LAGRANGIAN MECHANICS

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CONTENTS

INTRODUCTION

My original set of lectures on Mechanics was divided into three parts:

Lagrangian Mechanics

Hamiltonian Mechanics

Equivariant Mechanics .

The present text is an order of magnitude expansion of the first part and is differential geometric in character, the arena being the tangent bundle rather than the cotangent bundle. I have covered what I think are the basics. Points of detail are not swept under the rug but I have made an effort not to get bogged down in minutiae. Numerous examples have also been included.

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§1. FLOWS

Let M be a connected C^∞ manifold of dimension n. Fix a vector field X on on M -- then the image of a maximal integral curve of X is called a trajectory of X. The trajectories of X are connected, immersed submanifolds of M. They form a partition of M and their dimension is either 0 or 1 (the trajectories of dimension 0 are the points of M where the vector field X vanishes).

A first integral for X is an $f \in C^\infty(M):Xf = 0$.

[Note: The set of first integrals for X is a subring $C_X^\infty(M)$ of $C^\infty(M)$.]

1.1 LEMMA In order that f be a first integral for X it is necessary and sufficient that f be constant on the trajectories of X.

Recall now that there exists an open subset $D(X) \subset \underline{R} \times M$ and a differentiable function $\phi_X:D(X) \to M$ such that for each $x \in M$, the map $t \to \phi_X(t,x)$ is the trajectory of X with $\phi_X(0,x) = x$.

1. $\forall\ x \in M$,

$$I_x(X) = \{t \in \underline{R}: (t,x) \in D(X)\}$$

is an open interval containing the origin and is the domain of the trajectory which passes through x.

2. $\forall\ t \in \underline{R}$,

$$D_t(X) = \{x \in M: (t,x) \in D(X)\}$$

is open in M and the map

$$\phi_t, x \to \phi_X(t,x)$$

is a diffeomorphism $D_t(X) \to D_{-t}(X)$ with inverse ϕ_{-t}.

N.B. If (t,x) and $(s,\phi_X(t,x))$ are elements of $D(X)$, then $(s + t,x)$ is an element of $D(X)$ and

$$\phi_X(s,\phi_X(t,x)) = \phi_X(s + t,x),$$

i.e.,

$$\phi_s \circ \phi_t(x) = \phi_{s + t}(x).$$

One calls ϕ_X the flow of X and X its infinitesimal generator.

[Note: X is said to be complete if $D(X) = \underline{R} \times M$. When this is the case, each $\phi_t: M \to M$ is a diffeomorphism and the assignment

$$\begin{cases} \underline{R} \times M \\ \qquad\qquad \to t\cdot x = \phi_t(x) \\ (t,x) \end{cases}$$

is an action of $\underline{R}$ on M. Therefore $\phi_0 = \mathrm{id}_M$, $\phi_{-t} = \phi_t^{-1}$.]

1.2 EXAMPLE Take $M = \underline{R}$, $X = x^2 \frac{\partial}{\partial x}$ — then $D(X) = \{(t,x) \in \underline{R} \times \underline{R}: 1 - tx > 0\}$ and $\phi_X(t,x) = \frac{x}{1 - tx}$, thus X is not complete.

1.3 REMARK Every compactly supported vector field on M is complete.

1.4 <u>LEMMA</u> Suppose that X is a vector field on M -- then $\exists$ a strictly positive C^∞ function f on M such that fX is complete.

A <u>one parameter local group of diffeomorphisms</u> of M is a pair (U,ϕ) subject to the following assumptions:

1. U is an open subset of $\underline{R} \times M$ containing $\{0\} \times M$ such that $\forall\ x \in M$, $(\underline{R} \times \{x\}) \cap U$ is connected.

2. $\phi:U \to M$ is a C^∞ map such that $\phi(0,x) = x$ and

$$\phi(s,\phi(t,x)) = \phi(s + t,x).$$

E.g.: The pair $(D(X),\phi_X)$ determined by a vector field X is a one parameter local group of diffeomorphisms of M.

In practice, reference to U is ordinarily omitted and the one parameter local group of diffeomorphisms of M is denoted by $\{\phi_t\}$.

[Note: One also drops the appelation "local" if $U = \underline{R} \times M$.]

1.5 <u>LEMMA</u> Suppose that $\{\phi_t\}$ is a local one parameter group of diffeomorphisms of M -- then there exists a unique vector field X on M such that

$$(D(X),\phi_X) \supset (U,\phi).$$

[Note: Per $\{\phi_t\}$, X is its <u>infinitesimal generator</u> and $\forall\ f \in C^\infty(M)$,

$$(Xf)(x) = \lim_{t \to 0} \frac{f(\phi_t(x)) - f(x)}{t}.]$$

§2. TENSOR ANALYSIS

Let M be a connected C^∞ manifold of dimension n,

$$\mathcal{D}(M) = \bigoplus_{p,q=0}^{\infty} \mathcal{D}_q^p(M)$$

its tensor algebra.

[Note: Here, $\mathcal{D}_0^0(M) = C^\infty(M)$, $\mathcal{D}_0^1(M) = \mathcal{D}^1(M)$, the derivations of $C^\infty(M)$ (a.k.a. the vector fields on M), and $\mathcal{D}_1^0(M) = \mathcal{D}_1(M)$, the linear forms on $\mathcal{D}^1(M)$ (viewed as a module over $C^\infty(M)$).

2.1 <u>REMARK</u> By definition, $\mathcal{D}_q^p(M)$ is the $C^\infty(M)$-module of all $C^\infty(M)$-multilinear maps

$$\overbrace{\mathcal{D}_1(M) \times \cdots \times \mathcal{D}_1(M)}^{p} \times \overbrace{\mathcal{D}^1(M) \times \cdots \times \mathcal{D}^1(M)}^{q} \to C^\infty(M).$$

Its elements are the tensors of <u>type (p,q)</u>.

In what follows, all operations will be defined globally. However, for computational purposes, it is important to have at hand their local expression as well, meaning the form they take on a connected open set $U \subset M$ equipped with coordinates $x^1,\ldots,x^n$, or still, on a chart.

Let $T \in \mathcal{D}_q^p(M)$ — then locally

$$T = T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \left(\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}}\right) \otimes (dx^{j_1} \otimes \cdots \otimes dx^{j_q}),$$

where

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

$$= T(dx^{i_1}, \ldots, dx^{i_p}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_q}}) \in C^\infty(U)$$

are the <u>components</u> of T.

Under a change of coordinates, the components of T satisfy the <u>tensor transformation rule</u>:

$$T^{i_1' \cdots i_p'}{}_{j_1' \cdots j_q'}$$

$$= \frac{\partial x^{i_1'}}{\partial x^{i_1}} \cdots \frac{\partial x^{i_p'}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j_1'}} \cdots \frac{\partial x^{j_q}}{\partial x^{j_q'}} T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} .$$

2.2 <u>EXAMPLE</u> The <u>Kronecker tensor</u> is the tensor K of type (1,1) defined by $K(\Lambda,X) = \Lambda(X)$, hence

$$K^i{}_j = K(dx^i, \frac{\partial}{\partial x^j}) = \delta^i{}_j.$$

Given $f \in C^\infty(U)$, write

$$\frac{\partial f}{\partial x^i} = f_{,i}.$$

2.3 EXAMPLE Let $X,Y \in \mathcal{D}^1(M)$ -- then locally

$$\left[\begin{array}{ll} X = X^i \dfrac{\partial}{\partial x^i} & (X^i = \langle X, dx^i \rangle) \\ \\ Y = Y^j \dfrac{\partial}{\partial x^j} & (Y^j = \langle Y, dx^j \rangle) \end{array}\right.$$

=>

$$[X,Y] = (X^i Y^j{}_{,i} - Y^i X^j{}_{,i}) \frac{\partial}{\partial x^j} .$$

[Note: The bracket

$$[\ ,\ ]: \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M)$$

is $\underline{R}$-bilinear but not $C^\infty(M)$-bilinear. In fact,

$$[fX,gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.]$$

A type preserving $\underline{R}$-linear map

$$D: \mathcal{D}(M) \to \mathcal{D}(M)$$

which commutes with contractions is said to be a <u>derivation</u> if $\forall\ T_1, T_2 \in \mathcal{D}(M)$,

$$D(T_1 \otimes T_2) = DT_1 \otimes T_2 + T_1 \otimes DT_2.$$

The set of all derivations of $\mathcal{D}(M)$ forms a Lie algebra over $\underline{R}$, the bracket operation being defined by

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

2.4 <u>REMARK</u> For any $f \in C^\infty(M)$ and any $T \in \mathcal{D}(M)$, $fT = f \otimes T$, so $D(fT) = f(DT) + (Df)T$. In particular: D is a derivation of $C^\infty(M)$, hence is represented on $C^\infty(M)$ by a vector field.

2.5 <u>LEMMA</u> Let $D:\mathcal{D}(M) \to \mathcal{D}(M)$ be a derivation -- then $\forall\, T \in \mathcal{D}_q^p(M)$,

$$D[T(\Lambda^1,\ldots,\Lambda^p,X_1,\ldots,X_q)]$$

$$= (DT)(\Lambda^1,\ldots,\Lambda^p,X_1,\ldots,X_q)$$

$$+ \sum_{i=1}^{p} T(\Lambda^1,\ldots,D\Lambda^i,\ldots,\Lambda^p,X_1,\ldots,X_q)$$

$$+ \sum_{j=1}^{q} T(\Lambda^1,\ldots,\Lambda^p,X_1,\ldots,DX_j,\ldots,X_q).$$

[Note: This shows that D is known as soon as it is known on $C^\infty(M)$, $\mathcal{D}^1(M)$, and $\mathcal{D}_1(M)$. But for $\omega \in \mathcal{D}_1(M)$,

$$(D\omega)(X) = D[\omega(X)] - \omega(DX),$$

so functions and vector fields suffice.]

2.6 <u>EXAMPLE</u> There is a canonical identification

$$\mathcal{D}_1^1(M) \approx \mathrm{Hom}_{C^\infty(M)}(\mathcal{D}^1(M),\mathcal{D}^1(M)),$$

namely $T \to \hat{T}$, where

$$\hat{T}X(\Lambda) = T(\Lambda,X).$$

This said, let $D:\mathcal{D}(M) \to \mathcal{D}(M)$ be a derivation -- then

$$T \in \mathcal{D}_1^1(M) \Rightarrow DT \in \mathcal{D}_1^1(M),$$

thus it makes sense to form $\widehat{DT}$ and we claim that

$$(\widehat{DT})(X) = D\hat{T}X - \hat{T}(DX).$$

In fact,

$$(\widehat{DT})(X)(\Lambda) = (DT)(\Lambda,X)$$

$$= D[T(\Lambda,X)] - T(D\Lambda,X) - T(\Lambda,DX).$$

On the other hand,

$$(D\hat{T}X)(\Lambda) - \hat{T}(DX)(\Lambda)$$

$$= D[\hat{T}X(\Lambda)] - \hat{T}X(D\Lambda) - \hat{T}(DX)(\Lambda)$$

$$= D[T(\Lambda,X)] - T(D\Lambda,X) - T(\Lambda,DX).$$

2.7 <u>THEOREM</u> Suppose given a vector field X and an $\underline{R}$-linear map $\delta:\mathcal{D}^1(M) \to \mathcal{D}^1(M)$ such that

$$\delta(fY) = (Xf)Y + f\delta(Y)$$

for all $f \in C^\infty(M)$, $Y \in \mathcal{D}^1(M)$ -- then there exists a unique derivation

$$D:\mathcal{D}(M) \to \mathcal{D}(M)$$

such that

$$D|C^\infty(M) = X \text{ and } D|\mathcal{D}^1(M) = \delta.$$

PROOF Define D on $\mathcal{D}_1(M)$ by

$$(D\omega)(Y) = X[\omega(Y)] - \omega(\delta Y)$$

and extend to all of $\mathcal{D}(M)$ via 2.5.

§3. *LIE DERIVATIVES*

Let M be a connected C^∞ manifold of dimension n.

3.1 LEMMA One may attach to each $X \in \mathcal{D}^1(M)$ a derivation

$$L_X:\mathcal{D}(M) \to \mathcal{D}(M)$$

called the Lie derivative w.r.t. X. It is characterized by the properties

$$L_X f = Xf, \ L_X Y = [X,Y].$$

PROOF In the notation of 2.7, define $\delta:\mathcal{D}^1(M) \to \mathcal{D}^1(M)$ by

$$\delta(Y) = [X,Y].$$

Then

$$\begin{aligned}\delta(fY) &= [X,fY]\\ &= f[X,Y] + (Xf)Y \quad (\text{cf. } 2.3)\\ &= (Xf)Y + f[X,Y]\\ &= (Xf)Y + f\delta(Y).\end{aligned}$$

3.2 EXAMPLE Let $T \in \mathcal{D}_1^1(M)$ -- then in the notation of 2.6,

$$(L_X\hat{T})(Y) = [X,\hat{T}Y] - \hat{T}[X,Y],$$

where

$$L_X\hat{T} \equiv \widehat{L_X T}.$$

Owing to 2.5, $\forall\ T \in \mathcal{D}_q^p(M)$,

$$X[T(\Lambda^1,\ldots,\Lambda^p,X_1,\ldots,X_q)]$$

$$= (L_X T)(\Lambda^1,\ldots,\Lambda^p,X_1,\ldots,X_q)$$

$$+ \sum_{i=1}^{p} T(\Lambda^1,\ldots,L_X\Lambda^i,\ldots,\Lambda^p,X_1,\ldots,X_q)$$

$$+ \sum_{j=1}^{q} T(\Lambda^1,\ldots,\Lambda^p,X_1,\ldots,L_X X_j,\ldots,X_q).$$

[Note: If $\omega \in \mathcal{D}_1(M)$, then

$$(L_X\omega)(Y) = X\omega(Y) - \omega([X,Y]).]$$

Locally,

$$(L_X T)^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$= X^a T^{i_1\cdots i_p}{}_{j_1\cdots j_q,a}$$

$$- X^{i_1}{}_{,a}\, T^{a i_2\cdots i_p}{}_{j_1\cdots j_q} - \cdots$$

$$+ X^a{}_{,j_1}\, T^{i_1\cdots i_p}{}_{a j_2\cdots j_q} + \cdots.$$

[Note: From the definitions,

$$\left[\begin{array}{l} L_X \dfrac{\partial}{\partial x^i} = - X^a{}_{,i} \dfrac{\partial}{\partial x^a} \\[2ex] L_X dx^i = X^i{}_{,a} dx^a. \end{array}\right]$$

3.3 <u>REMARK</u> The symbol

$$(L_X T)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

is usually abbreviated to

$$L_X T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}.$$

3.4 <u>EXAMPLE</u> Let K be the Kronecker tensor (cf. 2.2) — then

$$L_X K = 0.$$

Indeed,

$$\begin{aligned} L_X K^i{}_j &= X^a \delta^i{}_{j,a} - X^i{}_{,a} \delta^a{}_j + X^a{}_{,j} \delta^i{}_a \\ &= 0 - X^i{}_{,j} + X^i{}_{,j} \\ &= 0. \end{aligned}$$

3.5 <u>THEOREM</u> Fix an $X \in \mathcal{D}^1(M)$ — then $\forall\ T \in \mathcal{D}^p_q(M)$,

$$\frac{d}{dt} \phi_t^* T \Big|_{t=t_0} = \phi_{t_0}^* L_X T.$$

[Note: The tacit assumption is that $D_{t_0}(X)$ is nonempty, the relation being valid in $D_{t_0}(X)$. Accordingly, if X is complete,

$$\frac{d}{dt} \phi_t^* T = \phi_t^* L_X T.]$$

3.6 <u>EXAMPLE</u> Take X complete -- then

$$\phi_t^* X = X \ \forall \ t.$$

[In fact,

$$\frac{d}{dt} \phi_t^* X = \phi_t^* L_X X$$

$$= \phi_t^* [X,X] = 0.$$

But $\phi_0^* X = \mathrm{id}_M^* X = X$.]

Consider now the exterior algebra $\Lambda^* M$ -- then L_X induces a derivation of $\Lambda^* M$:

$$L_X(\alpha \wedge \beta) = L_X \alpha \wedge \beta + \alpha \wedge L_X \beta.$$

3.7 <u>RAPPEL</u> ι_X is the interior product w.r.t. X, so

$$\iota_X : \Lambda^* M \to \Lambda^* M$$

is an antiderivation of degree -1. Explicitly, $\forall \ \alpha \in \Lambda^p M$,

$$\iota_X \alpha (X_1, \ldots, X_{p-1}) = \alpha(X, X_1, \ldots, X_{p-1}).$$

And one has

$$\iota_X(\alpha_1 \wedge \alpha_2) = \iota_X\alpha_1 \wedge \alpha_2 + (-1)^p\alpha_1 \wedge \iota_X\alpha_2.$$

Properties: (1) $\iota_X \circ \iota_X = 0$; (2) $\iota_X \circ \iota_Y + \iota_Y \circ \iota_X = 0$; (3) $\iota_{X+Y} = \iota_X + \iota_Y$; (4) $\iota_{fX} = f\iota_X$.

We have

- $L_X = \iota_X \circ d + d \circ \iota_X$.
- $\iota_{[X,Y]} = L_X \circ \iota_Y - \iota_Y \circ L_X$.

Therefore

$$\left[\begin{array}{l} L_X \circ d = d \circ L_X \\ \\ L_X \circ \iota_X = \iota_X \circ L_X. \end{array}\right.$$

3.8 <u>EXAMPLE</u> $\forall\ f \in C^\infty(M)$,

$$L_{fX}\alpha = fL_X\alpha + df \wedge \iota_X\alpha.$$

[For

$$\begin{aligned} L_{fX}\alpha &= \iota_{fX}d\alpha + d\iota_{fX}\alpha \\ &= f\iota_X d\alpha + d(f\iota_X\alpha) \\ &= f\iota_X d\alpha + df \wedge \iota_X\alpha + fd\iota_X\alpha \end{aligned}$$

$$= f(\iota_X d + d\iota_X)\alpha + df \wedge \iota_X\alpha$$

$$= f L_X\alpha + df \wedge \iota_X\alpha.]$$

If $\phi:N \to M$ is a diffeomorphism, then

$$\begin{cases} \phi^* L_X\alpha = L_{\phi^* X}\phi^*\alpha \\ \phi^*\iota_X\alpha = \iota_{\phi^* X}\phi^*\alpha. \end{cases}$$

If $\Phi:N \to M$ is a C^∞ map and if X is Φ-related to Y, then

$$\begin{cases} \Phi^* L_X\alpha = L_Y\Phi^*\alpha \\ \Phi^*\iota_X\alpha = \iota_Y\Phi^*\alpha. \end{cases}$$

[Note: Recall that

$$X \in \mathcal{D}^1(M) \ \& \ Y \in \mathcal{D}^1(N)$$

are said to be <u>Φ-related</u> if

$$d\Phi(Y_y) = X_{\Phi(y)} \quad \forall\, y \in Y$$

or, equivalently, if

$$Y(f \circ \Phi) = Xf \circ \Phi$$

for all $f \in C^\infty(M)$.]

§4. *TANGENT BUNDLES*

Let M be a connected C^∞ manifold of dimension n,

$$\pi_M : TM \to M$$

its tangent bundle -- then the sections $\mathcal{D}^1(M)$ of TM are the vector fields on M.

<u>N.B.</u> Suppose that $(U, \{x^1, \ldots, x^n\})$ is a chart on M -- then

$$((\pi_M)^{-1}U, \{q^1, \ldots, q^n, v^1, \ldots, v^n\})$$

is a chart on TM.

[Note: Here

$$\begin{cases} q^i = x^i \circ \pi_M \\ \\ v^i = dx^i \end{cases} \qquad (i = 1, \ldots, n).$$

And, under a compatible change of coordinates,

$$\begin{cases} \dfrac{\partial}{\partial \tilde{q}^i} = \dfrac{\partial q^j}{\partial \tilde{q}^i} \dfrac{\partial}{\partial q^j} + \dfrac{\partial v^j}{\partial \tilde{q}^i} \dfrac{\partial}{\partial v^j} \\ \\ \dfrac{\partial}{\partial \tilde{v}^i} = \dfrac{\partial v^j}{\partial \tilde{v}^i} \dfrac{\partial}{\partial v^j}, \end{cases}$$

where

$$\tilde{v}^i = \frac{\partial \tilde{q}^i}{\partial q^j} v^j$$

$\Rightarrow$

$$\frac{\partial \tilde{v}^i}{\partial v^j} = \frac{\partial \tilde{q}^i}{\partial q^j}.]$$

If $f:M \to N$ is a C^∞ map, then there is a commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ {\scriptstyle \pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[f]{} & N \end{array} .$$

4.1 <u>EXAMPLE</u> We have

$$\begin{array}{ccc} TTM & \xrightarrow{T\pi_M} & TM \\ {\scriptstyle \pi_{TM}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_M} \\ TM & \xrightarrow[\pi_M]{} & M \end{array} .$$

[Note: Local coordinates on the open subset $\pi_{TM}^{-1}((\pi_M)^{-1}U)$ of TTM are as follows: $q^i \equiv q^i \circ \pi_{TM}$, $v^i \equiv v^i \circ \pi_{TM}$, dq^i, dv^i.]

