $D^{(s)}(\tilde{\Lambda}_p)$ is selfadjoint, hence

$$\overline{D^{(s)}(\tilde{\Lambda}_p)} = D^{(s)}(\tilde{\Lambda}_p)^{T}.]$$

---

LEMMA We have

$$U^{(m,s)}(\tilde{\Lambda},0)\,\underset{\sim}{\alpha}(p,\sigma)\,U^{(m,s)}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\,\underset{\sim}{\alpha}(\tilde{\Lambda}p,\tau)$$

and

$$U^{(m,s)}(\tilde{\Lambda},0)\,\underset{\sim}{\gamma}(p,\sigma)\,U^{(m,s)}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\,\underset{\sim}{\gamma}(\tilde{\Lambda}p,\tau).$$

[To establish the first of these relations, consider

$$\sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}_p) U^{(m,s)}(\tilde{\Lambda},0) \underset{\sim}{a}(p,\tau) U^{(m,s)}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}_p)\left(\sum_{\rho} D^{(s)}_{\tau\rho}((\tilde{\Lambda}_p)^{-1}\tilde{\Lambda}^{-1}\tilde{\Lambda}_{\tilde{\Lambda}p}) \underset{\sim}{a}(\tilde{\Lambda}p,\rho)\right).$$

Next

$$D^{(s)}_{\tau\rho}((\tilde{\Lambda}_p)^{-1}\tilde{\Lambda}^{-1}\tilde{\Lambda}_{\tilde{\Lambda}p})$$

$$= \sum_{\tau'}\sum_{\rho'} D^{(s)}_{\tau\tau'}((\tilde{\Lambda}_p)^{-1}) D^{(s)}_{\tau'\rho'}(\tilde{\Lambda}^{-1}) D^{(s)}_{\rho'\rho}(\tilde{\Lambda}_{\tilde{\Lambda}p}).$$

But

$$\sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}_p) D^{(s)}_{\tau\tau'}((\tilde{\Lambda}_p)^{-1}) = \delta_{\sigma\tau'}.$$

We are therefore left with

$$\sum_{\tau'}\sum_{\rho'} \delta_{\sigma\tau'} D^{(s)}_{\tau'\rho'}(\tilde{\Lambda}^{-1}) \sum_{\rho} D^{(s)}_{\rho'\rho}(\tilde{\Lambda}_{\tilde{\Lambda}p}) \underset{\sim}{a}(\tilde{\Lambda}p,\rho)$$

or still,

$$\sum_{\rho'} D^{(s)}_{\sigma\rho'}(\tilde{\Lambda}^{-1}) \underset{\sim}{\alpha}(\tilde{\Lambda}p,\rho')$$

$$= \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1}) \underset{\sim}{\alpha}(\tilde{\Lambda}p,\tau),$$

as desired.]

To specify the field map $\underset{\sim}{\varphi}_{(m,s)}$, it suffices to specify its components $\underset{\sim}{\varphi}_{\sigma}$ ($\sigma$=s,s-1,...,-s+1,-s) and verify that they possess the transformation property per $U^{(m,s)}(\tilde{\Lambda},a)$.

Definition: Let

$$\underset{\sim}{\varphi}_{\sigma}(x) = \frac{1}{\sqrt{2}}\,\frac{1}{(2\pi)^{3/2}}\int_{X_m}[e^{-\sqrt{-1}\langle p,x\rangle}\underset{\sim}{\alpha}(p,\sigma) + e^{\sqrt{-1}\langle p,x\rangle}\underset{\sim}{\gamma}(p,\sigma)]d\mu_m(p).$$

---

LEMMA We have

$$U^{(m,s)}(\tilde{\Lambda},0)\,\underset{\sim}{\varphi}_{\sigma}(x)U^{(m,s)}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\,\underset{\sim}{\varphi}_{\tau}(\tilde{\Lambda}x).$$

[This lemma is a trivial consequence of the preceding lemma.]

---

It remains to incorporate the translations. To this end, note that

$$\begin{cases} U^{(m,s)}(I,a)\underset{\sim}{a}(p,\sigma)U^{(m,s)}(I,a)^{-1} = e^{-\sqrt{-1}\langle a,p\rangle}\underset{\sim}{a}(p,\sigma) \\ U^{(m,s)}(I,a)\underset{\sim}{c}(p,\sigma)U^{(m,s)}(I,a)^{-1} = e^{\sqrt{-1}\langle a,p\rangle}\underset{\sim}{c}(p,\sigma) \end{cases}$$

$\Longrightarrow$

$$\begin{cases} U^{(m,s)}(I,a)\underset{\sim}{\alpha}(p,\sigma)U^{(m,s)}(I,a)^{-1} = e^{-\sqrt{-1}\langle a,p\rangle}\underset{\sim}{\alpha}(p,\sigma) \\ U^{(m,s)}(I,a)\underset{\sim}{\gamma}(p,\sigma)U^{(m,s)}(I,a)^{-1} = e^{\sqrt{-1}\langle a,p\rangle}\underset{\sim}{\gamma}(p,\sigma). \end{cases}$$

---

LEMMA We have

$$U^{(m,s)}(I,a)\,\underset{\sim}{\varphi}_{\sigma}(x)U^{(m,s)}(I,a)^{-1} = \underset{\sim}{\varphi}_{\sigma}(x+a).$$

[In fact,

$$\int_{X_m}[e^{-\sqrt{-1}\langle p,x\rangle}U^{(m,s)}(I,a)\underset{\sim}{\alpha}(p,\sigma)U^{(m,s)}(I,a)^{-1}$$

$$+ e^{\sqrt{-1}\langle p,x\rangle} U^{(m,s)}(I,a)\underset{\sim}{\gamma}(p,\sigma)U^{(m,s)}(I,a)^{-1}]d\mu_m(p)$$

$$= \int_{X_m} [e^{-\sqrt{-1}\langle p,x+a\rangle}\underset{\sim}{\alpha}(p,\sigma)$$

$$+ e^{\sqrt{-1}\langle p,x+a\rangle}\underset{\sim}{\gamma}(p,\sigma)]d\mu_m(p),$$

which is tantamount to the assertion.]

---

Since

$$U^{(m,s)}(\tilde{\Lambda},a) = U^{(m,s)}(I,a)U^{(m,s)}(\tilde{\Lambda},0),$$

it follows that

$$U^{(m,s)}(\tilde{\Lambda},a)\underset{\sim}{\varphi}_\sigma(x)U^{(m,s)}(\tilde{\Lambda},a)^{-1}$$

$$= U^{(m,s)}(I,a)\left(\sum_\tau D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\underset{\sim}{\varphi}_\tau(\tilde{\Lambda}x)\right)U^{(m,s)}(I,a)^{-1}$$

$$= \sum_\tau D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})U^{(m,s)}(I,a)\underset{\sim}{\varphi}_\tau(\tilde{\Lambda}x)U^{(m,s)}(I,a)^{-1}$$

$$= \sum_\tau D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\underset{\sim}{\varphi}_\tau(\tilde{\Lambda}x+a),$$

thereby exhibiting the transformation property per $U^{(m,s)}(\tilde{\Lambda},a)$.

Remark: The field components obey the Klein-Gordon equation

$$(\Box^2 + m^2)\underset{\sim}{\varphi}_\sigma = 0$$

but that's the extent of it.

[Note: As Weinberg puts it "any field equation except the Klein-Gordon equation is nothing but a confession that the field contains superfluous components."]

Let $f \longleftrightarrow \{f_\sigma\}$ be an element of $\mathcal{S}(\underset{\sim}{R}^4; \underset{\sim}{C}^{2s+1})$ and assume that the $f_\sigma$ are real. Put

$$\begin{cases} A_f(p,\sigma) = \sqrt{2\pi} \sum_\tau \overline{D^{(s)}_{\tau\sigma}(\widetilde{\Lambda}_p)} \hat{f}_\tau(p) \\ C_f(p,\sigma) = \sqrt{2\pi} \sum_\tau \{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\tau\sigma} \hat{f}_\tau(p). \end{cases}$$

Definition: Let

$$\underset{\sim}{\varphi}_{(m,s)}(f) = \frac{1}{\sqrt{2}}(\underset{\sim}{a}(A_f) + \underset{\sim}{c}(C_f)).$$

---

LEMMA We have

$$\underset{\sim}{\varphi}_{(m,s)}(f) = \sum_\sigma \underset{\sim}{\varphi}_\sigma(f_\sigma),$$

where

$$\underset{\sim}{\varphi}_\sigma(f_\sigma) = \int_{\underset{\sim}{R}^4} f_\sigma(x)\,\underset{\sim}{\varphi}_\sigma(x)\,dx.$$

[The integral

$$\int_{\underset{\sim}{R}^4} f_\sigma(x)\,\underset{\sim}{\varphi}_\sigma(x)\,dx$$

equals

$$\frac{\sqrt{2\pi}}{\sqrt{2}}\Big(\int_{X_m} \underset{\sim}{\alpha}(p,\sigma)\frac{1}{(2\pi)^2}\Big(\int_{\underset{\sim}{R}^4} e^{-\sqrt{-1}\langle p,x\rangle} f_\sigma(x)\,dx\Big)\,d\mu_m(p)$$

$$+ \int_{X_m} \underset{\sim}{\gamma}(p,\sigma)\frac{1}{(2\pi)^2}\Big(\int_{\underset{\sim}{R}^4} e^{\sqrt{-1}\langle p,x\rangle} f_\sigma(x)\,dx\Big)\,d\mu_m(p)\Big)$$

or still,

$$\frac{\sqrt{2\pi}}{\sqrt{2}} \Big( \int_{X_m} \underset{\sim}{\alpha}(p,\sigma)\overline{\hat{f}_\sigma(p)}\,d\mu_m(p) + \int_{X_m} \underset{\sim}{\gamma}(p,\sigma)\hat{f}_\sigma(p)\,d\mu_m(p)\Big)$$

or still,

$$\frac{\sqrt{2\pi}}{\sqrt{2}} \Big( \int_{X_m} \sum_\tau \underset{\sim}{a}(p,\tau) D^{(s)}_{\sigma\tau}(\tilde{\Lambda}_p)\overline{\hat{f}_\sigma(p)}\,d\mu_m(p) + \int_{X_m} \sum_\tau \underset{\sim}{c}(p,\tau)\left\{D^{(s)}(\tilde{\Lambda}_p)C^{-1}\right\}_{\sigma\tau} \hat{f}_\sigma(p)\,d\mu_m(p)\Big).$$

On the other hand,

$$\frac{1}{\sqrt{2}}(\underset{\sim}{a}(A_f) + \underset{\sim}{c}(A_f))$$

$$= \frac{1}{\sqrt{2}} \Big( \sum_\sigma \int_{X_m} \underset{\sim}{a}(p,\sigma)\overline{A_f(p,\sigma)}\,d\mu_m(p) + \sum_\sigma \int_{X_m} \underset{\sim}{c}(p,\sigma)C_f(p,\sigma)\,d\mu_m(p)\Big)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} \Big( \sum_\sigma \int_{X_m} \underset{\sim}{a}(p,\sigma) \sum_\tau \overline{D_{\tau\sigma}(\tilde{\Lambda}_p)\hat{f}_\tau(p)}\,d\mu_m(p) + \sum_\sigma \int_{X_m} \underset{\sim}{c}(p,\sigma) \sum_\tau \left\{D^{(s)}(\tilde{\Lambda}_p)C^{-1}\right\}_{\tau\sigma} \hat{f}_\tau(p)\,d\mu_m(p)\Big)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} \Big( \sum_\tau \int_{X_m} \sum_\sigma \underset{\sim}{a}(p,\sigma)\, D^{(s)}_{\tau\sigma}(\tilde{\Lambda}_p)\overline{\hat{f}_\tau(p)}\,d\mu_m(p)$$

$$+ \sum_{\tau} \int_{X_m} \sum_{\sigma} \underset{\sim}{c}(p,\sigma) \left\{ D^{(s)}(\tilde{\Lambda}_p) C^{-1} \right\}_{\tau\sigma} \hat{f}_{\tau}(p) d\mu_m(p))$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} \left( \sum_{\sigma} \int_{X_m} \sum_{\tau} \underset{\sim}{a}(p,\tau) D^{(s)}_{\sigma\tau}(\tilde{\Lambda}_p) \overline{\hat{f}_{\sigma}(p)} d\mu_m(p) \right.$$

$$+ \sum_{\sigma} \int_{X_m} \sum_{\tau} \underset{\sim}{c}(p,\tau) \left\{ D^{(s)}(\tilde{\Lambda}_p) C^{-1} \right\}_{\sigma\tau} \hat{f}_{\sigma}(p) d\mu_m(p))$$

$$= \sum_{\sigma} \int_{\underset{\sim}{R}^4} f_{\sigma}(x) \underset{\sim}{\varphi}_{\sigma}(x) dx.$$

Therefore

$$\underset{\sim}{\varphi}_{(m,s)}(f) = \sum_{\sigma} \underset{\sim}{\varphi}_{\sigma}(f_{\sigma}).]$$

Remark: The field map $\underset{\sim}{\varphi}_{(m,s)} : \mathcal{S}(\underset{\sim}{R}^4; \underset{\sim}{C}^{2s+1}) \longrightarrow \underline{\text{End}}\, \mathcal{D}$ is obviously linear. That it is also $\tilde{\mathcal{O}}^{\uparrow}_{+}$-equivariant follows from the transformation property of the $\underset{\sim}{\varphi}_{\sigma}$ per $U^{(m,s)}(\tilde{\Lambda}, a)$.

There remains the task of verifying W1-W6. Of these, only W5 and W6 require proof, and I shall deal only with W6, leaving W5 on the backburner.

The symmetric and antisymmetric cases can be dealt with simultaneously if we adopt the convention that

$$[A,B]_{\pm} = AB \pm BA.$$

[Note:

$$[A,B]_{-} = - [B,A]_{-}$$

while

$$[A,B]_{+} = [B,A]_{+} .]$$

---

LEMMA We have

$$[\underset{\sim}{\varphi}_{\sigma}(x),\ \underset{\sim}{\varphi}_{\sigma'}(x')]_{\pm} = 0$$

unless $\sigma' = -\sigma$ and in this case,

$$[\underset{\sim}{\varphi}_{\sigma}(x), \underset{\sim}{\varphi}_{-\sigma}(x')]_{\pm} = \mp \frac{(-1)^{s-\sigma}}{\sqrt{-1}} \Delta_m(x-x').$$

[Dropping the constants, it suffices to consider the sum of

$$\int_{X_m}\int_{X_m} e^{-\sqrt{-1}\langle p,x\rangle} e^{\sqrt{-1}\langle p',x'\rangle} [\underset{\sim}{\alpha}(p,\sigma), \underset{\sim}{\gamma}(p',\sigma')]_{\pm} d\mu_m(p) d\mu_m(p')$$

and

$$\int_{X_m}\int_{X_m} e^{\sqrt{-1}\langle p,x\rangle} e^{-\sqrt{-1}\langle p',x'\rangle} [\underset{\sim}{\gamma}(p,\sigma), \underset{\sim}{\alpha}(p',\sigma')]_{\pm} d\mu_m(p) d\mu_m(p').$$

From the definitions,

$$[\alpha(p,\sigma), \underset{\sim}{\gamma}(p',\sigma')]_{\pm}$$

$$= \sum_{\tau}\sum_{\tau'} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}_p) \left\{D^{(s)}(\tilde{\Lambda}_{p'})C^{-1}\right\}_{\sigma'\tau'} [\underset{\sim}{a}(p,\tau), \underset{\sim}{c}(p',\tau')]_{\pm}$$

$$= \sum_{\tau}\sum_{\tau'} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}_p) \left\{D^{(s)}(\tilde{\Lambda}_{p'})C^{-1}\right\}_{\sigma'\tau'} \delta(p'-p)\, \delta_{\tau'\tau}$$

$$= \sum_{\rho} D^{(s)}_{\sigma\rho}(\tilde{\Lambda}_p) \left\{D^{(s)}(\tilde{\Lambda}_{p'})C^{-1}\right\}_{\sigma'\rho} \delta(p'-p).$$

Integrating w.r.t. p', the first term thus becomes

$$\int_{X_m} e^{-\sqrt{-1}\langle p,x-x'\rangle} \sum_{\rho} D^{(s)}_{\sigma\rho}(\tilde{\Lambda}_p) \left\{D^{(s)}(\tilde{\Lambda}_p)C^{-1}\right\}_{\sigma'\rho} d\mu_m(p).$$

Next

$$[\underset{\sim}{\gamma}(p,\sigma), \underset{\sim}{\alpha}(p',\sigma')]_{\pm}$$

$$= \sum_{\tau} \sum_{\tau'} \{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\sigma\tau} D^{(s)}_{\sigma'\tau'}(\widetilde{\Lambda}_{p'})[\underset{\sim}{c}(p,\tau),\underset{\sim}{a}(p',\tau')]_{\pm}$$

$$= \sum_{\tau} \sum_{\tau'} \{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\sigma\tau} D^{(s)}_{\sigma'\tau'}(\widetilde{\Lambda}_{p'}) \pm \delta(p-p') \quad \delta_{\tau\tau'}$$

$$= \sum_{\rho} \{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\sigma\rho} D^{(s)}_{\sigma'\rho}(\widetilde{\Lambda}_{p'}) \pm \delta(p-p').$$

Integrating w.r.t. p, the second term thus becomes

$$\pm \int_{X_m} e^{\sqrt{-1}\langle p',x-x'\rangle} \sum_{\rho} \{D^{(s)}(\widetilde{\Lambda}_{p'})C^{-1}\}_{\sigma\rho} D^{(s)}_{\sigma'\rho}(\widetilde{\Lambda}_{p'})d\mu_m(p')$$

or still,

$$\pm \int_{X_m} e^{\sqrt{-1}\langle p,x-x'\rangle} \sum_{\rho} \{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\sigma\rho} D^{(s)}_{\sigma'\rho}(\widetilde{\Lambda}_p)d\mu_m(p).$$

We are therefore left with

$$\int_{X_m} [\, e^{-\sqrt{-1}\langle p,x-x'\rangle} \sum_{\rho} D^{(s)}_{\sigma\rho}(\widetilde{\Lambda}_p)\{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\sigma'\rho}$$

$$\pm \, e^{\sqrt{-1}\langle p,x-x'\rangle} \sum_{\rho} \{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\sigma\rho} D^{(s)}_{\sigma'\rho}(\widetilde{\Lambda}_p)]\, d\mu_m(p)$$

or still,

$$\int_{X_m} [e^{-\sqrt{-1}\langle p,x-x'\rangle} \left(D^{(s)}(\widetilde{\Lambda}_p)\{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}^{T}\right)_{\sigma\sigma'}$$

$$\pm \, e^{\sqrt{-1}\langle p,x-x'\rangle} \left(\{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\} D^{(s)}(\widetilde{\Lambda}_p)^{T}\right)_{\sigma\sigma'}]d\mu_m(p)$$

or still,

$$\int_{X_m} [e^{-\sqrt{-1}\langle p,x-x'\rangle}\left(D^{(s)}(\tilde{\Lambda}_p)CD^{(s)}(\tilde{\Lambda}_p)^T\right)_{\sigma\sigma'}$$

$$\pm\ e^{\sqrt{-1}\langle p,x-x'\rangle}\left(D^{(s)}(\tilde{\Lambda}_p)C^{-1}D^{(s)}(\tilde{\Lambda}_p)^T\right)_{\sigma\sigma'}]d\mu_m(p).$$

Here we have used the fact that $C^{-1}=C^T \Rightarrow (C^{-1})^T=C$. It is also true that

$$C^2 = (-1)^{2s}.$$

So

$$\begin{cases} s \text{ integral} \Rightarrow C^{-1} = C \\ s \text{ half-integral} \Rightarrow C^{-1} = -C. \end{cases}$$

In this connection, recall that when s is integral, it is a question of the commutator

$$[\underset{\sim}{\varphi}_\sigma(x), \underset{\sim}{\varphi}_{\sigma'}(x')]_-$$

and when s is half-integral, it is a question of the anticommutator

$$[\underset{\sim}{\varphi}_\sigma(x), \varphi_{\sigma'}(x')]_+ .$$

We are therefore left with

$$\int_{X_m}\left(D^{(s)}(\tilde{\Lambda}_p)CD^{(s)}(\tilde{\Lambda}_p)^T\right)_{\sigma\sigma'}$$

$$\times [e^{-\sqrt{-1}\langle p,x-x'\rangle} - e^{\sqrt{-1}\langle p,x-x'\rangle}]d\mu_m(p).$$

It remains to explicate

$$\left(D^{(s)}(\tilde{\Lambda}_p)CD^{(s)}(\tilde{\Lambda}_p)^T\right)_{\sigma\sigma'}.$$

To this end, we shall use the fact that

$$D^{(s)}(\widetilde{\Lambda}_p)^{T} = CD^{(s)}(\widetilde{\Lambda}_q)C^{-1}.$$

Accordingly,

$$\begin{aligned}
& D^{(s)}(\widetilde{\Lambda}_p)CD^{(s)}(\widetilde{\Lambda}_p)^{T} \\
&= D^{(s)}(\widetilde{\Lambda}_p)CCD^{(s)}(\widetilde{\Lambda}_q)C^{-1} \\
&= (-1)^{2s}\, D^{(s)}(\widetilde{\Lambda}_p)D^{(s)}(\widetilde{\Lambda}_q)C^{-1} \\
&= (-1)^{2s}\, D^{(s)}(\widetilde{\Lambda}_p\widetilde{\Lambda}_q)C^{-1} \\
&= (-1)^{2s}\, D^{(s)}(I)C^{-1} \\
&= (-1)^{2s}\, C^{-1} = (-1)^{2s}C^{T}
\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}
& \left( D^{(s)}(\widetilde{\Lambda}_p)CD^{(s)}(\widetilde{\Lambda}_p)^{T} \right)_{\sigma\sigma'} \\
&= (-1)^{2s}\,(C^{T})_{\sigma\sigma'} \\
&= (-1)^{2s}\,(C)_{\sigma'\sigma} \\
&= (-1)^{2s}\,(-1)^{s+\sigma'}\,\delta_{\sigma(-\sigma')} .
\end{aligned}$$

This shows that there is no contribution unless $\sigma' = -\sigma$, in which case we are left with

$$(-1)^{2s}\,(-1)^{s-\sigma} = \mp\,(-1)^{s-\sigma} .$$

Reintroducing the constants then leads at once to the assertion.]

Remark: As a reality check, take s=0 -- then the only possibility for $\sigma$ is $\sigma=0$, hence

$$[\underset{\sim}{\varphi}_0(x), \underset{\sim}{\varphi}_0(x')]_- = \frac{1}{\sqrt{-1}} \Delta_m(x-x'),$$

which agrees with our earlier conclusions since $\underset{\sim}{\varphi}_0 = \underset{\sim}{\varphi}_m$.

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LEMMA We have

$$[\underset{\sim}{\varphi}_\sigma(x), \underset{\sim}{\varphi}_{\sigma'}(x')^*]_\pm = \partial^{(s)}_{\sigma,\sigma'} \Delta_m(x-x'),$$

where $\partial^{(s)}_{\sigma,\sigma'}$ is a differential operator in the $x_\mu$.

[Dropping the constants, it suffices to consider the sum of

$$\int_{X_m}\int_{X_m} e^{-\sqrt{-1}\langle p,x\rangle} e^{\sqrt{-1}\langle p',x'\rangle} [\underset{\sim}{\alpha}(p,\sigma), \underset{\sim}{\alpha}^*(p',\sigma')]_\pm \, d\mu_m(p) d\mu_m(p')$$

and

$$\int_{X_m}\int_{X_m} e^{\sqrt{-1}\langle p,x\rangle} e^{-\sqrt{-1}\langle p',x'\rangle} [\underset{\sim}{\gamma}(p,\sigma), \underset{\sim}{\gamma}^*(p',\sigma')]_\pm \, d\mu_m(p) d\mu_m(p').$$

From the definitions,

$$\begin{aligned}
&[\underset{\sim}{\alpha}(p,\sigma), \underset{\sim}{\alpha}^*(p',\sigma')]_\pm \\
&= \sum_\tau \sum_{\tau'} D^{(s)}_{\sigma\tau}(\widetilde{\Lambda}_p) D^{(s)}_{\tau'\sigma'}(\widetilde{\Lambda}_{p'}) [\underset{\sim}{a}(p,\tau), \underset{\sim}{c}(p',\tau')]_\pm \\
&= \sum_\tau \sum_{\tau'} D^{(s)}_{\sigma\tau}(\widetilde{\Lambda}_p) D^{(s)}_{\tau'\sigma'}(\widetilde{\Lambda}_{p'}) \delta(p'-p)\, \delta_{\tau'\tau} \\
&= \sum_\rho D^{(s)}_{\sigma\rho}(\widetilde{\Lambda}_p) D^{(s)}_{\rho\sigma'}(\widetilde{\Lambda}_{p'}) \delta(p'-p).
\end{aligned}$$

Integrating w.r.t. p', the first term thus becomes

$$\int_{X_m} e^{-\sqrt{-1}\langle p,x-x'\rangle} \sum_{\rho} D^{(s)}_{\sigma\rho}(\widetilde{\Lambda}_p) D^{(s)}_{\rho\sigma'}(\widetilde{\Lambda}_p) d\mu_m(p)$$

$$= \int_{X_m} e^{-\sqrt{-1}\langle p,x-x'\rangle} (D^{(s)}(\widetilde{\Lambda}_p) D^{(s)}(\widetilde{\Lambda}_p))_{\sigma\sigma'} d\mu_m(p).$$

Next

$$[\gamma(p,\sigma), \gamma^*(p',\sigma')]_{\pm}$$

$$= \sum_{\tau}\sum_{\tau'} \{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\sigma\tau} \overline{\{D^{(s)}(\widetilde{\Lambda}_{p'})C^{-1}\}}_{\sigma'\tau'} [c(p,\tau), a(p',\tau')]_{\pm}$$

$$= \sum_{\tau}\sum_{\tau'} \{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\sigma\tau} \overline{\{D^{(s)}(\widetilde{\Lambda}_{p'})C^{-1}\}}_{\sigma'\tau'} \pm \delta(p-p')\,\delta_{\tau\tau'}$$

$$= \sum_{\rho} \{D^{(s)}(\widetilde{\Lambda}_p)C^{-1}\}_{\sigma\rho} \overline{\{D^{(s)}(\widetilde{\Lambda}_{p'})C^{-1}\}}_{\sigma'\rho} \pm \delta(p-p').$$

Integrating w.r.t. p, the second term thus becomes

$$\pm \int_{X_m} e^{\sqrt{-1}\langle p',x-x'\rangle} \sum_{\rho} \{D^{(s)}(\widetilde{\Lambda}_{p'})C^{-1}\}_{\sigma\rho} \overline{\{D^{(s)}(\widetilde{\Lambda}_{p'})C^{-1}\}}_{\sigma'\rho}\, d\mu_m(p')$$

$$= \pm \int_{X_m} e^{\sqrt{-1}\langle p,x-x'\rangle} ((D^{(s)}(\widetilde{\Lambda}_p)C^{-1})\cdot(D^{(s)}(\widetilde{\Lambda}_p)C^{-1})^*)_{\sigma\sigma'} d\mu_m(p)$$

$$= \pm \int_{X_m} e^{\sqrt{-1}\langle p,x-x'\rangle} (D^{(s)}(\widetilde{\Lambda}_p)D^{(s)}(\widetilde{\Lambda}_p))_{\sigma\sigma'}\, d\mu_m(p).$$

Here we have used the fact that $D^{(s)}(\widetilde{\Lambda}_p)$ is selfadjoint. We are therefore left with

$$\int_{X_m} (D^{(s)}(\widetilde{\Lambda}_p)D^{(s)}(\widetilde{\Lambda}_p))_{\sigma\sigma'}$$

$$\times [e^{-\sqrt{-1}\langle p,x-x'\rangle} \pm e^{\sqrt{-1}\langle p,x-x'\rangle}]d\mu_m(p).$$

It remains to explicate

$$(D^{(s)}(\widetilde{\Lambda}_p)D^{(s)}(\widetilde{\Lambda}_p))_{\sigma\sigma'}.$$

But this is not difficult:

$$(D^{(s)}(\widetilde{\Lambda}_p)D^{(s)}(\widetilde{\Lambda}_p))_{\sigma\sigma'}$$

$$= \sum T^{(s)}_{\sigma\sigma'}(k_1,\ldots,k_{2s})p_{k_1}\cdots p_{k_{2s}},$$

where the sum is over all $k_1,\ldots,k_{2s}$ $(=0,1,2,3)$. If s is integral, then we use the minus sign and

$$\int_{X_m} p_{k_1}\cdots p_{k_{2s}}[e^{-\sqrt{-1}\langle p,x-x'\rangle} - e^{\sqrt{-1}\langle p,x-x'\rangle}]d\mu_m(p)$$

$$= D_{k_1}\cdots D_{k_{2s}}\int_{X_m}[e^{-\sqrt{-1}\langle p,x-x'\rangle} - e^{\sqrt{-1}\langle p,x-x'\rangle}]d\mu_m(p),$$

while if s is half-integral, then we use the plus sign and

$$\int_{X_m} p_{k_1}\cdots p_{k_{2s}}[e^{-\sqrt{-1}\langle p,x-x'\rangle} + e^{\sqrt{-1}\langle p,x-x'\rangle}]d\mu_m(p)$$

$$= D_{k_1}\cdots D_{k_{2s}}\int_{X_m}[e^{-\sqrt{-1}\langle p,x-x'\rangle} - e^{\sqrt{-1}\langle p,x-x'\rangle}]d\mu_m(p).$$

Of course, the point is that the minus sign does appear in the end, which in turn leads to the introduction of $\Delta_m$ and, finally, to the contention of the lemma.]

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Remark: The following simple formalities have been employed.

$$(1)\quad \sqrt{-1}\,\frac{d}{dx_0}\,[e^{-\sqrt{-1}\langle p,x\rangle} - e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= \sqrt{-1}\,[(-\sqrt{-1}\,p_0)e^{-\sqrt{-1}\langle p,x\rangle} - (\sqrt{-1}\,p_0)e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= [p_0\, e^{-\sqrt{-1}\langle p,x\rangle} + p_0\, e^{\sqrt{-1}\langle p,x\rangle}]$$

and

$$\sqrt{-1}\,\frac{d}{dx_0}\,[p_0\, e^{-\sqrt{-1}\langle p,x\rangle} + p_0\, e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= [p_0^2\, e^{-\sqrt{-1}\langle p,x\rangle} - p_0^2\, e^{\sqrt{-1}\langle p,x\rangle}]$$

$$(2)\quad -\sqrt{-1}\,\frac{d}{dx_\mu}\,[e^{-\sqrt{-1}\langle p,x\rangle} - e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= -\sqrt{-1}\,[(\sqrt{-1}\,p_\mu)e^{-\sqrt{-1}\langle p,x\rangle} - (-\sqrt{-1}\,p_\mu)e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= [p_\mu\, e^{-\sqrt{-1}\langle p,x\rangle} + p_\mu\, e^{\sqrt{-1}\langle p,x\rangle}]$$

and

$$-\sqrt{-1}\,\frac{d}{dx_\mu}\,[p_\mu\, e^{-\sqrt{-1}\langle p,x\rangle} + p_\mu\, e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= [p_\mu^2\, e^{-\sqrt{-1}\langle p,x\rangle} - p_\mu^2\, e^{\sqrt{-1}\langle p,x\rangle}]$$

$$(3)\quad \left(\sqrt{-1}\,\frac{d}{dx_0}\right)\left(-\sqrt{-1}\,\frac{d}{dx_\mu}\right)[e^{-\sqrt{-1}\langle p,x\rangle} - e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= \sqrt{-1}\,\frac{d}{dx_0}\,[p_\mu\, e^{-\sqrt{-1}\langle p,x\rangle} + p_\mu\, e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= [p_0 p_\mu\, e^{-\sqrt{-1}\langle p,x\rangle} - p_0 p_\mu\, e^{\sqrt{-1}\langle p,x\rangle}]$$

$$(4)\quad \left(-\sqrt{-1}\,\frac{d}{dx_\mu}\right)\left(-\sqrt{-1}\,\frac{d}{dx_\nu}\right)[e^{-\sqrt{-1}\langle p,x\rangle} - e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= -\sqrt{-1}\,\frac{d}{dx_\mu}\,[p_\nu\, e^{-\sqrt{-1}\langle p,x\rangle} + p_\nu\, e^{\sqrt{-1}\langle p,x\rangle}]$$

$$= [p_\mu p_\nu\, e^{-\sqrt{-1}\langle p,x\rangle} - p_\mu p_\nu\, e^{\sqrt{-1}\langle p,x\rangle}].$$

Example: Take s=1/2 -- then $D^{(1/2)}(\tilde{\Lambda}_p)D^{(1/2)}(\tilde{\Lambda}_p) = \tilde{\Lambda}_p^2 = \frac{1}{m}\underset{\sim}{p}$

$$= \frac{1}{m}\begin{pmatrix} p_0+p_3 & p_1 - \sqrt{-1}\,p_2 \\ p_1+\sqrt{-1}\,p_2 & p_0 - p_3 \end{pmatrix}.$$

Therefore

$$[\underset{\sim}{\varphi}_\sigma(x), \underset{\sim}{\varphi}_{\sigma'}(x')^*]_+$$

$$= \frac{1}{m}\int_{X_m} \underset{\sim}{p}_{\sigma\sigma'}[e^{-\sqrt{-1}\langle p,x-x'\rangle} + e^{\sqrt{-1}\langle p,x-x'\rangle}]\,d\mu_m(p)$$

$$= \frac{1}{m}\,\partial_{\sigma\sigma'}\,\frac{1}{\sqrt{-1}}\,\Delta_m(x-x').$$

Example: Take s=1 -- then $D^{(1)}(\tilde{\Lambda}_p)D^{(1)}(\tilde{\Lambda}_p) = D^{(1)}(\tilde{\Lambda}_p^2)$

$$= \frac{1}{m^2}\,[m^2 + 2\underset{\sim}{p}\cdot\underset{\sim}{S}(2\underset{\sim}{p}\cdot\underset{\sim}{S}+p_0)],$$

where

$$S_1 = \sqrt{\tfrac{1}{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_2 = \sqrt{\tfrac{1}{2}}\begin{bmatrix} 0 & -\sqrt{-1} & 0 \\ \sqrt{-1} & 0 & -\sqrt{-1} \\ 0 & \sqrt{-1} & 0 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

so one can compute

$$[\underset{\sim}{\varphi}_\sigma(x), \underset{\sim}{\varphi}_{\sigma'}(x')]_-$$

explicitly in this situation too.

Remark: It is easy to allow for antiparticles (proceed as in the spin zero case).

Instead of working with $D^{(s)}=D^{(s,0)}$, we could just as well have worked with $\bar{D}^{(s)}=D^{(0,s)}$, realized as

$$\tilde{\Lambda} \rightarrow D^{(s)}(\tilde{\Lambda}^{-1})^* .$$

Recall here that

$$\overline{D^{(s)}(\tilde{\Lambda})} = CD^{(s)}(\tilde{\Lambda}^{-1})^* C^{-1}.$$

One can then introduce another 2s+1 component field $\bar{\underset{\sim}{\varphi}}_{(m,s)}$ with the property that if $\overline{U}^{(m,s)}$ is the associated irreducible unitary representation of $\widetilde{\mathcal{G}}^{\uparrow}_{+}$, then

$$\overline{U}^{(m,s)}(\widetilde{\Lambda},a)\ \overline{\underset{\sim}{\varphi}}_{\sigma}(x)\overline{U}^{(m,s)}(\widetilde{\Lambda},a)^{-1}$$

$$= \sum_{\tau} \overline{D}^{(s)}_{\sigma\tau}\ (\widetilde{\Lambda}^{-1})\ \overline{\underset{\sim}{\varphi}}_{\tau}(\widetilde{\Lambda}x+a).$$

The (s,0) field $\underset{\sim}{\varphi}_{(m,s)}$ and the (0,s) field $\overline{\underset{\sim}{\varphi}}_{(m,s)}$ can now be combined into a single 2(2s+1) component field

$$\begin{bmatrix} \underset{\sim}{\varphi}_{(m,s)} \\ \overline{\underset{\sim}{\varphi}}_{(m,s)} \end{bmatrix},$$

where, needless to say,

$$\Pi = D^{(s)} \oplus \overline{D}^{(s)},$$

but there is a difficulty in that the vacuum is no longer unique up to phase (the underlying Hilbert space is a direct sum, not a tensor product).

Remark: Taking s=1/2, one can produce from the preceding the Dirac field. However, I shall omit this and give a different construction later on.

<u>Massless Particles</u> The free relativistic particle of spin $s > 0$ and mass zero carries the structure of a vector QFT. Its type depends on the helicity $\lambda = \pm s$.

Rappel: In nature, $\exists$ two kinds of massless particles.

<u>I</u>: There are those which exist in the helicity states $\pm\lambda$. E.g.: The photon ($\lambda = \pm 1$) and the graviton ($\lambda = \pm 2$).

<u>II</u>: There are those which exist in a single helicity state $\lambda$ but admit an antiparticle existing in the single helicity state $-\lambda$. E.g.: The neutrino ($\lambda = -1/2$) and the antineutrino ($\lambda = 1/2$).

Notation: As usual,

$$\begin{cases} \tilde{\mathcal{L}}^{\uparrow}_{+} \to \mathcal{L}^{\uparrow}_{+} & (\tilde{\Lambda} \to \Lambda) \\ \tilde{\mathcal{P}}^{\uparrow}_{+} \to \mathcal{P}^{\uparrow}_{+} & ((\tilde{\Lambda}, a) \to (\Lambda, a)). \end{cases}$$

(1) Assign to each $p \in X_0$ the <u>boost</u> $\Lambda_p \in \mathcal{L}^{\uparrow}_{+}$ : $\Lambda_p(1,0,0,1) = p$. Thus $\Lambda_p$ is the result of sending $(1,0,0,1)$ to $(|\underset{\sim}{p}|,0,0,|\underset{\sim}{p}|)$ followed by the rotation $R(\hat{p})$ (in the plane containing $\hat{p}$ and the z-axis) which takes the z-axis into $\hat{p} = \underset{\sim}{p}/|\underset{\sim}{p}|$.

(2) Assign to each $p \in X_0$ the <u>Wigner angle</u> $\Theta(\tilde{\Lambda}, p) \in [0, 2\pi[$. Thus

$$(\tilde{\Lambda}_p)^{-1} \tilde{\Lambda} \tilde{\Lambda}_{\tilde{\Lambda}^{-1} p} \in E,$$

where

$$E = \left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & z \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix} : 0 \le \theta < 2\pi,\ z \in \underset{\sim}{C} \right\},$$

so it is a matter of picking off the angle in the decomposition of

$(\tilde{\Lambda}_p)^{-1}\tilde{\Lambda}\tilde{\Lambda}_{\tilde{\Lambda}^{-1}p}$ into its diagonal and off-diagonal components.

Fact: Given $n \in \underset{\sim}{Z}$, $\exists$ an irreducible unitary representation $U^n$ of $\tilde{\mathcal{P}}_+^{\uparrow}$ on $L^2(X_0, \mu_0)$, viz.

$$(U^n(\tilde{\Lambda},a)f)(p)$$
$$= e^{\sqrt{-1}\langle a,p\rangle} e^{\sqrt{-1}\, n\, \Theta(\tilde{\Lambda},p)} f(\tilde{\Lambda}^{-1}p).$$

Turning now to quantum field theory, one has to start by specifying the Hilbert space, which we shall do by taking for $\mathcal{H}$ the symmetric Fock space over $L^2(X_0, \mu_0)$ if s is integral or the antisymmetric Fock space over $L^2(X_0, \mu_0)$ if s is half-integral.

[Note: Here, of course, the underlying unitary representation of $\tilde{\mathcal{P}}_+^{\uparrow}$ on $\mathcal{H}$ is derived from $U^{2\lambda}$.]

There are various possibilities for $\Pi$. Before getting into this, introduce operators

$$\begin{cases} \underset{\sim}{a}(p,\lambda) \\ \underset{\sim}{c}(p,\lambda) \end{cases}$$

satisfying the usual conditions ($\lambda$ being merely a label in this context). Obviously,

$$U^{2\lambda}(\tilde{\Lambda},0)\underset{\sim}{c}(p,\lambda)U^{2\lambda}(\tilde{\Lambda},0)^{-1}$$

$$= \underline{\exp}(\sqrt{-1}\, 2\lambda\, \Theta((\tilde{\Lambda}_{\tilde{\Lambda}p})^{-1}\tilde{\Lambda}\tilde{\Lambda}_p))\underset{\sim}{c}(\tilde{\Lambda}p,\lambda)$$

and

$$U^{2\lambda}(\tilde{\Lambda},0)\underset{\sim}{a}(p,\lambda)U^{2\lambda}(\tilde{\Lambda},0)^{-1}$$

$$= \underline{\exp}(\sqrt{-1}\,2\lambda\,\Theta\,((\tilde{\Lambda}_p)^{-1}\tilde{\Lambda}^{-1}\tilde{\Lambda}_{\tilde{\Lambda}p}))\underset{\sim}{a}(\tilde{\Lambda}p,\lambda).$$

Definition: If our particle is left-handed, i.e., if $\lambda = -s$, put

$$\underset{\sim}{\Psi}^{+}_{L,\sigma}(x) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int_{X_0} e^{-\sqrt{-1}\langle p,x\rangle}$$

$$\times\, D^{(s)}_{\sigma\lambda}(\tilde{\Lambda}_p)\underset{\sim}{a}(p,\lambda)d\mu_0(p)$$

but if our particle is right-handed, i.e., if $\lambda = s$, put

$$\underset{\sim}{\Psi}^{+}_{R,\sigma}(x) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int_{X_0} e^{-\sqrt{-1}\langle p,x\rangle}$$

$$\times\, \bar{D}^{(s)}_{\sigma\lambda}(\tilde{\Lambda}_p)\underset{\sim}{a}(p,\lambda)d\mu_0(p).$$

[Note: As usual, $\sigma$ ranges over $s, s-1, \ldots, -s+1, -s$.]