Let $X \in \mathcal{D}^1(TM)$ — then

$$X:TM \to TTM$$

and $\pi_{TM} \circ X = \mathrm{id}_{TM}$. Locally,

$$X = A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial v^i} .$$

4.2 <u>EXAMPLE</u> Consider the one parameter group of diffeomorphisms $\phi_t:TM \to TM$ defined by $\phi_t(x,X_x) = (x,e^tX_x)$ ($X_x \in T_xM$) -- then its infinitesimal generator $\Delta \in \mathcal{D}^1(TM)$ is called the <u>dilation vector field</u> on TM. Locally, ϕ_t sends $(q^1,\ldots,q^n,v^1,\ldots,v^n)$ to $(q^1,\ldots,q^n,e^tv^1,\ldots,e^tv^n)$, so locally,

$$\Delta = v^i \frac{\partial}{\partial v^i}.$$

Denote by T^2M the submanifold of TTM consisting of those points whose images under π_{TM} and $T\pi_M$ are one and the same -- then $\Gamma \in \mathcal{D}^1(TM)$ is said to be <u>second order</u> provided $\Gamma TM \subset T^2M$ or still, if $T\pi_M \circ \Gamma = \mathrm{id}_{TM}$. Locally, therefore, a second order Γ has the form

$$v^i \frac{\partial}{\partial q^i} + C^i \frac{\partial}{\partial v^i}.$$

[Note: To ascertain the transformation rule for the C^i, write

$$\tilde{v}^i \frac{\partial}{\partial \tilde{q}^i} + \tilde{C}^i \frac{\partial}{\partial \tilde{v}^i}$$

$$= \tilde{v}^i\Big(\frac{\partial q^j}{\partial \tilde{q}^i} \frac{\partial}{\partial q^j} + \frac{\partial v^j}{\partial \tilde{q}^i} \frac{\partial}{\partial v^j}\Big) + \tilde{C}^i \frac{\partial v^j}{\partial \tilde{v}^i} \frac{\partial}{\partial v^j}$$

$$= v^j \frac{\partial}{\partial q^j} + \Big(\frac{\partial v^j}{\partial \tilde{q}^i} \tilde{v}^i + \frac{\partial v^j}{\partial \tilde{v}^i} \tilde{C}^i\Big) \frac{\partial}{\partial v^j}$$

$$= v^i \frac{\partial}{\partial q^i} + \Big(\frac{\partial v^i}{\partial \tilde{q}^j} \tilde{v}^j + \frac{\partial v^i}{\partial \tilde{v}^j} \tilde{C}^j\Big) \frac{\partial}{\partial v^i}$$

=>

$$C^i = \frac{\partial v^i}{\partial \tilde{q}^j} \tilde{v}^j + \frac{\partial v^i}{\partial \tilde{v}^j} \tilde{C}^j$$

or still,

$$C^i = \frac{\partial v^i}{\partial \tilde{q}^j} \tilde{v}^j + \frac{\partial q^i}{\partial \tilde{q}^j} \tilde{C}^j.]$$

4.3 <u>REMARK</u> Suppose that $\Gamma \in \mathcal{D}^1(TM)$ is second order — then an integral curve γ of Γ is a solution to

$$\frac{dq^i}{dt} = v^i, \ \frac{dv^i}{dt} = C^i$$

or still, is a solution to

$$\frac{d^2q^i}{dt^2} = C^i,$$

from which the term "second order".

Given an $X \in \mathcal{D}^1(M)$, let $\{\phi_t\}$ be the one parameter local group of diffeomorphisms of M associated with X — then $\{T\phi_t\}$ is a one parameter local group of diffeomorphisms of TM. Denote its infinitesimal generator by X^T (cf. 1.5) — then X^T is called the <u>lift</u> of X to TM. Locally, if

$$X = X^i \frac{\partial}{\partial x^i},$$

then

$$X^T = (X^i \circ \pi_M) \frac{\partial}{\partial q^i} + v^j (X^i_{,j} \circ \pi_M) \frac{\partial}{\partial v^i}.$$

Example:

$$\left(\frac{\partial}{\partial x^i}\right)^T = \frac{\partial}{\partial q^i}.$$

[Note: Let $s_{TM}:TTM \to TTM$ be the canonical involution -- then

$$\pi_{TM} \circ s_{TM} = T\pi_M.$$

So, $\forall\ X \in \mathcal{D}^1(M)$,

$$\begin{aligned}\pi_{TM} \circ s_{TM} \circ TX &= T\pi_M \circ TX \\ &= T(\pi_M \circ X) \\ &= T(\mathrm{id}_M) \\ &= \mathrm{id}_{TM}\end{aligned}$$

=>

$$s_{TM} \circ TX \in \mathcal{D}^1(TM).$$

And, in fact,

$$s_{TM} \circ TX = X^T.]$$

4.4 <u>LEMMA</u> $\forall\ X \in \mathcal{D}^1(M)$,

$$[\Delta, X^T] = 0.$$

4.5 <u>LEMMA</u> Let $X, Y \in \mathcal{D}^1(M)$ — then

$$[X^T, Y^T] = [X, Y]^T.$$

Given an $X \in \mathcal{D}^1(M)$, define a one parameter group of diffeomorphisms $\phi_t:TM \to TM$ by

$$\phi_t(x,V_x) = (x,V_x + tX_x) \quad (V_x \in T_xM)$$

and let X^V be its infinitesimal generator (cf. 1.5) -- then X^V is called the <u>vertical lift</u> of X to TM. Locally, if

$$X = X^i \frac{\partial}{\partial x^i},$$

then

$$X^V = (X^i \circ \pi_M) \frac{\partial}{\partial v^i}.$$

Example:

$$\left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial v^i}.$$

4.6 <u>LEMMA</u> $\forall X \in \mathcal{D}^1(M)$,

$$[\Delta, X^V] = -X^V.$$

4.7 <u>LEMMA</u> Let $X,Y \in \mathcal{D}^1(M)$ -- then

$$[X^V, Y^V] = 0.$$

4.8 <u>LEMMA</u> Let $X,Y \in \mathcal{D}^1(M)$ -- then

$$[X^V, Y^T] = [X,Y]^V.$$

Let $\phi:M \to M$ be a diffeomorphism -- then $T\phi:TM \to TM$ is a diffeomorphism and there is a commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{T\phi} & TM \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow[\phi]{} & M \end{array} .$$

[Note: Classically, $T\phi$ is called a <u>point transformation</u>.]

4.9 <u>LEMMA</u> Let $\phi:M \to M$ be a diffeomorphism -- then for any second order $\Gamma \in \mathcal{D}^1(TM)$, $(T\phi)_*\Gamma$ is second order.

<u>PROOF</u> In fact,

$$\begin{aligned} &T\pi_M \circ (T\phi)_*\Gamma \\ &= T\pi_M \circ TT\phi \circ \Gamma \circ (T\phi)^{-1} \\ &= T(\pi_M \circ T\phi) \circ \Gamma \circ (T\phi)^{-1} \\ &= T(\phi \circ \pi_M) \circ \Gamma \circ (T\phi)^{-1} \\ &= T\phi \circ T\pi_M \circ \Gamma \circ (T\phi)^{-1} \\ &= T\phi \circ \mathrm{id}_{TM} \circ (T\phi)^{-1} \\ &= T\phi \circ (T\phi)^{-1} \\ &= \mathrm{id}_{TM}. \end{aligned}$$

§5. *THE VERTICAL MORPHISM*

Let M be a connected C^∞ manifold of dimension n,

$$\pi : E \to M$$

a vector bundle — then π is a surjective submersion and the kernel of

$$T\pi : TE \to TM$$

is called the <u>vertical tangent bundle</u> of E, denoted VE.

5.1 <u>REMARK</u> Take a point $p \in E$ and put $x = \pi(p)$ — then the fiber $E_x = \pi^{-1}(x)$ is a submanifold of E containing p, hence $T_p E_x \subset T_p E$ and, in fact, $T_p E_x$ is precisely the kernel of $T\pi_p : T_p E \to T_x M$. Let us also note that TE_x can be identified with $E_x \times E_x$, so VE can be identified with $E \times_M E$, the latter being defined by the pullback square

$$\begin{array}{ccc} E \times_M E & \xrightarrow{pr_2} & E \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\pi]{} & M . \end{array}$$

There is a commutative diagram

$$\begin{array}{ccc} TE & \xrightarrow{T\pi} & TM \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ E & \xrightarrow[\pi]{} & M \end{array}$$

and a pullback square

$$\begin{tikzcd} E \times_M TM \arrow[r] \arrow[d] & TM \arrow[d, "\pi_M"] \\ E \arrow[r, "\pi"'] & M \end{tikzcd},$$

thus there is an arrow

$$TE \to E \times_M TM.$$

5.2 <u>LEMMA</u> The sequence

$$0 \to VE \to TE \to E \times_M TM \to 0$$

is exact.

Now take $E = TM$ -- then a <u>vertical vector field</u> is a section of VTM. Accordingly, to say that $X \in \mathcal{D}^1(TM)$ is vertical amounts to saying that

$$T\pi_M \circ X = 0$$

or still,

$$X(f \circ \pi_M) = 0 \ \forall \ f \in C^\infty(M).$$

Therefore the bracket of two vertical vector fields is again vertical. Locally, the vertical vector fields on TM have the form

$$B^i \frac{\partial}{\partial v^i}.$$

N.B. $\forall\, X \in \mathcal{D}^1(M)$, X^V is vertical but not every vertical vector field is a vertical lift (e.g., Δ).

5.3 LEMMA If $\Gamma \in \mathcal{D}^1(TM)$ is second order, then for every $X \in \mathcal{D}^1(M)$, the bracket $[\Gamma, X^T]$ is a vertical vector field.

PROOF It need only be shown that $\forall\, f \in C^\infty(M)$,

$$L_{[\Gamma, X^T]}(f \circ \pi_M) = 0.$$

But

$$\begin{aligned} &L_{[\Gamma, X^T]}(f \circ \pi_M) \\ &\quad = L_\Gamma(L_{X^T}(f \circ \pi_M)) - L_{X^T}(L_\Gamma(f \circ \pi_M)) \\ &\quad = L_\Gamma((Xf) \circ \pi_M) - L_{X^T}(L_\Gamma(f \circ \pi_M)), \end{aligned}$$

which reduces matters to the equality

$$L_{X^T}(L_\Gamma(f \circ \pi_M)) = L_\Gamma((Xf) \circ \pi_M).$$

Working locally, write

$$X = X^i \frac{\partial}{\partial x^i}.$$

Then

$$(Xf) \circ \pi_M = (X^i \circ \pi_M) \frac{\partial (f \circ \pi_M)}{\partial q^i}$$

$\Rightarrow$

$$L_\Gamma((Xf) \circ \pi_M)$$

$$= v^j \frac{\partial}{\partial q^j} ((X^i \circ \pi_M) \frac{\partial (f \circ \pi_M)}{\partial q^i}).$$

On the other hand,

$$X^\top = (X^i \circ \pi_M) \frac{\partial}{\partial q^i} + v^k (X^i_{,k} \circ \pi_M) \frac{\partial}{\partial v^i}$$

$\Rightarrow$

$$L_{X^\top}(L_\Gamma(f \circ \pi_M))$$

$$= L_{X^\top}(v^j \frac{\partial (f \circ \pi_M)}{\partial q^j})$$

$$= v^j (X^i \circ \pi_M) \frac{\partial}{\partial q^j} \frac{\partial (f \circ \pi_M)}{\partial q^i}$$

$$+ v^k (X^i_{,k} \circ \pi_M) \frac{\partial}{\partial v^i} (v^j \frac{\partial (f \circ \pi_M)}{\partial q^j})$$

$$= v^j (X^i \circ \pi_M) \frac{\partial}{\partial q^j} \frac{\partial (f \circ \pi_M)}{\partial q^i}$$

$$+ v^j (X^i_{,j} \circ \pi_M) \frac{\partial (f \circ \pi_M)}{\partial q^i}$$

$$= v^j \frac{\partial}{\partial q^j} ((X^i \circ \pi_M) \frac{\partial (f \circ \pi_M)}{\partial q^i}).$$

[Note: For a completely different proof, see 5.19.]

Bearing in mind that

$$VTM \approx TM \times_M TM,$$

consider the exact sequence

$$0 \rightarrow TM \times_M TM \xrightarrow{\mu} TTM \xrightarrow{\nu} TM \times_M TM \rightarrow 0$$

provided by 5.2 — then

$$\begin{cases} \pi_{TM} \circ \mu = pr_1 \\ \\ pr_1 \circ \nu = \pi_{TM}. \end{cases}$$

5.4 <u>LEMMA</u> $\forall\, X \in \mathcal{D}^1(TM)$, $\mu \circ \nu \circ X \in \mathcal{D}^1(TM)$.

<u>PROOF</u> In fact,

$$\begin{aligned} &\pi_{TM} \circ \mu \circ \nu \circ X \\ &\quad = pr_1 \circ \nu \circ X \\ &\quad = \pi_{TM} \circ X \\ &\quad = \text{id}_{TM}. \end{aligned}$$

Put

$$SX = \mu \circ \nu \circ X \qquad (X \in \mathcal{D}^1(TM)).$$

Then

$$S:\mathcal{D}^1(TM) \to \mathcal{D}^1(TM)$$

is called the <u>vertical morphism</u>.

<u>N.B.</u> It is clear that

$$S \in \operatorname{Hom}_{C^\infty(TM)}(\mathcal{D}^1(TM), \mathcal{D}^1(TM)).$$

Therefore S can also be regarded as an element of $\mathcal{D}_1^1(TM)$.

5.5 <u>LEMMA</u> $S^2 = 0$ and

$$\operatorname{Ker} S = \operatorname{Im} S,$$

the vertical vector fields on TM.

5.6 <u>LEMMA</u> Locally,

$$S(A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial v^i}) = A^i \frac{\partial}{\partial v^i}.$$

[Note: If S is thought of as lying in $\mathcal{D}_1^1(TM)$, then its local expression is

$$\frac{\partial}{\partial v^i} \otimes dq^i.]$$

5.7 <u>LEMMA</u> $\forall\ X \in \mathcal{D}^1(M)$,

$$SX^T = X^V.$$

5.8 <u>REMARK</u> Let $\Gamma \in \mathcal{D}^1(TM)$ -- then Γ is second order iff $S\Gamma = \Delta$.

[Note: The set $SO(TM)$ of second order vector fields on TM is an affine space whose translation group is the set of vertical vector fields in $\mathcal{D}^1(TM)$.]

The vertical morphism does not respect the structure of $\mathcal{D}^1(TM)$ as a Lie algebra. Instead:

5.9 <u>LEMMA</u> $\forall\ X,Y \in \mathcal{D}^1(TM)$,

$$[SX,SY] = S[SX,Y] + S[X,SY].$$

<u>PROOF</u> It will be enough to consider the following possibilities.

- Both X & Y are vertical lifts.
- Both X & Y are lifts.
- X is a vertical lift and Y is a lift.

Since S annihilates vertical vector fields,

$$\begin{cases} SX^V = 0 \\ SY^V = 0, \end{cases}$$

which settles the first possibility. Turning to the second,

$$\begin{aligned} [SX^T,SY^T] &= [X^V,Y^V] \quad (\text{cf. } 5.7) \\ &= 0 \quad (\text{cf. } 4.7). \end{aligned}$$

And (cf. 4.8)

$$\begin{cases} S[SX^T,Y^T] = S[X^V,Y^T] = S[X,Y]^V = 0 \\ S[X^T,SY^T] = S[X^T,Y^V] = S[Y,X]^V = 0. \end{cases}$$

Finally,

$$S[X^V, Y^T] = S[X,Y]^V = 0$$

while

$$\begin{cases} S[SX^V, Y^T] = S[0, Y^T] = 0 \\ S[X^V, SY^T] = S[X^V, Y^V] = 0. \end{cases}$$

5.10 <u>REMARK</u> Aanalogously, $\forall\ X \in \mathcal{D}^1(TM)$,

$$SX = S[\Delta, X] + [SX, \Delta].$$

By definition,

$$(L_X S)(Y) = [X, SY] - S[X,Y] \quad \text{(cf. 3.2)}.$$

Therefore

$$S \circ L_X S + L_X S \circ S = 0.$$

Proof:

$$S((L_X S)(Y)) + (L_X S)(SY)$$

$$= S([X,SY] - S[X,Y]) + [X, S^2 Y] - S[X,SY]$$

$$= S[X,SY] - S[X,SY]$$

$$= 0.$$

[Note: Recall that $S^2 = 0$ (cf. 5.5).]

Consequently,

$$\begin{aligned}(L_{SX}S)(Y) &= [SX,SY] - S[SX,Y] \\ &= S[X,SY] \quad \text{(cf. 5.9)} \\ &= S((L_X S)(Y)) \\ &= -(L_X S)(SY),\end{aligned}$$

i.e.,

$$L_{SX}S = \begin{cases} S \circ L_X S \\ \\ -L_X S \circ S. \end{cases}$$

5.11 LEMMA We have

$$L_\Delta S = -S.$$

5.12 EXAMPLE For any $\Gamma \in \mathcal{D}^1(TM)$ of second order,

$$\begin{aligned}S &= -L_\Delta S \\ &= -L_{S\Gamma}S \quad \text{(cf. 5.8)} \\ &= -S \circ L_\Gamma S = L_\Gamma S \circ S.\end{aligned}$$

5.13 <u>LEMMA</u> $\forall\ X \in \mathcal{D}^1(M)$,

$$L_{X^V} S = 0.$$

5.14 <u>EXAMPLE</u> If $X \in \mathcal{D}^1(M)$ and $\Gamma \in \mathcal{D}^1(TM)$ is second order, then

$$S[X^V,\Gamma] = X^V.$$

Indeed,

$$L_{X^V} S = 0 \qquad (\text{cf. } 5.13)$$

=>

$$\begin{aligned} S[X^V,\Gamma] &= [X^V,S\Gamma] \\ &= [X^V,\Delta] \qquad (\text{cf. } 5.8) \\ &= X^V \qquad (\text{cf. } 4.6). \end{aligned}$$

5.15 <u>LEMMA</u> Fix $\Gamma \in \mathcal{D}^1(TM)$ of second order and suppose that $X \in \mathcal{D}^1(TM)$ is vertical -- then

$$(L_\Gamma S)(X) = X.$$

<u>PROOF</u> There is no loss of generality in working with a vertical lift:

$$\begin{aligned} (L_\Gamma S)(X^V) &= [\Gamma,SX^V] - S[\Gamma,X^V] \\ &= [\Gamma,0] + S[X^V,\Gamma] \qquad (\text{cf. } 5.5) \\ &= X^V \qquad (\text{cf. } 5.14). \end{aligned}$$

5.16 LEMMA Fix $\Gamma \in \mathcal{D}^1(TM)$ of second order and suppose that

$$(L_\Gamma S)(X) = X.$$

Then X is vertical.

PROOF In fact,

$$SX = S((L_\Gamma S)(X))$$

$$= -(L_\Gamma S)(SX)$$

$$= -SX \quad \text{(cf. 5.5 and 5.15)}$$

$$\Rightarrow$$

$$SX = 0.$$

Therefore $X \in \text{Ker } S$, hence X is vertical (cf. 5.5).

Write $\mathcal{V}(TM)$ for the vertical subspace of $\mathcal{D}^1(TM)$. Combining 5.15 and 5.16 then leads to the following important conclusion.

5.17 SCHOLIUM If $\Gamma \in \mathcal{D}^1(TM)$ is second order, then the operator

$$L_\Gamma S:\mathcal{D}^1(TM) \to \mathcal{D}^1(TM)$$

has eigenvalue +1 with $\mathcal{V}(TM)$ as eigenspace.

5.18 LEMMA $\forall\ X \in \mathcal{D}^1(M)$,

$$L_{X^T} S = 0.$$

5.19 <u>EXAMPLE</u> If $X \in \mathcal{D}^1(M)$ and $\Gamma \in \mathcal{D}^1(TM)$ is second order, then

$$S[X^T,\Gamma] = 0 \quad (\text{cf. } 5.3).$$

Indeed,

$$L_{X^T} S = 0 \quad (\text{cf. } 5.18)$$

$$\Rightarrow$$

$$\begin{aligned} S[X^T,\Gamma] &= [X^T,S\Gamma] \\ &= [X^T,\Delta] \quad (\text{cf. } 5.8) \\ &= 0 \quad (\text{cf. } 4.4). \end{aligned}$$

5.20 <u>LEMMA</u> For any second order $\Gamma \in \mathcal{D}^1(TM)$,

$$(L_\Gamma S)^2$$

is the identity operator.

<u>PROOF</u> In view of 5.17, $(L_\Gamma S)^2$ is the identity on vertical vector fields, thus it suffices to show that

$$(L_\Gamma S)^2(X^T) = X^T \quad (X \in \mathcal{D}^1(M)).$$

To begin with,

$$\begin{aligned} (L_\Gamma S)(X^T) &= [\Gamma,SX^T] - S[\Gamma,X^T] \\ &= [\Gamma,X^V] + S[X^T,\Gamma] \quad (\text{cf. } 5.7) \\ &= [\Gamma,X^V] \quad (\text{cf. } 5.19). \end{aligned}$$

But

$$
\begin{aligned}
S(X^{\mathrm{T}} + [\Gamma, X^{\mathrm{V}}]) \\
&= SX^{\mathrm{T}} + S[\Gamma, X^{\mathrm{V}}] \\
&= X^{\mathrm{V}} - S[X^{\mathrm{V}}, \Gamma] \quad \text{(cf. 5.7)} \\
&= X^{\mathrm{V}} - X^{\mathrm{V}} \quad \text{(cf. 5.14)} \\
&= 0
\end{aligned}
$$

=>

$$X^{\mathrm{T}} + [\Gamma, X^{\mathrm{V}}] \in \mathcal{V}(TM) \quad \text{(cf. 5.5)}$$

=>

$$X^{\mathrm{T}} + (L_\Gamma S)(X^{\mathrm{T}}) \in \mathcal{V}(TM)$$

=>

$$
\begin{aligned}
(L_\Gamma S)(X^{\mathrm{T}} + (L_\Gamma S)(X^{\mathrm{T}})) \\
&= X^{\mathrm{T}} + (L_\Gamma S)(X^{\mathrm{T}}) \quad \text{(cf. 5.15)}
\end{aligned}
$$

=>

$$(L_\Gamma S)^2(X^{\mathrm{T}}) = X^{\mathrm{T}}.$$

Maintaining the assumption that $\Gamma \in \mathcal{D}^1(TM)$ is second order, put

$$V_\Gamma = \frac{1}{2}(I + L_\Gamma S), \; H_\Gamma = \frac{1}{2}(I - L_\Gamma S).$$

Then

$$\begin{cases} V_\Gamma^2 = V_\Gamma \\ \\ H_\Gamma^2 = H_\Gamma \end{cases}, \quad \begin{cases} V_\Gamma \circ H_\Gamma = 0 \\ \\ H_\Gamma \circ V_\Gamma = 0 \end{cases}, \quad V_\Gamma + H_\Gamma = I.$$

And, as has been seen above,

$$V_\Gamma \mathcal{D}^1(TM) = \mathcal{V}(TM).$$

On the other hand, we call $H_\Gamma \mathcal{D}^1(TM)$ the <u>horizontal subspace</u> of $\mathcal{D}^1(TM)$ determined by Γ and denote it by $\mathcal{H}_\Gamma(TM)$. Therefore

$$\mathcal{D}^1(TM) = \mathcal{V}(TM) \oplus \mathcal{H}_\Gamma(TM).$$

5.21 <u>REMARK</u> Since

$$\begin{aligned} (L_\Gamma S)(\Gamma) &= [\Gamma, S\Gamma] - S[\Gamma,\Gamma] \\ &= [\Gamma,\Delta] \quad \text{(cf. 5.8)}, \end{aligned}$$

it follows that Γ is horizontal iff $[\Delta,\Gamma] = \Gamma$.

[Note: The difference

$$[\Delta,\Gamma] - \Gamma$$

is called the <u>deviation</u>. It is necessarily vertical:

$$\begin{aligned} S([\Delta,\Gamma] - \Gamma) &= S[\Delta,\Gamma] - S\Gamma \\ &= \Delta - \Delta = 0. \end{aligned}$$

Here

$$S[\Delta,\Gamma] = -S((L_\Gamma S)(\Gamma))$$

$$= S\Gamma \quad (\text{cf. } 5.12)$$

$$= \Delta \quad (\text{cf. } 5.8).$$

Locally,

$$\left[\begin{array}{l} \Delta = v^i \dfrac{\partial}{\partial v^i} \\[2em] \Gamma = v^i \dfrac{\partial}{\partial q^i} + C^i \dfrac{\partial}{\partial v^i} \end{array}\right.$$

=>

$$[\Delta,\Gamma] = v^i \frac{\partial}{\partial v^i} + (v^i \frac{\partial C^j}{\partial v^i} - C^j) \frac{\partial}{\partial v^j} .$$

So

$$[\Delta,\Gamma] = \Gamma$$

<=>

$$v^i \frac{\partial C^j}{\partial v^i} = 2C^j \quad (j = 1,\ldots,n).]$$

Given $X \in \mathcal{D}^1(M)$, put

$$X^h = H_\Gamma X^\intercal,$$

thus

$$X^h = \frac{1}{2} (X^\intercal - (L_\Gamma S)(X^\intercal))$$

$$= \frac{1}{2} (X^{T} - [\Gamma, X^{V}])$$

$$= \frac{1}{2} (X^{T} + [X^{V}, \Gamma]),$$

and, by definition, X^h is the <u>horizontal lift</u> of X to TM. Locally, if

$$X = X^i \frac{\partial}{\partial x^i}$$

and

$$\Gamma = v^i \frac{\partial}{\partial q^i} + C^i \frac{\partial}{\partial v^i},$$

then

$$X^h = (X^i \circ \pi_M) (\frac{\partial}{\partial x^i})^h,$$

where

$$(\frac{\partial}{\partial x^i})^h = \frac{\partial}{\partial q^i} + \frac{1}{2} \frac{\partial C^j}{\partial v^i} \frac{\partial}{\partial v^j}.$$

5.22 <u>REMARK</u> In general,

$$X^{T} \neq X^{V} + X^{h}.$$

To see this, observe that $\forall f \in C^{\infty}(M)$,

$$\left[\begin{array}{l} (fX)^{V} = (f \circ \pi_M) X^{V} \\ \\ (fX)^{h} = (f \circ \pi_M) X^{h}, \end{array}\right.$$

but, generically,

$$(fX)^{T} \neq (f \circ \pi_M)X^{T}.$$

[Note: Locally, matters are manifest.]

5.23 <u>LEMMA</u> $\forall\ X \in \mathcal{D}^1(M)$,

$$SX^h = X^v.$$

<u>PROOF</u> We have

$$SX^h = \frac{1}{2}\,(SX^{T} + S[X^v,\Gamma])$$

$$= \frac{1}{2}\,(X^v + S[X^v,\Gamma]) \qquad (\text{cf. } 5.7)$$

$$= \frac{1}{2}\,(X^v + X^v) \qquad (\text{cf. } 5.14)$$

$$= X^v.$$

5.24 <u>REMARK</u> Let

$$J_\Gamma = S + \frac{1}{2}\,(L_\Gamma(L_\Gamma S)) \circ V_\Gamma.$$

Then $\forall\ X \in \mathcal{D}^1(M)$,

$$\begin{cases} J_\Gamma X^h = X^v \\ J_\Gamma X^v = -\,X^h. \end{cases}$$

5.25 <u>LEMMA</u> Let $X,Y \in \mathcal{D}^1(M)$ — then

$$S[X^h,Y^h] = [X,Y]^v.$$

[Note: In general, $[X,Y]^h \neq [X^h,Y^h]$ but

$$\begin{aligned} S([X,Y]^h - [X^h,Y^h]) &= S[X,Y]^h - S[X^h,Y^h] \\ &= [X,Y]^v - S[X^h,Y^h] \quad \text{(cf. 5.23)} \\ &= [X,Y]^v - [X,Y]^v \\ &= 0, \end{aligned}$$

so

$$[X,Y]^h - [X^h,Y^h] \in \mathcal{V}(TM).]$$

There is one final point, namely for any diffeomorphism $\phi:M \to M$,

$$(T\phi)_* \circ S = S \circ (T\phi)_*.$$

Take now a $\Gamma \in \mathcal{SO}(TM)$ — then $(T\phi)_*\Gamma \in \mathcal{SO}(TM)$ (cf. 4.9), so (cf. 5.8)

$$\begin{aligned} \Delta = S\Gamma &= S(T\phi)_*\Gamma \\ &= (T\phi)_*S\Gamma = (T\phi)_*\Delta. \end{aligned}$$

§6. VERTICAL DIFFERENTIATION

Let M be a connected C^∞ manifold of dimension n,

$$S:\mathcal{D}^1(TM) \rightarrow \mathcal{D}^1(TM)$$

the vertical morphism -- then S operates by duality on Λ^*TM, call it S^*, thus

$$S^*f = f \quad (f \in C^\infty(TM))$$

and

$$S^*\alpha(X_1,\ldots,X_p) = \alpha(SX_1,\ldots,SX_p) \quad (\alpha \in \Lambda^p TM).$$

[Note: Locally,

$$S^*(dq^i) = 0,\ S^*(dv^i) = dq^i.]$$

<u>N.B.</u> $\forall\ f \in C^\infty(TM)$,

$$df = \frac{\partial f}{\partial q^i}\, dq^i + \frac{\partial f}{\partial v^i}\, dv^i$$

=>

$$S^*(df) = \frac{\partial f}{\partial v^i}\, dq^i.$$

Given $X \in \mathcal{D}^1(TM)$, define $\iota_X S^*$ by

$$(\iota_X S^*)(\alpha) = \iota_X(S^*\alpha).$$

6.1 <u>LEMMA</u> We have

$$\iota_X S^* = S^* \circ \iota_{SX}.$$

PROOF On elements of $C^\infty(TM)$, this is obvious, so let $\alpha \in \Lambda^p TM$ $(p > 0)$ -- then

$$(\iota_X S^*)(\alpha)(X_1,\ldots,X_{p-1})$$

$$= \iota_X(S^*\alpha)(X_1,\ldots,X_{p-1})$$

$$= S^*\alpha(X,X_1,\ldots,X_{p-1})$$

$$= \alpha(SX,SX_1,\ldots,SX_{p-1})$$

$$= (\iota_{SX}\alpha)(SX_1,\ldots,SX_{p-1})$$

$$= S^*(\iota_{SX}\alpha)(X_1,\ldots,X_{p-1}).$$

[Note: Therefore

$$X \in \text{Ker } S(= V(TM)) \Rightarrow \iota_X S^* = 0.$$

In particular:

$$\iota_\Delta S^* = 0.]$$

Let

$$\delta_S f = 0 \quad (f \in C^\infty(TM))$$

and for $p > 0$, put

$$(\delta_S\alpha)(X_1,\ldots,X_p) = \sum_{i=1}^{p} \alpha(X_1,\ldots,SX_i,\ldots,X_p).$$

[Note: Locally,

$$\delta_S(dq^i) = 0, \ \delta_S(dv^i) = dq^i.]$$

N.B. $\forall \ f \in C^\infty(TM)$,

$$df = \frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial v^i} dv^i$$

=>

$$\delta_S(df) = \frac{\partial f}{\partial v^i} dq^i.$$

[Note: Globally,

$$\delta_S(df) = S^*(df).]$$

6.2 LEMMA We have

$$\begin{cases} \delta_S \circ S^* = 0 \\ S^* \circ \delta_S = 0. \end{cases}$$

6.3 LEMMA $\forall \ X \in \mathcal{D}^1(TM)$,

$$\iota_X \circ \delta_S - \delta_S \circ \iota_X = \iota_{SX}.$$

PROOF On elements of $C^\infty(TM)$, this is obvious, so let $\alpha \in \Lambda^p TM$ $(p > 0)$ -- then

$$(\iota_X(\delta_S \alpha))(X_1, \ldots, X_{p-1})$$

$$- (\delta_S(\iota_X\alpha))(X_1,\ldots,X_{p-1})$$

$$= (\delta_S\alpha)(X,X_1,\ldots,X_{p-1})$$

$$- \sum_{i=1}^{p-1} (\iota_X\alpha)(X_1,\ldots,SX_i,\ldots,X_{p-1})$$

$$= \alpha(SX,X_1,\ldots,X_{p-1}) + \sum_{i=1}^{p-1} \alpha(X,X_1,\ldots,SX_i,\ldots,X_{p-1})$$

$$- \sum_{i=1}^{p-1} \alpha(X,X_1,\ldots,SX_i,\ldots,X_{p-1})$$

$$= \alpha(SX,X_1,\ldots,X_{p-1})$$

$$= (\iota_{SX}\alpha)(X_1,\ldots,X_{p-1}).$$

6.4 <u>LEMMA</u> We have

$$\delta_S \circ L_\Delta - L_\Delta \circ \delta_S = \delta_S.$$

Define now

$$d_S : \Lambda^* TM \to \Lambda^* TM$$

by

$$d_S = \delta_S \circ d - d \circ \delta_S.$$

[Note: Locally,

$$d_S(dq^i) = 0, \; d_S(dv^i) = 0.]$$

N.B. $\forall\, f \in C^\infty(TM)$,

$$df = \frac{\partial f}{\partial q^i}\, dq^i + \frac{\partial f}{\partial v^i}\, dv^i$$

$\Rightarrow$

$$d_S f = (\delta_S \circ d - d \circ \delta_S) f$$

$$= \delta_S(df)$$

$$= \frac{\partial f}{\partial v^i}\, dq^i.$$

[Note: Globally,

$$d_S f = S^*(df), \; d_S(df) = -\, d(S^*(df)).]$$

6.5 LEMMA d_S is an antiderivation of Λ^*TM of degree 1.

PROOF Write

$$d_S = [\delta_S, d]$$

and observe that δ_S is a derivation of Λ^*TM of degree 0 while d is an antiderivation of Λ^*TM of degree 1.

6.6 LEMMA We have

$$d \circ d_S + d_S \circ d = 0.$$

PROOF In fact,

$$d \circ d_S + d_S \circ d$$

$$= d \circ (\delta_S \circ d - d \circ \delta_S) + (\delta_S \circ d - d \circ \delta_S) \circ d$$

$$= d \circ \delta_S \circ d - d \circ \delta_S \circ d$$

$$= 0.$$

6.7 LEMMA $\forall\, f \in C^{\infty}(TM)$,

$$\delta_S dS^*df = 0.$$

PROOF Bearing in mind that the LHS is a 2-form, let $X,Y \in \mathcal{D}^1(TM)$ — then

$$(\delta_S dS^*df)(X,Y)$$

$$= (dS^*df)(SX,Y) + (dS^*df)(X,SY)$$

$$= L_{SX}((S^*df)(Y))$$

$$- L_Y((S^*df)(SX)) - (S^*df)([SX,Y])$$

$$+ L_X((S^*df)(SY))$$

$$- L_{SY}((S^*df)(X)) - (S^*df)([X,SY])$$

$$= L_{SX}(df(SY))$$

$$- L_Y(df(S^2X)) - df(S[SX,Y])$$

$$+ L_X(df(S^2Y))$$

$$- L_{SY}(df(SX)) - df(S[X,SY])$$

$$= L_{SX}(L_{SY}f) - L_{S[SX,Y]}f$$

$$- L_{SY}(L_{SX}f) - L_{S[X,SY]}f$$

$$= ((SX)(SY) - S[SX,Y]$$

$$- (SY)(SX) - S[X,SY])f$$

$$= ([SX,SY] - S[SX,Y] - S[X,SY])f$$

$$= 0 \text{ (cf. 5.9)}.$$

[Note: Recall that $S^2 = 0$ (cf. 5.5).]

6.8 <u>LEMMA</u> We have

$$d_S^2 = 0.$$

<u>PROOF</u> It suffices to show that $\forall\ f \in C^\infty(TM)$,

$$\begin{cases} d_S^2 f = 0 \\ d_S^2(df) = 0. \end{cases}$$

But

$$d_S^2 f = d_S d_S f$$

$$= d_S S^* df$$

$$= (\delta_S \circ d - d \circ \delta_S) S^* df$$

$$= \delta_S dS^* df \quad \text{(cf. 6.2)}$$

$$= 0 \quad \text{(cf. 6.7)}.$$

And then (cf. 6.6)

$$d_S^2(df) = d_S(d_S df)$$

$$= - d_S(dd_S f)$$

$$= d(d_S^2 f)$$

$$= 0.$$

6.9 LEMMA We have

$$S^* \circ d_S = 0 \text{ and } d_S \circ S^* = S^* \circ d.$$

Moreoever,

$$\delta_S \circ d_S = d_S \circ \delta_S.$$

6.10 LEMMA We have

$$\begin{cases} \iota_\Delta \circ d_S + d_S \circ \iota_\Delta = \delta_S \\ \\ d_S \circ L_\Delta - L_\Delta \circ d_S = d_S. \end{cases}$$

PROOF To discuss the first relation, let $f \in C^\infty(TM)$ -- then

$$\begin{aligned} &(\iota_\Delta \circ d_S + d_S \circ \iota_\Delta)f \\ &\quad = \iota_\Delta d_S f \\ &\quad = \iota_\Delta S^* f \\ &\quad = 0 \quad (\text{cf. } 6.1). \end{aligned}$$

And

$$\begin{aligned} &(\iota_\Delta \circ d_S + d_S \circ \iota_\Delta)df \\ &\quad = \iota_\Delta d_S(df) + d_S(\Delta f) \\ &\quad = \iota_\Delta(- d(S^*(df))) + S^*(d(L_\Delta f)) \\ &\quad = (- L_\Delta + d\iota_\Delta)(S^*(df)) + \delta_S d(L_\Delta f) \\ &\quad = - L_\Delta \delta_S(df) + \delta_S L_\Delta(df) \\ &\quad = (\delta_S \circ L_\Delta - L_\Delta \circ \delta_S)(df) \\ &\quad = \delta_S(df) \quad (\text{cf. } 6.4). \end{aligned}$$

6.11 <u>REMARK</u> The analog of the identity

$$L_X = \iota_X \circ d + d \circ \iota_X$$

per d_S is the relation

$$L_{SX} + [\delta_S, L_X] = \iota_X \circ d_S + d_S \circ \iota_X.$$

6.12 <u>REMARK</u> Let

$$T \in \mathrm{Hom}_{C^\infty(TM)}(\mathcal{D}^1(TM), \mathcal{D}^1(TM)).$$

Defining δ_T in the obvious way, put

$$d_T = \delta_T \circ d - d \circ \delta_T.$$

Then

$$d \circ d_T + d_T \circ d = 0$$

but, in general, $d_T^2 \neq 0$. On the other hand, $\forall\ X \in \mathcal{D}^1(TM)$

$$L_X \circ d_T - d_T \circ L_X = d_{L_X T}.$$

E.g.: Take $T = S$, $X = \Delta$ — then

$$L_\Delta \circ d_S - d_S \circ L_\Delta = d_{L_\Delta S}$$

$$= d_{-S} \quad \text{(cf. 5.11)}$$

=>

$$d_S \circ L_\Delta - L_\Delta \circ d_S = d_S \quad \text{(cf. 6.10)}.$$

[Note: If T is the identity map, then

$$\delta_T\alpha = p\alpha \qquad (\alpha \in \Lambda^p TM).$$

Therefore

$$\begin{aligned} d_T\alpha &= \delta_T d\alpha - d\delta_T\alpha \\ &= (p+1)d\alpha - pd\alpha \\ &= d\alpha, \end{aligned}$$

so $d_T = d$.]

The image $S^*(\Lambda^* TM)$ is called the vector space of <u>horizontal differential forms</u> on TM. It is d_S-stable (cf. 6.9).

<u>N.B.</u> $\forall\ f \in C^\infty(TM)$, $d_S f$ is horizontal. In fact, $d_S f = S^*(df)$.

6.13 <u>LEMMA</u> Suppose that α is horizontal -- then

$$\left[\begin{aligned} &\iota_\Delta\alpha = 0 \\ &\delta_S\alpha = 0. \end{aligned}\right.$$

<u>PROOF</u> Write $\alpha = S^*\beta$ -- then

$$\left[\begin{aligned} &\iota_\Delta\alpha = \iota_\Delta S^* \ = 0\beta = 0 \qquad (\text{cf. } 6.1) \\ &\delta_S\alpha = \delta_S S^*\beta = 0\beta = 0 \qquad (\text{cf. } 6.2). \end{aligned}\right.$$

Let $\alpha \in \Lambda^1 TM$ -- then α is horizontal iff locally,

$$\alpha = a_i(q^1,\ldots,q^n,v^1,\ldots,v^n)dq^i.$$

So, $\forall\ \omega \in \Lambda^1 M$, $(\pi_M)^*\omega$ is horizontal and

$$d_S((\pi_M)^*\omega) = 0.$$

6.14 <u>LEMMA</u> Let $\alpha \in \Lambda^1 TM$ -- then α is horizontal iff $\alpha(X) = 0$ for all vertical vector fields X on TM.

§7. THE FIBER DERIVATIVE

Let M be a connected C^∞ manifold of dimension n,

$$\pi_M^*:T^*M \rightarrow M$$

its cotangent bundle — then the sections $\mathcal{D}_1(M)$ of T^*M are the 1-forms on M, i.e., $\Lambda^1 M$.

<u>N.B.</u> Suppose that $(U,\{x^1,\ldots,x^n\})$ is a chart on M -- then

$$((\pi_M^*)^{-1}U,\{q^1,\ldots,q^n,p_1,\ldots,p_n\})$$

is a chart on T^*M.

[Note: Here

$$\left[\begin{array}{l} q^i = x^i \circ \pi_M^* \\ \\ p_i = \dfrac{\partial}{\partial x^i} \end{array}\right. \qquad (i = 1,\ldots,n).]$$

Denote by $h\Lambda^1 TM$ the vector space of horizontal 1-forms on TM and consider the pullback square

$$\begin{array}{ccc} TM \times_M T^*M & \xrightarrow{pr_2} & T^*M \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi_M^*} \\ TM & \xrightarrow[\pi_M]{} & M \end{array} .$$

Then one can identify $h\Lambda^1 TM$ with the sections of pr_1, thus there is an isomorphism $\alpha \to F_\alpha = pr_2 \circ \alpha$ from $h\Lambda^1 TM$ to the vector space of fiber preserving C^∞ functions $TM \to T^*M$:

$$
\begin{array}{ccc}
TM & \xrightarrow{F_\alpha} & T^*M \\
{\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_M^*} \\
M & = & M
\end{array}
\quad .
$$

[Note: For more details and a generalization, cf. 13.4.]

Locally, if

$$\alpha = a_i dq^i,$$

then

$$q^i \circ F_\alpha = q^i, \ p_i \circ F_\alpha = a_i.$$

Let Θ be the fundamental 1-form on T^*M.

7.1 <u>LEMMA</u> $\forall\ \alpha \in h\Lambda^1 TM$,

$$F_\alpha^* \Theta = \alpha.$$

[Locally,

$$\Theta = p_i dq^i,$$

so

$$F_\alpha^*(p_i dq^i) = (p_i \circ F_\alpha) d(q^i \circ F_\alpha)$$

$$= a_i dq^i$$

$$= \alpha.]$$

Given an $f \in C^\infty(TM)$, the 1-form $d_S f$ is horizontal: $d_S f \in h\Lambda^1 TM$. Put $Ff = F_{d_S f}$ -- then $Ff: TM \to T^*M$ is the fiber derivative of f. The correspondence $f \to Ff$ is linear and $Ff = Fg$ iff $\exists\, h \in C^\infty(M)$: $f - g = h \circ \pi_M$.

Locally,

$$d_S f = \frac{\partial f}{\partial v^i}\, dq^i,$$

thus locally,

$$q^i \circ Ff = q^i,\ p_i \circ Ff = \frac{\partial f}{\partial v^i}.$$

[Note: Invariantly, Ff sends T_xM to T_x^*M via the prescription

$$Ff(x,X_x)(Y_x) = \frac{d}{dt} f(x,X_x + tY_x)\Big|_{t=0} \quad (X_x, Y_x \in T_xM).]$$

7.2 REMARK Ff is fiber preserving but Ff need not be linear on fibers.

[Note: Ff is a diffeomorphism iff Ff is bijective on fibers.]

Each $X \in \mathcal{D}^1(T^*M)$, i.e., each section $X: T^*M \to TT^*M$, induces a fiber preserving C^∞ function $F_X: T^*M \to TM$, viz. $F_X = T\pi_M^* \circ X$. To a given $H \in C^\infty(T^*M)$, there corresponds a vector field X_H on T^*M characterized by the condition $\iota_{X_H}\Omega = -\,dH$.

Put $FH = F_{X_H}$ -- then $FH:T^*M \rightarrow TM$ is the <u>fiber derivative</u> of H.

[Note: Locally,

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} .$$

Therefore, along an integral curve of X_H, we have

$$\left[\begin{array}{l} \dfrac{dq^i}{dt} = \dfrac{\partial H}{\partial p_i} \\[2ex] \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q^i}, \end{array}\right.$$

the equations of Hamilton.]

§8. LAGRANGIANS

Let M be a connected C^∞ manifold of dimension n -- then a lagrangian is simply any element $L \in C^\infty(TM)$. This said, put

$$\left[\begin{array}{l} \theta_L = d_S L \\ \\ \omega_L = d\theta_L. \end{array}\right.$$

N.B. From the definitions,

$$(FL)^*\Theta = (F_{d_S L})^*\Theta$$

$$= d_S L \quad \text{(cf. 7.1)}.$$

Accordingly, if $\Omega = d\Theta$, then

$$(FL)^*\Omega = \omega_L.$$

[Note: Recall that the pair (T^*M,Ω) is a symplectic manifold.]