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LEMMA We have

$$U^{2\lambda}(\tilde{\Lambda},0)\,\underset{\sim}{\Psi}^{+}_{L,\sigma}(x)U^{2\lambda}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\,\underset{\sim}{\Psi}^{+}_{L,\tau}(\tilde{\Lambda}x)$$

and

$$U^{2\lambda}(\tilde{\Lambda},0)\,\underset{\sim}{\Psi}^{+}_{R,\sigma}(x)U^{2\lambda}(\tilde{\Lambda},0)^{-1}$$

$$= \sum_{\tau} \bar{D}^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\,\underset{\sim}{\Psi}^{+}_{R,\tau}(\tilde{\Lambda}x)$$

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Remark: In the massive case, $\Pi = \begin{cases} D^{(s)} \\ \bar{D}^{(s)} \end{cases}$ was automatically embedded in the theory through the very definition of $\begin{cases} U^{(m,s)} \\ \bar{U}^{(m,s)} \end{cases}$.

This is not true here: $\Pi = \begin{cases} D^{(s)} \\ \bar{D}^{(s)} \end{cases}$ has been introduced solely to get the transformation property of the fields. Our choice is the simplest but one could equally well have worked with

$$\begin{cases} \text{[left]} \quad (s + \frac{1}{2}, \frac{1}{2}), \ (s + 1, 1), \ldots \\ \text{[right]} \ (\frac{1}{2}, s + \frac{1}{2}), \ (1, s + 1), \ldots . \end{cases}$$

To continue, it is necessary to differentiate between possibilities I and II. In what follows, I shall concentrate on I (the discussion of II entails the introduction of antiparticles).

Definition: Assuming that we are working with particles of type I, put

$$\underset{\sim}{\Psi}^{-}_{L,\sigma}(x) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int_{X_0} e^{\sqrt{-1}\langle p,x\rangle}$$

$$\times D^{(s)}_{\sigma\lambda}(\widetilde{\Lambda}_p)\underset{\sim}{c}(p,-\lambda)d\mu_0(p)$$

and

$$\underset{\sim}{\Psi}^{-}_{R,\sigma}(x) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int_{X_0} e^{\sqrt{-1}\langle p,x\rangle}$$

$$\times \bar{D}^{(s)}_{\sigma\lambda}(\widetilde{\Lambda}_p)\underset{\sim}{c}(p,-\lambda)d\mu_0(p).$$

[Note: The lemma also applies to these fields (with $U^{2\lambda}$ replaced by $U^{-2\lambda}$).]

Remark: It is a fact that

$$\left\{\begin{array}{l} D^{(s)}_{\sigma\lambda}(\tilde{\Lambda}_p) = D^{(s)}_{\sigma\lambda}(R(\hat{p}))(|\underset{\sim}{p}|)^s \\ \bar{D}^{(s)}_{\sigma\lambda}(\tilde{\Lambda}_p) = \bar{D}^{(s)}_{\sigma\lambda}(R(\hat{p}))(|\underset{\sim}{p}|)^s \end{array}\right. \qquad (\lambda = \pm s).$$

On the other hand,

$$\bar{D}^{(s)}_{\sigma\lambda}(R(\hat{p})) = D^{(s)}_{\sigma\lambda}(R(\hat{p})).$$

Therefore

$$\left\{\begin{array}{l} \underset{\sim}{\Psi}^{+}_{L,\sigma} \\ \underset{\sim}{\Psi}^{+}_{R,\sigma} \end{array}\right. \quad \& \quad \left\{\begin{array}{l} \underset{\sim}{\Psi}^{-}_{L,\sigma} \\ \underset{\sim}{\Psi}^{-}_{R,\sigma} \end{array}\right.$$

actually involve rotations only.

To take advantage of this circumstance, we shall modify our definitions slightly:

$$\underset{\sim}{\Psi}^{+}_{L,\sigma}(x) = \frac{1}{\sqrt{2}}\,\frac{1}{(2\pi)^{3/2}}\int_{X_0}(2|\underset{\sim}{p}|)^s\, e^{-\sqrt{-1}\langle p,x\rangle}$$

$$\times\, D^{(s)}_{\sigma\lambda}(R(\hat{p}))\underset{\sim}{a}(p,\lambda)\,d\mu_0(p),$$

$$\underset{\sim}{\Psi}^{+}_{R,\sigma}(x) = \frac{1}{\sqrt{2}}\,\frac{1}{(2\pi)^{3/2}}\int_{X_0}(2|\underset{\sim}{p}|)^s\, e^{-\sqrt{-1}\langle p,x\rangle}$$

$$\times\, D^{(s)}_{\sigma\lambda}(R(\hat{p}))\underset{\sim}{a}(p,\lambda)\,d\mu_0(p)$$

and

$$\underset{\sim}{\Psi}^{-}_{L,\sigma}(x) = \frac{1}{\sqrt{2}}\,\frac{1}{(2\pi)^{3/2}}\int_{X_0}(2|\underset{\sim}{p}|)^s\, e^{\sqrt{-1}\langle p,x\rangle}$$

$$\times\, D^{(s)}_{\sigma\lambda}(R(\hat{p}))\underset{\sim}{c}(p,-\lambda)\,d\mu_0(p),$$

$$\underset{\sim}{\Psi}^{-}_{R,\sigma}(x) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int_{X_0} (2|\underset{\sim}{p}|)^s e^{\sqrt{-1}\langle p,x\rangle}$$

$$\times D^{(s)}_{\sigma\lambda}(R(\hat{p}))\underset{\sim}{c}(p,-\lambda)d\mu_0(p).$$

[Note: Bear in mind that, despite appearances, the formulas for

$$\begin{cases} \underset{\sim}{\Psi}^{+}_{L,\sigma} \\ \underset{\sim}{\Psi}^{+}_{R,\sigma} \end{cases} \& \begin{cases} \underset{\sim}{\Psi}^{-}_{L,\sigma} \\ \underset{\sim}{\Psi}^{-}_{R,\sigma} \end{cases}$$

are not identical. Indeed, when the particle is left-handed $\lambda = -s$ but when the particle is right-handed, $\lambda = s$.]

The next step is to combine

$$\begin{cases} \underset{\sim}{\Psi}^{+}_{L,\sigma} \ \& \ \underset{\sim}{\Psi}^{-}_{L,\sigma} \\ \underset{\sim}{\Psi}^{+}_{R,\sigma} \ \& \ \underset{\sim}{\Psi}^{-}_{R,\sigma} \end{cases},$$

which necessitates the introduction of tensor products.

Convention: Assume henceforth that s is integral.

[Note: When s is half-integral, certain technical modifications in the overall setup have to be made but the final conclusions are similar.]

The Hilbert space $\mathcal{H}$ is the same for any $\lambda$. However, to emphasize the underlying unitary representation of $\widetilde{\mathcal{P}}^{\uparrow}_{+}$, write $\mathcal{H}_\lambda$ when $U^{2\lambda}$ is operating and $\mathcal{H}_{-\lambda}$ when $U^{-2\lambda}$ is operating. This done, pass to $\mathcal{H}_\lambda \otimes \mathcal{H}_{-\lambda}$, call it $\mathcal{H}$, then let $\mathcal{D} = \mathcal{D}_\lambda \otimes \mathcal{D}_{-\lambda}$, $\Omega = \Omega_\lambda \otimes \Omega_{-\lambda}$, $U = U^{2\lambda} \otimes U^{-2\lambda}$, and

$$\begin{cases} \underset{\sim}{\Psi}_{L,\sigma} = \underset{\sim}{\psi}^{+}_{L,\sigma} \otimes I_{-\lambda} + I_{\lambda} \otimes \underset{\sim}{\psi}^{-}_{L,\sigma} \\ \underset{\sim}{\Psi}_{R,\sigma} = \underset{\sim}{\psi}^{+}_{R,\sigma} \otimes I_{-\lambda} + I_{\lambda} \otimes \underset{\sim}{\psi}^{-}_{R,\sigma} . \end{cases}$$

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LEMMA We have

$$\begin{cases} \underset{\sim}{\Psi}^{*}_{R,\sigma} = \sum_{\tau} C_{\sigma\tau} \underset{\sim}{\Psi}_{L,\tau} \\ \underset{\sim}{\Psi}^{*}_{L,\sigma} = (-1)^{2s} \sum_{\tau} C_{\sigma\tau} \underset{\sim}{\Psi}_{R,\tau} . \end{cases}$$

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Accordingly, it suffices to consider in detail just the $\underset{\sim}{\Psi}_{L,\sigma}$ and their adjoints. The definition of the field map proceeds along the usual lines so I'll omit it. For the record, though, note that

$$\begin{aligned} & U(\tilde{\Lambda},0) \underset{\sim}{\Psi}_{L,\sigma}(x)\, U(\tilde{\Lambda},0)^{-1} \\ & \quad = U^{2\lambda}(\tilde{\Lambda},0)\, \underset{\sim}{\psi}^{+}_{L,\sigma}(x)\, U^{2\lambda}(\tilde{\Lambda},0)^{-1} \otimes I_{-\lambda} \\ & \quad + I_{\lambda} \otimes U^{-2\lambda}(\tilde{\Lambda},0)\, \underset{\sim}{\psi}^{-}_{L,\sigma}(x)\, U^{-2\lambda}(\tilde{\Lambda},0)^{-1} \\ & \quad = \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\, \underset{\sim}{\psi}^{+}_{L,\tau}(\tilde{\Lambda}x) \otimes I_{-\lambda} \\ & \quad + I_{\lambda} \otimes \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\, \underset{\sim}{\psi}^{-}_{L,\tau}(\tilde{\Lambda}x) \\ & = \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1}) (\underset{\sim}{\psi}^{+}_{L,\tau}(\tilde{\Lambda}x) \otimes I_{-\lambda} + I_{\lambda} \otimes \underset{\sim}{\psi}^{-}_{L,\tau}(\tilde{\Lambda}x)) \\ & = \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\, \underset{\sim}{\Psi}_{L,\tau}(\tilde{\Lambda}x). \end{aligned}$$

Since

$$U(I,a) \underset{\sim}{\Psi}_{L,\sigma}(x) U(I,a)^{-1} = \underset{\sim}{\Psi}_{L,\sigma}(x+a),$$

it follows that

$$U(\tilde{\Lambda},a) \underset{\sim}{\Psi}_{L,\sigma}(x) U(\tilde{\Lambda},a)^{-1}$$

$$= \sum_{\tau} D^{(s)}_{\sigma\tau}(\tilde{\Lambda}^{-1})\, \underset{\sim}{\Psi}_{L,\tau}(\tilde{\Lambda}x+a).$$

Remark: The field components obey the Klein-Gordon equation

$$\Box^2\, \underset{\sim}{\Psi}_{L,\sigma}(x) = 0.$$

In addition, it can be shown that

$$[s\,(\frac{\partial}{\partial t}) - \underset{\sim}{S}^{(s)} \cdot \nabla\,]\, \underset{\sim}{\Psi}_{L}(x) = 0.$$

There remains the task of verifying W1-W6. Of these, only W5 and W6 require proof but, as in the massive case, I'll omit W5 and focus on W6.

---

LEMMA We have

$$[\,\underset{\sim}{\Psi}_{L,\sigma}(x),\ \underset{\sim}{\Psi}_{L,\sigma'}(x')\,] = 0.$$

[On the basis of the definitions, this is immediate.]

---

LEMMA We have

$$[\,\underset{\sim}{\Psi}_{L,\sigma}(x),\ \underset{\sim}{\Psi}_{L,\sigma'}(x')^*\,] = \partial^{(s)}_{\sigma,\sigma'}\, \Delta_0(x-x'),$$

where $\partial^{(s)}_{\sigma,\sigma'}$ is a differential operator in the $x_\mu$.

[It suffices to consider the sum of

$$\frac{1}{2(2\pi)^3} \int_{X_0} \int_{X_0} (2|\underset{\sim}{p}|)^s (2|\underset{\sim}{p}'|)^s\, e^{-\sqrt{-1}\langle p,x\rangle}\, e^{\sqrt{-1}\langle p',x'\rangle}$$

$$\times\, D^{(s)}_{\sigma\lambda}(R(\hat{p}))\, \overline{D^{(s)}_{\sigma'\lambda}(R(\hat{p}'))}\, [\underset{\sim}{a}(p,\lambda), \underset{\sim}{c}(p',\lambda)] \otimes I_{-\lambda}\, d\mu_0(p)\, d\mu_0(p')$$

and

$$\frac{1}{2(2\pi)^3}\int_{X_0}\int_{X_0}(2|\mathbf{p}|)^s(2|\mathbf{p}'|)^s\, e^{\sqrt{-1}\langle p,x\rangle}\, e^{-\sqrt{-1}\langle p',x'\rangle}$$

$$\times D^{(s)}_{\sigma\lambda}(R(\hat{p}))\overline{D^{(s)}_{\sigma'\lambda}(R(\hat{p}'))I}_{\lambda}\otimes[\mathbf{c}(p,-\lambda),\mathbf{a}(p',-\lambda)]d\mu_0(p)d\mu_0(p').$$

Employing the commutation relations

$$\begin{cases}[\mathbf{a}(p,\lambda),\mathbf{c}(p',\lambda)] = \delta(p'-p)I_{\lambda}\\[\mathbf{c}(p,-\lambda),\mathbf{a}(p',-\lambda)] = \delta(p-p')I_{-\lambda}\end{cases}$$

reduces the sum to

$$\frac{1}{2(2\pi)^3}\int_{X_0}(2|\mathbf{p}|)^{2s}\, D^{(s)}_{\sigma\lambda}(R(\hat{p}))\overline{D^{(s)}_{\sigma'\lambda}(R(\hat{p}))}$$

$$[e^{-\sqrt{-1}\langle p,x-x'\rangle} - e^{\sqrt{-1}\langle p,x-x'\rangle}]d\mu_0(p),$$

the $I_{\lambda}\otimes I_{-\lambda}$ having been dropped from the notation. But the factor

$$(2|\mathbf{p}|)^{2s}\, D^{(s)}_{\sigma\lambda}(R(\hat{p}))\overline{D^{(s)}_{\sigma'\lambda}(R(p))} \quad (\lambda=-s)$$

is a sum of terms of the form

$$C_{k_1\cdots k_{2s}}\, p_{k_1}\cdots p_{k_{2s}},$$

where $k_1,\ldots,k_{2s}$ range over 0,1,2,3. Defining $\partial^{(s)}_{\sigma,\sigma'}$ in the obvious way then leads to the assertion.]

The Dirac Equation Working in $\mathcal{H} = L^2(R^3;\underset{\sim}{C}^4)$, the evolution equation of this theory has the form

$$\sqrt{-1}\,\hbar\,\frac{\partial}{\partial t}\,\psi(t,\underset{\sim}{x}) = H\,\psi(t,\underset{\sim}{x}),$$

where

$$\psi(t,x) = \begin{pmatrix} \psi_1(t,\underset{\sim}{x}) \\ \vdots \\ \psi_4(t,\underset{\sim}{x}) \end{pmatrix} \in \underset{\sim}{C}^4$$

and

$$H = -\sqrt{-1}\,\hbar\,c\,\underset{\sim}{\alpha}\cdot\nabla + \beta mc^2,$$

a $4 \times 4$ matrix differential operator -- the Dirac operator. By construction, $\underset{\sim}{\alpha}$ is a triple $(\alpha_1, \alpha_2, \alpha_3)$ of hermitian $4 \times 4$ matrices while $\beta$ is a hermitian $4 \times 4$ matrix, subject to the relations

$$\begin{cases} \alpha_i\alpha_j + \alpha_j\alpha_i = 2\,\delta_{ij}I \\ \alpha_i\beta + \beta\alpha_i = 0 \\ \beta^2 = I. \end{cases}$$

There are various choices which realize these conditions. For example, one can take

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Here, as usual,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

thus

$$H = \begin{pmatrix} mc^2\, I & -\sqrt{-1}\,\hbar\, c\, \underset{\sim}{\sigma}\cdot\nabla \\ -\sqrt{-1}\,\hbar\, c\,\underset{\sim}{\sigma}\cdot\nabla & -mc^2\, I \end{pmatrix}.$$

Assume henceforth that $c=1$ and $\hbar=1$.

---

LEMMA H is essentially selfadjoint on $\mathcal{S}(\underset{\sim}{R}^3;\underset{\sim}{C}^4)$ and selfadjoint on $W^{1,2}(\underset{\sim}{R}^3;\underset{\sim}{C}^4)$. Its spectrum is purely absolutely continuous and is given by

$$\sigma(H) = ]-\infty,-m] \cup [m,+\infty[.$$

[Note: Recall that in general $W^{k,p}(\underset{\sim}{R}^3;\underset{\sim}{C}^4)$ stands for the Sobolev space consisting of those $\underset{\sim}{C}^4$-valued $L^p$-functions whose distributional derivatives of order $\leq k$ are also $L^p$.]

---

The domain of H is "configuration space". Under Fourier transformation, H is sent to "momentum space", where as a multiplication operator it assumes the form

$$h(\underset{\sim}{p}) = \begin{pmatrix} m & \underset{\sim}{\sigma}\cdot\underset{\sim}{p} \\ \underset{\sim}{\sigma}\cdot\underset{\sim}{p} & -m \end{pmatrix}.$$

For each $\underset{\sim}{p}$, this is a hermitian $4 \times 4$ matrix with eigenvalues

$$\left\{\begin{matrix} \lambda_1(\underset{\sim}{p}) \\ \lambda_2(\underset{\sim}{p}) \end{matrix}\right. = \mu(\underset{\sim}{p}) \quad \& \quad \left\{\begin{matrix} \lambda_3(\underset{\sim}{p}) \\ \lambda_4(\underset{\sim}{p}) \end{matrix}\right. = -\mu(\underset{\sim}{p}).$$

Here, as earlier,

$$\mu(\underset{\sim}{p}) = \sqrt{m^2 + |\underset{\sim}{p}|^2}.$$

The unitary operator which diagonalizes $h(\underset{\sim}{p})$ is then

$$u(\underset{\sim}{p}) = \frac{(m+\mu(\underset{\sim}{p}))I + \beta\underset{\sim}{\alpha}\cdot\underset{\sim}{p}}{\sqrt{2\mu(\underset{\sim}{p})(m+\mu(\underset{\sim}{p}))}} .$$

In fact,

$$u(\underset{\sim}{p})h(\underset{\sim}{p})u(\underset{\sim}{p})^{-1} = \beta\mu(\underset{\sim}{p}),$$

so

$$\mathcal{W} = u \circ FT$$

converts H into an operator of multiplication by the diagonal matrix

$$(\mathcal{W}H\mathcal{W}^{-1})(\underset{\sim}{p}) = \beta\mu(\underset{\sim}{p}).$$

Remark: The transformation

$$U_{FW} = (FT)^{-1} \circ \mathcal{W}$$

is called the <u>Foldy-Wouthuysen transformation</u>. One has

$$U_{FW}H\,U_{FW}^{-1} = \begin{pmatrix} \sqrt{-\Delta+m^2} & 0 \\ 0 & -\sqrt{-\Delta+m^2} \end{pmatrix},$$

where $\sqrt{-\Delta+m^2}$ is the inverse Fourier transform of multiplication by $\mu(\underset{\sim}{p})$.

In the Hilbert space $\mathcal{W}L^2(\underset{\sim}{R}^3;\underset{\sim}{C}^4)$, the two upper components of a wavefunction have positive energy while the two lower components have negative energy. Accordingly, we define the <u>subspace of positive energy</u> $\mathcal{H}_{pos} \subset \mathcal{H}$ as the subspace spanned by the

$$\psi_{pos} \equiv \mathcal{W}^{-1}\,\tfrac{1}{2}(I+\beta)\mathcal{W}\psi \qquad (\psi \in \mathcal{H})$$

and we define the <u>subspace of negative energy</u> $\mathcal{H}_{neg} \subset \mathcal{H}$ as the subspace spanned by the

$$\psi_{neg} \equiv \mathcal{W}^{-1} \tfrac{1}{2} (I-\beta) \mathcal{W} \psi \qquad (\psi \in \mathcal{H}).$$

Obviously,

$$\mathcal{H} = \mathcal{H}_{pos} \oplus \mathcal{H}_{neg} ,$$

the associated orthogonal projections being

$$\begin{cases} P_{pos} = \mathcal{W}^{-1} \tfrac{1}{2} (I+\beta) \mathcal{W} \\ P_{neg} = \mathcal{W}^{-1} \tfrac{1}{2} (I-\beta) \mathcal{W} \end{cases} .$$

But these projections do not determine superselection rules. This is because $\exists$ observables which do not commute with $P_{pos}$ & $P_{neg}$ (see below).

[Note: H is a positive operator on $\mathcal{H}_{pos}$ and a negative operator on $\mathcal{H}_{neg}$.]

Remark: The <u>standard position operator</u> is $\underset{\sim}{x}=(x_1,x_2,x_3)$ (i.e., multiplication by $x_i$, an observable if there ever was one). But $\underset{\sim}{x}$ mixes up the positive and negative energy states in a very complex manner (this effect is the origin of the Zitterbewegung). There is, however, another position operator that leaves $\mathcal{H}_{pos}$ and $\mathcal{H}_{neg}$ invariant, viz. the <u>Newton-Wigner position operator</u>:

$$\underset{\sim}{x}_{NW} = U_{FW}^{-1} \underset{\sim}{x} U_{FW}.$$

Still, it too has its problems.

Let

$$U_t = \underline{\exp}[-\sqrt{-1}\, Ht]$$

and write

$$\psi(t,\underset{\sim}{x}) = (U_t\,\psi)(\underset{\sim}{x}) = \psi_t(\underset{\sim}{x}).$$

---

LEMMA Suppose that $\psi \in \mathcal{S}(R^3;\underset{\sim}{C}^4)$ -- then

$$\psi(t,\underset{\sim}{x}) = \int_{\underset{\sim}{R}^3} K(t,\underset{\sim}{x}-\underset{\sim}{y})\,\psi(\underset{\sim}{y})d^3y \quad (t\neq 0),$$

where

$$K(t,\underset{\sim}{x}) = \left(\frac{\partial}{\partial t} - \underset{\sim}{\alpha}\cdot\nabla - \sqrt{-1}\,\beta m\right)\Delta_m(t,\underset{\sim}{x}).$$

---

Remark: As we know,

$$|t| < |\underset{\sim}{x}| \Rightarrow \Delta_m(t,\underset{\sim}{x}) = 0.$$

So, for fixed $t>0$, $\Delta_m(t,\underset{\sim}{x})$ is supported by $\{\underset{\sim}{x}: |\underset{\sim}{x}| \leq |t|\}$. Consequently, if at time t=0 the support of $\psi$ is contained inside a sphere of radius r, then $\psi_t$ must vanish outside of

$$\{\underset{\sim}{x}: |\underset{\sim}{x}| \leq r\} + \{\underset{\sim}{x}: |\underset{\sim}{x}| \leq t\}$$

$$\subset \{\underset{\sim}{x}: |\underset{\sim}{x}| \leq r+t\}.$$

Since for us c=1, this can be interpreted as saying that $\psi$ propagates at most with the speed of light.

---

THEOREM (Hegerfeldt) Fix a Hilbert space $\mathcal{H}$. Suppose that H is selfadjoint and positive and A is positive -- then for any unit vector $\psi \in \mathcal{H}$, either

$$\langle e^{-\sqrt{-1}Ht}\psi, Ae^{-\sqrt{-1}Ht}\psi\rangle \neq 0$$

for almost all t and the set of such t is open and dense or

$$\langle e^{-\sqrt{-1}Ht}\psi , Ae^{-\sqrt{-1}Ht}\psi \rangle = 0 \ \forall t.$$

---

Remark: This theorem can be used to prove that if $\psi \neq 0$ is in $\begin{cases} \mathcal{H}_{pos} \\ \mathcal{H}_{neg} \end{cases}$, then the support of $\psi$ is all of $\underset{\sim}{R}^3$.

Return now to the Dirac equation:

$$\sqrt{-1}\,\frac{\partial \psi}{\partial t} = \frac{1}{\sqrt{-1}}\left(\alpha_1 \frac{\partial \psi}{\partial x_1} + \alpha_2 \frac{\partial \psi}{\partial x_2} + \alpha_3 \frac{\partial \psi}{\partial x_3}\right) + \beta m \psi .$$

Multiply through by $\beta$ and put $\gamma_0 = \beta$, $\gamma_i = \beta \gamma_i$ (i=1,2,3) ($\Rightarrow \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$) -- then we have

$$\sqrt{-1}\left(\gamma_0 \frac{\partial}{\partial t} + \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} + \gamma_3 \frac{\partial}{\partial x_3}\right)\psi - m\psi = 0$$

or still,

$$\sqrt{-1}\left(\gamma_0 \frac{\partial}{\partial t} + \underset{\sim}{\gamma} \cdot \nabla\right)\psi - m\psi = 0.$$

[Note: If we had worked instead with

$$\alpha_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} , \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} ,$$

then the Dirac equation would be

$$\sqrt{-1}\left(\gamma_0 \frac{\partial}{\partial t} - \underset{\sim}{\gamma} \cdot \nabla\right)\psi - m\psi = 0.]$$

---

**LEMMA** If $\psi$ satisfies the Dirac equation, then $\psi$ satisfies the Klein-Gordon equation.

[In fact,

$$\left(m+\sqrt{-1}\left(\gamma_0 \frac{\partial}{\partial t} + \underset{\sim}{\gamma} \cdot \nabla\right)\right)\left(m-\sqrt{-1}\left(\gamma_0 \frac{\partial}{\partial t} + \underset{\sim}{\gamma} \cdot \nabla\right)\right)$$

$$= (\Box^2 + m^2)I.]$$

---

<u>The Dirac Field</u> This field does not involve an irreducible unitary representation of $\tilde{\mathcal{P}}_+^\uparrow$ but rather a unitary representation of $\tilde{\mathcal{P}}_+^\uparrow$ with two irreducible components.

Agreeing to view a function $f:\underset{\sim}{R}^4 \to \underset{\sim}{C}^4$ as a column vector, $f^T:\underset{\sim}{R}^4 \to \underset{\sim}{C}^4$ is then a row vector. By definition, the <u>adjoint</u> $f^\dagger$ of $f$ is $f^\dagger = \bar{f}^T \gamma_0$, where

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

so

$$f^\dagger = (\bar{f}_3, \bar{f}_4, \bar{f}_1, \bar{f}_2).$$

[Note: This $\gamma_0$ is not the $\gamma_0$ of the previous section.]

Denote now by $\mathcal{H}(m,1/2)$ the Hilbert space consisting of those measurable functions $f:X_m \to \underset{\sim}{C}^4$ for which the integral

$$\langle f,f \rangle = \frac{1}{m} \int_{X_m} f^\dagger(p)\,\gamma(p) f(p)\,d\mu_m(p)$$

is finite. Here

$$\gamma(p) = \gamma_0 p_0 + \underset{\sim}{\gamma} \cdot \underset{\sim}{p} = \gamma_0 p_0 + \sum_\mu \gamma_\mu p_\mu$$

and, as earlier,

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ -\sigma_\mu & 0 \end{pmatrix}.$$

Remark: $\forall\ p \in X_m$, $\gamma_0\,\gamma(p)$ is positive definite, hence $\langle f,f \rangle \geq 0$ and $f=0$ iff $\langle f,f \rangle = 0$. This is seen as follows. Let

$$\begin{cases} \underset{\sim}{p} = p_0\, I + \underset{\sim}{\sigma}\cdot\underset{\sim}{p} \\ \widetilde{p} = p_0\, I - \underset{\sim}{\sigma}\cdot\underset{\sim}{p}\,. \end{cases}$$

Then

$$\begin{cases} \underline{\mathrm{tr}}(\underset{\sim}{p}) = p_0 \\ \underline{\det}(\underset{\sim}{p}) = p_0^2 - |\underset{\sim}{p}|^2\,, \end{cases}$$

so $\underset{\sim}{p}$ is positive definite if $p \in X_m$. On the other hand,

$$\widetilde{p} = \underset{\sim}{q} \quad (q = (p_0, -\underset{\sim}{p}))\,,$$

thus $\widetilde{p}$ is also positive definite if $p \in X_m$. Using this notation, we have

$$\begin{aligned} \gamma_0\,\gamma(p) &= p_0\, I + \sum_{\mu} \gamma_0 \gamma_\mu\, p_\mu \\ &= p_0\, I + \sum_{\mu} \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} p_\mu \\ &= \begin{pmatrix} \widetilde{p} & 0 \\ 0 & \underset{\sim}{p} \end{pmatrix}. \end{aligned}$$

Therefore, $\forall\, p \in X_m$, $\gamma_0\,\gamma(p)$ is positive definitive. Consequently, $\forall\, p \in X_m$,

$$\begin{aligned} & f^{\dagger}(p)\,\gamma(p) f(p) \\ &= \bar{f}^{T}(p)\,\gamma_0\,\gamma(p) f(p) \\ &= \langle f(p), \gamma_0\,\gamma(p) f(p)\rangle \geq 0. \end{aligned}$$

[Note: It is also easy to check that $\overline{\langle f,g\rangle} = \langle g,f\rangle$. Thus let

$$\langle u,v\rangle = \bar{u}^{T}\gamma_0 v \quad (u,v \in \underset{\sim}{C}^4).$$

Then

$$\overline{\langle u,v\rangle} = \langle v,u\rangle$$

and

$$\begin{aligned}
\langle \gamma(p)u,v\rangle &= (\overline{\gamma(p)u})^{T}\gamma_0 v \\
&= \bar{u}^{T}\,\overline{\gamma(p)}^{T}\gamma_0 v \\
&= \bar{u}^{T}\gamma(p)^{*}\gamma_0 v \\
&= \bar{u}^{T}(\gamma_0 p_0 - \sum_{\mu}\gamma_{\mu}p_{\mu})\gamma_0 v \\
&= \bar{u}^{T}\gamma_0\gamma_0(\gamma_0 p_0 - \sum_{\mu}\gamma_{\mu}p_{\mu})\gamma_0 v \\
&= \bar{u}^{T}\gamma_0(\gamma_0 p_0 - \sum_{\mu}\gamma_0\gamma_{\mu}\gamma_0 p_{\mu})v \\
&= \bar{u}^{T}\gamma_0(\gamma_0 p_0 + \sum_{\mu}\gamma_{\mu}p_{\mu})v \\
&= \bar{u}^{T}\gamma_0(\gamma(p)v) \\
&= \langle u, \gamma(p)v\rangle .
\end{aligned}$$

Therefore

$$\begin{aligned}
\overline{\langle f,g\rangle} &= \frac{1}{m}\int_{X_m}\overline{f^{\dagger}(p)\,\gamma(p)g(p)}\,d\mu_m(p) \\
&= \frac{1}{m}\int_{X_m}\overline{\langle f(p),\gamma(p)g(p)\rangle}\,d\mu_m(p) \\
&= \frac{1}{m}\int_{X_m}\overline{\langle \gamma(p)f(p),g(p)\rangle}\,d\mu_m(p) \\
&= \frac{1}{m}\int_{X_m}\langle g(p),\gamma(p)f(p)\rangle\,d\mu_m(p) \\
&= \frac{1}{m}\int_{X_m}g^{\dagger}(p)\,\gamma(p)f(p)\,d\mu_m(p)
\end{aligned}$$

$$= \langle g,f\rangle .]$$

Fact: $\forall$ m > 0, $\exists$ a unitary representation $W^{(m,1/2)}$ of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}(m,1/2)$. Explicitly:

$$(W^{(m,1/2)}(\widetilde{\Lambda},a)f)(p)$$

$$= e^{\sqrt{-1}\langle a,p\rangle} D_{1/2,1/2}(\widetilde{\Lambda})f(\widetilde{\Lambda}^{-1}p),$$

where

$$D_{1/2,1/2} = D^{(1/2,0)} \oplus D^{(0,1/2)},$$

i.e.,

$$D_{1/2,1/2}(\widetilde{\Lambda}) = \begin{pmatrix} \widetilde{\Lambda} & 0 \\ 0 & (\widetilde{\Lambda}^{-1})^* \end{pmatrix} .$$

In this connection, recall that

$$\overline{\widetilde{\Lambda}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\widetilde{\Lambda}^{-1})^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

[Note: To check unitarity, we shall use the relations

$$\begin{cases} D_{1/2,1/2}(\widetilde{\Lambda})\,\gamma(p)D_{1/2,1/2}(\widetilde{\Lambda})^{-1} = \gamma(\Lambda p) \\ D_{1/2,1/2}(\widetilde{\Lambda})^*\,\gamma_0 = \gamma_0\, D_{1/2,1/2}(\widetilde{\Lambda})^{-1} . \end{cases}$$

Thus

$$\langle W^{(m,1/2)}(\widetilde{\Lambda},a)f,\ W^{(m,1/2)}(\widetilde{\Lambda},a)f \rangle$$

$$= \frac{1}{m}\int_{X_m} (W^{(m,1/2)}(\widetilde{\Lambda},a)f)^\dagger(p)\,\gamma(p)W^{(m,1/2)}(\widetilde{\Lambda},a)f(p)\,d\mu_m(p)$$

$$= \frac{1}{m}\int_{X_m} \bar{f}^{\,T}(\widetilde{\Lambda}^{-1}p)D_{1/2,1/2}(\widetilde{\Lambda})^*\,\gamma_0\,\gamma(p)D_{1/2,1/2}(\widetilde{\Lambda})f(\widetilde{\Lambda}^{-1}p)\,d\mu_m(p)$$

$$= \frac{1}{m} \int_{X_m} \bar{f}^{T}(\tilde{\Lambda}^{-1}p)\, \gamma_0 D_{1/2,1/2}(\tilde{\Lambda})^{-1} \gamma(p) D_{1/2,1/2}(\tilde{\Lambda}) f(\tilde{\Lambda}^{-1}p)\, d\mu_m(p)$$

$$= \frac{1}{m} \int_{X_m} \bar{f}^{T}(\tilde{\Lambda}^{-1}p)\, \gamma_0\, \gamma(\tilde{\Lambda}^{-1}p) f(\tilde{\Lambda}^{-1}p)\, d\mu_m(p)$$

$$= \frac{1}{m} \int_{X_m} \bar{f}^{T}(p)\, \gamma_0\, \gamma(p) f(p)\, d\mu_m(p)$$

$$= \frac{1}{m} \int_{X_m} f^{\dagger}(p)\, \gamma(p) f(p)\, d\mu_m(p)$$

$$= \langle f,f \rangle .]$$

¶This representation is not irreducible since there is an orthogonal decomposition

$$\mathcal{H}(m,1/2) = \mathcal{H}(m,1/2,+) \oplus \mathcal{H}(m,1/2,-)$$

into two invariant subspaces on which $\tilde{\mathcal{G}}_{+}^{\uparrow}$ does act irreducibly:

$$W^{(m,1/2)} = U^{(m,1/2,+)} \oplus U^{(m,1/2,-)}.$$

The orthogonal projections

$$\begin{cases} P_m^{+}: \mathcal{H}(m,1/2) \longrightarrow \mathcal{H}(m,1/2,+) \\ \\ P_m^{-}: \mathcal{H}(m,1/2) \longrightarrow \mathcal{H}(m,1/2,-) \end{cases}$$

are

$$\begin{cases} P_m^{+} = \dfrac{m+\gamma}{2m} \\ \\ P_m^{-} = \dfrac{m-\gamma}{2m} \end{cases} .$$

Here it is understood that $\gamma$ stands for multiplication by $\gamma(p)$, an operation which defines a selfadjoint operator having a discrete spectrum with two eigenvalues $\pm m$:

$$\gamma f^{\pm} = \pm m f \qquad (f^{\pm} \in \mathcal{H}(m,1/2,\pm)).$$

[Note: Under Fourier transformation,

$$\sqrt{-1}\,\left(\gamma_0 \frac{\partial}{\partial t} - \underset{\sim}{\gamma}\cdot\nabla\right)$$

goes over to multiplication by $\gamma(p)$, thus the elements of $\mathcal{H}(m,1/2,+)$ are solutions to the Dirac equation.]

Remark: On $\mathcal{H}(m,1/2,+)$, the inner product is

$$\begin{aligned}\langle f^+, g^+\rangle &= \frac{1}{m}\int_{X_m} (f^+)^\dagger(p)\,\gamma(p)g^+(p)\,d\mu_m(p)\\ &= \frac{1}{m}\int_{X_m} (f^+)^\dagger(p)\,(mg^+(p))\,d\mu_m(p)\\ &= \int_{X_m} (f^+)^\dagger(p)g^+(p)\,d\mu_m(p).\end{aligned}$$

Similar comments apply to $\mathcal{H}(m,1/2,-)$ except that the inner product has a minus sign in front of $\int_{X_m}$.

[Note: Take $f^+ = g^+$ and consider

$$(f^+)^\dagger(p)f^+(p)$$

or still,

$$\langle f^+(p),\ \gamma_0 f^+(p)\rangle .$$

To see what this really is, unravel the relation $\gamma(p)f^+(p)=mf^+(p)$ to get

$$\gamma_0 p_0 f^+(p) = mf^+(p) - \sum_{\mu}\gamma_{\mu}p_{\mu}f^+(p)$$

$\Longrightarrow$

$$\langle f^+(p),\ \gamma_0 p_0 f^+(p)\rangle$$

$$= m\,\langle f^+(p), f^+(p)\rangle - \langle f^+(p),\ \sum_\mu \gamma_\mu p_\mu f^+(p)\rangle\ .$$

The matrices $\gamma_1, \gamma_2, \gamma_3$ are skew hermitian, hence

$$\begin{aligned}&\langle f^+(p),\ \sum_\mu \gamma_\mu p_\mu f^+(p)\rangle\\ &= \langle \sum_\mu \gamma_\mu^* p_\mu f^+(p), f^+(p)\rangle\\ &= -\langle \sum_\mu \gamma_\mu p_\mu f^+(p), f^+(p)\rangle\\ &= -\overline{\langle f^+(p),\ \sum_\mu \gamma_\mu p_\mu f^+(p)\rangle}\ ,\end{aligned}$$

so

$$\langle f^+(p),\ \sum_\mu \gamma_\mu p_\mu f^+(p)\rangle$$

is pure imaginary. Since the other expressions are real, we can divide by $p_0$ to obtain

$$\langle f^+(p), \gamma_0 f^+(p)\rangle = \frac{m}{p_0}\ \langle f^+(p), f^+(p)\rangle\ .$$

Therefore

$$\langle f^+, f^+\rangle = m \int_{X_m} \frac{\langle f^+(p), f^+(p)\rangle}{p_0}\, d\mu_m(p).]$$

Remark: Define vector bundles $B(m,1/2,\pm)$ by

$$\{(p,v):\ p \in X_m, v \in \underset{\sim}{C}^4:\ \gamma(p)v = \pm mv\}$$

with projection

$$(p,v) \longrightarrow p.$$

The square integrable sections of $B(m,1/2,\pm)$ are those $f^{\pm}: X_m \longrightarrow \underset{\sim}{C}^4$ such that

$$m \int_{X_m} \frac{\langle f^{\pm}(p), f^{\pm}(p)\rangle}{p_0} d\mu_m(p) < +\infty ,$$

i.e., the elements of $\mathcal{H}(m,1/2,\pm)$. This means that we are dealing with a certain system of imprimitivity which, on general grounds, is equivalent to the one associated with the representation of the stability group $\underset{\sim}{SU}(2)$ of the fiber at $(m,0,0,0)$. On the other hand, $U^{(m,1/2)}$ arises from the system of imprimitivity implicit in the method of the "little group" per $D^{1/2}$. Claim:

$$U^{(m,1/2,\pm)} \equiv W^{(m,1/2)} \mid \mathcal{H}(m,1/2,\pm) \approx U^{(m,1/2)} .$$

To prove this, it need only be shown that the two representations of $\underset{\sim}{SU}(2)$ are equivalent.

(+) In the relation

$$\gamma_0 p_0 v + \sum_{\mu} \gamma_{\mu} p_{\mu} v = v \quad (v \in \underset{\sim}{C}^4),$$

feed in $(m,0,0,0)$ to get $\gamma_0 v = v$, i.e.,

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} v' \\ v'' \end{pmatrix} = \begin{pmatrix} v' \\ v'' \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} v'' \\ v' \end{pmatrix} = \begin{pmatrix} v' \\ v'' \end{pmatrix} .$$

Therefore the fiber at $(m,0,0,0)$ in $B(m,1/2,+)$ is the subspace of all vectors in $\underset{\sim}{C}^4$ of the form $\begin{pmatrix} u \\ u \end{pmatrix}$ $(u \in \underset{\sim}{C}^2)$. But

$$\forall \tilde{\Lambda} \in \underset{\sim}{SU}(2),$$

$$D_{1/2,1/2}(\tilde{\Lambda}) = \begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & (\tilde{\Lambda}^{-1})^* \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix},$$

thus the action on the fiber at (m,0,0,0) is

$$\begin{pmatrix} u \\ u \end{pmatrix} \rightarrow D_{1/2,1/2}(\tilde{\Lambda}) \begin{pmatrix} u \\ u \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{\Lambda} u \\ \tilde{\Lambda} u \end{pmatrix}$$

$$= \begin{pmatrix} D^{1/2}(\tilde{\Lambda})u \\ D^{1/2}(\tilde{\Lambda})u \end{pmatrix},$$

which is equivalent to the usual action of $\underset{\sim}{SU}(2)$ on $\underset{\sim}{C}^2$, i.e., to $D^{1/2}$.