8.1 LEMMA We have

$$\delta_S \omega_L = 0.$$

PROOF In fact,

$$-\delta_S \omega_L = -\delta_S d d_S L$$

$$= \delta_S d_S dL \quad \text{(cf. 6.6)}$$

$$= d_S \delta_S dL \quad (\text{cf. } 6.9)$$

$$= d_S(\delta_S \circ d - d \circ \delta_S)L$$

$$= d_S^2 L$$

$$= 0 \quad (\text{cf. } 6.8).$$

Let

$$\text{Ker } \omega_L = \{X \in \mathcal{D}^1(TM) : \iota_X \omega_L = 0\}.$$

Then ω_L is symplectic iff Ker ω_L = 0 .

8.2 LEMMA ω_L is symplectic iff FL is a local diffeomorphism.

PROOF If ω_L is symplectic, then

$$FL:(TM,\omega_L) \rightarrow (T^*M,\Omega)$$

is a canonical transformation, hence is a local diffeomorphism. And conversely... .

L is said to be nondegenerate if ω_L is symplectic; otherwise, L is said to be degenerate.

8.3 EXAMPLE Take $M = \underline{R}$ -- then

$$\begin{cases} L(q,v) = q \\ L(q,v) = v \end{cases}$$

are both degenerate. For

$$\theta_L = \frac{\partial L}{\partial v} dq$$

so in either case, $\omega_L = 0$.

8.4 EXAMPLE Let g be a semiriemannian structure on M and take for L the function

$$(x,X_x) \to \frac{1}{2} g_x(X_x,X_x) \quad (X_x \in T_xM).$$

Then

$$FL(x,X_x)(x,Y_x) = g_x(X_x,Y_x) \quad (Y_x \in T_xM).$$

I.e.:

$$FL = g^{\flat},$$

thus $FL:TM \to T^*M$ is a diffeomorphism, so L is nondegenerate (cf. 8.2).

[Note: Suppose that $X \in \mathcal{D}^1(M)$ is an infinitesimal isometry of g, i.e., $L_X g = 0$. Working locally, write

$$L(q^1,\ldots,q^n,v^1,\ldots,v^n) = \frac{1}{2} (g_{ij} \circ \pi_M)v^i v^j.$$

Then

$$\begin{aligned}
2X^{\mathsf{T}}L = {} & (X^a g_{ij,a} \circ \pi_M)v^i v^j \\
& + (g_{ij} \circ \pi_M)(X^{\mathsf{T}}v^i)v^j + (g_{ij} \circ \pi_M)v^i(X^{\mathsf{T}}v^j) \\
= {} & (X^a g_{ij,a} \circ \pi_M)v^i v^j \\
& + (g_{ij} \circ \pi_M)(v^k X^i_{,k} \circ \pi_M)v^j + (g_{ij} \circ \pi_M)v^i(v^\ell X^j_{,\ell} \circ \pi_M)
\end{aligned}$$

$$= (X^a g_{ij,a} \circ \pi_M) v^i v^j$$

$$+ (g_{kj} \circ \pi_M)(X^k{}_{,i} \circ \pi_M) v^i v^j + (g_{i\ell} \circ \pi_M)(X^\ell{}_{,j} \circ \pi_M) v^i v^j$$

$$= (L_X g_{ij} \circ \pi_M) v^i v^j$$

$$= 0.$$

Therefore

$$X^T L = 0.]$$

There is a local criterion for nondegeneracy which is useful in practice.

8.5 LEMMA L is nondegenerate iff for all coordinate systems $\{q^1, \ldots, q^n, v^1, \ldots, v^n\}$,

$$\det \left[\frac{\partial^2 L}{\partial v^i \partial v^j} \right] \neq 0$$

everywhere.

PROOF On general grounds, ω_L is symplectic iff ω_L^n is a volume form. Locally,

$$\theta_L = \frac{\partial L}{\partial v^i} dq^i,$$

hence locally,

$$\omega_L = \frac{\partial^2 L}{\partial q^i \partial v^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dv^i \wedge dq^j.$$

But this implies that

$$\omega_L^n = \pm\, n!\ \det \left[\frac{\partial^2 L}{\partial v^i \partial v^j} \right] dv^1 \wedge \cdots \wedge dv^n \wedge dq^1 \wedge \cdots \wedge dq^n,$$

thus ω_L^n is a volume form iff

$$\det \left[\frac{\partial^2 L}{\partial v^i \partial v^j} \right] \neq 0$$

everywhere.

8.6 <u>EXAMPLE</u> Take $M = \underline{R}^n$ and define $L:\underline{R}^{2n} \to \underline{R}$ by

$$L(q^1,\ldots,q^n,v^1,\ldots,v^n) = \sum_{i=1}^{n} m_i \frac{(v^i)^2}{2} - V(q^1,\ldots,q^n),$$

where the $m_i \in \underline{R}$ are constants and $V \in C^\infty(\underline{R}^n)$ -- then

$$\det \left[\frac{\partial^2 L}{\partial v^i \partial v^j} \right] = m_1 \cdots m_n,$$

so L is nondegenerate iff $m_1 \neq 0,\ldots,m_n \neq 0$.

Given L, put

$$E_L = \Delta L - L.$$

Then E_L is the <u>energy function</u> attached to L.

8.7 <u>LEMMA</u> We have

$$\iota_\Delta \omega_L = d_S E_L.$$

PROOF Since θ_L is horizontal,

$$\iota_\Delta \theta_L = 0 \quad \text{(cf. 6.13)}.$$

Therefore

$$\begin{aligned}
\iota_\Delta \omega_L &= \iota_\Delta d\theta_L \\
&= (L_\Delta - d \circ \iota_\Delta)\theta_L \\
&= L_\Delta \theta_L \\
&= L_\Delta d_S L \\
&= (d_S \circ L_\Delta - d_S)L \quad \text{(cf. 6.10)} \\
&= d_S(\Delta - 1)L \\
&= d_S E_L.
\end{aligned}$$

Let

$$D_L = \{X \in \mathcal{D}^1(TM) : \iota_X \omega_L = - dE_L\}.$$

Then L is said to admit global dynamics if D_L is nonempty.

8.8 EXAMPLE Take $M = \underline{R}$ (cf. 8.3).

- If $L(q,v) = q$, then $\omega_L = 0$, $E_L = - L (\Delta L = v \frac{\partial q}{\partial v} = 0)$, thus D_L is empty.

• If $L(q,v) = v$, then $\omega_L = 0$, $E_L = 0$ $(\Delta L = v \frac{\partial v}{\partial v} = v)$, thus $D_L = \mathcal{D}^1(\underline{R}^2)$.

8.9 LEMMA Let $X \in D_L$ — then $L_X \omega_L = 0$.

PROOF One has only to write

$$\begin{aligned} L_X \omega_L &= (\iota_X \circ d + d \circ \iota_X)\omega_L \\ &= 0 + d(-\,dE_L) \\ &= 0. \end{aligned}$$

8.10 REMARK E_L is a first integral for any $X \in D_L$. Proof: $XE_L = \langle X, dE_L \rangle = -\langle X, \iota_X \omega_L \rangle = -\,\omega_L(X,X) = 0$.

8.11 LEMMA If L admits global dynamics, then

$$\langle \mathrm{Ker}\ \omega_L, -\,dE_L \rangle = 0.$$

8.12 LEMMA If L is nondegenerate, then L admits global dynamics: $\exists$ a (unique) $\Gamma_L \in \mathcal{D}^1(TM)$ such that

$$\iota_{\Gamma_L} \omega_L = -\,dE_L.$$

And Γ_L is second order.

PROOF The existence (and uniqueness) of Γ_L is implied by the assumption that

ω_L is symplectic. As for the claim that Γ_L is second order, to begin with

$$\iota_{\Gamma_L} \circ \delta_S - \delta_S \circ \iota_{\Gamma_L} = \iota_{S\Gamma_L} \quad \text{(cf. 6.3)}.$$

Therefore

$$\delta_S \iota_{\Gamma_L} \omega_L = (\iota_{\Gamma_L} \circ \delta_S - \iota_{S\Gamma_L})\omega_L$$

$$= - \iota_{S\Gamma_L} \omega_L \quad \text{(cf. 8.1)}.$$

But

$$\iota_\Delta \omega_L = d_S E_L \quad \text{(cf. 8.7)}$$

$$= (d_S + d \circ \delta_S) E_L$$

$$= \delta_S dE_L$$

$$= - \delta_S \iota_{\Gamma_L} \omega_L$$

$$= \iota_{S\Gamma_L} \omega_L.$$

Since ω_L is symplectic, it follows that

$$S\Gamma_L = \Delta,$$

thus Γ_L is second order (cf. 5.8).

[Note: Working locally, write

$$\Gamma_L = v^i \frac{\partial}{\partial q^i} + C^i \frac{\partial}{\partial v^i} .$$

Put

$$W(L) = [W_{ij}(L)],$$

where

$$W_{ij}(L) = \frac{\partial^2 L}{\partial v^i \partial v^j} .$$

Then $W(L)$ is invertible (cf. 8.5) and

$$C^i = (W(L)^{-1})^{ij} \left(\frac{\partial L}{\partial q^j} - \frac{\partial^2 L}{\partial v^j \partial q^k} v^k\right).]$$

E.g.: In the setting of 8.6, suppose that $m_1 = 1,\ldots,m_n = 1$ -- then L is nondegenerate and

$$\Gamma_L = v^i \frac{\partial}{\partial q^i} - \frac{\partial V}{\partial q^i} \frac{\partial}{\partial v^i} .$$

Here is another illustration. Take $M = \underline{R}$, fix nonzero constants m,g,ℓ and put

$$L(q,v) = \frac{m}{2} \ell^2 v^2 + mg\ell \cos q.$$

Then

$$\frac{\partial^2 L}{\partial v \partial v} = m\ell^2, \ \frac{\partial L}{\partial q} = - mg\ell \sin q$$

$$\Rightarrow$$

$$C = (m\ell^2)^{-1} (- mg\ell \sin q)$$

$$= - \frac{g}{\ell} \sin q.$$

8.13 <u>LEMMA</u> If Γ is second order, then for any L,

$$\Delta L = \iota_\Gamma \theta_L.$$

<u>PROOF</u> We have

$$\begin{aligned}\iota_\Gamma \theta_L &= \theta_L(\Gamma)\\ &= d_S L(\Gamma)\\ &= S^*(dL)(\Gamma)\\ &= dL(S\Gamma)\\ &= dL(\Delta) \qquad (\text{cf. } 5.8)\\ &= \Delta L.\end{aligned}$$

8.14 <u>LEMMA</u> If Γ is second order, then

$$\iota_\Gamma \omega_L = -dE_L \iff L_\Gamma \theta_L = dL.$$

<u>PROOF</u> Assume first that $L_\Gamma \theta_L = dL$ — then

$$\begin{aligned}\iota_\Gamma \omega_L &= \iota_\Gamma d\theta_L\\ &= (L_\Gamma - d \circ \iota_\Gamma)\theta_L\\ &= L_\Gamma \theta_L - d\iota_\Gamma \theta_L\\ &= dL - d\Delta L \qquad (\text{cf. } 8.13)\end{aligned}$$

$$= d(1 - \Delta)L$$

$$= - dE_L.$$

On the other hand,

$$\iota_\Gamma \omega_L = - dE_L$$

$$\Rightarrow$$

$$(L_\Gamma - d \circ \iota_\Gamma)\theta_L = d(L - \Delta L)$$

$$\Rightarrow$$

$$L_\Gamma \theta_L - d\Delta L = dL - d\Delta L \quad \text{(cf. 8.13)}$$

$$\Rightarrow$$

$$L_\Gamma \theta_L = dL.$$

Suppose that $\Gamma \in \mathcal{D}^1(TM)$ is second order -- then Γ is said to admit a lagrangian L if

$$L_\Gamma \theta_L = dL$$

or still,

$$\iota_\Gamma \omega_L = - dE_L.$$

[Note: The set of L for which $L_\Gamma \theta_L = dL$ is a vector space over $\underline{R}$.]

N.B. Locally,

$$\theta_L = \frac{\partial L}{\partial v^i} dq^i$$

=>

$$L_\Gamma \theta_L = L_\Gamma\left(\frac{\partial L}{\partial v^i}\right) dq^i + \frac{\partial L}{\partial v^i} L_\Gamma(dq^i)$$

$$= L_\Gamma\left(\frac{\partial L}{\partial v^i}\right) dq^i + \frac{\partial L}{\partial v^i} d(q^i(\Gamma))$$

$$= L_\Gamma\left(\frac{\partial L}{\partial v^i}\right) dq^i + \frac{\partial L}{\partial v^i} dv^i$$

=>

$$0 = L_\Gamma \theta_L - dL = \left(L_\Gamma\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i}\right) dq^i$$

=>

$$L_\Gamma\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0 \quad (i = 1,\ldots,n).$$

Write

$$\Gamma = v^j \frac{\partial}{\partial q^j} + C^j \frac{\partial}{\partial v^j}$$

and let γ be an integral curve of Γ so that

$$\left[\begin{array}{l} \dfrac{d(q^j(\gamma(t)))}{dt} = v^j(\gamma(t)) \\[2ex] \dfrac{d(v^j(\gamma(t)))}{dt} = C^j(\gamma(t)). \end{array}\right.$$

Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)\Big|_{\gamma(t)}$$

$$= \frac{\partial^2 L}{\partial v^i \partial q^j}\bigg|_{\gamma(t)} \frac{d}{dt} (q^j(\gamma(t)))$$

$$+ \frac{\partial^2 L}{\partial v^i \partial v^j}\bigg|_{\gamma(t)} \frac{d}{dt} v^j(\gamma(t))$$

$$= \frac{\partial^2 L}{\partial v^i \partial q^j}\bigg|_{\gamma(t)} v^j(\gamma(t))$$

$$+ \frac{\partial^2 L}{\partial v^i \partial v^j}\bigg|_{\gamma(t)} C^j(\gamma(t))$$

$$= L_\Gamma(\frac{\partial L}{\partial v^i})\bigg|_{\gamma(t)}.$$

I.e.: Along γ, the equations of Lagrange

$$\frac{d}{dt}(\frac{\partial L}{\partial v^i}) - \frac{\partial L}{\partial q^i} = 0 \qquad (i = 1,\ldots,n)$$

are satisfied.

8.15 <u>LEMMA</u> A second order Γ always admits a lagrangian.

<u>PROOF</u> Let $\omega \in \Lambda^1 M$ and put

$$L = \iota_\Gamma(\pi_M)^*\omega.$$

Then

$$\theta_L = d_S L$$

$$= d_S \iota_\Gamma(\pi_M)^*\omega.$$

Locally,

$$\omega = a_i dx^i$$

=>

$$(\pi_M)^*\omega = (a_i \circ \pi_M) dq^i$$

=>

$$\iota_\Gamma(\pi_M)^*\omega = (a_i \circ \pi_M) v^i$$

=>

$$d_S i_\Gamma(\pi_M)^*\omega = \frac{\partial(\iota_\Gamma(\pi_M)^*\omega)}{\partial v^i} dq^i$$

$$= (a_i \circ \pi_M) dq^i$$

=>

$$\theta_L = (\pi_M)^*\omega.$$

But

$$dL = d\iota_\Gamma(\pi_M)^*\omega$$

$$= (L_\Gamma - \iota_\Gamma \circ d)(\pi_M)^*\omega$$

$$= L_\Gamma(\pi_M)^*\omega - \iota_\Gamma d(\pi_M)^*\omega$$

$$= L_\Gamma\theta_L - \iota_\Gamma(\pi_M)^*d\omega$$

=>

$$L_\Gamma \theta_L - dL = \iota_\Gamma (\pi_M)^* d\omega.$$

So, if ω is closed, then L is a lagrangian for Γ.

8.16 REMARK Fix a second order Γ -- then the proof shows that each closed 1-form on M gives rise to a lagrangian for Γ. Lagrangians of this type are termed trivial and there may be no others. For instance, take $M = \underline{R}^2$ and consider

$$\Gamma = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + (q^1 + q^2) \frac{\partial}{\partial v^1} + (q^1 q^2) \frac{\partial}{\partial v^2} .$$

Then it can be shown that Γ does not admit a nontrivial lagrangian.

8.17 EXAMPLE Take $M = \underline{R}^2$ and let

$$\Gamma = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2}.$$

Then

$$L = \frac{1}{2} ((v^1)^2 + (v^2)^2)$$

is a lagrangian for Γ, necessarily nondegenerate (cf. 8.5). Now fix real numbers a,b,c and let

$$L = \frac{1}{2} (a(v^1)^2 + 2c(v^1 v^2) + b(v^2)^2).$$

We have

$$\theta_L = \frac{\partial L}{\partial v^1} dq^1 + \frac{\partial L}{\partial v^2} dq^2$$

$$= (av^1 + cv^2)dq^1 + (bv^2 + cv^1)dq^2$$

=>

$$\omega_L = (adv^1 + cdv^2) \wedge dq^1 + (bdv^2 + cdv^1) \wedge dq^2$$

=>

$$\iota_\Gamma \omega_L = - (adv^1 + cdv^2)dq^1(\Gamma) - (bdv^2 + cdv^1)dq^2(\Gamma)$$

$$= - (av^1 + cv^2)dv^1 - (bv^2 + cv^1)dv^2$$

$$= - dE_L.$$

Accordingly, L is a lagrangian for Γ which, in view of 8.5, is nondegenerate iff $ab - c^2 \neq 0$.

8.18 EXAMPLE Take $M = \underline{R}^2$ and let

$$\Gamma = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} - q^1 \frac{\partial}{\partial v^1} - q^2 \frac{\partial}{\partial v^2} .$$

Then

$$\left[\begin{array}{l} L_+ = \frac{1}{2}((v^1)^2 + (v^2)^2) - \frac{1}{2}((q^1)^2 + (q^2)^2) \\ \\ L_- = \frac{1}{2}((v^1)^2 - (v^2)^2) - \frac{1}{2}((q^1)^2 - (q^2)^2) \end{array}\right.$$

are both nondegenerate lagrangians for Γ.

[Note: Another possibility is

$$L = v^1 v^2 - q^1 q^2.]$$

8.19 <u>RAPPEL</u> A 1-form $\omega \in \Lambda^1 M$ determines a C^∞ function $\hat{\omega}:TM \to \underline{R}$, viz.

$$\hat{\omega}(x,X_x) = \omega_x(X_x) \quad (X_x \in T_x M).$$

[Note: For use below, observe that $\Delta\hat{\omega} = \hat{\omega}$ and $F\hat{\omega} = \omega \circ \pi_M$ $(TM \xrightarrow{\pi_M} M \xrightarrow{\omega} T^*M)$.]

8.20 <u>LEMMA</u> Suppose given nondegenerate lagrangians L, L'. Determine $\Gamma_L, \Gamma_{L'} \in \mathcal{D}^1(TM)$ per 8.12 -- then $\omega_L = \omega_{L'}$ and $\Gamma_L = \Gamma_{L'}$ iff $L' = L + \hat{\omega} + C$, where $\omega \in \Lambda^1 M$ is closed and C is a constant.

<u>PROOF</u> Assuming that $L' = L + \hat{\omega} + C$, we have

$$\begin{aligned} E_{L'} &= \Delta L' - L' \\ &= \Delta(L + \hat{\omega} + C) - (L + \hat{\omega} + C) \\ &= \Delta L - L + (\Delta\hat{\omega} - \hat{\omega}) - C \\ &= \Delta L - L + (\hat{\omega} - \hat{\omega}) - C \\ &= E_L - C. \end{aligned}$$

Next,

$$\begin{aligned} \omega_{L'} &= (FL')^*\Omega \\ &= (FL + F\hat{\omega})^*\Omega \\ &= (FL + \omega \circ \pi_M)^*\Omega \\ &= \omega_L + \pi_M^*(\omega^*\Omega). \end{aligned}$$

And

$$\omega^*\Omega = \omega^* d\Theta$$

$$= d\omega^*\Theta$$

$$= d\omega \quad \text{(cf. infra)}$$

$$= 0.$$

Consequently, $\omega_{L'} = \omega_L$. But

$$\begin{cases} \iota_{\Gamma_L}\omega_L = -\,dE_L \\ \\ \iota_{\Gamma_{L'}}\omega_{L'} = -\,dE_{L'}. \end{cases}$$

Since $E_{L'} = E_L - C$, it follows that

$$\iota_{\Gamma_L}\omega_L = \iota_{\Gamma_{L'}}\omega_{L'}$$

or still,

$$\iota_{\Gamma_L}\omega_L = \iota_{\Gamma_{L'}}\omega_L.$$

Therefore $\Gamma_L = \Gamma_{L'}$. The argument in the other direction is similar.

<u>N.B.</u> To check that $\omega^*\Theta = \Theta$, it suffices to work locally:

$$\omega = \langle\frac{\partial}{\partial x^i}, \omega\rangle dx^i$$

$$= (p_i \circ \omega)\,dx^i$$

$\Rightarrow$

$$\begin{aligned}\omega^*\Theta &= \omega^*(p_i dq^i)\\ &= (p_i \circ \omega)d(q^i \circ \omega)\\ &= (p_i \circ \omega)d(x^i \circ \pi_M^* \circ \omega)\\ &= (p_i \circ \omega)dx^i\\ &= \omega.\end{aligned}$$

Given $\alpha \in \Lambda^2 TM$, define $S \rfloor \alpha \in \mathcal{D}_2^0(TM)$ by

$$(S \rfloor \alpha)(X,Y) = \alpha(SX,Y).$$

8.21 <u>LEMMA</u> $\forall\ X \in \mathcal{D}^1(TM)$,

$$L_X(S \rfloor \alpha) = (L_X S) \rfloor \alpha + S \rfloor (L_X \alpha).$$

Assuming now that L is a nondegenerate lagrangian, we have

$$\begin{aligned}L_{\Gamma_L}(S \rfloor \omega_L) &= (L_{\Gamma_L} S) \rfloor \omega_L + S \rfloor (L_{\Gamma_L} \omega_L)\\ &= (L_{\Gamma_L} S) \rfloor \omega_L \quad \text{(cf. 8.9)}.\end{aligned}$$

On the other hand, according to 8.1,

$$\delta_S \omega_L = 0.$$

Therefore $S \rfloor \omega_L$ is symmetric, hence the same is true of $L_{\Gamma_L}(S \rfloor \omega_L)$ or still, of $(L_{\Gamma_L} S) \rfloor \omega_L$. So, $\forall\ X,Y \in \mathcal{D}^1(TM)$,

$$\omega_L((L_{\Gamma_L} S)(X),Y) + \omega_L(X,(L_{\Gamma_L} S)(Y)) = 0.$$

And this leads to the following conclusion.

8.22 LEMMA $\forall\ X,Y \in \mathcal{D}^1(TM)$,

$$\left[\begin{array}{l} \omega_L(V_{\Gamma_L} X,Y) + \omega_L(X,V_{\Gamma_L} Y) = \omega_L(X,Y) \\ \\ \omega_L(H_{\Gamma_L} X,Y) + \omega_L(X,H_{\Gamma_L} Y) = \omega_L(X,Y) \end{array}\right.$$

and

$$\omega_L(V_{\Gamma_L} X,Y) = \omega_L(X,H_{\Gamma_L} Y).$$

Consequently,

$$\left[\begin{array}{l} \omega_L(V_{\Gamma_L} X,V_{\Gamma_L} Y) = \omega_L(X,H_{\Gamma_L} \circ V_{\Gamma_L} Y) = 0 \\ \\ \omega_L(H_{\Gamma_L} X,H_{\Gamma_L} Y) = \omega_L(V_{\Gamma_L} \circ H_{\Gamma_L} X,Y) = 0. \end{array}\right.$$

N.B. X and Y are vertical iff

$$\left[\begin{array}{l} X = V_{\Gamma_L} X \\ \\ Y = V_{\Gamma_L} Y. \end{array}\right.$$

So, $\forall\ X,Y \in V(TM)$,

$$\iota_X \omega_L(Y) = 0,$$

which implies that $\iota_X \omega_L$ is horizontal (cf. 6.14).

8.23 LEMMA Given a horizontal 1-form α, define $X_\alpha \in \mathcal{D}^1(TM)$ by $\iota_{X_\alpha} \omega_L = \alpha$ -- then X_α is vertical.

PROOF $\forall\ Y \in \mathcal{D}^1(TM)$,

$$\omega_L(V_{\Gamma_L} X_\alpha, Y) + \omega_L(X_\alpha, V_{\Gamma_L} Y) = \omega_L(X_\alpha, Y)$$

or still,

$$\omega_L(V_{\Gamma_L} X_\alpha, Y) + \alpha(V_{\Gamma_L} Y) = \omega_L(X_\alpha, Y)$$

or still,

$$\omega_L(V_{\Gamma_L} X_\alpha, Y) = \omega_L(X_\alpha, Y) \qquad \text{(cf. 6.14)}$$

=>

$$V_{\Gamma_L} X_\alpha = X_\alpha.$$

I.e.: X_α is vertical.

Therefore the map

$$X \to \iota_X \omega_L$$

from vertical vector fields on TM to horizontal 1-forms on TM is a linear isomorphism.

A lagrangian L is nondegenerate provided FL is a local diffeomorphism (cf. 8.2) but there are important circumstances when FL is actually a diffeomorphism (cf. 8.4).

[Note: Take $M = \underline{R}$ and let $L(q,v) = e^v$ -- then L is nondegenerate but $FL:\underline{R}^2 \to \underline{R}^2$ is not surjective, hence is not a diffeomorphism.]

8.24 <u>LEMMA</u> Suppose that FL is a diffeomorphism. Put $H = E_L \circ (FL)^{-1}$ -- then

$$FH:T^*M \to TM$$

is a diffeomorphism and $FH = (FL)^{-1}$. One has

$$\begin{cases} (FL)_*\Gamma_L = X_H \\ (FH)_*X_H = \Gamma_L. \end{cases}$$

Furthermore, the trajectories of Γ_L are in a one-to-one correspondence with the trajectories of X_H and they coincide when projected to M.

[Note: Explicated,

$$\begin{cases} (FL)_*\Gamma_L = TFL \circ \Gamma_L \circ (FL)^{-1} \\ (FH)_*X_H = TFH \circ X_H \circ (FH)^{-1} \end{cases}$$

$\Rightarrow$

$$\begin{cases} TFL \circ \Gamma_L = X_H \circ FL \\ TFH \circ X_H = \Gamma_L \circ FH.] \end{cases}$$

Locally, $FL(q^1,\ldots,q^n,v^1,\ldots,v^n)$ is given by

$$q^i \circ FL = q^i,\ p_i \circ FL = \frac{\partial L}{\partial v^i}\ .$$

To calculate H in local coordinates, write

$$\begin{aligned} H &= E_L \circ (FL)^{-1} \\ &= \Delta L \circ (FL)^{-1} - L \circ (FL)^{-1} \\ &= (v^i \frac{\partial L}{\partial v^i}) \circ (FL)^{-1} - L \circ (FL)^{-1} \\ &= (v^i \circ (FL)^{-1})(\frac{\partial L}{\partial v^i} \circ (FL)^{-1}) - L \circ (FL)^{-1} \\ &= p_i(v^i \circ (FL)^{-1}) - L \circ (FL)^{-1}. \end{aligned}$$

Abuse the notation and let $v^i \equiv v^i \circ (FL)^{-1}$ — then, since $q_i = q_i \circ (FL)^{-1}$, we have

$$\begin{aligned} &H(q^1,\ldots,q^n,p_1,\ldots,p_n) \\ &\quad = p_i v^i - L(q^1,\ldots,q^n,v^1,\ldots,v^n), \end{aligned}$$

the traditional expression.

APPENDIX

The equations of Lagrange

$$\frac{d}{dt}\,(\frac{\partial L}{\partial v^i}) - \frac{\partial L}{\partial q^i} = 0 \qquad (i = 1,\ldots,n)$$

are tied to the q^i and the v^i but there are situations where a change of variable is advantageous.

If $(U, \{x^1, \ldots, x^n\})$ is a chart on M, then

$$((\pi_M)^{-1}U, \{q^1, \ldots, q^n, v^1, \ldots, v^n\})$$

is a chart on TM. In §4, we took v^i to be dx^i viewed as a function on the fibers, i.e., $v^i = \hat{dx}^i$ (cf. 8.19). However, instead of using the dx^i, we could just as well work with any other set $\{\alpha^1, \ldots, \alpha^n\}$ of 1-forms on U, say

$$\alpha^i = f^i{}_j dx^j \quad (f^i{}_j \in C^\infty(U)),$$

subject to the requirement that

$$\alpha^1 \wedge \ldots \wedge \alpha^n \neq 0$$

which forces functional independence of the $\hat{\alpha}^i$ $(\equiv (f^i{}_j \circ \pi_M) v^j)$.

<u>N.B.</u> Put

$$\bar{v}^i = \hat{\alpha}^i.$$

Then in classical terminology, the v^i are <u>velocities</u> and the $\bar{v}^i$ are <u>quasivelocities</u>.

Define functions $\bar{f}^i{}_j \in C^\infty(U)$ by

$$dx^i = \bar{f}^i{}_j \alpha^j.$$

Then the matrices $[\bar{f}^i{}_j]$ and $[f^i{}_j]$ are inverses of one another.

A.1 <u>LEMMA</u> We have

$$\left[\begin{array}{l} \dfrac{\partial}{\partial v^i} = (f^j{}_i \circ \pi_M) \dfrac{\partial}{\partial \bar{v}^j} \\ \\ \dfrac{\partial}{\partial \bar{v}^i} = (\bar{f}^j{}_i \circ \pi_M) \dfrac{\partial}{\partial v^j} . \end{array}\right.$$

A.2 <u>EXAMPLE</u> Locally,

$$\begin{aligned} \Delta &= v^i \frac{\partial}{\partial v^i} \\ &= (\bar{f}^i{}_j \circ \pi_M)\bar{v}^j (f^k{}_i \circ \pi_M) \frac{\partial}{\partial \bar{v}^k} \\ &= \bar{v}^j \frac{\partial}{\partial \bar{v}^j} . \end{aligned}$$

To minimize confusion, let

$$\bar{q}^i = q^i .$$

Then

$$((\pi_M)^{-1}U, \{\bar{q}^1, \ldots, \bar{q}^n, \bar{v}^1, \ldots, \bar{v}^n\})$$

is a chart on TM.

A.3 <u>LEMMA</u> We have

$$\frac{\partial}{\partial \bar{q}^i} = \frac{\partial}{\partial q^i} + \left(\frac{\partial}{\partial q^i} (\bar{f}^j{}_k \circ \pi_M)\right)(f^k{}_\ell \circ \pi_M) v^\ell \frac{\partial}{\partial v^j} .$$

A.4 <u>EXAMPLE</u> Take $M = \underline{R}$ and suppose that $\alpha = \phi dx$ $(\phi > 0)$. Let $F \in C^\infty(\underline{R}^2)$ -- then

$$F(\bar{q},\bar{v}) = F(q,\phi(q)v),$$

so

$$\frac{\partial F}{\partial \bar{q}} = \frac{\partial F}{\partial q} + \frac{\partial}{\partial q}\left(\frac{1}{\phi}\right)\phi\left(v\,\frac{\partial F}{\partial v}\right)$$

$$= \frac{\partial F}{\partial q} - \frac{\phi'}{\phi}\left(v\,\frac{\partial F}{\partial v}\right).$$

E.g., consider

$$F(q,v) = \frac{1}{2}v^2.$$

Then

$$F(\bar{q},\bar{v}) = \frac{1}{2}\left(\frac{\bar{v}}{\phi(\bar{q})}\right)^2$$

=>

$$\frac{\partial F}{\partial \bar{q}} = \frac{1}{2}(\bar{v})^2\,\frac{d}{d\bar{q}}\,\phi(\bar{q})^{-2}$$

$$= \frac{1}{2}(\phi(q)v)^2\,(-\,2\,\phi(q)^{-3}\,\phi'(q))$$

$$= -\frac{\phi'}{\phi}v^2$$

$$= -\frac{\phi'}{\phi}\left(v\,\frac{\partial F}{\partial v}\right).$$

A.5 <u>LEMMA</u> We have

$$\left[\frac{\partial}{\partial \bar{v}^i},\frac{\partial}{\partial \bar{v}^j}\right] = 0.$$

Now put

$$\bar{X}_i = (\bar{f}^k{}_i \circ \pi_M) \frac{\partial}{\partial \bar{q}^k}$$

A.6 LEMMA We have

$$[\bar{X}_i, \frac{\partial}{\partial \bar{v}^j}] = 0.$$

Define functions

$$\gamma^k_{ij} \in C^\infty((\pi_M)^{-1}U)$$

by

$$\gamma^k_{ij} = (\bar{f}^\ell{}_i \circ \pi_M)(\bar{f}^m{}_j \circ \pi_M)(\frac{\partial}{\partial \bar{q}^m} (f^k{}_\ell \circ \pi_M) - \frac{\partial}{\partial \bar{q}^\ell} (f^k{}_m \circ \pi_M)).$$

A.7 LEMMA We have

$$[\bar{X}_i, \bar{X}_j] = \gamma^k_{ij} \bar{X}_k.$$

N.B. The set

$$\{\bar{X}_1, \ldots, \bar{X}_n, \frac{\partial}{\partial \bar{v}^1}, \ldots, \frac{\partial}{\partial \bar{v}^n}\}$$

is a basis for

$$\mathcal{D}^1((\pi_M)^{-1}U).$$

A.8 EXAMPLE Take $M = \underline{R}^3$ and use spherical coordinates:

$$\begin{cases} q^1 = r \ (r > 0) \\ q^2 = \theta \ (0 < \theta < \pi) \\ q^3 = \phi \ (0 < \phi < 2\pi). \end{cases}$$

Let

$$\begin{cases} \bar{v}^1 = v^1 \\ \bar{v}^2 = rv^2 \\ \bar{v}^3 = r \sin\theta \, v^3. \end{cases}$$

Then

$$[f^i{}_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \sin\theta \end{bmatrix}$$

and

$$[\bar{f}^i{}_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & 1/r \sin\theta \end{bmatrix}.$$

Therefore

$$\begin{cases} \bar{X}_1 = \dfrac{\partial}{\partial \bar{q}^1} \\ \bar{X}_2 = \dfrac{1}{r} \dfrac{\partial}{\partial \bar{q}^2} \\ \bar{X}_3 = \dfrac{1}{r \sin\theta} \dfrac{\partial}{\partial \bar{q}^3}. \end{cases}$$

And

$$[\bar{X}_1,\bar{X}_2] = -\frac{1}{r}\bar{X}_2,\ [\bar{X}_1,\bar{X}_3] = -\frac{1}{r}\bar{X}_3,\ [\bar{X}_2,\bar{X}_3] = -\frac{\cot\theta}{r}\bar{X}_3.$$

Consequently, the nonzero γ_{ij}^k are

$$\begin{cases} \gamma_{12}^2 = -\gamma_{21}^2 = -\dfrac{1}{r} \\ \gamma_{13}^3 = -\gamma_{31}^3 = -\dfrac{1}{r} \\ \gamma_{23}^3 = -\gamma_{32}^3 = -\dfrac{\cot\theta}{r}. \end{cases}$$

A.9 <u>EXAMPLE</u> Take $M = \underline{SO}(3)$ and let

$$\begin{cases} q^1 = \phi \\ q^2 = \theta \\ q^3 = \psi \end{cases}$$

be the local chart corresponding to the 3-1-3 rotation sequence (see the Appendix). Put

$$\begin{cases} \bar{v}^1 = v_\phi \sin\theta\sin\psi + v_\theta\cos\psi \\ \bar{v}^2 = v_\phi \sin\theta\cos\psi - v_\theta\sin\psi \\ \bar{v}^3 = v_\phi\cos\theta + v_\psi. \end{cases}$$

Then

$$[f^i_{\ j}] = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}$$

and

$$[\bar{f}^i_{\ j}] = \begin{bmatrix} \sin\psi/\sin\theta & \cos\psi/\sin\theta & 0 \\ \cos\psi & -\sin\psi & 0 \\ -\cos\theta\sin\psi/\sin\theta & -\cos\theta\cos\psi/\sin\theta & 1 \end{bmatrix}.$$

Therefore

$$\begin{cases} \bar{X}_1 = (\sin\psi/\sin\theta)\dfrac{\partial}{\partial\bar{q}^1} + \cos\psi\dfrac{\partial}{\partial\bar{q}^2} - (\cos\theta\sin\psi/\sin\theta)\dfrac{\partial}{\partial\bar{q}^3} \\ \bar{X}_2 = (\cos\psi/\sin\theta)\dfrac{\partial}{\partial\bar{q}^1} - \sin\psi\dfrac{\partial}{\partial\bar{q}^2} - (\cos\theta\cos\psi/\sin\theta)\dfrac{\partial}{\partial\bar{q}^3} \\ \bar{X}_3 = \dfrac{\partial}{\partial\bar{q}^3}\,. \end{cases}$$

Here

$$[\bar{X}_i, \bar{X}_j] = \varepsilon_{ijk}\bar{X}_k,$$

thus

$$\gamma^k_{ij} = \varepsilon_{ijk}.$$

Suppose that $L \in C^\infty(TM)$ is a lagrangian.

A.10 LEMMA Locally,

$$\theta_L = \frac{\partial L}{\partial \bar{v}^i} (f^i{}_j \circ \pi_M) d\bar{q}^j.$$

PROOF In fact,

$$\theta_L = d_S L$$

$$= S^*(dL)$$

$$= S^*(\frac{\partial L}{\partial \bar{q}^i} d\bar{q}^i + \frac{\partial L}{\partial \bar{v}^i} d\bar{v}^i)$$

$$= \frac{\partial L}{\partial \bar{q}^i} S^*(d\bar{q}^i) + \frac{\partial L}{\partial \bar{v}^i} S^*(d\bar{v}^i)$$

$$= \frac{\partial L}{\partial \bar{q}^i} S^*(dq^i) + \frac{\partial L}{\partial \bar{v}^i} S^*(d\hat{\alpha}^i)$$

$$= \frac{\partial L}{\partial \bar{v}^i} S^*(d((f^i{}_j \circ \pi_M) v^j))$$

$$= \frac{\partial L}{\partial \bar{v}^i} S^*(d(f^i{}_j \circ \pi_M) v^j + (f^i{}_j \circ \pi_M) dv^j)$$

$$= \frac{\partial L}{\partial \bar{v}^i} S^*(\frac{\partial (f^i{}_j \circ \pi_M)}{\partial q^k} v^j dq^k + \frac{\partial (f^i{}_j \circ \pi_M)}{\partial v^k} v^j dv^k)$$

$$+ \frac{\partial L}{\partial \bar{v}^i} S^*((f^i{}_j \circ \pi_M) dv^j)$$

$$= \frac{\partial L}{\partial \bar{v}^i} (f^i{}_j \circ \pi_M) S^*(dv^j)$$

$$= \frac{\partial L}{\partial \bar{v}^i} (f^i{}_j \circ \pi_M) dq^j$$

$$= \frac{\partial L}{\partial \bar{v}^i} (f^i{}_j \circ \pi_M) d\bar{q}^j.$$

[Note: Obviously,

$$(f^i{}_j \circ \pi_M) d\bar{q}^j = \pi_M^*(\alpha^i).]$$

A.11 <u>LEMMA</u> Locally,

$$\omega_L = (f^k{}_j \circ \pi_M) \frac{\partial^2 L}{\partial \bar{q}^i \partial \bar{v}^k} d\bar{q}^i \wedge d\bar{q}^j$$

$$+ \frac{1}{2} \left(\frac{\partial}{\partial \bar{q}^i} (f^k{}_j \circ \pi_M) - \frac{\partial}{\partial \bar{q}^j} (f^k{}_i \circ \pi_M)\right) \frac{\partial L}{\partial \bar{v}^k} d\bar{q}^i \wedge d\bar{q}^j$$

$$+ (f^k{}_i \circ \pi_M) \frac{\partial^2 L}{\partial \bar{v}^j \partial \bar{v}^k} d\bar{v}^j \wedge d\bar{q}^i.$$

[Note: Write

$$\begin{cases} d\bar{q}^i = (\bar{f}^i{}_\ell \circ \pi_M) \pi_M^*(\alpha^\ell) \\ d\bar{q}^j = (\bar{f}^j{}_m \circ \pi_M) \pi_M^*(\alpha^m). \end{cases}$$

Then

$$\frac{1}{2} \left(\frac{\partial}{\partial \bar{q}^i} (f^k{}_j \circ \pi_M) - \frac{\partial}{\partial \bar{q}^j} (f^k{}_i \circ \pi_M)\right) \frac{\partial L}{\partial \bar{v}^k} d\bar{q}^i \wedge d\bar{q}^j$$

$$= \frac{1}{2} \left((\bar{f}^j{}_m \circ \pi_M)(\bar{f}^i{}_\ell \circ \pi_M)\left(\frac{\partial}{\partial \bar{q}^i} (f^k{}_j \circ \pi_M) - \frac{\partial}{\partial \bar{q}^j} (f^k{}_i \circ \pi_M)\right)\right) \frac{\partial L}{\partial \bar{v}^k} \pi_M^*(\alpha^\ell) \wedge \pi_M^*(\alpha^m)$$

$$= \frac{1}{2} \gamma^k_{m\ell} \frac{\partial L}{\partial \bar{v}^k} \pi_M^*(\alpha^\ell) \wedge \pi_M^*(\alpha^m).]$$

If $\Gamma \in SO(TM)$ is second order, then

$$\Gamma = (\bar{f}^i{}_j \circ \pi_M)\bar{v}^j \frac{\partial}{\partial \bar{q}^i} + \bar{C}^i \frac{\partial}{\partial \bar{v}^i},$$

i.e.,

$$\Gamma = \bar{v}^i \bar{X}_i + \bar{C}^i \frac{\partial}{\partial \bar{v}^i}.$$

Indeed,

$$S\Gamma = (f^k{}_i \circ \pi_M)(\bar{f}^i{}_j \circ \pi_M)\bar{v}^j \frac{\partial}{\partial \bar{v}^k}$$

$$= \bar{v}^j \frac{\partial}{\partial \bar{v}^j}$$

$$= \Delta \text{ (cf. A.2).}$$

Assume henceforth that L is nondegenerate. Determine Γ_L per 8.12 -- then Γ_L is second order and along an integral curve γ of Γ_L, the equations of Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0 \quad (i = 1,\ldots,n)$$

are satisfied or still, passing from velocities to quasivelocities,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \bar{v}^j}\right) - \bar{v}^i \gamma^k_{ij} \frac{\partial L}{\partial \bar{v}^k} = \bar{X}_j L \quad (j = 1,\ldots,n).$$

A.12 <u>EXAMPLE</u> Take $M = \underline{R}^2$ and use polar coordinates:

$$\begin{cases} q^1 = r & (r > 0) \\ q^2 = \theta & (0 < \theta < 2\pi). \end{cases}$$

Put

$$\begin{cases} \bar{v}^1 = v^1 \\ \bar{v}^2 = r^2 v^2. \end{cases}$$

Then

$$[f^i{}_j] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

and

$$[\bar{f}^i{}_j] = \begin{bmatrix} 1 & 0 \\ 0 & 1/r^2 \end{bmatrix}.$$

In cartesian coordinates, let L be

$$\frac{1}{2}\,((\dot{x})^2 + (\dot{y})^2) - V(\sqrt{x^2+y^2})$$

which in polar coordinates is

$$\frac{1}{2}\,((\dot{r})^2 + r^2(\dot{\theta})^2) - V(r)$$

or, in terms of $\bar{q}^1, \bar{q}^2, \bar{v}^1, \bar{v}^2$:

$$\frac{1}{2}\,(\bar{v}^1)^2 + \frac{1}{2}\,\frac{(\bar{v}^2)^2}{(\bar{q}^1)^2} - V(\bar{q}^1).$$

Write

$$\begin{cases} T = \frac{1}{2}\,(\bar{v}^1)^2 + \frac{1}{2}\,\frac{(\bar{v}^2)^2}{(\bar{q}^1)^2} \\ F = -\,V' \quad (= -\,dV/dr). \end{cases}$$

Then the equations of motion are

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \bar{v}^1}\right) - \bar{v}^i \gamma^k_{i1} \dfrac{\partial T}{\partial \bar{v}^k} = \bar{X}_1 T + F \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \bar{v}^2}\right) - \bar{v}^i \gamma^k_{i2} \dfrac{\partial T}{\partial \bar{v}^k} = \bar{X}_2 T + 0 \end{cases}$$

that, when explicated, reduce to

$$\begin{cases} \dot{\bar{v}}^1 = \dfrac{(\bar{v}^2)^2}{(\bar{q}^1)^3} + F(\bar{q}^1) \\[2ex] \dot{\bar{v}}^2 = 0. \end{cases}$$

Therefore

$$\Gamma_L = \bar{v}^i \bar{X}_i + \left(\frac{(\bar{v}^2)^2}{(\bar{q}^1)^3} + F(\bar{q}^1)\right) \frac{\partial}{\partial \bar{v}^1} .$$

To return to $q^1 = r$, $q^2 = \theta$, $v^1 = \dot{r}$, $v^2 = \dot{\theta}$, note that

$$\begin{cases} \bar{X}_1 = \dfrac{\partial}{\partial r} - 2\,\dfrac{\dot{\theta}}{r}\,\dfrac{\partial}{\partial \dot{\theta}} . \\[2ex] \bar{X}_2 = \dfrac{1}{r^2}\,\dfrac{\partial}{\partial \theta} \end{cases} \qquad \text{(cf. A.3).}$$

Accordingly,

$$\Gamma_L = \dot{r}\left(\frac{\partial}{\partial r} - 2\,\frac{\dot{\theta}}{r}\,\frac{\partial}{\partial \dot{\theta}}\right) + r^2\dot{\theta}\left(\frac{1}{r^2}\,\frac{\partial}{\partial \theta}\right)$$

$$+ (r\dot{\theta}^2 + F(r))\,\frac{\partial}{\partial \dot{r}}$$

$$= \dot{r}\,\frac{\partial}{\partial r} + \dot{\theta}\,\frac{\partial}{\partial \theta} + (r\dot{\theta}^2 + F(r))\,\frac{\partial}{\partial \dot{r}} - 2\,\frac{\dot{r}\dot{\theta}}{r}\,\frac{\partial}{\partial \dot{\theta}} .$$

A.13 <u>EXAMPLE</u> Take M = <u>SO</u>(3) (cf. A.9). Suppose that locally,

$$L = \frac{1}{2}\,(I_1(\bar{v}^1)^2 + I_2(\bar{v}^2)^2 + I_3(\bar{v}^3)^2),$$

where the I_i are positive constants — then here

$$\frac{d}{dt}\,(\frac{\partial L}{\partial \bar{v}^j}) - \bar{v}^i \gamma_{ij}^k \frac{\partial L}{\partial \bar{v}^k} = 0 \qquad (j = 1,2,3)$$

or, equivalently,

$$\frac{d}{dt}\,(\frac{\partial L}{\partial \bar{v}^i}) - \bar{v}^j \gamma_{ji}^k \frac{\partial L}{\partial \bar{v}^k} = 0 \qquad (i = 1,2,3).$$

But

$$\gamma_{ji}^k = \varepsilon_{jik} = - \varepsilon_{ijk}.$$

Therefore

$$\left[\begin{array}{l} \dot{\bar{v}}^1 = \dfrac{(I_2-I_3)}{I_1}\,\bar{v}^2\bar{v}^3 \\[2ex] \dot{\bar{v}}^2 = \dfrac{(I_3-I_1)}{I_2}\,\bar{v}^3\bar{v}^1 \\[2ex] \dot{\bar{v}}^3 = \dfrac{(I_1-I_2)}{I_3}\,\bar{v}_1\bar{v}_2. \end{array}\right.$$

[Note: These relations are instances of Euler's equations (see the Appendix).]