(-) This time $\gamma_0 v = -v$, so the fiber at (m,0,0,0) in B(m,1/2,-) is the subspace of all vectors in $\underset{\sim}{C}^4$ of the form $\begin{pmatrix} u \\ -u \end{pmatrix}$ $(u \in \underset{\sim}{C}^2)$ and one can proceed as above.

Put

$$C = \sqrt{-1}\, \gamma_0 \gamma_2 .$$

Then C is real, $C^T = C^{-1} = -C$, and $C^2 = -I$. Explicitly:

$$C = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \qquad \left( \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right).$$

In addition:

$$\begin{cases} C\,\gamma_0\,C^{-1} = -\gamma_0 \\ C\,\gamma_1\,C^{-1} = \gamma_1 \\ C\,\gamma_2\,C^{-1} = -\gamma_2 \\ C\,\gamma_3\,C^{-1} = \gamma_3 . \end{cases}$$

Definition: The <u>charge conjugation</u> on $\mathcal{H}(m,1/2)$ is the arrow $f \rightarrow f_C$, where

$$f_C = C(f^{\dagger})^T ,$$

i.e.,

$$f_C = C(\bar{f}^{\,T}\gamma_0)^T$$

$$= C\,\gamma_0^T\,\bar{f} = C\,\gamma_0\,\bar{f}.$$

Observation: $(f_C)_C = f$. In fact,

$$(f_C)_C = C\,\gamma_0\,\overline{f_C}$$

$$= C\,\gamma_0\,C\,\gamma_0\,f$$

$$= \sqrt{-1}\,\gamma_0\gamma_2\gamma_0 \cdot \sqrt{-1}\,\gamma_0\gamma_2\gamma_0\,f$$

$$= (\sqrt{-1})^2\,(\gamma_0\gamma_2\gamma_0)^2\,f$$

$$= (\sqrt{-1})^2\,(-\gamma_2)^2\,f$$

$$= (-1)\,\gamma_2^2\,f$$

$$= (-1)\,(-I)\,f$$

$$= f.$$

---

LEMMA The charge conjugation is antiunitary:

$$\langle f_C, g_C \rangle = \overline{\langle f, g \rangle} \ .$$

[The RHS equals

$$\frac{1}{m} \int_{X_m} f^{\top}(p)\, \gamma_0\, \overline{\gamma(p)}\ \overline{g(p)}\ d\mu_m(p).$$

As for the LHS, it equals

$$\frac{1}{m} \int_{X_m} f_C^{\dagger}(p)\, \gamma(p)\ g_C(p)\ d\mu_m(p).$$

But

$$\begin{aligned} f_C^{\dagger} &= (C\,\gamma_0 \bar{f})^{\dagger} \\ &= \overline{(C\,\gamma_0 \bar{f}\,)}^{\top} \gamma_0 \\ &= (C\,\gamma_0 f)^{\top}\ \gamma_0 \\ &= f^{\top} (C\,\gamma_0)^{\top}\ \gamma_0 \\ &= f^{\top}\ \gamma_0^{\top}\ \ C^{\top} \gamma_0 \\ &= f^{\top}\ \gamma_0\ C^{-1} \gamma_0. \end{aligned}$$

We must therefore examine

$$\frac{1}{m} \int_{X_m} f^{\top}(p)\, \gamma_0\ C^{-1}\, \gamma_0\ \gamma(p)\ \ C\, \gamma_0\ \overline{g(p)}\ d\mu_m(p),$$

the claim being that

$$C^{-1}\gamma_0\,\gamma(p)\,C\,\gamma_0 = \overline{\gamma(p)}$$

$$\equiv \gamma_0 p_0 + \gamma_1 p_1 - \gamma_2 p_2 + \gamma_3 p_3.$$

But

(1) $C^{-1}\gamma_0\gamma_0\,C\,\gamma_0 = \gamma_0$;

(2) $C^{-1}\gamma_0\gamma_1\,C\,\gamma_0$

$= C^{-1}\gamma_0\,C\cdot C^{-1}\gamma_1\,C\cdot\gamma_0$

$= -\gamma_0\cdot\gamma_1\cdot\gamma_0$

$= -(\gamma_0\gamma_1\gamma_0) = \gamma_1$;

(3) $C^{-1}\gamma_0\gamma_2\,C\gamma_0$

$= C^{-1}\gamma_0\,C\cdot C^{-1}\gamma_2\,C\cdot\gamma_0$

$= -\gamma_0\cdot -\gamma_2\cdot\gamma_0$

$= \gamma_0\gamma_2\gamma_0 = -\gamma_2$;

(4) $C^{-1}\gamma_0\gamma_3\,C\gamma_0$

$= C^{-1}\gamma_0\,C\cdot C^{-1}\gamma_3\,C\cdot\gamma_0$

$= -\gamma_0\cdot\gamma_3\cdot\gamma_0$

$= -(\gamma_0\gamma_3\gamma_0) = \gamma_3.$

This establishes the claim, from which the lemma.]

---

LEMMA The charge conjugation is quasiintertwining:

$$W^{(m,1/2)}(\tilde{\Lambda},-a)f_C = (W^{(m,1/2)}(\tilde{\Lambda},a)f)_C.$$

[Evaluated at p, the RHS is

$$C\gamma_0\,\overline{e^{\sqrt{-1}\langle a,p\rangle}\,D_{1/2,1/2}(\tilde{\Lambda})f(\tilde{\Lambda}^{-1}p)}$$

$$= C\gamma_0\,e^{-\sqrt{-1}\langle a,p\rangle}\,\overline{D_{1/2,1/2}(\tilde{\Lambda})}\,\overline{f(\tilde{\Lambda}^{-1}p)}$$

and the LHS is

$$e^{-\sqrt{-1}\langle a,p\rangle}\,D_{1/2,1/2}(\tilde{\Lambda})C\gamma_0\,\overline{f(\tilde{\Lambda}^{-1}p)}.$$

The issue is therefore the equality of

$$D_{1/2,1/2}(\tilde{\Lambda})C\gamma_0$$

and

$$C\gamma_0\,\overline{D_{1/2,1/2}(\tilde{\Lambda})}.$$

But

$$C\,D_{1/2,1/2}(\tilde{\Lambda})C^{-1} = D_{1/2,1/2}(\tilde{\Lambda}^{-1})^T$$

or still,

$$C^{-1}\,D_{1/2,1/2}(\tilde{\Lambda})C = D_{1/2,1/2}(\tilde{\Lambda}^{-1})^T,$$

$C^{-1}$ being $-C$. Write

$$D_{1/2,1/2}(\tilde{\Lambda})C\gamma_0$$

$$= C\gamma_0\cdot(C\gamma_0)^{-1}\,D_{1/2,1/2}(\tilde{\Lambda})C\gamma_0$$

$$= C\gamma_0\cdot\gamma_0\,C^{-1}\,D_{1/2,1/2}(\tilde{\Lambda})C\gamma_0$$

$$= C\gamma_0 \cdot \gamma_0 D_{1/2,1/2}(\tilde{\Lambda}^{-1})^T \gamma_0.$$

By definition,

$$D_{1/2,1/2}(\tilde{\Lambda}) = \begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & (\tilde{\Lambda}^{-1})^* \end{pmatrix}$$

$\Rightarrow$

$$D_{1/2,1/2}(\tilde{\Lambda}^{-1})^T = \begin{pmatrix} (\tilde{\Lambda}^{-1})^T & 0 \\ 0 & \bar{\tilde{\Lambda}} \end{pmatrix}$$

$\Rightarrow$

$$\gamma_0 D_{1/2,1/2}(\tilde{\Lambda}^{-1})^T \gamma_0 = \begin{pmatrix} \bar{\tilde{\Lambda}} & 0 \\ 0 & (\tilde{\Lambda}^{-1})^T \end{pmatrix}.$$

It remains only to note that

$$\overline{D_{1/2,1/2}(\tilde{\Lambda})} = \begin{pmatrix} \bar{\tilde{\Lambda}} & 0 \\ 0 & (\tilde{\Lambda}^{-1})^T \end{pmatrix}.]$$

---

Remark: The experts claim that $W^{(m,1/2)}$ commutes with the charge conjugation but, as we have seen above, the experts are wrong. To run a reality check, take $\tilde{\Lambda} = I$ -- then $W^{(m,1/2)}(I,a)$ is multiplication by the character $\chi_a : p \longrightarrow e^{\sqrt{-1}\langle a,p\rangle}$. Can it be that $\chi_a f_C = (\chi_a f)_C$? Well,

$$\chi_a f_C = (\chi_a f)_C$$

$\Rightarrow$

$$(\chi_a f_C)_C = ((\chi_a f)_C)_C = \chi_a f.$$

On the other hand,

$$(\chi_a f_C)_C = C\gamma_0 \overline{\chi_a} \overline{f_C}$$

$$= \overline{\chi}_a \, C\gamma_0 \overline{f_C}$$

$$= \overline{\chi}_a (f_C)_C = \overline{\chi}_a f,$$

hence

$$\overline{\chi}_a f = \chi_a f,$$

an impossibility.

Another point is this: The charge conjugation sends $\mathcal{H}(m,1/2,+)$ to $\mathcal{H}(m,1/2,-)$. For suppose that $\gamma(p)f = mf$ -- then $(\gamma(p)f)_C = mf_C$. And:

$$(\gamma(p)f)_C$$

$$= C\gamma_0 \overline{\gamma(p)f}$$

$$= C\gamma_0[\gamma_0 p_0 + \gamma_1 p_1 - \gamma_2 p_2 + \gamma_3 p_3]\overline{f}$$

$$= C\gamma_0[\gamma_0 p_0 + \gamma_1 p_1 - \gamma_2 p_2 + \gamma_3 p_3](C\gamma_0)^{-1} C\gamma_0 \overline{f}$$

$$= C\gamma_0[\gamma_0 p_0 + \gamma_1 p_1 - \gamma_2 p_2 + \gamma_3 p_3]\gamma_0^{-1} C^{-1} f_C$$

$$= C[\gamma_0 p_0 - \gamma_1 p_1 + \gamma_2 p_2 - \gamma_3 p_3] C^{-1} f_C$$

$$= [-\gamma_0 p_0 - \gamma_1 p_1 - \gamma_2 p_2 - \gamma_3 p_3] f_C$$

$$= -\gamma(p) f_C$$

$$\Rightarrow \quad \gamma(p) f_C = -mf_C.$$

Remark: The charge conjugation is antiunitary, quasi intertwining, and sends $\mathcal{H}(m,1/2,+)$ to $\mathcal{H}(m,1/2,-)$. The restriction of $W^{(m,1/2)}$ to $\mathcal{H}(m,1/2,+)$ is $U^{(m,1/2,+)} \approx U^{(m,1/2)}$. Define now a unitary representation $\check{U}^{(m,1/2,-)}$ of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}(m,1/2,-)$ by

$$(\check{U}^{(m,1/2,-)}(\tilde{\Lambda},a)f)(p)$$

$$= e^{-\sqrt{-1}\langle a,p\rangle} D_{1/2,1/2}(\tilde{\Lambda}) f(\tilde{\Lambda}^{-1}p).$$

Then $\check{U}^{(m,1/2,-)}$ can be identified with the contragredient to $U^{(m,1/2,-)}$ or still, with the contragredient to $U^{(m,1/2)}$. In this connection, it is necessary to keep in mind that $\bar{D}^{1/2} = \check{D}^{1/2} \approx D^{1/2}$ and recall the rules

$$\begin{cases} \check{U} = \bar{U} \\ \overline{U^L} \approx U^{\bar{L}} . \end{cases}$$

Let T be the restriction of the charge conjugation to $\mathcal{H}(m,1/2,+)$ -- then

$$\check{U}^{(m,1/2)} \circ T = T \circ U^{(m,1/2)} ,$$

which is in agreement with the general fact that a unitary representation is always related to its contragredient by an antiunitary intertwining operator.

We shall now associate with the foregoing a vector QFT of type $D_{1/2,1/2}$, taking for $\mathcal{H}$ the antisymmetric Fock space over $\mathcal{H}(m,1/2)$, i.e.,

$$\mathcal{H} = \mathcal{F}_a(\mathcal{H}(m,1/2)).$$

In this situation, an element $\psi \in \mathcal{H}$ is a string $\psi = \{\psi_0, \psi_1, \ldots\}$, where

$$\Psi_n = \Psi_n(p_1, \sigma_1, e_1; \cdots; p_n, \sigma_n, e_n) \quad (\sigma_i = \pm \tfrac{1}{2},\ e_i = \pm)$$

is antisymmetric w.r.t. permutation of triples

$$(p_i, \sigma_i, e_i) \longleftrightarrow (p_j, \sigma_j, e_j).$$

Proceeding per usual, one can then attach to a given $(p, \sigma, e)$ operators

$$\begin{cases} \underset{\sim}{a}(p, \sigma, e) \\ \underset{\sim}{c}(p, \sigma, e). \end{cases}$$

Properties:

(1) $\underset{\sim}{a}(p, \sigma, e)^* = \underset{\sim}{c}(p, \sigma, e)$ & $\underset{\sim}{c}(p, \sigma, e)^* = \underset{\sim}{a}(p, \sigma, e)$;

(2) $\{\underset{\sim}{a}(p, \sigma, e), \underset{\sim}{a}(p', \sigma', e')\} = 0$ & $\{\underset{\sim}{c}(p, \sigma, e), \underset{\sim}{c}(p', \sigma', e')\} = 0$;

(3) $\{\underset{\sim}{a}(p, \sigma, e), \underset{\sim}{c}(p', \sigma', e')\} = \delta(p'-p)\ \delta_{\sigma'\sigma}\ \delta_{e'e}$.

Definition: Let

$$\underset{\sim}{\chi}_{(m,1/2)}(x) = \frac{\sqrt{m}}{(2\pi)^{3/2}} \int_{X_m} [e^{-\sqrt{-1}\langle p,x\rangle} \underset{\sim}{\mu}(p) + e^{\sqrt{-1}\langle p,x\rangle} \underset{\sim}{\nu}(p)]\, d\mu_m(p),$$

where

$$\begin{cases} \underset{\sim}{\mu}(p) = \sum\limits_{\sigma} \underset{\sim}{a}(p, \sigma, +) u(p, \sigma) \\ \underset{\sim}{\nu}(p) = \sum\limits_{\sigma} \underset{\sim}{c}(p, \sigma, -) v(p, \sigma). \end{cases}$$

Here, $u(p, \sigma)$ is the standard plane wave solution to the momentum space Dirac equation:

$$\gamma(p) u(p, \sigma) = m u(p, \sigma),$$

and $v(p,\sigma) = u(p,\sigma)_C$.

[Note: Thus $\underset{\sim}{\chi}_{(m,1/2)}$ has four components $\underset{\sim}{\chi}_k$.]

It is a trivial consequence of the definitions that

$$\{\underset{\sim}{\chi}_k(x), \underset{\sim}{\chi}_{k'}(x')\} = 0.$$

As for

$$\{\underset{\sim}{\chi}_k(x), \underset{\sim}{\chi}_{k'}(x')^*\},$$

it will be more convenient to discuss

$$\{\underset{\sim}{\chi}_k(x), \underset{\sim}{\chi}_{k'}(x')^\dagger\},$$

where

$$\underset{\sim}{\chi}_{(m,1/2)}(x)^\dagger = \underset{\sim}{\chi}_{(m,1/2)}(x)^* \gamma_0$$

$$= \frac{\sqrt{m}}{(2\pi)^{3/2}} \int_{X_m} [e^{\sqrt{-1}\langle p,x\rangle} \underset{\sim}{\mu}(p)^\dagger + e^{-\sqrt{-1}\langle p,x\rangle} \underset{\sim}{\nu}(p)^\dagger] d\mu_m(p)$$

and

$$\begin{cases} \underset{\sim}{\mu}(p)^\dagger = \sum_\sigma \underset{\sim}{c}(p,\sigma,+)\ u(p,\sigma)^\dagger \\ \underset{\sim}{\nu}(p)^\dagger = \sum_\sigma \underset{\sim}{a}(p,\sigma,-)\ v(p,\sigma)^\dagger. \end{cases}$$

Fact:

$$\begin{cases} \sum_\sigma u(p,\sigma)_k\ u(p,\sigma)^\dagger_{k'} = \left(\dfrac{\gamma(p)+m}{2m}\right)_{kk'} \\ \sum_\sigma v(p,\sigma)_k\ v(p,\sigma)^\dagger_{k'} = \left(\dfrac{\gamma(p)-m}{2m}\right)_{kk'} \end{cases}.$$

---

<u>LEMMA</u> We have

$$\{\underset{\sim}{\chi}_k(x), \underset{\sim}{\chi}_{k'}(x')\} = (\sqrt{-1}\partial + m)_{kk'} \frac{1}{\sqrt{-1}} \Delta_m(x-x'),$$

where

$$\partial = \gamma_0 \frac{\partial}{\partial x_0} - \sum_{\mu} \gamma_{\mu} \frac{\partial}{\partial x_{\mu}} \quad .$$

[It suffices to consider the sum of

$$\frac{m}{(2\pi)^3} \int_{X_m} \int_{X_m} e^{-\sqrt{-1}\langle p,x\rangle} e^{\sqrt{-1}\langle p',x'\rangle} \{ \underset{\sim}{\mu}(p)_k, \underset{\sim}{\mu}(p')^{\dagger}_{k'} \} \, d\mu_m(p) d\mu_m(p')$$

and

$$\frac{m}{(2\pi)^3} \int_{X_m} \int_{X_m} e^{\sqrt{-1}\langle p,x\rangle} e^{-\sqrt{-1}\langle p',x'\rangle} \{ \underset{\sim}{\nu}(p)_k, \nu(p')^{\dagger}_{k'} \} \, d\mu_m(p) d\mu_m(p') .$$

From the definitions,

$$\{ \underset{\sim}{\mu}(p)_k, \underset{\sim}{\mu}(p')^{\dagger}_{k'} \}$$

$$= \sum_{\sigma} \sum_{\sigma'} u(p,\sigma)_k \, u(p',\sigma')^{\dagger}_{k'} \, \{ \underset{\sim}{a}(p,\sigma,+), \underset{\sim}{c}(p',\sigma',+) \}$$

$$= \sum_{\sigma} \sum_{\sigma'} u(p,\sigma)_k \, u(p',\sigma')^{\dagger}_{k'} \;\; \delta(p'-p) \, \delta_{\sigma'\sigma}$$

$$= \sum_{\sigma} u(p,\sigma)_k \, u(p',\sigma)^{\dagger}_{k'} \;\; \delta(p'-p) .$$

Integrating w.r.t. p', the first term thus becomes

$$\frac{m}{(2\pi)^3} \int_{X_m} e^{-\sqrt{-1}\langle p,x-x'\rangle} \sum_{\sigma} u(p,\sigma)_k \, u(p,\sigma)^{\dagger}_{k'} \, d\mu_m(p)$$

$$= \frac{m}{(2\pi)^3} \int_{X_m} e^{-\sqrt{-1}\langle p,x-x'\rangle} \left( \frac{\gamma(p)+m}{2m} \right)_{kk'} d\mu_m(p) .$$

Next

$$\{ \underset{\sim}{\nu}(p)_k, \underset{\sim}{\nu}(p')^{\dagger}_{k'} \}$$

$$= \sum_{\sigma} \sum_{\sigma'} v(p,\sigma)_k \, v(p',\sigma')^{\dagger}_{k'} \{ \underset{\sim}{c}(p,\sigma,-), \underset{\sim}{a}(p',\sigma',-) \}$$

$$= \sum_{\sigma} \sum_{\sigma'} v(p,\sigma)_k \, v(p',\sigma')^{\dagger}_{k'} \; \delta(p-p') \; \delta_{\sigma\sigma'}$$

$$= \sum_{\sigma} v(p,\sigma)_k \, v(p',\sigma')^{\dagger}_{k'} \quad \delta(p-p').$$

Integrating w.r.t. p, the second term thus becomes

$$\frac{m}{(2\pi)^3} \int_{X_m} e^{\sqrt{-1}\langle p',x-x'\rangle} \sum_{\sigma} v(p',\sigma)_k \, v(p',\sigma)^{\dagger}_{k'} \, d\mu_m(p')$$

$$= \frac{m}{(2\pi)^3} \int_{X_m} e^{\sqrt{-1}\langle p,x-x'\rangle} \left( \frac{\gamma(p)-m}{2m} \right)_{kk'} d\mu_m(p).$$

We are therefore left with

$$\frac{m}{(2\pi)^3} \int_{X_m} [e^{-\sqrt{-1}\langle p,x-x'\rangle} \left( \frac{\gamma(p)+m}{2m} \right)_{kk'}$$

$$+ e^{\sqrt{-1}\langle p,x-x'\rangle} \left( \frac{\gamma(p)-m}{2m} \right)_{kk'} ] \, d\mu_m(p).$$

But

$$\int_{X_m} e^{-\sqrt{-1}\langle p,x-x'\rangle} \gamma(p) d\mu_m(p)$$

$$= \sqrt{-1}\,\partial \int_{X_m} e^{-\sqrt{-1}\langle p,x-x'\rangle} d\mu_m(p)$$

and

$$\int_{X_m} e^{\sqrt{-1}\langle p,x-x'\rangle}\,\gamma(p)\,d\mu_m(p)$$

$$= \sqrt{-1}\,\partial \int_{X_m} -e^{\sqrt{-1}\langle p,x-x'\rangle}\,d\mu_m(p),$$

so what remains is

$$(\sqrt{-1}\,\partial+m)_{kk'}\ \frac{1}{2(2\pi)^3}\int_{X_m}[e^{-\sqrt{-1}\langle p,x-x'\rangle}-e^{\sqrt{-1}\langle p,x-x'\rangle}]\,d\mu_m(p)$$

$$= (\sqrt{-1}\,\partial+m)_{kk'}\ \frac{1}{\sqrt{-1}}\ \Delta_m(x-x').$$

Hence the assertion.]