§9. SYMMETRIES

Let M be a connected C^∞ manifold of dimension n. Given a second order $\Gamma \in \mathcal{D}^1(TM)$, put

$$\mathcal{D}^1_\Gamma(TM) = \{X \in \mathcal{D}^1(TM) : S[X,\Gamma] = 0\}.$$

[Note: Locally, the elements of $\mathcal{D}^1_\Gamma(TM)$ have the form

$$X = A^i \frac{\partial}{\partial q^i} + (\Gamma A^i) \frac{\partial}{\partial v^i} .]$$

9.1 <u>LEMMA</u> Define

$$\pi_\Gamma : \mathcal{D}^1(TM) \to \mathcal{D}^1_\Gamma(TM)$$

by

$$\pi_\Gamma(X) = X + S[\Gamma,X].$$

Then π_Γ is a projection of $\mathcal{D}^1(TM)$ onto $\mathcal{D}^1_\Gamma(TM)$ with kernel $V(TM)$.

[To check that $\pi_\Gamma(X)$ really is in $\mathcal{D}^1_\Gamma(TM)$, write

$$\begin{aligned} &S[\pi_\Gamma(X),\Gamma] \\ &\quad = S[X + S[\Gamma,X],\Gamma] \\ &\quad = S[X,\Gamma] + S[S[\Gamma,X],\Gamma] \\ &\quad = S[X,\Gamma] + [S[\Gamma,X],S\Gamma] \\ &\qquad\qquad - S[[\Gamma,X],S\Gamma] \quad (\text{cf. } 5.9) \end{aligned}$$

$$= S[X,\Gamma] + [S[\Gamma,X],\Delta]$$

$$- S[[\Gamma,X],\Delta] \quad \text{(cf. 5.8)}$$

$$= S[X,\Gamma] + [S[\Gamma,X],\Delta]$$

$$+ S[\Delta,[\Gamma,X]]$$

$$= S[X,\Gamma] + S[\Gamma,X] \quad \text{(cf. 5.10)}$$

$$= 0.]$$

9.2 <u>LEMMA</u> Define a multiplication

$$C^\infty(TM) \times \mathcal{D}^1_\Gamma(TM) \to \mathcal{D}^1_\Gamma(TM)$$

by

$$f*X = fX + (\Gamma f)SX \quad (= \pi_\Gamma(fX)).$$

Then $\mathcal{D}^1_\Gamma(TM)$ is a module over $C^\infty(TM)$.

[Note: So, while $\mathcal{D}^1_\Gamma(TM)$ is not stable under the usual multiplication by elements of $C^\infty(TM)$, it is stable under the usual multiplication by elements of $C^\infty_\Gamma(TM)$ (the subring of $C^\infty(TM)$ consisting of the first integrals for Γ) (cf. §1).]

The elements of $\mathcal{D}^1_\Gamma(TM)$ are called the <u>pseudosymmetries</u> of Γ, a <u>symmetry</u> of Γ being an $X \in \mathcal{D}^1(TM)$ such that $[X,\Gamma] = 0$.

[Note: Trivially, a symmetry of Γ is a pseudosymmetry of Γ.]

9.3 EXAMPLE Let $X \in \mathcal{D}^1(M)$ -- then

$$S[X^T,\Gamma] = 0 \quad \text{(cf. 5.19)}.$$

Therefore $X^T \in \mathcal{D}^1_\Gamma(TM)$, hence X^T is a pseudosymmetry of Γ.

A point symmetry of Γ is an $X \in \mathcal{D}^1(M)$ such that

$$[X^T,\Gamma] = 0.$$

So, strictly speaking, a point symmetry is not a symmetry... .

9.4 REMARK Agreeing to call a vector field on TM projectable if it is π_M-related to a vector field on M, the definitions then imply that the projectable symmetries of Γ are precisely the lifts of the point symmetries of Γ.

9.5 LEMMA If X is a symmetry of Γ and if $f \in C^\infty_\Gamma(TM)$, then $Xf \in C^\infty_\Gamma(TM)$.

PROOF For

$$\begin{aligned} 0 = [\Gamma,X]f &= \Gamma(Xf) - X(\Gamma f) \\ &= \Gamma(Xf). \end{aligned}$$

Suppose now that L is a nondegenerate lagrangian -- then ω_L is symplectic so for any $f \in C^{\infty}(TM)$, $\exists$ a unique vector field $X_f \in \mathcal{D}^1(TM)$ such that

$$\iota_{X_f}\omega_L = df.$$

9.6 <u>LEMMA</u> If f is a first integral for Γ_L, then X_f is a symmetry of Γ_L.

<u>PROOF</u> Write

$$\iota_{[X_f,\Gamma_L]}\omega_L = -\iota_{[\Gamma_L,X_f]}\omega_L$$

$$= -(L_{\Gamma_L} \circ \iota_{X_f} - \iota_{X_f} \circ L_{\Gamma_L})\omega_L$$

$$= -L_{\Gamma_L}\iota_{X_f}\omega_L \quad \text{(cf. 8.9)}$$

$$= -L_{\Gamma_L} df$$

$$= -dL_{\Gamma_L} f$$

$$= -d\Gamma_L f$$

$$= 0.$$

Therefore

$$[X_f,\Gamma_L] = 0.$$

9.7 <u>REMARK</u> If $X \in \mathcal{D}^1(TM)$ is a symmetry of Γ_L, then

$$\iota_X\iota_{\Gamma_L}\omega_L \in C^{\infty}_{\Gamma_L}(TM).$$

Proof:

$$L_{\Gamma_L}(\iota_X \iota_{\Gamma_L} \omega_L)$$

$$= - L_{\Gamma_L}(\iota_X dE_L)$$

$$= - L_{\Gamma_L}(L_X E_L)$$

$$= (L_{[X,\Gamma_L]} - L_X L_{\Gamma_L}) E_L$$

$$= - L_X L_{\Gamma_L} E_L$$

$$= 0 \quad \text{(cf. 8.10)}.$$

[Note: It may very well happen that

$$\iota_X \iota_{\Gamma_L} \omega_L$$

vanishes identically.]

An <u>infinitesimal symmetry</u> of L is a vector field $X \in \mathcal{D}^1(M)$ such that

$$X^T L = 0.$$

I.e.:

$$L \in C^\infty_{X^T}(TM).$$

[Note: It will be shown below that

$$[X^T, \Gamma_L] = 0 \quad \text{(cf. 9.14)}.$$

Accordingly, an infinitesimal symmetry of L is a point symmetry of Γ_L.]

9.8 THEOREM (Noether) If X is an infinitesimal symmetry of L, then $X^V L$ is a first integral for Γ_L.

PROOF In fact,

$$\begin{aligned}
L_{\Gamma_L}(\iota_{X^T}\theta_L) &= \iota_{[\Gamma_L, X^T]}\theta_L + \iota_{X^T}(L_{\Gamma_L}\theta_L) \\
&= \iota_0\theta_L + \iota_{X^T}dL \quad \text{(cf. 8.14)} \\
&= X^T L \\
&= 0.
\end{aligned}$$

Therefore $\iota_{X^T}\theta_L$ is a first integral for Γ_L. But

$$\begin{aligned}
\iota_{X^T}\theta_L &= \iota_{X^T}d_S L \\
&= \iota_{X^T}S^*(dL) \\
&= S^* \circ \iota_{SX^T}(dL) \quad \text{(cf. 6.1)} \\
&= S^* \circ \iota_{X^V}(dL) \quad \text{(cf. 5.7)} \\
&= S^*(dL(X^V)) \\
&= dL(X^V)
\end{aligned}$$

$$= X^V L.$$

9.9 EXAMPLE Take $M = \underline{R}^3$ and let

$$L(q^1,q^2,q^3,v^1,v^2,v^3) = \frac{1}{2} \sum_{i=1}^{3} (v^i)^2 - V(q^2,q^3).$$

Put $X = \frac{\partial}{\partial x^1}$ -- then $X^T = \frac{\partial}{\partial q^1}$, so $X^T L = 0$. Since $X^V = \frac{\partial}{\partial v^1}$, it follows that $X^V L = v^1$ is a first integral for Γ_L (conservation of linear momentum along the x^1-axis).

9.10 EXAMPLE Take $M = \underline{R}^3$ and let

$$L(q^1,q^2,q^3,v^1,v^2,v^3) = \frac{1}{2} \left(\sum_{i=1}^{3} (v^i)^2 - \sum_{i=1}^{3} (q^i)^2 \right).$$

Put $X = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$ -- then

$$X^T = q^1 \frac{\partial}{\partial q^2} - q^2 \frac{\partial}{\partial q^1} - v^2 \frac{\partial}{\partial v^1} + v^1 \frac{\partial}{\partial v^2}$$

$$\Rightarrow$$

$$\begin{aligned} X^T L &= - q^1 q^2 + q^2 q^1 - v^2 v^1 + v^1 v^2 \\ &= 0. \end{aligned}$$

But here

$$X^V = q^1 \frac{\partial}{\partial v^2} - q^2 \frac{\partial}{\partial v^1} .$$

And this means that

$$X^V L = q^1 v^2 - q^2 v^1$$

is a first integral for Γ_L (conservation of angular momentum around the x^3-axis).

As will become apparent, one need not work exclusively with the lifts to TM of vector fields on M.

9.11 <u>EXAMPLE</u> Take $M = \underline{R}^3$ and let

$$L(q^1,q^2,q^3,v^1,v^2,v^3) = \frac{1}{2} \sum_{i=1}^{3} (v^i)^2.$$

Put

$$X = f(v^1,v^2,v^3) \frac{\partial}{\partial q^1}.$$

Obviously, $XL = 0$. In addition,

$$\Gamma_L = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + v^3 \frac{\partial}{\partial q^3}$$

$\Rightarrow$

$$[X,\Gamma_L] = 0.$$

The argument employed in 9.8 then implies that $\iota_X \theta_L$ is a first integral for Γ_L. But

$$\iota_X \theta_L = \iota_X \left(\frac{\partial L}{\partial v^i} dq^i\right)$$

$$= \frac{\partial L}{\partial v^i} \iota_X dq^i$$

$$= v^1 f(v^1, v^2, v^3).$$

I.e.: $v^1 f(v^1, v^2, v^3)$ is a first integral for Γ_L. Of course, the lagrangian at hand represents the free particle, so any function of the velocity had better be a "constant of the motion".

9.12 LEMMA If X is an infinitesimal symmetry of L, then

$$L_{X^T}\theta_L = 0.$$

PROOF We have

$$L_{X^T}\theta_L = L_{X^T}d_S L$$

$$= d_S L_{X^T} L + d_{L_{X^T} S} L \quad \text{(cf. 6.12)}$$

$$= d_S 0 + d_0 L \quad \text{(cf. 5.18)}$$

$$= 0.$$

[Note: Therefore

$$L_{X^T}\omega_L = L_{X^T}d\theta_L$$

$$= dL_{X^T}\theta_L$$

$$= 0.]$$

9.13 LEMMA If X is an infinitesimal symmetry of L, then $X^T E_L = 0$.

<u>PROOF</u> For

$$\iota_{\Gamma_L}\omega_L = -\,dE_L$$

$$\Rightarrow$$

$$\iota_{\Gamma_L}\omega_L(X^T) = -\,dE_L(X^T)$$

$$= -\,X^T E_L.$$

And

$$\iota_{\Gamma_L}\omega_L(X^T) = \omega_L(\Gamma_L, X^T)$$

$$= d\theta_L(\Gamma_L, X^T)$$

$$= (L_{\Gamma_L}\theta_L)(X^T) - (L_{X^T}\theta_L)(\Gamma_L) + \theta_L([\Gamma_L, X^T])$$

$$= (L_{\Gamma_L}\theta_L)(X^T) + \theta_L([\Gamma_L, X^T]) \quad \text{(cf. 9.12)}$$

$$= dL(X^T) + \theta_L([\Gamma_L, X^T]) \quad \text{(cf. 8.14)}$$

$$= X^T L + \theta_L([\Gamma_L, X^T])$$

$$= \theta_L([\Gamma_L, X^T]).$$

But $[\Gamma_L, X^T]$ is vertical (cf. 5.3) and θ_L is horizontal, hence $\theta_L([\Gamma_L, X^T]) = 0$ (cf. 6.14).

9.14 <u>LEMMA</u> If X is an infinitesimal symmetry of L, then X^T is a symmetry of Γ_L.

<u>PROOF</u> Simply note that

$$\iota_{[X^T,\Gamma_L]}\omega_L = (L_{X^T} \circ \iota_{\Gamma_L} - \iota_{\Gamma_L} \circ L_{X^T})\omega_L$$

$$= L_{X^T}(- dE_L) - \iota_{\Gamma_L}(L_{X^T}\omega_L)$$

$$= - dL_{X^T}E_L - \iota_{\Gamma_L} 0 \quad (\text{cf. } 9.12)$$

$$= - d(X^T E_L)$$

$$= 0 \quad (\text{cf. } 9.13).$$

9.15 <u>REMARK</u> Let $X \in \mathcal{D}^1(TM)$. Assume:

$$\begin{cases} dL_X\theta_L = 0 \\ dXE_L = 0. \end{cases}$$

Then

$$[X,\Gamma_L] = 0.$$

Proof:

$$\iota_{[X,\Gamma_L]}\omega_L = (L_X \circ \iota_{\Gamma_L} - \iota_{\Gamma_L} \circ L_X)\omega_L$$

$$= L_X(- dE_L) - \iota_{\Gamma_L} L_X d\theta_L$$

$$= - d\mathcal{L}_X E_L - \iota_{\Gamma_L} d\mathcal{L}_X \theta_L$$

$$= 0.$$

A <u>Noether symmetry</u> of Γ_L is a vector field $X \in \mathcal{D}^1(M)$ such that $\mathcal{L}_{X^T}\theta_L$ is exact (say $\mathcal{L}_{X^T}\theta_L = df$, where $f \in C^\infty(TM)$) and $X^T E_L = 0$.

[Note: A Noether symmetry X of Γ_L is necessarily a point symmetry of Γ_L:

$$[X^T, \Gamma_L] = 0 \quad \text{(cf. 9.15).]}$$

9.16 <u>LEMMA</u> If X is a Noether symmetry of Γ_L, then $f - X^V L$ is a first integral for Γ_L.

<u>PROOF</u> To begin with,

$$\iota_{X^T}\omega_L = \iota_{X^T} d\theta_L$$

$$= \mathcal{L}_{X^T}\theta_L - d\iota_{X^T}\theta_L$$

$$= df - dX^V L$$

$$= d(f - X^V L).$$

Therefore

$$\Gamma_L(f - X^V L) = d(f - X^V L)(\Gamma_L)$$

$$= (\iota_{X^T}\omega_L)(\Gamma_L)$$

$$= \omega_L(X^T, \Gamma_L)$$

$$= - \omega_L(\Gamma_L, X^T)$$

$$= - \iota_{\Gamma_L}\omega_L(X^T)$$

$$= dE_L(X^T)$$

$$= X^T E_L$$

$$= 0.$$

Suppose that X is an infinitesimal symmetry of L — then

$$\begin{cases} L_{X^T}\theta_L = 0 & \text{(cf. 9.12)} \\ X^T E_L = 0 & \text{(cf. 9.13).} \end{cases}$$

So X is a Noether symmetry of Γ_L and 9.8 is a special case of 9.16 (take f = 0).

9.17 <u>REMARK</u> If X is a point symmetry of Γ_L such that

$$\begin{cases} L_{X^T}\theta_L = 0 \\ X^T E_L = 0, \end{cases}$$

then X is an infinitesimal symmetry of L. To see this, start by writing

$$L_{X^T}(\iota_{\Gamma_L}\theta_L) = \iota_{[X^T,\Gamma_L]}\theta_L + \iota_{\Gamma_L}(L_{X^T}\theta_L)$$

$$= \iota_0\theta_L + \iota_{\Gamma_L}0$$

$$= 0.$$

Next

$$X^T E_L = 0 \Rightarrow X^T(\Delta L - L) = 0$$

$$\Rightarrow X^T\Delta L = X^T L.$$

So

$$0 = L_{X^T}(\iota_{\Gamma_L}\theta_L)$$

$$= L_{X^T}\Delta L \quad \text{(cf. 8.13)}$$

$$= X^T\Delta L$$

$$= X^T L.$$

A <u>Cartan symmetry</u> of Γ_L is a vector field $X \in \mathcal{D}^1(TM)$ such that $L_X\theta_L$ is exact (say $L_X\theta_L = df$, where $f \in C^\infty(TM)$) and $XE_L = 0$.

[Note: A Cartan symmetry X of Γ_L is necessarily a symmetry of Γ_L:

$$[X,\Gamma_L] = 0 \quad \text{(cf. 9.15).]}$$

N.B. The lift of a Noether symmetry of Γ_L is a Cartan symmetry of Γ_L. In the other direction, the projection of a projectable Cartan symmetry of Γ_L is a Noether symmetry of Γ_L (cf. 9.4).

9.18 EXAMPLE Γ_L is a Cartan symmetry of Γ_L (which, in general, is not projectable). Proof:

$$\left[\begin{array}{l} L_{\Gamma_L}\theta_L = dL \quad \text{(cf. 8.14)} \\ \\ \Gamma_L E_L = 0 \quad \text{(cf. 8.10)}. \end{array}\right.$$

9.19 REMARK The lift of a point symmetry of Γ_L need not be a Cartan symmetry of Γ_L (cf. 9.24).

9.20 LEMMA If X is a Cartan symmetry of Γ_L, then $f - (SX)L$ is a first integral for Γ_L.

[Argue as in 9.16, observing that

$$\begin{aligned} \iota_X\omega_L &= \iota_X d\theta_L \\ &= L_X\theta_L - d\iota_X\theta_L \\ &= df - d\iota_X S^*(dL) \\ &= df - dS^*(dL)(X) \end{aligned}$$

$$= d(f - (SX)L).]$$

Consider the following setup. Suppose $\exists\ f \in C^{\infty}(TM)$:

$$\begin{cases} L_X\theta_L = df \\ XL = \Gamma_L f. \end{cases}$$

Then

$$f - (SX)L$$

is a first integral for Γ_L. In fact,

$$\begin{aligned}
\Gamma_L(f - (SX)L) &= d(f - (SX)L)(\Gamma_L) \\
&= (\iota_X\omega_L)(\Gamma_L) \\
&= XE_L \\
&= X(\Delta L - L) \\
&= X(\iota_{\Gamma_L}\theta_L - L) \qquad \text{(cf. 8.13)} \\
&= X\iota_{\Gamma_L}\theta_L - \Gamma_L f \\
&= \iota_{\Gamma_L} L_X\theta_L - \Gamma_L f \\
&= \iota_{\Gamma_L} df - \Gamma_L f
\end{aligned}$$

$$= df(\Gamma_L) - \Gamma_L f$$

$$= \Gamma_L f - \Gamma_L f$$

$$= 0.$$

N.B. X is a Cartan symmetry of Γ_L. Thus put

$$F = f - (SX)L.$$

Then

$$\left[\begin{array}{l} \iota_{\Gamma_L}\omega_L = -\,dE_L \\ \\ \iota_X\omega_L = dF. \end{array}\right.$$

And

$$XE_L = \iota_X dE_L$$

$$= -\,\iota_X\iota_{\Gamma_L}\omega_L$$

$$= \iota_{\Gamma_L}\iota_X\omega_L$$

$$= \iota_{\Gamma_L} dF$$

$$= \Gamma_L F$$

$$= 0$$

$\Rightarrow$

$$XE_L = 0.$$

9.21 <u>EXAMPLE</u> Here is a realization of the foregoing procedure. Take $M = \underline{R}^3 - \{0\}$ and put

$$\begin{cases} |q| = ((q^1)^2 + (q^2)^2 + (q^3)^2)^{1/2} \\ |v| = ((v^1)^2 + (v^2)^2 + (v^3)^2)^{1/2}. \end{cases}$$

Let

$$L = \frac{1}{2}(|v|^2) + \frac{K}{|q|} \quad (K \neq 0).$$

Then

$$\begin{cases} \theta_L = v^i dq^i \\ \omega_L = dv^i \wedge dq^i, \end{cases}$$

hence L is nondegenerate,

$$E_L = \frac{|v|^2}{2} - \frac{K}{|q|},$$

and

$$\Gamma_L = v^i \frac{\partial}{\partial q^i} - \frac{Kq^i}{|q|^3} \frac{\partial}{\partial v^i}.$$

Define vector fields $X_k \in \mathcal{D}^1(TM)$ $(k = 1,2,3)$ by

$$X_k = -(2q^k v^i - v^k q^i - (q \cdot v)\delta^{ki}) \frac{\partial}{\partial q^i}$$

$$- (K(|q|^2 \delta^{ki} - q^k q^i)/|q|^3 - |v|^2 \delta^{ki} + v^k v^i) \frac{\partial}{\partial v^i},$$

where

$$q \cdot v = q^1 v^1 + q^2 v^2 + q^3 v^3.$$

One can check that $[X_k, \Gamma_L] = 0$, thus X_k is a symmetry of Γ_L which is not a lift of a vector field on M. Set

$$f_k = (q\cdot v)v^k - (|v|^2 + K/|q|)q^k.$$

Since

$$\begin{cases} L_{X_k}\theta_L = df_k \\ X_k L = \Gamma_L f_k, \end{cases}$$

the conclusion is that

$$\begin{aligned} f_k &- (SX)L \\ &= (|v|^2 - K/|q|)q^k - (q\cdot v)v^k \end{aligned}$$

is a first integral for Γ_L.