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To complete the picture, one has to write down the field map (which is easy) and check that it has the required properties (which is also easy).

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LEMMA We have

$$\sqrt{-1}\,(\gamma_0\frac{\partial}{\partial t}-\underset{\sim}{\gamma}\cdot\nabla)\,\underset{\sim}{\chi}_{(m,1/2)} - m\,\underset{\sim}{\chi}_{(m,1/2)}=0.$$

[This is because

$$\sqrt{-1}\,\partial\int_{X_m} e^{-\sqrt{-1}\langle p,x\rangle}\,\underset{\sim}{u}(p)\,d\mu_m(p)$$

$$= \int_{X_m} e^{-\sqrt{-1}\langle p,x\rangle}\sum_{\sigma}\underset{\sim}{a}(p,\sigma,+)\,\gamma(p)\,u(p,\sigma)\,d\mu_m(p)$$

$$= m \int_{X_m} e^{-\sqrt{-1}\,\langle p,x\rangle}\,\underset{\sim}{\mu}(p)\,d\mu_m(p)$$

and

$$\sqrt{-1}\,\partial \int_{X_m} e^{\sqrt{-1}\,\langle p,x\rangle}\,\underset{\sim}{\nu}(p)\,d\mu_m(p)$$

$$= \int_{X_m} -e^{\sqrt{-1}\,\langle p,x\rangle} \sum_{\sigma} \underset{\sim}{c}(p,\sigma,-)\;\gamma(p)\,v(p,\sigma)\,d\mu_m(p)$$

$$= m \int_{X_m} e^{\sqrt{-1}\,\langle p,x\rangle}\,\underset{\sim}{\nu}(p)\,d\mu_m(p).]$$

<u>Krein Spaces</u> These are pairs $(\mathcal{H}, T)$, where $\mathcal{H}$ is a separable Hilbert space and $T: \mathcal{H} \to \mathcal{H}$ is a unitary involution: $T^2 = I$.

[Note: Let $\mathcal{H}_{\pm} = \{x \in \mathcal{H} : Tx = \pm x\}$ -- then $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and the operators

$$P_{\pm} = \frac{1}{2}(I \pm T)$$

are the orthogonal projections onto $\mathcal{H}_{\pm}$. In the supersymmetric context, T is called a <u>grading operator</u>.]

Put

$$\langle x, y \rangle_T = \langle x, Ty \rangle \quad (x, y \in \mathcal{H}).$$

Definition: A densely defined linear automorphism $U: \mathcal{D} \to \mathcal{D}$ is said to be <u>T-unitary</u> if

$$\langle Ux, Uy \rangle_T = \langle x, y \rangle_T \quad \forall\, x, y \in \mathcal{D}.$$

[Note: In general, U is unbounded. However, U does admit closure. To see this, take $x \in \mathcal{D}$, $y \in T\mathcal{D} : y = Tz$ $(z \in \mathcal{D})$ -- then

$$\begin{aligned}\langle Ux, y \rangle &= \langle Ux, Tz \rangle \\ &= \langle Ux, z \rangle_T \\ &= \langle x, U^{-1}z \rangle_T \\ &= \langle x, TU^{-1}Ty \rangle,\end{aligned}$$

so $U^*$ exists, $TU^{-1}T \subset U^*$, and $T\mathcal{D} \subset \mathrm{Dom}_{U^*}$. In particular: $\underline{\mathrm{Dom}_{U^*}}$ is dense, hence U admits closure. Therefore U is bounded if $\mathcal{D} = \mathcal{H}$ (closed graph theorem).]

Example: Take $\mathcal{H} = L^2(X_0, \mu_0; \underset{\sim}{C}^4)$ and define T by the prescription

$$(Tf)_\mu = \begin{cases} -f_0 & (\mu=0) \\ f_\mu & (\mu=1,2,3). \end{cases}$$

Then T is a unitary involution. Here

$$\langle f,g\rangle_T = -\int_{X_0} \overline{f_0(p)}\, g_0(p)\,d\mu_0(p) + \sum_\mu \int_{X_0} \overline{f_\mu(p)}\, g_\mu(p)\,d\mu_0(p)$$

or still,

$$\langle f,g\rangle_T = \int_{X_0} \sum_{\mu,\nu} \overline{f_\mu(p)}(-g_{\mu\nu})g_\nu(p)\,d\mu_0(p).$$

Define now a representation U of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$ by

$$(U(\Lambda,a)f)_\mu(p) = e^{\sqrt{-1}\,\langle a,p\rangle} \sum_\sigma \Lambda_{\mu\sigma} f_\sigma(\Lambda^{-1}p).$$

Then

$$\begin{aligned}
&\langle U(\Lambda,a)f, U(\Lambda,a)g\rangle_T \\
&= \int_{X_0} \sum_{\mu,\nu} \overline{(U(\Lambda,a)f)_\mu(p)}\,(-g_{\mu\nu})(U(\Lambda,a)g)_\nu(p)\,d\mu_0(p) \\
&= \int_{X_0} \sum_{\mu,\nu} \sum_\sigma \Lambda_{\mu\sigma} \overline{f_\sigma(\Lambda^{-1}p)}\,(-g_{\mu\nu}) \sum_\tau \Lambda_{\nu\tau}\, g_\tau(\Lambda^{-1}p)\,d\mu_0(p) \\
&= \int_{X_0} \sum_{\sigma,\tau} \overline{f_\sigma(p)} \left(\sum_{\mu,\nu} \Lambda_{\mu\sigma}(-g_{\mu\nu})\Lambda_{\nu\tau}\right) g_\tau(p)\,d\mu_0(p) \\
&= \int_{X_0} \sum_{\sigma,\tau} \overline{f_\sigma(p)}\,(-g_{\sigma\tau})g_\tau(p)\,d\mu_0(p)
\end{aligned}$$

$$= \langle f,g \rangle_T,$$

which shows that the $U(\Lambda,a)$ are T-unitary.

[Note: The symmetric Fock space over $L^2(X_0, \mu_0; \underset{\sim}{C}^4)$ is again a Krein space. On the finite particle subspaces, a given $U(\Lambda,0)$ is bounded but this bound depends on n and, e.g., for boosts, tends to $+\infty$ as $n \to +\infty$.]

<u>States</u> According to our original definition, a state on $\mathcal{A}$ is a positive linear functional $\omega$ such that $\omega(I)=1$. We shall now weaken this.

Definition: A <u>state</u> on $\mathcal{A}$ is a linear functional $\omega$ which is continuous and hermitian normalized by $\omega(I)=1$.

[Note: Recall that a positive linear functional is automatically continuous and hermitian.]

Suppose given a state $\omega$ on $\mathcal{A}$. Put

$$N_{\omega} = \{ g : \omega(f \times g)=0 \ \ \forall f \in \mathcal{A} \} .$$

Then $N_{\omega}$ is a closed left ideal in $\mathcal{A}$.

The prescription

$$\langle [f],[g] \rangle = \omega(f^* \times g)$$

is a

--- separately continuous

--- nondegenerate

--- sesquilinear

--- hermitian

form on the quotient $\mathcal{D} = \mathcal{A}/N_{\omega}$.

Define now a linear map $\underset{\sim}{\varphi} : \mathcal{A} \to \underline{\text{End}}\,\mathcal{D}$ by $\underset{\sim}{\varphi}(f)[g]=[f \times g]$ and let $\Omega_0=[1]$ -- then $\mathcal{D}=\{ \underset{\sim}{\varphi}(f)\Omega_0 : f \in \mathcal{A} \}$ and $\langle \Omega_0,\Omega_0 \rangle =1$.

---

<u>LEMMA</u> The arrow $f \to \underset{\sim}{\varphi}(f)$ is a representation of $\mathcal{A}$ by linear operators on $\mathcal{D}$ such that

$$\langle \underset{\sim}{\varphi}(f)[g],[h] \rangle = \langle [g], \underset{\sim}{\varphi}(f^*)[h] \rangle .$$

---

Remark: If $\omega$ is $\mathcal{P}_+^{\uparrow}$-invariant, then $\mathcal{P}_+^{\uparrow}$ can be represented on

$\mathcal{D}$ by writing

$$(\Lambda, a)\cdot[f] = [(\Lambda, a)\cdot f].$$

This action respects $\subset , \supset$ , i.e.,

$$\subset(\Lambda, a)\cdot[f],\ (\Lambda, a)\cdot[g]\supset\ =\ \subset[f],[g]\supset .$$

Hilbert spaces make their appearance in the theory through the following assumption.

HSSC: $\exists$ a continuous Hilbert seminorm $p_{\mathcal{W}}$ on $\mathcal{S}$ such that

$$|\mathcal{W}(f^* \times g)| \leq p_{\mathcal{W}}(f) p_{\mathcal{W}}(g).$$

[Note: A Hilbert seminorm is a seminorm that is derived from an inner product.]

Remark: Obviously, $p_{\mathcal{W}}(g)=0 \Rightarrow g \in N_{\mathcal{W}}$ but conceivably, $\exists\ g \in N_{\mathcal{W}}$ : $p_{\mathcal{W}}(g) \neq 0$. To circumvent this difficulty, introduce a new seminorm $p'_{\mathcal{W}}$ on $\mathcal{S}$ by

$$p'_{\mathcal{W}}(f) = \inf_{g \in N_{\mathcal{W}}} p_{\mathcal{W}}(f+g).$$

Then $\underline{\ker}\ p'_{\mathcal{W}} = N_{\mathcal{W}}$ and one can check that $p'_{\mathcal{W}}$ is a continuous Hilbert seminorm on $\mathcal{S}$ such that

$$|\mathcal{W}(f^* \times g)| \leq p'_{\mathcal{W}}(f) p'_{\mathcal{W}}(g).$$

Accordingly, there is no loss of generality in supposing that $\underline{\ker}\ p_{\mathcal{W}} = N_{\mathcal{W}}$, hence that the associated inner product $\langle\ ,\ \rangle_{\mathcal{W}}$ is positive definite on $\mathcal{D}$.

---

LEMMA If $\exists$ a sequence of continuous Hilbert seminorms $p_n$ on

$\mathcal{S}(\underset{\sim}{R}^{4n})$ such that

$$|\mathcal{W}_{n+m}(f_n^* \times g_m)| \leq p_n(f_n)p_m(g_m)$$

for all $f_n \in \mathcal{S}(\underset{\sim}{R}^{4n})$, $g_m \in \mathcal{S}(\underset{\sim}{R}^{4m})$, then the HSSC obtains.

[Let

$$\langle f,g \rangle = \sum_n (n+1)^2 \langle f_n, g_n \rangle_n .$$

Then

$$|\mathcal{W}(f^* \times g)| = \Big| \sum_{n,m} \mathcal{W}_{n+m}(f_n^* \times g_m) \Big|$$

$$\leq \sum_{n,m} |\mathcal{W}_{n+m}(f_n^* \times g_m)|$$

$$\leq \sum_{n,m} p_n(f_n)p_m(g_m)$$

$$\leq \Big(\sum_n p_n(f_n)\Big) \Big(\sum_m p_m(g_m)\Big)$$

$$\leq \Big(\sum_n (n+1)p_n(f_n) \frac{1}{(n+1)}\Big) \Big(\sum_m (m+1)p_m(g_m) \frac{1}{(m+1)}\Big)$$

$$\leq \Big(\sum_n (n+1)^2 p_n(f_n)^2\Big)^{1/2} \Big(\sum_n \frac{1}{(n+1)^2}\Big)^{1/2}$$

$$\times \Big(\sum_m (m+1)^2 p_m(g_m)^2\Big)^{1/2} \Big(\sum_m \frac{1}{(m+1)^2}\Big)^{1/2}$$

$$\leq C \sqrt{\langle f,f \rangle} \sqrt{\langle g,g \rangle} .]$$

---

Maintaining the assumption that the HSSC is in force, let $\mathcal{H}$ be the completion of $\mathcal{D}$ per the inner product $\langle\, , \rangle_{\mathcal{W}}$ associated with

$p_{\omega}$ -- then the form $\subset,\supset : \mathcal{D} \times \mathcal{D} \to \underset{\sim}{C}$ extends by continuity to a form $\subset,\supset : \mathcal{H} \times \mathcal{H} \to \underset{\sim}{C}$. As such, $\subset,\supset$ is sesquilinear and hermitian (but possibly degenerate). Thanks to the Riesz representation theorem, $\exists$ a bounded linear operator $T: \mathcal{H} \to \mathcal{H}$ characterized by the relation

$$\subset x,y \supset \ = \langle Tx,y\rangle_{\omega} \qquad (x,y \in \mathcal{H}).$$

Observation: T is selfadjoint.

[In fact,

$$\begin{aligned} \langle Tx,y\rangle_{\omega} &= \subset x,y \supset \\ &= \overline{\subset y,x \supset} \\ &= \overline{\langle Ty,x\rangle}_{\omega} = \langle x,Ty\rangle_{\omega} .] \end{aligned}$$

Remark: Matters can always be arranged so as to ensure that T is one-to-one. Thus let $P_T$ be the orthogonal projection $\mathcal{H} \to \mathcal{H}_T \equiv (\underline{\ker}\ T)^{\perp}$. Since $\subset,\supset$ is nondegenerate on $\mathcal{D}$, $\mathcal{D} \cap \underline{\ker}\ T = \{0\}$, hence the arrow of restriction $\begin{cases} \mathcal{D} \to P_T\mathcal{D} \\ x \to P_T x \end{cases}$

has a trivial kernel ($P_T x=0 \Rightarrow (1-P_T)x=x \Rightarrow x \in \underline{\ker}\ T$). The whole setup can then be transferred to $P_T\mathcal{D} \subset \mathcal{H}_T$ ($P_T\mathcal{D}$ is dense in $\mathcal{H}_T$). By construction, $T|\mathcal{H}_T$ is one-to-one ($x \in \mathcal{H}_T$ and $Tx=0 \Rightarrow x \in \mathcal{H}_T^{\perp} \Rightarrow$ $x=0$) (of course T, being selfadjoint, does leave $\mathcal{H}_T$ invariant). Finally, $\forall\ x,y \in \mathcal{H}_T$,

$$\subset x,y \supset \ = \langle Tx,y\rangle_{\omega}$$

and now $\subset , \supset : \mathcal{H}_T \times \mathcal{H}_T \to \underset{\sim}{C}$ is nondegenerate.

To summarize, under HSSC, $\mathcal{D}$ is a pre-Hilbert space with completion $\mathcal{H}$. In addition, $\exists$ an injective bounded selfadjoint operator $T: \mathcal{H} \to \mathcal{H}$ such that

$$\subset x,y \supset = \langle Tx,y \rangle_{\omega} \quad \forall\, x,y \in \mathcal{H} .$$

Example: Consider the field operators $\underset{\sim}{\varphi}(f)$:

$$\subset \underset{\sim}{\varphi}(f)[g],[h] \supset = \subset [g], \underset{\sim}{\varphi}(f^*)[h] \supset$$

$\Rightarrow$

$$\begin{aligned} \langle T \underset{\sim}{\varphi}(f)[g],[h] \rangle_{\omega} &= \langle T[g], \underset{\sim}{\varphi}(f^*)[h] \rangle_{\omega} \\ &= \langle [g], T \underset{\sim}{\varphi}(f^*)[h] \rangle_{\omega} . \end{aligned}$$

Therefore $T\underset{\sim}{\varphi}(f)$ has an adjoint and

$$(T \underset{\sim}{\varphi}(f))^* | \mathcal{D} = T \underset{\sim}{\varphi}(f^*) .$$

Definition: $\omega$ satisfies the <u>Krein condition</u> if under HSSC, T is surjective (hence invertible).

[Note: $T^{-1}$ is symmetric (hence $T^{-1}$ is bounded). Thus write

$$\begin{cases} x = Ta \\ y = Tb \end{cases} \text{ -- then } \begin{cases} \langle T^{-1}x,y \rangle_{\omega} = \langle a,Tb \rangle_{\omega} \\ \langle x,T^{-1}y \rangle_{\omega} = \langle Ta,b \rangle_{\omega} \end{cases} \text{ and } T^{-1} \text{ is symmetric.]}$$

Remark: In the presence of the Krein condition, suppose given two inner products $\begin{cases} \langle\ ,\ \rangle_1 \\ \langle\ ,\ \rangle_2 \end{cases}$ on $\mathcal{H}$ such that $\subset x,y \supset = \begin{cases} \langle T_1 x,y \rangle_1 \\ \langle T_2 x,y \rangle_2 \end{cases}$ .

Then the norms $\begin{matrix} \|\cdot\|_1 \\ \|\cdot\|_2 \end{matrix}$ determine the same topology on $\mathcal{H}$ . Indeed,

$$\langle x,x\rangle_2 = \langle x, T_2T_2^{-1}x\rangle_2$$

$$= [x, T_2^{-1}x]$$

$$= \langle x, T_1T_2^{-1}x\rangle_1$$

$\Rightarrow$

$$\|x\|_2 \leq \|T_1T_2^{-1}\| \cdot \|x\|_1$$

and vice-versa.

---

LEMMA Assuming that the Krein condition is in force, $\exists$ an equivalent inner product on $\mathcal{H}$ in which $T^2=I$.

[Since $|T|$ $(=\sqrt{T^2})$ is strictly positive ($T$ is invertible), the prescription

$$\langle x,y\rangle = \langle x, |T|\, y\rangle_\omega$$

is an inner product on $\mathcal{H}$. It remains only to note that

$$[x,y] = \langle Tx,y\rangle_\omega$$

$$= \langle x, T\,|T|^{-1}\,(|T|y)\rangle_\omega$$

$$= \langle x, \underline{\mathrm{sgn}}\ T\, y\rangle,$$

where as usual, $\underline{\mathrm{sgn}}\ T = T|T|^{-1}$ (of course $(\underline{\mathrm{sgn}}\ T)^2 = T^2 \cdot |T|^{-2} = T^2 \cdot T^{-2} = I$).]

---

We shall assume henceforth that $T^2=I$. The pair $(\mathcal{H}, T)$ is therefore a Krein space. Accordingly, let us write $\langle\ ,\ \rangle$ in place of $\langle\ ,\ \rangle_\omega$ and $\langle\ ,\ \rangle_T$ in place of $[\ ,\ ]$.

Example: Suppose that $\mathcal{W}$ is $\mathcal{P}_+^\uparrow$-invariant. Put $U(\Lambda,a)[f] = (\Lambda,a)\cdot[f]$ -- then

$$\langle U(\Lambda,a)[f],\ U(\Lambda,a)[g]\rangle_T = \langle [f],\ [g]\rangle_T,$$

i.e., the $U(\Lambda,a)$ are T-unitary.

[Note: Bear in mind that the $U(\Lambda,a)$ are unbounded in general.]

There are certain circumstances under which the Krein condition arises naturally. Thus let $\alpha : \mathcal{S} \to \mathcal{S}$ be an automorphism with the following properties:

(1) $\mathcal{W}((\alpha(\alpha(f)))^* \times g) = \mathcal{W}(f^* \times g)$;

(2) $\mathcal{W}(\alpha(f)^* \times f) \geq 0$;

(3) $\mathcal{W}(\alpha(f)^* \times g) = \mathcal{W}(f^* \times \alpha(g))$.

Then

$$\langle f,g\rangle_\alpha = \mathcal{W}(\alpha(f)^* \times g)$$

is an inner product on $\mathcal{S}$, so

$$p_\alpha(f) = \mathcal{W}(\alpha(f)^* \times f)^{1/2}$$

is a Hilbert seminorm on $\mathcal{S}$ with $N_{\mathcal{W}} \subset \underline{\ker}\ p_\alpha$ $(g \in N_{\mathcal{W}} \Longleftrightarrow \mathcal{W}(f \times g) = 0$ $\forall\ f \in \mathcal{S} \Rightarrow \mathcal{W}(\alpha(g)^* \times g) = 0 \Rightarrow p_\alpha(g) = 0)$.