[Note: This lagrangian is the one that figures in the Kepler problem and what is being said is that the so-called Lenz vector is conserved.]

9.22 <u>LEMMA</u> If f is a first integral for Γ_L, then X_f is a Cartan symmetry of Γ_L (cf. 9.6).

<u>PROOF</u> We have

$$\begin{aligned} df = \iota_{X_f}\omega_L &= \iota_{X_f}d\theta_L \\ &= (L_{X_f} - d \circ \iota_{X_f})\theta_L \end{aligned}$$

$$\Rightarrow$$

$$L_{X_f}\theta_L = d(f + \theta_L(X_f)).$$

And

$$X_f E_L = (\iota_{X_f} \omega_L)(\Gamma_L)$$

$$= df(\Gamma_L)$$

$$= \Gamma_L f$$

$$= 0.$$

9.23 <u>REMARK</u> Given a Cartan symmetry X of Γ_L, put

$$F = f - (SX)L.$$

Then F is a first integral for Γ_L (cf. 9.20) and

$$\iota_X \omega_L = df \Rightarrow X = X_F.$$

So far we have worked with a fixed nonsingular lagrangian L. However, as has been seen in §8 (cf. 8.17 and 8.18), distinct nonsingular lagrangians L and L' can give rise to the same dynamics in that

$$\Gamma_L = \Gamma_{L'} .$$

In turn, this leads to differing descriptions of the symmetries and first integrals.

9.24 <u>EXAMPLE</u> Take $M = \underline{R}^3$ and let

$$\Gamma = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + v^3 \frac{\partial}{\partial q^3} - q^1 \frac{\partial}{\partial v^1} - q^2 \frac{\partial}{\partial v^2} - q^3 \frac{\partial}{\partial v^3} .$$

Then

$$L = \frac{1}{2} ((v^1)^2 + (v^2)^2 + (v^3)^2 - (q^1)^2 - (q^2)^2 - (q^3)^2)$$

and

$$L' = \frac{1}{2} ((v^1)^2 + (v^2)^2 - (v^3)^2 - (q^1)^2 - (q^2)^2 + (q^3)^2)$$

are both nondegenerate lagrangians for Γ:

$$\left[\begin{array}{l} \Gamma_L = \Gamma \\ \\ \Gamma_{L'} = \Gamma. \end{array}\right.$$

Moreover,

$$\left[\begin{array}{l} X_1 = x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2} \\ \\ X_2 = x^3 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^3} \\ \\ X_3 = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \end{array}\right.$$

are infinitesimal symmetries of L, thus by 9.8 lead to the first integrals

$$\left[\begin{array}{l} q^2 v^3 - q^3 v^2 \\ q^3 v^1 - q^1 v^3 \\ q^1 v^2 - q^2 v^1 \end{array}\right.$$

for Γ. On the other hand,

$$\left[\begin{aligned} X_1' &= x^3 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^3} \\ X_2' &= x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3} \\ X_3' &= x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \end{aligned}\right.$$

are infinitesimal symmetries of L', thus by 9.8 lead to the first integrals

$$\left[\begin{aligned} & q^3 v^1 - q^1 v^3 \\ & q^3 v^2 - q^2 v^3 \\ & q^1 v^2 - q^2 v^1 \end{aligned}\right.$$

for Γ.

[Note: X_1' and X_2' are point symmetries of $\Gamma_{L'}$ (cf. 9.14) or still, are point symmetries of Γ_L. Therefore

$$\left[\begin{aligned} (X_1')^T E_L &= 2(q^1 q^3 + v^1 v^3) \\ (X_2')^T E_L &= 2(q^2 q^3 + v^2 v^3) \end{aligned}\right.$$

are first integrals for Γ_L (cf. 9.7) (or directly). But neither $(X_1')^T$ nor $(X_2')^T$ is a Cartan symmetry of Γ_L.]

According to 6.12, $\forall X \in \mathcal{D}^1(M)$,

$$L_{X^T} \circ d_S - d_S \circ L_{X^T} = d_{L_{X^T}S} .$$

But

$$L_{X^T}S = 0 \quad \text{(cf. 5.18)}.$$

Therefore

$$L_{X^T} \circ d_S = d_S \circ L_{X^T}.$$

Consequently,

$$\begin{aligned} L_{X^T}\theta_L &= L_{X^T}d_S L \\ &= d_S L_{X^T}L \\ &= \theta_{X^T L}. \end{aligned}$$

And then

$$\begin{aligned} L_{X^T}\omega_L &= L_{X^T}d\theta_L \\ &= dL_{X^T}\theta_L \\ &= d\theta_{X^T L} \\ &= \omega_{X^T L}. \end{aligned}$$

[Note: Our standing assumption is that L is nondegenerate but, in general, $X^T L$ will be degenerate.]

9.25 <u>LEMMA</u> $\forall\, X \in \mathcal{D}^1(M)$,

$$\iota_{[X^T,\Gamma_L]}\theta_L = 0.$$

<u>PROOF</u> Indeed

$$\begin{aligned}\iota_{[X^T,\Gamma_L]}\theta_L &= d_S L([X^T,\Gamma_L]) \\ &= S^* dL([X^T,\Gamma_L]) \\ &= dL(S[X^T,\Gamma_L]) \\ &= dL(0) \quad (\text{cf. } 5.19) \\ &= 0.\end{aligned}$$

[Note: This result enables one to simplify the proof of 9.8, there being no need to appeal to 9.14 to force

$$\iota_{[\Gamma_L,X^T]}\theta_L = 0$$

since 9.25 implies that this is automatic.]

9.26 <u>LEMMA</u> $\forall\, X \in \mathcal{D}^1(M)$,

$$\iota_{[\Gamma_L, X^T]}\omega_L = \iota_{\Gamma_L}\omega_{X^T L} + dE_{X^T L}.$$

PROOF First

$$\begin{aligned}
\iota_{[\Gamma_L, X^T]}\omega_L &= \iota_{[\Gamma_L, X^T]} d\theta_L \\
&= L_{[\Gamma_L, X^T]}\theta_L - d\iota_{[\Gamma_L, X^T]}\theta_L \\
&= L_{[\Gamma_L, X^T]}\theta_L \quad \text{(cf. 9.25)} \\
&= L_{\Gamma_L} L_{X^T}\theta_L - L_{X^T} L_{\Gamma_L}\theta_L \\
&= L_{\Gamma_L}\theta_{X^T L} - L_{X^T} dL \quad \text{(cf. 8.14)}.
\end{aligned}$$

Next, write

$$\begin{aligned}
L_{X^T} dL &= dX^T L \\
&= dX^T L - d\Delta X^T L + d\Delta X^T L \\
&= d(1 - \Delta) X^T L + d\Delta X^T L \\
&= - dE_{X^T L} + d\Delta X^T L.
\end{aligned}$$

Therefore

$$\iota_{[\Gamma_L, X^T]}\omega_L = L_{\Gamma_L}\theta_{X^T L} - d\Delta X^T L + dE_{X^T L}.$$

But

$$[\Delta, X^{\mathsf{T}}] = 0 \quad (\text{cf. } 4.4),$$

so

$$L_{\Gamma_L}\theta_{X^{\mathsf{T}}L} - d\Delta X^{\mathsf{T}}L$$

$$= L_{\Gamma_L}\theta_{X^{\mathsf{T}}L} - dX^{\mathsf{T}}\Delta L$$

$$= L_{\Gamma_L}\theta_{X^{\mathsf{T}}L} - dX^{\mathsf{T}}\iota_{\Gamma_L}\theta_L \quad (\text{cf. } 8.13).$$

Finally

$$\iota_{\Gamma_L}\omega_{X^{\mathsf{T}}L} = \iota_{\Gamma_L}L_{X^{\mathsf{T}}}\omega_L$$

$$= \iota_{\Gamma_L}L_{X^{\mathsf{T}}}d\theta_L$$

$$= \iota_{\Gamma_L}dL_{X^{\mathsf{T}}}\theta_L$$

$$= (L_{\Gamma_L} - d \circ \iota_{\Gamma_L})L_{X^{\mathsf{T}}}\theta_L$$

$$= L_{\Gamma_L}\theta_{X^{\mathsf{T}}L} - d\iota_{\Gamma_L}L_{X^{\mathsf{T}}}\theta_L$$

$$= L_{\Gamma_L}\theta_{X^{\mathsf{T}}L} + d(\iota_{[X^{\mathsf{T}},\Gamma_L]} - L_{X^{\mathsf{T}}} \circ \iota_{\Gamma_L})\theta_L$$

$$= L_{\Gamma_L}\theta_{X^{\mathsf{T}}L} - dX^{\mathsf{T}}\iota_{\Gamma_L}\theta_L \quad (\text{cf. } 9.25).$$

Now recall that, by definition, Γ_L admits the lagrangian $X^T L$ provided

$$\iota_{\Gamma_L} \omega_{X^T L} = - dE_{X^T L}$$

which, in view of 9.6, will be the case iff

$$[X^T, \Gamma_L] = 0.$$

I.e.: Iff X is a point symmetry of Γ_L.

§10. *MECHANICAL SYSTEMS*

Let M be a connected C^∞ manifold of dimension n — then an (autonomous) mechanical system $\mathcal{M}$ is a triple (M,T,Π), where $T \in C^\infty(TM)$ and Π is a horizontal 1-form on TM.

One calls

M — the configuration space

TM — the velocity phase space

n — the number of degrees of freedom.

10.1 REMARK Recall that the horizontal 1-forms on TM are in a one-to-one correspondence with the fiber preserving C^∞ functions $TM \to T^*M$ (cf. §7). In the context of a mechanical system, either entity is termed an (external) force field.

10.2 EXAMPLE Let L be a lagrangian. Take $\Pi = 0$ — then the triple $(M,L,0)$ is a mechanical system.

A mechanical system $\mathcal{M}$ is said to be nondegenerate if

$$\omega_T = dd_S T$$

is symplectic.

Suppose that $\mathcal{M}$ is nondegenerate — then $\exists$ a unique vector field $\Gamma_{\mathcal{M}} \in \mathcal{D}^1(TM)$ such that

$$\iota_{\Gamma_{\mathcal{M}}} \omega_T = d(T - \Delta T) + \Pi \ (= -dE_T + \Pi).$$

And, as the notation suggests, Γ_M is second order (cf. 8.12) (note that $\delta_S \Pi = 0$ (cf. 6.13)).

N.B. Working locally, write $\Pi = \Pi_i dq^i$ -- then along an integral curve γ of Γ_M, the equations of Lagrange

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v^i}\right) - \frac{\partial T}{\partial q^i} = \Pi_i \qquad (i = 1,\ldots,n)$$

with forces are satisfied.

10.3 EXAMPLE Take $M = \underline{R}^3$ and

$$\begin{cases} T = \frac{m}{2}\left((v^1)^2 + (v^2)^2 + (v^3)^2\right) \quad (m > 0) \\ \\ \Pi = \Pi_1 dq^1 + \Pi_2 dq^2 + \Pi_3 dq^3. \end{cases}$$

Then the mechanical system (M,T,Π) represents the motion of a particle of mass $m > 0$ in $\underline{R}^3$ under the influence of a force field Π. Here

$$\Gamma_M = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + v^3 \frac{\partial}{\partial q^3}$$

$$+ \frac{\Pi_1}{m} \frac{\partial}{\partial v^1} + \frac{\Pi_2}{m} \frac{\partial}{\partial v^2} + \frac{\Pi_3}{m} \frac{\partial}{\partial v^3}$$

and the integral curves of Γ_M are the solutions to

$$\frac{d^2 q^i}{dt^2} = \frac{\Pi_i}{m} \qquad (i = 1,2,3).$$

[Note: In the above, it is understood that q^1, q^2, q^3 are the usual cartesian coordinates. Matters change if we use spherical coordinates: $\tilde{q}^1 = r$ $(r > 0)$, $\tilde{q}^2 = \theta$ $(0 < \theta < \pi)$, $\tilde{q}^3 = \phi$ $(0 < \phi < 2\pi)$, so

$$\begin{cases} q^1 = \tilde{q}^1 \sin \tilde{q}^2 \cos \tilde{q}^3 \\ q^2 = \tilde{q}^1 \sin \tilde{q}^2 \sin \tilde{q}^3 \\ q^3 = \tilde{q}^1 \cos \tilde{q}^2. \end{cases}$$

Thus now

$$T = \frac{m}{2} \left((\tilde{v}^1)^2 + (\tilde{q}^1)^2 (\tilde{v}^2)^2 + (\tilde{q}^1)^2 (\tilde{v}^3)^2 (\sin \tilde{q}^2)^2 \right)$$

and

$$\Pi = \tilde{\Pi}_1 d\tilde{q}^1 + \tilde{\Pi}_2 d\tilde{q}^2 + \tilde{\Pi}_3 d\tilde{q}^3.$$

The tensor transformation rule of §2 can then be used to compute the $\tilde{\Pi}_i$ in terms of Π_i. To illustrate,

$$\tilde{\Pi}_3 = \frac{\partial q^1}{\partial \tilde{q}^3} \Pi_1 + \frac{\partial q^2}{\partial \tilde{q}^3} \Pi_2 + \frac{\partial q^3}{\partial \tilde{q}^3} \Pi_3$$

$$= (- \tilde{q}^1 \sin \tilde{q}^2 \sin \tilde{q}^3) \Pi_1 + (\tilde{q}^1 \sin \tilde{q}^2 \cos \tilde{q}^3) \Pi_2 .]$$

A nondegenerate mechanical system $\mathcal{M} = (M, T, \Pi)$ is said to be <u>conservative</u> if $\exists\, V \in C^\infty(M)$:

$$\Pi = - d(V \circ \pi_M) \; (= - \pi_M^*(dV)).$$

In this situation, we have

$$\begin{aligned}\iota_{\Gamma_M}\omega_T &= d(T - \Delta T) + \Pi \\ &= d(T - \Delta T) - d(V \circ \pi_M) \\ &= d(T - V \circ \pi_M - \Delta T) \\ &= d(L - \Delta L) \\ &= - dE_L,\end{aligned}$$

where

$$L = T - V \circ \pi_M.$$

Thus Π has disappeared and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0 \qquad (i = 1,\ldots,n).$$

But this puts us right back into §8 (with L nondegenerate) (evidently, $\omega_L = \omega_T$ and $\Gamma_L = \Gamma_M$).

Typically, $T = \frac{1}{2} g$, where g is a semiriemannian structure on M (cf. 8.4), hence

$$\Delta T = 2T \ (\Rightarrow E_T = \Delta T - T = T).$$

10.4 <u>LEMMA</u> Suppose that T is nondegenerate and $\Delta T = 2T$ -- then

$$L_\Delta \omega_T = \omega_T.$$

PROOF In fact,

$$L_\Delta \omega_T = L_\Delta dd_S T$$

$$= d(L_\Delta d_S T)$$

$$= d(d_S \circ L_\Delta - d_S)T \quad \text{(cf. 6.10)}$$

$$= 2dd_S T - dd_S T$$

$$= dd_S T$$

$$= \omega_T.$$

10.5 LEMMA Suppose that T is nondegenerate and $\Delta T = 2T$ -- then

$$[\Delta, \Gamma_T] = \Gamma_T,$$

thus the deviation of Γ_T vanishes.

PROOF For

$$\iota_{[\Delta,\Gamma_T]}\omega_T = (L_\Delta \circ \iota_{\Gamma_T} - \iota_{\Gamma_T} \circ L_\Delta)\omega_T$$

$$= L_\Delta(-\, dE_T) - \iota_{\Gamma_T}\omega_T \quad \text{(cf. 10.4)}$$

$$= -\, d\Delta E_T + dE_T$$

$$= d(E_T - \Delta E_T)$$

$$= d(T - 2T)$$

$$= - dE_T$$

$$= \iota_{\Gamma_T}\omega_T$$

=>

$$[\Delta,\Gamma_T] = \Gamma_T.$$

Take $T = \frac{1}{2}g$. Given a chart $(U,\{x^1,\ldots,x^n\})$ on M, let $\{\Gamma^i{}_{k\ell}\}$ be the connection coefficients per the metric connection ∇ determined by g.

10.6 <u>LEMMA</u> Locally,

$$\Gamma_T = v^i \frac{\partial}{\partial q^i} - ((\Gamma^i{}_{k\ell} \circ \pi_M)v^k v^\ell) \frac{\partial}{\partial v^i}.$$

[Note: The projection $\pi_M: TM \to M$ sets up a one-to-one correspondence between the (maximal) integral curves of Γ_T and the (maximal) geodesics of (M,g).]

10.7 <u>REMARK</u> The set $SO(TM)$ of second order vector fields on TM is an affine space whose translation group is the set of vertical vector fields in $\mathcal{D}^1(TM)$ (cf. 5.8). Choose Γ_T as its origin -- then Γ_T determines a bijection

$$SO(TM) \to V(TM),$$

viz.

$$\Gamma \to \Gamma - \Gamma_T.$$

Now consider

$$L = T - V \circ \pi_M.$$

Then

$$E_L = \Delta L - L$$

$$= \Delta(T - V \circ \pi_M) - (T - V \circ \pi_M)$$

$$= T + V \circ \pi_M.$$

[Note: Here

$$FL = F(T - V \circ \pi_M)$$

$$= FT - F(V \circ \pi_M)$$

$$= FT$$

$$= g^\flat \quad \text{(cf. 8.4)}.$$

Therefore FL is a diffeomorphism, hence 8.24 is applicable, and

$$H = \frac{1}{2} T \circ g^\# + V \circ \pi_M^*.]$$

10.8 <u>LEMMA</u> We have

$$\Gamma_L = \Gamma_T - (\text{grad } V)^V.$$

[Note: Locally,

$$\text{grad } V = (g^{ij} \frac{\partial V}{\partial x^j}) \frac{\partial}{\partial x^i}$$

=>

$$(\text{grad } V)^{v} = ((g^{ij} \frac{\partial V}{\partial x^j}) \circ \pi_M) \frac{\partial}{\partial v^i} .]$$

10.9 <u>REMARK</u> Suppose that $X \in \mathcal{D}^1(M)$ is an infinitesimal isometry of g such that $XV = 0$ -- then $X^T L = 0$ (cf. 8.4), thus X is an infinitesimal symmetry of L and so $X^v L$ is a first integral for Γ_L (cf. 9.8). Explicated,

$$X^v L: TM \to \underline{R}$$

is the function $g(X,\_\_)$. Locally,

$$X^v L = X^i \circ \pi_M \frac{\partial}{\partial v^i} \frac{1}{2} ((g_{k\ell} \circ \pi_M) v^k v^\ell)$$

$$= X^i \circ \pi_M \frac{1}{2} ((g_{k\ell} \circ \pi_M) \frac{\partial v^k}{\partial v^i} v^\ell + (g_{k\ell} \circ \pi_M) v^k \frac{\partial v^\ell}{\partial v^i})$$

$$= X^i \circ \pi_M \frac{1}{2} ((g_{i\ell} \circ \pi_M) v^\ell + (g_{ki} \circ \pi_M) v^k)$$

$$= X^i \circ \pi_M \frac{1}{2} ((g_{ij} \circ \pi_M) v^j + (g_{ji} \circ \pi_M) v^j)$$

$$= X^i \circ \pi_M (g_{ij} \circ \pi_M) v^j.$$

[Note: For a case in point, consider 9.9.]

10.10 <u>LEMMA</u> Suppose that $\Gamma \in \mathcal{D}^1(TM)$ is second order. Define $\Pi_\Gamma \in \Lambda^1 TM$ by

$$\Pi_\Gamma = \iota_\Gamma \omega_T + dT.$$

Then Π_Γ is horizontal.