And:

$$|\mathcal{W}(\alpha(f)^* \times g)|^2 \leq \mathcal{W}(\alpha(f)^* \times f)\,\mathcal{W}(\alpha(g)^* \times g)$$

$\Rightarrow$

$$\begin{aligned} &|\mathcal{W}(f^* \times g)|^2 \\ &= |\mathcal{W}((\alpha(\alpha(f)))^* \times g)|^2 \\ &\leq \mathcal{W}((\alpha(\alpha(f)))^* \times \alpha(f))\,\mathcal{W}(\alpha(g)^* \times g) \end{aligned}$$

$$= \omega(f^* \times \alpha(f))\,\omega(\alpha(g)^* \times g)$$

$$= \omega(\alpha(f)^* \times f)\,\omega(\alpha(g)^* \times g)$$

$$= p_\alpha(f)^2\, p_\alpha(g)^2.$$

This shows that the HSSC holds if in addition $p_\alpha$ is continuous, as we suppose. Here, ker $p_\alpha = N_\omega$, so there is no need to pass to $p'_\alpha$.

Denote still by $\langle\ ,\ \rangle_\alpha$ the positive definite inner product on $\mathcal{D}$ associated with $p_\alpha$, $\mathcal{H}$ the corresponding completion.

---

LEMMA $\mathcal{H}$ admits the structure of a Krein space with $T\mathcal{D} = \mathcal{D}$ and $T\Omega_0 = \Omega_0$.

[Define $T: \mathcal{D} \to \mathcal{D}$ by $T[f] = [\alpha(f)]$ -- then T is welldefined and $T^2[f] = [\alpha(\alpha(f))] = [f]$. It is easy to check that

$$\langle T[f], [g]\rangle_\alpha = \langle [f], T[g]\rangle_\alpha$$

and

$$[\![ [f], [g] ]\!] = \langle T[f], [g]\rangle_\alpha .$$

Finally, T is continuous. In fact,

$$[f_n] \to 0 \Rightarrow \|[f_n]\|_\alpha^2 = \omega(\alpha(f_n)^* \times f_n) \to 0$$

$$\Rightarrow$$

$$\|T[f_n]\|_\alpha^2 = \omega((\alpha(\alpha(f_n)))^* \times \alpha(f_n))$$

$$= \omega(f_n^* \times \alpha(f_n))$$

$$= \omega(\alpha(f_n)^* \times f_n) \to 0.]$$

---

Example: Work with the Borchers algebra $\mathcal{S}^4$ generated by

$\mathcal{S}(\underset{\sim}{R}^4;\underset{\sim}{C}^4)$. Put

$$W_{\mu\nu}(x,y) = -\frac{g_{\mu\nu}}{2(2\pi)^3}\int_{X_0} e^{-\sqrt{-1}\langle p,x-y\rangle}\,d\mu_0(p).$$

Define a state $\mathcal{W}$ on $\mathcal{S}^4$ as follows:

$$\mathcal{W}_0 = 1,\ \mathcal{W}_{2n+1} = 0,$$

$$\mathcal{W}_2(f\times g) = \sum_{\mu,\nu}\int_{\underset{\sim}{R}^4}\int_{\underset{\sim}{R}^4} f_\mu(x)g_\nu(y)W_{\mu\nu}(x,y)\,dxdy,$$

$$\mathcal{W}_{2n}(f_1\times\cdots\times f_{2n})$$

$$= \sum_{i,j}\prod_{k=1}^{n}\mathcal{W}_2(f_{i_k}\times f_{j_k}),$$

where the sum is over all partitions of $\{1,\ldots,2n\}$ into n disjoint pairs $(i_1,j_1),\ldots,(i_n,j_n)$ with $i_k < j_k$. Recall now the unitary involution $T:L^2(X_0,\mu_0;\underset{\sim}{C}^4) \rightarrow L^2(X_0,\mu_0;\underset{\sim}{C}^4)$ given by

$$(Tf)_\mu = \begin{cases} -f_0 & (\mu=0) \\ f_\mu & (\mu=1,2,3). \end{cases}$$

Let $\alpha$ be the extension of $T\,|\,\mathcal{S}(\underset{\sim}{R}^4;\underset{\sim}{C}^4)$ to all of $\mathcal{S}^4$ -- then $\alpha$ is an automorphism. Moreover, $\alpha^2=1$,

$$\mathcal{W}_2(\alpha(f)^*\times f) = \sum_\mu 2\pi\int_{X_0} |\hat{f}_\mu(p)|^2\,d\mu_0(p) \geq 0,$$

and

$$\mathcal{W}_2(\alpha(f)^* \times g) = \mathcal{W}_2(f^* \times \alpha(g)).$$

Therefore all the assumptions are satisfied.

[Note: The $\mathcal{W}_n$ are the correlation functions of free QED in the Feynman gauge.]

In the case of gauge theories, not all the Wightman axioms are satisfied. Basically there is a conflict between locality (= micro-causality) and positivity. Examination of specific cases reveals that it is best to keep locality but jettison positivity.

It is not difficult to isolate the essential ingredients. Thus return to our state $\mathcal{W}$, assume that it is $\mathcal{P}_+^\uparrow$-invariant, and impose the Krein condition. Locality is then achieved by supposing that $I_{loc} \subset N_{\mathcal{W}}$. The other assumption is the spectral property, viz. that the support of $\hat{W}_n$ is contained in $\overline{V}_+ \times \cdots \times \overline{V}_+$ (n-1 factors). Here $W_n$ is the tempered distribution on $\underset{\sim}{R}^{4n-4}$ with

$$\mathcal{W}_n(x_1,\ldots,x_n) = W_n(x_1-x_2,\ldots,x_{n-1}-x_n).$$

[Note: The uniqueness of the vacuum is not part of the setup. For example, it might happen that T commutes with the U(I,a), yet $T\Omega_0 \not\in \underset{\sim}{C}\Omega_0$.]

<u>The Gupta-Bleuler Construction</u> As we have seen, the pair $(L^2(X_0,\mu_0;\underset{\sim}{C}^4),T)$ is a Krein space. Here $(Tf)_\mu = \begin{cases} -f_0 & (\mu=0) \\ f_\mu & (\mu=1,2,3) \end{cases}$

and

$$\langle f,g\rangle_T = -\int_{X_0} \overline{f_0(p)}g_0(p)d\mu_0(p)$$

$$+\sum_\mu \int_{X_0} \overline{f_\mu(p)}g_\mu(p)d\mu_0(p).$$

---

LEMMA Fix a $p\in\underset{\sim}{R}^4$ in the light cone : $p_0^2 = p_1^2 + p_2^2 + p_3^2$ $(p_0>0)$. Suppose that $\langle p,a\rangle = 0$ -- then $\langle a,a\rangle \leq 0$.

[Since $\langle \Lambda a, \Lambda a\rangle = \langle a,a\rangle$ $(\Lambda \in \mathcal{L}_+^\uparrow)$, we can assume without loss of generality that $p=(1,0,0,1)$, hence $0 = \langle p,a\rangle = a_0^2 - a_3^2$

$$\Rightarrow \qquad \langle a,a\rangle = a_0^2 - a_1^2 - a_2^2 - a_3^2 = -a_1^2 - a_2^2 \leq 0.]$$

---

It follows from the lemma that

$$\sum_\mu p_\mu f_\mu(p) = 0 \quad \text{a.e} .$$

$$\Rightarrow$$

$$\langle f,f\rangle_T \geq 0.$$

The f which satisfy this auxiliary condition constitute a closed subspace GB of $L^2(X_0,\mu_0;\underset{\sim}{C}^4)$. Denote by $GB_0$ the closed subspace of GB made up of those f for which $\langle f,f\rangle_T = 0$ -- then the completion of the quotient $GB/GB_0$ is a Hilbert space that in the Gupta-Bleuler formalism describes the one-photon states.

To generalize these considerations, take for $\mathcal{H}$ the symmetric Fock space over $L^2(X_0, \mu_0; \underset{\sim}{C}^4)$ and extend T to $\mathcal{H}$ in the obvious way -- then the pair $(\mathcal{H}, T)$ is a Krein space. Our objective now will be to construct a quantum field $\underset{\sim}{A} = \{\underset{\sim}{A}_\mu\}$ which transforms according to the standard representation of $\mathcal{L}_+^\uparrow$ on $\underset{\sim}{C}^4$, i.e.,

$$U(\Lambda, a)\underset{\sim}{A}_\mu(x)U(\Lambda, a)^{-1} = \sum_\nu (\Lambda^{-1})_{\mu\nu} \underset{\sim}{A}_\nu(\Lambda x + a).$$

Physically, $\underset{\sim}{A}$ is a gauge for the free electric-magnetic field but we are no longer dealing with a QFT in the sense of Wightman. Instead, it is the more general framework of the preceding section that is relevant.

An element $\Psi \in \mathcal{H}$ is a string $\Psi = \{\Psi_0, \Psi_1, \ldots\}$, where

$$\Psi_n = \Psi_n(p_1, \mu_1; \cdots; p_n, \mu_n)$$

is symmetric w.r.t. permutations of pairs:

$$(p_i, \mu_i) \longleftrightarrow (p_j, \mu_j).$$

Given $(p, \mu)$, define operators $\begin{cases} \underset{\sim}{a}(p, \mu) \\ \underset{\sim}{c}(p, \mu) \end{cases}$ by

$$(\underset{\sim}{a}(p, \mu)\Psi)_n (p_1, \mu_1; \cdots; p_n, \mu_n)$$

$$= \sqrt{n+1}\, \Psi_{n+1} (p, \mu; p_1, \mu_1; \cdots; p_n, \mu_n)$$

$$(\underset{\sim}{c}(p, \mu)\Psi)_n (p_1, \mu_1; \cdots; p_n, \mu_n)$$

$$= -\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(p - p_j)\, g_{\mu\mu_j}\, \Psi_{n-1} (p_1, \mu_1; \cdots; \widehat{p_j, \mu_j}; \cdots; p_n, \mu_n).$$

Remark: The fact that the metric tensor figures in the definition of the creation operator serves to shift the focus to $\langle\ ,\ \rangle_T$. Thus let $\dagger$ stand for the adjoint per $\langle\ ,\ \rangle_T$ -- then

$$\underset{\sim}{c}(p,\mu) = \underset{\sim}{a}(p,\mu)^{\dagger} = T\,\underset{\sim}{a}(p,\mu)^{*}T.$$

On the other hand,

$$\begin{cases} [\underset{\sim}{a}(p,\mu),\ \underset{\sim}{a}(q,\nu)] = 0 \\ [\underset{\sim}{c}(p,\mu),\ \underset{\sim}{c}(q,\nu)] = 0 \end{cases}$$

and

$$[\underset{\sim}{a}(p,\mu),\ \underset{\sim}{c}(q,\nu)] = \delta(q-p)(-g_{\mu\nu})\cdot I,$$

as to be expected.

Given f, define operators $\begin{cases} \underset{\sim}{a}(f) \\ \underset{\sim}{c}(f) \end{cases}$ by

$$\begin{cases} \underset{\sim}{a}(f) = \sum\limits_{\mu} \int\limits_{X_0} \underset{\sim}{a}(p,\mu)\overline{f(p,\mu)}\,d\mu_0(p) \\ \underset{\sim}{c}(f) = \sum\limits_{\mu} \int\limits_{X_0} \underset{\sim}{c}(p,\mu)f(p,\mu)\,d\mu_0(p). \end{cases}$$

Then

$$[\underset{\sim}{a}(f),\underset{\sim}{c}(g)] = \langle f,g\rangle_T\cdot I.$$

It has been noted earlier that there is a representation U of $\mathcal{P}_{+}^{\uparrow}$ on $L^2(X_0,\mu_0;C^4)$ by T-unitary operators:

$$(U(\Lambda,a)f)_{\mu}(p) = e^{\sqrt{-1}\langle a,p\rangle} \sum_{\sigma} \Lambda_{\mu\sigma} f_{\sigma}(\Lambda^{-1}p).$$

Extend U to a representation of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$ (but omit the cap pi from the notation) -- then the $U(\Lambda,0)$ are, in general, unbounded.

From the definitions,

$$U(\Lambda,0)\underset{\sim}{c}(p,\mu)U(\Lambda,0)^{-1} = \sum_\nu (\Lambda^{-1})_{\mu\nu}\underset{\sim}{c}(\Lambda p,\nu).$$

Here, it is necessary to bear in mind that $\Lambda^{-1} = G\Lambda^T G$ and use the relation

$$\delta(p - \Lambda^{-1}q) = \delta(\Lambda p-q).$$

Therefore (since $U(\Lambda,0)^\dagger = U(\Lambda,0)^{-1}$)

$$U(\Lambda,0)\underset{\sim}{a}(p,\mu)U(\Lambda,0)^{-1} = \sum_\nu (\Lambda^{-1})_{\mu\nu}\underset{\sim}{a}(\Lambda p,\nu).$$

Definition: Let

$$\underset{\sim}{A}_\mu(x) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int_{X_0}[e^{-\sqrt{-1}\langle p,x\rangle}\underset{\sim}{a}(p,\mu)$$
$$+ e^{\sqrt{-1}\langle p,x\rangle}\underset{\sim}{c}(p,\mu)]d\mu_0(p).$$

Properties:

(1) $\langle \underset{\sim}{A}_\mu(x)\phi, \psi\rangle_T = \langle\phi, \underset{\sim}{A}_\mu(x)\psi\rangle_T$;

(2) $T\underset{\sim}{A}_\mu(x)T = -\sum_\nu g_{\mu\nu}\underset{\sim}{A}_\nu(x)$;

(3) $\Box^2\underset{\sim}{A}_\mu(x) = 0.$

---

LEMMA We have

$$U(\Lambda,a)\underset{\sim}{A}_\mu(x)U(\Lambda,a)^{-1} = \sum_\nu (\Lambda^{-1})_{\mu\nu}\underset{\sim}{A}_\nu(\Lambda x+a).$$

---

LEMMA We have

$$[\underset{\sim}{A}_\mu(x), \underset{\sim}{A}_\nu(y)] = -g_{\mu\nu}\cdot\frac{1}{\sqrt{-1}}\Delta_0(x-y).$$

---

Remark: It is not difficult to check that

$$\langle \Omega_0, \underset{\sim}{A}_\mu(x)\underset{\sim}{A}_\nu(y)\Omega_0\rangle_T$$

$$= W_{\mu\nu}(x,y)$$

$$= -\frac{g_{\mu\nu}}{2(2\pi)^3}\int_{X_0} e^{-\sqrt{-1}\langle p, x-y\rangle} d\mu_0(p).$$

The auxiliary condition introduced at the beginning for the elements of $L^2(X_0,\mu_0;\underset{\sim}{C}^4)$ can be extended to the elements of

$$\mathcal{H} = \mathcal{F}_s\left(L^2(X_0,\mu_0;\underset{\sim}{C}^4)\right).$$

Using it, one can define as before closed subspaces $\mathcal{H}_{GB}$ and $\mathcal{H}_{GB_0}$ of $\mathcal{H}$, the completion of the quotient $\mathcal{H}_{GB}/\mathcal{H}_{GB_0}$ then being the physical Hilbert space $\mathcal{H}_{ph}$.

While it is not true that the $\underset{\sim}{A}_\mu(x)$ leave $\mathcal{H}_{GB}$ invariant, the formal combination

$$\underset{\sim}{F}_{\mu\nu}(x) = \partial_\mu \underset{\sim}{A}_\nu(x) - \partial_\nu \underset{\sim}{A}_\mu(x)$$

does. Therefore $\underset{\sim}{F} = \{\underset{\sim}{F}_{\mu\nu}\}$ is a quantum field, the <u>free electric-magnetic field</u>. It transforms according to the rule

$$U(\Lambda,a)\underset{\sim}{F}_{\mu\nu}(x)U(\Lambda,a)^{-1} = \sum_{\sigma,\tau}(\Lambda^{-1})_{\mu\sigma}(\Lambda^{-1})_{\nu\tau}\underset{\sim}{F}_{\sigma\tau}(\Lambda x+a).$$

Moreover, the field components satisfy the (free) Maxwell equations

$$\begin{cases} \partial_\mu \underset{\sim}{F}_{\nu\rho} + \partial_\nu \underset{\sim}{F}_{\rho\mu} + \partial_\rho \underset{\sim}{F}_{\mu\nu} = 0 \\ \sum_\mu \partial_\mu \underset{\sim}{F}^{\mu\nu} = 0. \end{cases}$$

[Note: The second relation is not an operator identity on $\mathcal{H}_{GB}$! Rather, it holds only in the weak sense, i.e.,

$$\sum_{\mu} \partial_{\mu} \langle \phi , \underset{\sim}{F}^{\mu\nu}(x) \psi \rangle_{T} = 0 \quad \forall \phi, \psi \in \mathcal{H}_{GB}.]$$

<u>Nuclear Spaces</u> Let X be a Hausdorff LCTVS. Suppose that $\{p_\alpha\}$ is a directed collection of defining seminorms, i.e., $\forall \alpha, \forall \beta, \exists \gamma$ :

$$\begin{cases} p_\alpha \leq p_\gamma \\ p_\beta \leq p_\gamma \end{cases}.$$ Put $N_\alpha = \underline{\ker}\, p_\alpha$, $X_\alpha = X/N_\alpha$.

Assume: The $p_\alpha$ are Hilbert seminorms.

Definition: X is <u>nuclear</u> if $\forall \alpha\ \exists \beta: \alpha \leq \beta$ and $f_{\alpha,\beta}: \overline{X}_\beta \to \overline{X}_\alpha$ is Hilbert-Schmidt.

Example: $\mathcal{S}(\underset{\sim}{R}^n)$ is nuclear, as is its dual $\mathcal{S}'(\underset{\sim}{R}^n)$.

Notation: If X and Y are nuclear, then $X\hat{\otimes}Y$ is their completed tensor product (hence is universal w.r.t. continuous bilinear maps or still, is universal w.r.t. separately continuous bilinear maps).

Fact: X,Y nuclear $\Rightarrow X\hat{\otimes}Y$ nuclear.

Example: $\mathcal{S}(\underset{\sim}{R}^n) \hat{\otimes} \mathcal{S}(\underset{\sim}{R}^m) \approx \mathcal{S}(\underset{\sim}{R}^{n+m})$ and $\mathcal{S}'(\underset{\sim}{R}^n) \hat{\otimes} \mathcal{S}'(\underset{\sim}{R}^m) \approx \mathcal{S}'(\underset{\sim}{R}^{n+m})$ (Schwartz kernel theorem).

Henceforth we shall assume that X is a nuclear Fréchet space equipped with a continuous involution $*$.

Definition: The <u>Borchers algebra</u> $\mathfrak{A}_X$ attached to X is

$$\bigoplus_0^\infty (\hat{\otimes}^n X).$$

[Note: Therefore $\mathfrak{A}_X$ is a nuclear LF-space.]

Let $\mathcal{U}$ be a Hausdorff LCTVS, $\subset , \supset$ a form on $\mathcal{U}$ which we take to be

--- separately continuous

--- nondegenerate

--- sesquilinear

--- hermitian.

Definition: A <u>representation</u> of $\mathfrak{A}_X$ on $\mathcal{V}$ is a homomorphism $\pi : \mathfrak{A}_X \to \underline{\text{End}}\, \mathcal{V}$ such that

$$\langle v_1, \pi(a)v_2 \rangle = \langle \pi_1(a^*)v_1, v_2 \rangle$$

and for which the arrow

$$\begin{cases} \mathfrak{A}_X \times \mathcal{V} \to \mathcal{V} \\ (a,v) \to \pi(a)v \end{cases}$$

is separately continuous.

[Note: Representations $\begin{cases} \pi_1 \\ \pi_2 \end{cases}$ of $\mathfrak{A}_X$ on $\begin{cases} \mathcal{V}_1 \\ \mathcal{V}_2 \end{cases}$ are <u>equivalent</u> if $\exists$ a form preserving topological isomorphism $W: \mathcal{V}_1 \to \mathcal{V}_2$ that intertwines $\pi_1$ & $\pi_2$.]

Definition: A <u>state</u> on $\mathfrak{A}_X$ is a linear functional $\omega$ which is continuous and hermitian normalized by $\omega(I)=1$.

Suppose given a state $\omega$ on $\mathfrak{A}_X$. Put

$$N_\omega = \{ y: \omega(xy)=0 \ \forall\, x \in \mathfrak{A}_X \}.$$

Then $N_\omega$ is a closed left ideal in $\mathfrak{A}_X$.

The quotient

$$\mathcal{D}_X = \mathfrak{A}_X / N_\omega$$

is nuclear and the prescription

$$\langle [x], [y] \rangle = \omega(x^*y)$$

is a form on $\mathcal{D}_X$ possessing the properties enumerated above.

One can then represent $\mathfrak{A}_X$ on $\mathcal{D}_X$ in the obvious way: $\pi(x)[y]=[xy]$. And, with $\Omega_0=[1]$, $\mathcal{D}_X=\{\pi(x)\Omega_0 : x\in\mathfrak{A}_X\}$.

[Note: The relation

$$\mathcal{W}(x) = \langle\Omega_0, \pi(x)\Omega_0\rangle$$

connects $\mathcal{W}$ and $\pi$.]

Remark: The arrow

$$\text{state} \longrightarrow \text{representation}$$

is called the <u>GNS construction</u>.

<u>The Bongaarts Construction</u> The homogeneous Maxwell equation

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

is equivalent to the existence of functions $A_\mu$ from which the $F_{\mu\nu}$ can be obtained by differentiation:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

The field $\{F_{\mu\nu}\}$ does not determine the potential $\{A_\mu\}$ uniquely but rather only up to a gauge transformation. In our setting, the situation is similar. Thus roughly speaking, each QFT for the field tensor, say $\{\mathcal{H}_F, \Omega_F, F_{\mu\nu}\}$, gives rise to a set of triples $\{\mathcal{H}_A, \Omega_A, A_\mu\}$, any one such being termed a gauge for $\{\mathcal{H}_F, \Omega_F, F_{\mu\nu}\}$. However, $\mathcal{H}_F$ and $\mathcal{H}_A$ are different spaces, so the formula $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ makes no sense as an operator relation. This issue (and others) can be clarified by invoking the theory of Borchers algebras and their states.

Let $X_F$ be the subspace of $\mathcal{S}(\underset{\sim}{R}^4; \underset{\sim}{C}^4 \otimes \underset{\sim}{C}^4)$ consisting of those f whose components $f^{\mu\nu}$ are antisymmetric -- then $X_F$ is a nuclear Fréchet space and we shall write $\mathfrak{A}_F = \mathfrak{A}_{X_F}$. Denote by $X_F^0$ the linear subspace of $X_F$ whose elements are those f such that $f^{\mu\nu} = \partial_\rho \psi^{\mu\nu\rho}$ ($\psi^{\mu\nu\rho}$ antisymmetric and rapidly decreasing). Let $I_F^0$ be the closed $*$-ideal generated by $X_F^0$ in $\mathfrak{A}_F$, that is, in each $\hat{\otimes}^n X_F$, form the closed linear span of the tensor products

$$\begin{cases} X_F^0 \otimes X_F \otimes \cdots \otimes X_F \\ \\ X_F \otimes X_F \otimes \cdots \otimes X_F^0 \end{cases}$$

and then take their direct sum.

Definition: An <u>F-theory</u> is a positive state in $\mathfrak{A}_F'$ which annihilates $I_F^0$.

Let $\omega_F$ be an F-theory -- then by definition

$$\omega_F(x) = 0 \quad \forall x \in I_F^0.$$

Moreover, via the GNS construction, $\omega_F$ determines a nuclear space $\mathcal{D}_F$ (which is also a pre-Hilbert space), a cyclic unit vector $\Omega_F$, and a map $\underset{\sim}{\varphi}_F: \mathfrak{A}_F \to \underline{\text{End}}\, \mathcal{D}_F$.

[Note: At the moment, we do not require that $\omega_F$ possess any additional property like, e.g., $\mathcal{P}_+^\uparrow$-invariance.]

Let $X_A$ be the space $\mathcal{S}(\underset{\sim}{R}^4;\underset{\sim}{C}^4)$ -- then $X_A$ is a nuclear Fréchet space and we shall write $\mathfrak{A}_A = \mathfrak{A}_{X_A}$. Denote by $X_A^0$ the linear subspace of $X_A$ whose elements are those f such that $f^\mu = \partial_\nu \psi^{\mu\nu}$ ( $\psi^{\mu\nu}$ antisymmetric and rapidly decreasing). Let $\mathfrak{A}_A^{ph}$ be the closed subalgebra of $\mathfrak{A}_A$ generated by $X_A^0$.

Definition: An <u>A-theory</u> is a state in $\mathfrak{A}_A'$.

Let $\omega_A$ be an A-theory -- then, via the GNS construction, $\omega_A$ determines a nuclear space $\mathcal{D}_A$, a cyclic unit vector $\Omega_A$, and a map $\underset{\sim}{\varphi}_A: \mathfrak{A}_A \to \underline{\text{End}}\, \mathcal{D}_A$.

[Note: No positivity requirement has been imposed on $\omega_A$, thus the pair ( $\mathcal{D}_A$, $\langle\ ,\ \rangle$ ) is not necessarily a pre-Hilbert space.]

Fact: We have

$$\begin{cases} X_F^0 = \{f: \partial_\mu f^{\mu\nu} = 0\} \\ X_A^0 = \{f: \partial_\mu f^\mu = 0\}. \end{cases}$$

Define now a continuous linear map $d: X_F \longrightarrow X_A$ by the prescription

$$f^{\mu\nu} \longrightarrow 2\partial_\nu f^{\mu\nu}$$

POINCARÉ LEMMA The kernel of d is $X_F^0$ and the image of d is $X_A^0$.

Extend d to a continuous $*$-homomorphism $\Theta_d: \mathfrak{A}_F \longrightarrow \mathfrak{A}_A$, hence

$$\Theta_d = \bigoplus_0^\infty (\hat{\otimes}^n d).$$

LEMMA The kernel of $\Theta_d$ is $I_F^0$ and the image of $\Theta_d$ is $\mathfrak{A}_A^{ph}$.

The transpose $\Theta_d'$ sends $\mathfrak{A}_A'$ to $\mathfrak{A}_F'$. Its kernel is the annihilator of $\mathfrak{A}_A^{ph}$ and its image is the annihilator of $I_F^0$.

Let $\omega_F$ be an F-theory -- then the fiber

$$(\Theta_d')^{-1}(\omega_F)$$

is not empty. Suppose, therefore, that

$$\omega_F = \Theta_d'(\omega_A) \equiv \omega_A \circ \Theta_d.$$

Question: What can be said about $\omega_A$?

First of all, $\omega_A$ is necessarily normalized:

$$\begin{aligned}\omega_A(I) &= \omega_A \circ \Theta_d(I)\\ &= \omega_F(I) = 1.\end{aligned}$$

Next, $\omega_F$ is positive, hence hermitian. While $\omega_A$ need not be hermitian, one can get around this by considering instead

$$\frac{1}{2}\,[\mathcal{W}_A(x) + \overline{\mathcal{W}_A(x^*)}].$$

In fact,

$$\begin{aligned} &cc \circ \mathcal{W}_A \circ * \circ \Theta_d \\ &= cc \circ \mathcal{W}_A \circ \Theta_d \circ * \\ &= cc \circ \mathcal{W}_F \circ * \\ &= \mathcal{W}_F. \end{aligned}$$

The situation as regards positivity is not so simple: There is no guarantee that $\mathcal{W}_A$ can be chosen positive (ditto for $\mathcal{P}_+^\uparrow$-invariance).

In what follows, it will be assumed that $\mathcal{W}_A$ is hermitian, thus is an A-theory. We then have

$$\begin{cases} \mathcal{W}_F \longrightarrow \{\mathcal{D}_F, \Omega_F, \underset{\sim}{\varphi}_F\} \\ \mathcal{W}_A \longrightarrow \{\mathcal{D}_A, \Omega_A, \underset{\sim}{\varphi}_A\} \end{cases}.$$

Definition: The triple $\{\mathcal{D}_A, \Omega_A, \underset{\sim}{\varphi}_A\}$ is a gauge for $\{\mathcal{D}_F, \Omega_F, \underset{\sim}{\varphi}_F\}$.

[Note: It is also said that an A-theory in the fiber $(\Theta_d^!)^{-1}(\mathcal{W}_F)$ is a gauge for $\mathcal{W}_F$.]

Put

$$\mathcal{D}_A^{ph} = \{\underset{\sim}{\varphi}_A(x)\Omega_A : x \in \mathfrak{A}_A^{ph}\}.$$

---

LEMMA The assignment

$$\underset{\sim}{\varphi}_A(\Theta_d x)\Omega_A \longrightarrow \underset{\sim}{\varphi}_F(x)\Omega_F \qquad (x \in \mathfrak{A}_F)$$

defines a linear isometric map W from $\mathcal{D}_A^{ph}$ onto $\mathcal{D}_F$ such that $W\Omega_A = \Omega_F$.

[To check that W is well defined, one has to show that

$$\underset{\sim}{\varphi}_A(\Theta_d x)\Omega_A = 0 \Rightarrow \underset{\sim}{\varphi}_F(x)\Omega_F = 0.$$

To see this, note first that $\forall\, x,y \in \mathfrak{A}_F$,

$$\begin{aligned} &\langle \underset{\sim}{\varphi}_F(x)\Omega_F, \underset{\sim}{\varphi}_F(y)\Omega_F \rangle \\ &= \omega_F(x^*y) \\ &= \omega_A \circ \Theta_d(x^*y) \\ &= \omega_A(\Theta_d(x^*)\Theta_d(y)) \\ &= \omega_A(\Theta_d(x)^*\Theta_d(y)) \\ &= \langle \underset{\sim}{\varphi}_A(\Theta_d x)\Omega_A, \underset{\sim}{\varphi}_A(\Theta_d y)\Omega_B \rangle . \end{aligned}$$

So

$$\begin{aligned} &\underset{\sim}{\varphi}_A(\Theta_d x)\Omega_A = 0 \\ \Rightarrow\ &\langle \underset{\sim}{\varphi}_F(x)\Omega_F, \underset{\sim}{\varphi}_F(y)\Omega_F \rangle = 0 \quad \forall\, y \in \mathfrak{A}_F, \end{aligned}$$

thus, by nondegeneracy,

$$\underset{\sim}{\varphi}_F(x)\Omega_F = 0.$$

That W is isometric is, of course, obvious.]

---

While the lemma implies that $\mathcal{D}_A^{ph}$ is an inner product space, there may still be elements $x \in \mathcal{D}_A^{ph}$ of zero length: $\langle x,x \rangle = 0$.

But $\subset x,x \supset = 0 \Rightarrow \subset Wx, Wx \supset = 0 \Rightarrow x \in \underline{\ker}\ W$. Consequently, the quotient $\mathcal{D}_A^{ph}/\underline{\ker}\ W\ (\approx \mathcal{D}_F)$ is a pre-Hilbert space.

Observation: Let $x \in \mathcal{D}_A^{ph}$ -- then $\subset x,x \supset = 0$ iff $\subset x,y \supset = 0$ $\forall\, y \in \mathcal{D}_A^{ph}$.

[Apply the Schwartz inequality

$$|\subset x,y \supset| \leq \sqrt{\subset x,x \supset}\ \sqrt{\subset y,y \supset}$$

to get the nontrivial implication.]

In practice, it is sometimes possible to choose $\omega_A$:

$$\mathcal{D}_A^{ph} = \mathcal{D}_A.$$

Since $\subset\ ,\ \supset$ is nondegenerate, $\subset x,x \supset = 0 \Rightarrow x = 0 \Rightarrow \underline{\ker}\ W = 0 \Rightarrow$ $\mathcal{D}_A \approx \mathcal{D}_F$. Therefore, in this situation, one can realize the field operators $\underset{\sim}{\varphi}_A$ and $\underset{\sim}{\varphi}_F$ on the same pre-Hilbert space. Accordingly, $\forall\ f \in X_F$,

$$\begin{aligned}
\underset{\sim}{F}(f) &= \int_{\underset{\sim}{R}^4} f^{\mu\nu}(x)\ \underset{\sim}{F}_{\mu\nu}(x)\,dx \\
&= \underset{\sim}{A}(df) \\
&= \int_{\underset{\sim}{R}^4} 2\,\partial_\nu f^{\mu\nu}(x)\underset{\sim}{A}_\mu(x)\,dx \\
&= \int_{\underset{\sim}{R}^4} \partial_\nu f^{\mu\nu}(x)\underset{\sim}{A}_\mu(x)\,dx \\
&\quad + \int_{\underset{\sim}{R}^4} \partial_\nu f^{\mu\nu}(x)\underset{\sim}{A}_\mu(x)\,dx
\end{aligned}$$

$$= \int_{\underset{\sim}{R}^4} \partial_\nu f^{\mu\nu}(x)\underset{\sim}{A}_\mu(x)dx$$

$$+ \int_{\underset{\sim}{R}^4} \partial_\mu f^{\nu\mu}(x)\underset{\sim}{A}_\nu(x)dx$$

$$= - \int_{\underset{\sim}{R}^4} f^{\mu\nu}(x)\,\partial_\nu\underset{\sim}{A}_\mu(x)dx$$

$$- \int_{\underset{\sim}{R}^4} f^{\nu\mu}(x)\,\partial_\mu\underset{\sim}{A}_\nu(x)dx$$

$$= \int_{\underset{\sim}{R}^4} f^{\mu\nu}(x)\,[\partial_\mu\underset{\sim}{A}_\nu(x) - \partial_\nu\underset{\sim}{A}_\mu(x)]dx$$

$\Longrightarrow$

$$\underset{\sim}{F}_{\mu\nu}(x) = \partial_\mu\underset{\sim}{A}_\nu(x) - \partial_\nu\underset{\sim}{A}_\mu(x).$$

[Note: The Coulomb ( = radiation) gauge for the free electric-magnetic field is an example of this setup. But there is a price to be paid: The relation

$$U(\Lambda,0)\underset{\sim}{A}_\mu(x)U(\Lambda,0)^{-1} = \sum_\nu (\Lambda^{-1})_{\mu\nu}\underset{\sim}{A}_\nu(\Lambda x)$$

fails to hold.]

<u>The Free Electric-Magnetic Field</u> In the previous section, we introduced the nuclear space $X_F$ and its associated Borchers algebra $\mathfrak{A}_F$. We shall now consider a particular F-theory $\mathcal{W}_F$ and its collection of gauges $\mathcal{W}_A$:

$$\mathcal{W}_F = \mathcal{W}_A \circ \Theta_d .$$

[Note: Recall that $\mathcal{W}_A$ is necessarily an A-theory, i.e., is a state on $\mathfrak{A}_A$. Therefore $\mathcal{W}_A$ is continuous, hermitian, and $\mathcal{W}_A(I)=1$.]

Notationally, it will be convenient to use superscripts rather than subscripts, i.e., replace $\mathcal{W}_F$, $\mathcal{W}_A$ by $\mathcal{W}^F$, $\mathcal{W}^A$.

Definition: The 2-point function of the free electric-magnetic field is

$$\mathcal{W}^F_{\mu_1 \nu_1 ; \mu_2 \nu_2}(x_1, x_2)$$

$$= \varepsilon^{\sigma_1 \tau_1}_{\mu_1 \nu_1} \varepsilon^{\sigma_2 \tau_2}_{\mu_2 \nu_2} \partial^1_{\sigma_1} \partial^2_{\sigma_2} \left( - g_{\tau_1 \tau_2} \frac{1}{\sqrt{-1}} \Delta_+(x_1 - x_2 ; 0) \right).$$

Explanations:

(1) $\varepsilon^{\sigma\tau}_{\mu\nu} = 0$ unless $\sigma, \tau$ is a permutation of $\mu, \nu$ and is then equal to $\pm 1$ according to the sign of the permutation.

(2) $\partial^j_\sigma = \partial / \partial x^\sigma_j$ ($\sigma=0,1,2,3$ & $j=1,2$) ($x_j \in \underset{\sim}{R}^4 \Rightarrow x_j = (x^0_j, x^1_j, x^2_j, x^3_j)$).

(3) $\Delta_+(a;0) = \dfrac{\sqrt{-1}}{2(2\pi)^3} \displaystyle\int_{X_0} e^{-\sqrt{-1}\langle p,a \rangle} d\mu_0(p)$.

Remark: Accordingly, $\forall f_1, f_2 \in X_F$,

$$\langle \mathcal{W}^F_2, f_1 \times f_2 \rangle$$

$$= \int_{\underset{\sim}{R}^4} \int_{\underset{\sim}{R}^4} f_1^{\mu_1 \nu_1}(x_1) f_2^{\mu_2 \nu_2}(x_2)\, \mathcal{W}^F_{\mu_1 \nu_1; \mu_2 \nu_2}(x_1, x_2)\, dx_1 dx_2 .$$

Example: Let

$$W_{\mu\nu}(x_1, x_2) = -\frac{g_{\mu\nu}}{2(2\pi)^3} \int_{X_0} e^{-\sqrt{-1}\langle p, x_1 - x_2 \rangle} d\mu_0(p) .$$

Then

$$\mathcal{W}^F_{\mu_1 \nu_1; \mu_2 \nu_2}$$

$$= \partial^1_{\mu_1} \partial^2_{\mu_2} W_{\nu_1 \nu_2} - \partial^1_{\mu_1} \partial^2_{\nu_2} W_{\nu_1 \mu_2}$$

$$- \partial^1_{\nu_1} \partial^2_{\mu_2} W_{\mu_1 \nu_2} + \partial^1_{\nu_1} \partial^2_{\nu_2} W_{\mu_1 \mu_2} .$$

Remark: While $W_{\mu\nu}$ is not unique, any other possibility has the form

$$W_{\mu\nu}(x_1, x_2) + \partial^1_\mu \phi^{(1)}_\nu (x_1, x_2) + \partial^2_\nu \phi^{(2)}_\mu (x_1, x_2),$$

where

$$\phi^{(1)}_\nu \;\&\; \phi^{(2)}_\mu$$

are tempered.

To complete the definition of $\mathcal{W}^F$, let

$$\mathcal{W}^F_0 = 1, \quad \mathcal{W}^F_{\mu_1 \nu_1; \cdots; \mu_{2n+1} \nu_{2n+1}} = 0$$

and

$$\mathcal{W}^F_{\mu_1 \nu_1; \cdots; \mu_{2n} \nu_{2n}} (x_1, \ldots, x_{2n})$$

$$= \sum \mathcal{W}^F_{\mu_{j_1}\nu_{j_1};\mu_{j_2}\nu_{j_2}}(x_{j_1},x_{j_2}) \cdots$$

$$\mathcal{W}^F_{\mu_{j_{2n-1}}\nu_{j_{2n-1}};\mu_{j_{2n}}\nu_{j_{2n}}}(x_{j_{2n-1}},x_{j_{2n}}).$$

Here the sum is over all permutations $j_1,\ldots,j_{2n}$ of $1,\ldots,2n$ with $j_1 < j_3 < \cdots < j_{2n-1}$ and $j_1 < j_2,\ldots,j_{2n-1} < j_{2n}$.

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LEMMA $\mathcal{W}^F$ is an F-theory.

[That $\mathcal{W}^F$ annihilates $I^0_F$ is more or less implicit in the definitions (details omitted). Let's check positivity. For this, it suffices to look at $\mathcal{W}^F_2$. Passing to Fourier transforms, we have

$$\langle \mathcal{W}^F_2, f_1 \times f_2 \rangle$$

$$= -8\pi \int_{X_0} g_{\mu_1\mu_2} p_{\nu_1} \hat{f}_1^{\mu_1\nu_1}(-p)\, p_{\nu_2} \hat{f}_2^{\mu_2\nu_2}(p)\, d\mu_0(p).$$

Now replace $f_1$ by $f_1^*$ $(= \overline{f_1})$ to get:

$$\langle \mathcal{W}^F_2, f_1^* \times f_2 \rangle$$

$$= 8\pi \int_{X_0} \sum_{j,k=1}^{3} M_{jk}(p)\, \overline{\phi_1^j(p)}\, \phi_2^k(p)\, d\mu_0(p),$$

where

$$\begin{cases} \phi_1^j(p) = p_\nu \, \hat{f}_1^{j\nu}(p) \\ \phi_2^k(p) = p_\nu \, \hat{f}_2^{j\nu}(p) \end{cases}$$

and

$$M_{jk}(p) = \delta_{jk} - \frac{p^j p^k}{p_0^2} \quad (p_0^2 = |\underset{\sim}{p}|^2).$$

The matrix $M_{jk}$ is obviously hermitian and using the fact that $p_0^2 = |\underset{\sim}{p}|^2$, one can check that it is idempotent, hence positive.]

Definition: The free electric-magnetic field $\underset{\sim}{F} = \{F_{\mu\nu}\}$ is the QFT determined by $\mathcal{W}^F$.

This field has all the usual properties, e.g., locality. It transforms according to the rule

$$U(\Lambda,a)\underset{\sim}{F}_{\mu\nu}(x)U(\Lambda,a)^{-1} = \sum_{\sigma,\tau} (\Lambda^{-1})_{\mu\sigma}(\Lambda^{-1})_{\nu\tau}\underset{\sim}{F}_{\sigma\tau}(\Lambda x+a)$$

and the Maxwell equations

$$\begin{cases} \partial_\mu \underset{\sim}{F}_{\nu\rho} + \partial_\nu \underset{\sim}{F}_{\rho\mu} + \partial_\rho \underset{\sim}{F}_{\mu\nu} = 0 \\ \sum_\mu \partial_\mu \underset{\sim}{F}^{\mu\nu} = 0 \end{cases}$$

obtain. Finally,

$$\mathcal{W}^F_{\mu_1\nu_1;\ldots;\mu_n\nu_n}(x_1,\ldots,x_n)$$

$$= \langle \Omega_F, \underset{\sim}{F}_{\mu_1\nu_1}(x_1)\cdots \underset{\sim}{F}_{\mu_n\nu_n}(x_n)\,\Omega_F\rangle .$$

Remark: There have been many investigations of the gauges $\mathcal{W}^A$ associated with $\mathcal{W}^F$. One important point is the fact that if $\mathcal{W}^A$ is $\mathcal{L}_+^\uparrow$-invariant, then the form $\subset , \supset$ cannot be positive, i.e., the pair $(\mathcal{D}_A, \subset,\supset)$ is not a pre-Hilbert space.