<u>PROOF</u> Bearing in mind 6.14 (and the fact that θ_T is horizontal), take $X \in \mathcal{D}^1(TM)$ vertical and write

$$\begin{aligned}
\iota_\Gamma \omega_T(X) &= \iota_\Gamma d\theta_T(X) \\
&= (L_\Gamma - d \circ \iota_\Gamma)\theta_T(X) \\
&= (L_\Gamma \theta_T)(X) - d\iota_\Gamma \theta_T(X) \\
&= \Gamma\theta_T(X) - \theta_T([\Gamma,X]) - d\Delta T(X) \qquad \text{(cf. 8.13)} \\
&= \Gamma 0 - d_S T([\Gamma,X]) - d\Delta T(X) \\
&= - dT(S[\Gamma,X]) - 2dT(X) \\
&= - dT(- X) - 2dT(X) \qquad \text{(cf. 5.15)} \\
&= - dT(X).
\end{aligned}$$

Therefore

$$\begin{aligned}
\Pi_\Gamma(X) &= \iota_\Gamma \omega_T(X) + dT(X) \\
&= - dT(X) + dT(X) \\
&= 0.
\end{aligned}$$

[Note: Locally,

$$\Gamma = v^i \frac{\partial}{\partial q^i} + C^i \frac{\partial}{\partial v^i},$$

so locally,

$$\Pi_\Gamma = (g_{ij} \circ \pi_M)(C^j + (\Gamma^j{}_{k\ell} \circ \pi_M) v^k v^\ell) dq^i.]$$

N.B. This result implies that one can attach to each second order Γ a nondegenerate mechanical system

$$M_\Gamma = (M, T, \Pi_\Gamma).$$

And, of course,

$$\Gamma_{M_\Gamma} = \Gamma.$$

If Γ_1, Γ_2 are second order and if $\Pi_{\Gamma_1} = \Pi_{\Gamma_2}$, then

$$\iota_{\Gamma_1} \omega_T = \iota_{\Gamma_2} \omega_T,$$

so $\Gamma_1 = \Gamma_2$.

On the other hand, if $\alpha \in h\Lambda^1 TM$, then $\exists$ a unique vertical X_α:

$$\iota_{X_\alpha} \omega_T = \alpha \quad \text{(cf. 8.23)}.$$

Since Γ_T is second order (cf. 8.12) and

$$\iota_{\Gamma_T} \omega_T = - dE_T = - dT,$$

it follows that

$$\iota_{X_\alpha + \Gamma_T}\omega_T + dT$$

$$= \iota_{X_\alpha}\omega_T + \iota_{\Gamma_T}\omega_T + dT$$

$$= \alpha - dT + dT = \alpha.$$

10.11 SCHOLIUM The map

$$\Gamma \to \Pi_\Gamma$$

sets up a one-to-one correspondence between the set of second order vector fields on TM and the set of horizontal 1-forms on TM.

Let $\gamma:I \to TM$ be a trajectory of Γ. Fix $t_1 < t_2$ in I -- then the work done by the force field Π_Γ during the time interval $[t_1,t_2]$ is

$$\int_{t_1}^{t_2} \gamma^*\Pi_\Gamma.$$

But

$$\Pi_\Gamma = \iota_\Gamma\omega_\Gamma + dT$$

=>

$$\Pi_\Gamma(\Gamma) = dT(\Gamma).$$

Therefore

$$\int_{t_1}^{t_2} \gamma^*\Pi_\Gamma = T\Big|_{\gamma(t_1)}^{\gamma(t_2)} .$$

10.12 <u>REMARK</u> If $\Pi_\Gamma = - d(V_\Gamma \circ \pi_M)$ for some $V_\Gamma \in C^\infty(M)$, then

$$\int_{t_1}^{t_2} \gamma^*\Pi_\Gamma = V_\Gamma \circ \pi_M \Big|_{\gamma(t_1)}^{\gamma(t_2)},$$

implying thereby that

$$T(\gamma(t_1)) + V_\Gamma \circ \pi_M(\gamma(t_1)) = T(\gamma(t_2)) + V_\Gamma \circ \pi_M(\gamma(t_2)).$$

Put

$$L_\Gamma = T - V_\Gamma \circ \pi_M.$$

Then

$$E_{L_\Gamma} = T + V_\Gamma \circ \pi_M$$

and, being constant along γ, is a first integral for Γ (cf. 1.1), which, in the present setting, is another way of looking at 8.10 ($\Gamma_{L_\Gamma} = \Gamma$).

§11. *FIBERED MANIFOLDS*

Let M be a connected C^∞ manifold of dimension n -- then a <u>fibration</u> is a surjective submersion $\pi:E \to M$ and the triple (E,M,π) is called a <u>fibered manifold</u>.

E.g.: Vector bundles over M are fibered manifolds.

<u>N.B.</u> A fibration $\pi:E \to M$ is necessarily an open map, thus is quotient (being surjective).

If

$$\begin{cases} \pi:E \to M \\ \pi':E' \to M' \end{cases}$$

are fibrations, then a <u>morphism</u>

$$(F,f):(E,M,\pi) \to (E',M',\pi')$$

is a pair of C^∞ functions

$$\begin{cases} F:E \to E' \\ f:M \to M' \end{cases}$$

such that $\pi' \circ F = f \circ \pi$.

[Note: Accordingly, $\forall\ x \in M$,

$$F(\pi^{-1}(x)) \subset (\pi')^{-1}(f(x)).]$$

A morphism

$$(F,f):(E,M,\pi) \to (E',M',\pi')$$

is an <u>isomorphism</u> if $\exists$ a morphism

$$(F',f'):(E',M',\pi') \to (E,M,\pi)$$

such that

$$\begin{cases} F' \circ F = \mathrm{id}_E \\ f' \circ f = \mathrm{id}_M. \end{cases}$$

One then says that (E,M,π) and (E',M',π') are <u>isomorphic</u>.

11.1 <u>LEMMA</u> If $\Phi:N \to M$ is a surjective C^∞ map of constant rank, then Φ is a submersion, hence is a fibration.

Suppose that $\pi:E \to M$ is a fibration — then the rank of π is constant, viz.

$$\mathrm{rk}\ \pi = \dim M.$$

So, $\forall\ x \in M$, the fiber $E_x = \pi^{-1}(x)$ is a closed submanifold of E with

$$\dim E_x = \dim E - \dim M.$$

[Note: In general, E_x is not connected.]

11.2 <u>EXAMPLE</u> Take $E = \underline{R}^2 - \{(0,0)\}$, $M = \underline{R}$, $\pi = \mathrm{pr}_1$ -- then π is a fibration. Here, $\pi^{-1}(x)$ $(x \neq 0)$ is connected but $\pi^{-1}(0)$ is not connected.

11.3 <u>LEMMA</u> Suppose that $\pi:E \to M$ is a surjective C^∞ map — then π is a fibration iff every point of E is in the image of a local section of π.

11.4 REMARK The set of sections of a fibration π may be empty. For example, consider

$$(T\underline{S}^2\backslash\{0\},\ \underline{S}^2,\ \pi_{\underline{S}^2}|T\underline{S}^2\backslash\{0\})$$

and recall that $\underline{S}^2$ does not admit a never vanishing vector field.

11.5 LEMMA If (E,M,π) is a fibered manifold and if $\Phi:N \to M$ is a C^∞ map, then there is a pullback square

$$\begin{array}{ccc} N \times_M E & \xrightarrow{pr_2} & E \\ pr_1 \downarrow & & \downarrow \pi \\ N & \xrightarrow[\Phi]{} & M \end{array}$$

and $(N \times_M E, N, pr_1)$ is a fibered manifold.

PROOF It is clear that pr_1 is surjective. To see that it is a submersion, fix $(y_0,p_0) \in N \times_M E$ and choose a local section $\sigma:U \to E$ such that $p_0 \in \sigma(U)$ (cf. 11.3) -- then $\Phi(y_0) = \pi(p_0) \in U$. Define $\tau:\Phi^{-1}(U) \to N \times_M E$ by $\tau(y) = (y,\sigma(\Phi(y)))$ to get a local section of pr_1 passing through (y_0,p_0). Therefore pr_1 is a fibration (cf. 11.3).

Suppose that $\pi:E \to M$ is a fibration -- then the kernel of

$$T\pi:TE \to TM$$

is called the <u>vertical tangent bundle</u> of E, denoted VE. What was said at the beginning of §5 for the special case when E was assumed to be a vector bundle is applicable in general, thus there is an exact sequence

$$0 \rightarrow VE \rightarrow TE \rightarrow E \times_M TM \rightarrow 0 \qquad (\text{cf. } 5.2)$$

of vector bundles over E.

11.6 <u>EXAMPLE</u> Consider T^2M, the submanifold of TTM consisting of those points whose images under π_{TM} and $T\pi_M$ are one and the same or still, the fixed points of the canonical involution $s_{TM}:TTM \rightarrow TTM$. Note that

$$\dim T^2M = 3n.$$

- Let

$$\pi^{21} = \pi_{TM}\big|T^2M.$$

Then π^{21} is a fibration, thus the triple (T^2M,TM,π^{21}) is a fibered manifold.

- Let

$$\pi^1 = \pi_M \circ \pi^{21}.$$

Then π^1 is a fibration, thus the triple (T^2M,M,π^1) is a fibered manifold.

This data then gives rise to exact sequences

$$\left[\begin{array}{l} 0 \rightarrow V^{21}T^2M \xrightarrow{\mu_{21}} TT^2M \xrightarrow{\nu_{21}} T^2M \times_{TM} TTM \rightarrow 0 \\ 0 \rightarrow V^1T^2M \xrightarrow{\mu_1} TT^2M \xrightarrow{\nu_1} T^2M \times_M TM \rightarrow 0. \end{array}\right.$$

Moreover, there are canonical isomorphisms

$$\left[\begin{array}{l} T^2M \times_M TM \xrightarrow{i_{21}} V^{21}T^2M \\ \\ T^2M \times_{TM} TTM \xrightarrow{i_1} V^1T^2M \end{array}\right.$$

of vector bundles over T^2M. Now put

$$\left[\begin{array}{l} S^{21} = \mu_1 \circ i_1 \circ \nu_{21} \\ \\ S^1 = \mu_{21} \circ i_{21} \circ \nu_1. \end{array}\right.$$

Then

$$\left[\begin{array}{l} \text{Ker } S^{21} = V^{21}T^2M = \text{Im } S^1 \\ \\ \text{Ker } S^1 = V^1T^2M = \text{Im } S^{21} \end{array}\right.$$

and

$$(S^{21})^3 = 0.$$

[Note: T^2M is the <u>acceleration phase space</u>. Local coordinates in T^2M are

$$(q^i, v^i, a^i) \quad (i = 1, \ldots, n).]$$

Let (E,M,π) be a fibered manifold -- then a <u>trivialization</u> of (E,M,π) is a pair (F,t), where $t:E \to M \times F$ is a diffeomorphism such that

$$pr_1 \circ t = \pi.$$

Schematically:

$$\begin{array}{ccc} E & \xrightarrow{t} & M \times F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle pr_1} \\ M & = & M. \end{array}$$

[Note: The triple $(M \times F, M, pr_1)$ is a fibered manifold and

$$(t, id_M) : (E, M, \pi) \to (M \times F, M, pr_1)$$

is an isomorphism.

<u>N.B.</u> A fibered manifold (E, M, π) is said to be <u>trivial</u> if it admits a trivialization.

Let (E, M, π) be a fibered manifold — then (E, M, π) is said to be <u>locally trivial</u> if $\forall\ x \in M$, $\exists$ a triple (U_x, F_x, t_x), where U_x is a neighborhood of x and $t_x : \pi^{-1}(U_x) \to U_x \times F_x$ is a diffeomorphism such that

$$pr_1 \circ t_x = \pi | \pi^{-1}(U_x).$$

E.g.: Vector bundles over M are locally trivial fibered manifolds.

11.7 <u>LEMMA</u> If (E, M, π) is a locally trivial fibered manifold, then $\exists\ F$: $\forall$ local trivialization (U_x, F_x, t_x) $(x \in M)$, F_x and F are diffeomorphic.

<u>N.B.</u> In general, therefore, a fibered manifold is not locally trivial (cf. 11.2).

11.8 <u>LEMMA</u> If (E,M,π) is a fibered manifold and if π is proper, then (E,M,π) is locally trivial.

11.9 <u>EXAMPLE</u> The Hopf map $\underline{S}^3 \to \underline{S}^2$ is the restriction to $\underline{S}^3$ of the arrow $\underline{R}^4 \to \underline{R}^3$ defined by the rule that sends (x^1,x^2,x^3,x^4) to

$$((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2,\ 2(x^1x^4 + x^2x^3),\ 2(x^2x^4 - x^1x^3)).$$

It is a proper fibration, hence is locally trivial (cf. 11.8).

§12. AFFINE BUNDLES

Let M be a connected C^∞ manifold of dimension n, $\pi:E \to M$ a vector bundle -- then an affine bundle modeled on (E,M,π) is a pair $((A,M,\rho),r)$, where $\rho:A \to M$ is a fibration and $r:A \times_M E \to A$ is a morphism of fibered manifolds over id_M such that $\forall\ x \in M$,

$$r_x:A_x \times E_x \to A_x$$

is a free and transitive action of the additive group of E_x on the set A_x (thus A_x is an affine space modelled on E_x).

[Note: The triple $(A \times_M E,A,\mathrm{pr}_1)$ is a fibered manifold (cf. 11.3), hence so is $(A \times_M E,M,\rho \circ \mathrm{pr}_1)$ and the requirement is that the diagram

$$\begin{array}{ccc} A \times_M E & \xrightarrow{r} & A \\ \mathrm{pr}_1 \downarrow & & \downarrow \rho \\ A & \xrightarrow[\rho]{} & M \end{array}$$

commute, i.e., that the diagram

$$\begin{array}{ccc} A \times_M E & \xrightarrow{r} & A \\ \rho \circ \mathrm{pr}_1 \downarrow & & \downarrow \rho \\ M & = & M \end{array}$$

commute.]

12.1 LEMMA The fibered manifold (A,M,ρ) is locally trivial.

PROOF Bearing in mind that (E,M,π) is locally trivial, fix $x \in M$ and choose (U_x,F_x,t_x) accordingly. Without loss of generality, it can be assumed that U_x is the domain of a local section σ of A (cf. 11.3). Let $a \in \rho^{-1}(U_x)$ -- then there exists a unique element $\phi(a) \in \pi^{-1}(\rho(a))$:

$$a = \sigma(\rho(a)) + \phi(a).$$

The correspondence

$$\begin{cases} \rho^{-1}(U_x) \to \pi^{-1}(U_x) \\ \quad a \to \phi(a) \end{cases}$$

is a diffeomorphism which can then be postcomposed with t_x.

N.B. Every vector bundle (E,M,π) "is" an affine bundle $((E,M,\pi),+)$,

$$+:E \times_M E \to E$$

being addition in the fibers of π.

12.2 EXAMPLE Consider the fibered manifold (T^2M,TM,π^{21}) (cf. 11.6) -- then the fibers of π^{21} are not vector spaces but they are affine spaces. To make this precise, introduce the vector bundle

$$\pi_V:VTM \to TM \quad (\pi_V = \pi_{TM}|VTM).$$

Take an $x \in TM$ and let

$$\left[\begin{array}{l} a \in (\pi^{21})^{-1}(x) \\ \\ v \in (\pi_V)^{-1}(x). \end{array}\right.$$

Then

$$a + v \in (\pi^{21})^{-1}(x)$$

and the action

$$(r_V)_x : (\pi^{21})^{-1}(x) \times (\pi_V)^{-1}(x) \to (\pi^{21})^{-1}(x)$$

is free and transitive. Since this can be globalized, it follows that

$$((T^2M, TM, \pi^{21}), r_V)$$

is an affine bundle modelled on

$$(VTM, TM, \pi_V).$$

Let $\Gamma(\rho)$ stand for the set of sections of (A,M,ρ). E.g.: $\Gamma(\pi^{21}) = SO(TM)$ (cf. 5.8).

12.3 <u>LEMMA</u> Each $s \in \Gamma(\rho)$ determines an isomorphism $\phi_s : A \to E$ of fibered manifolds over id_M:

$$\begin{tikzcd} A \arrow[r, "\phi_s"] \arrow[d, "\rho"'] & E \arrow[d, "\pi"] \\ M \arrow[r, equal] & M. \end{tikzcd}$$

PROOF Given $a \in A_x$, there exists a unique $\phi_s(a) \in E_x$:

$$a = s(x) + \phi_s(a) \quad (x \in M).$$

12.4 REMARK $\Gamma(\rho)$ is not empty. This is because: (1) The fibers of ρ are contractible and (2) M is a polyhedron, hence is a CW complex.

Affine bundles are the natural setting for the study of fiber derivatives (the considerations in §7 constitute a special case).

Suppose that

$$\begin{cases} ((A,M,\rho),r) \\ ((A',M,\rho'),r') \end{cases}$$

are affine bundles modelled on vector bundles

$$\begin{cases} \phi:E \to M \\ \phi':E' \to M \end{cases}$$

respectively. Let

$$\zeta:A \to A'$$

be a morphism of fibered manifolds over id_M -- then $T\zeta$ restricts to a morphism

$$V\zeta:VA \to VA'$$

of vector bundles over M and there is a factorization

$$\begin{tikzcd}
VA \arrow[r, "v_\zeta"] \arrow[rr, bend left, "V\zeta"] & A \times_{A'} VA' \arrow[r, "pr_2"] \arrow[d] & VA' \arrow[d] \\
& A \arrow[r, "\zeta"] \arrow[d, "\rho"'] & A' \arrow[d, "\rho'"] \\
& M \arrow[r, equal] & M.
\end{tikzcd}$$

Here

$$v_\zeta \in \mathrm{Hom}_A(VA, A \times_{A'} VA'),$$

thus determines an element

$$s_{v_\zeta} \in \sec \mathrm{Hom}_A(VA, A \times_{A'} VA').$$

But

- $\left[\begin{array}{l} VA \approx A \;{}_M E \\ \\ VA' \approx A' \times_M E' \end{array}\right.$

- $A \times_M E' \approx A \times_{A'} (A' \times_M E')$.

So we have a diagram

$$\begin{tikzcd}
VA \arrow[r, "v_\zeta"] \arrow[dd, leftrightarrow] & A \times_{A'} VA' \arrow[r, "pr_2"] \arrow[d, leftrightarrow] & VA' \arrow[dd, leftrightarrow] \\
& A \times_{A'} (A' \times_M E') \arrow[d, leftrightarrow] & \\
A \times_M E & A \times_M E' \arrow[r] & A' \times_M E',
\end{tikzcd}$$

from which an arrow

$$A \times_M E \xrightarrow{d_\zeta} A \times_M E'$$

that, being a morphism of vector bundles over A, gives rise in turn to an element

$$s_{d_\zeta} \in \sec \mathrm{Hom}_A(A \times_M E, A \times_M E').$$

And by construction,

$$\begin{tikzcd} \mathrm{Hom}_A(VA, A \times_{A'} VA') \arrow[r, phantom, "\simeq"] & \mathrm{Hom}_A(A \times_M E, A \times_M E') \\ A \arrow[u, "s_{V_\zeta}"] \arrow[r, equal] & A \arrow[u, "s_{d_\zeta}"'] \end{tikzcd} .$$

Now identify

$$\mathrm{Hom}_A(A \times_M E, A \times_M E')$$

with

$$A \times_M \mathrm{Hom}_M(E, E').$$

Then the arrow

$$A \xrightarrow{s_{d_\zeta}} \mathrm{Hom}_A(A \times_M E, A \times_M E')$$

$$\simeq A \times_M \mathrm{Hom}_M(E, E') \xrightarrow{pr_2} \mathrm{Hom}_M(E, E')$$

is a morphism of fibered manifolds over id_M, denote it by $F\zeta$:

$$\begin{array}{ccc} A & \xrightarrow{F\zeta} & \mathrm{Hom}_M(E,E') \\ \rho\downarrow & & \downarrow \\ M & = & M. \end{array}$$

Definition: $F\zeta$ is the <u>fiber derivative</u> of ζ.

[Note: Canonically,

$$\mathrm{Hom}_M(E,E') \approx E^* \otimes_M E'$$

or still, omitting M,

$$\mathrm{Hom}(E,E') \approx E^* \otimes E'.]$$

<u>N.B.</u> $\forall\ x \in M$,

$$\zeta_x : A_x \to A'_x.$$

Since A_x and A'_x are affine spaces, the derivative of ζ_x at a point $a_x \in A_x$ is a linear map $D\zeta_x(a_x):E_x \to E'_x$. And, in fact,

$$D\zeta_x(a_x) = F\zeta(a_x).$$

12.5 <u>REMARK</u> Since

$$F\zeta : A \to \mathrm{Hom}(E,E')$$

is a morphism of fibered manifolds over id_M, it makes sense to iterate the procedure and form $F^k\zeta$. E.g.: Take $k = 2$ -- then

$$F^2\zeta : A \to \mathrm{Hom}(E,\mathrm{Hom}(E,E'))$$

$$\approx \mathrm{Hom}(E \otimes E, E')$$

$$\approx E^* \otimes E^* \otimes E',$$

the <u>fiber hessian</u> of ζ.

Let $f \in C^\infty(A)$ -- then f can be viewed as a morphism

$$A \to M \times \underline{R}$$

of fibered manifolds over id_M and

$$Ff:A \to \mathrm{Hom}(E, M \times \underline{R}) = E^*.$$

12.6 <u>EXAMPLE</u> Take $A = TM$, $E = TM$, thus $E^* = T^*M$ and

$$Ff:TM \to T^*M$$

is the fiber derivative of f per §7.

In the above, let $\zeta = Ff$ (and $A' = E' = E^*$) -- then

$$VA \approx A \times_M E \xrightarrow{d_{Ff}} A \times_M E^*.$$

But

- $$\begin{cases} (VA)^* \approx A \times_M E^* \\ VE^* \approx E^* \times_M E^* \end{cases}$$

- $A \times_M E^* \approx A \times_{E^*} (E^* \times_M E^*)$.

Therefore

$$
\begin{aligned}
(VA)^* &\approx A \times_M E^* \\
&\approx A \underset{E^*}{\times} (E^* \times_M E^*) \\
&\approx A \underset{E^*}{\times} VE^* \\
&\xrightarrow{pr_2} VE^*.
\end{aligned}
$$

Call the resulting arrow

$$(VA)^* \to VE^*$$

b_{Ff} -- then b_{Ff} is an isomorphism on fibers (this being the case of pr_2). On the other hand, there is a morphism

$$WFf : VA \to (VA)^*$$

of vector bundles over A and from the definitions,

$$VFf = b_{Ff} \circ WFf.$$

Schematically:

$$
\begin{tikzcd}
VA \arrow[r, "VFf"] \arrow[d, "WFf"'] & VE^* \arrow[d, equal] \\
(VA)^* \arrow[r, "b_{Ff}"'] & VE^*.
\end{tikzcd}
$$

12.7 <u>REMARK</u> The fiber hessian F^2f is an arrow

$$A \to \mathrm{Hom}(E, E^*).$$

As such, it determines an arrow

$$A \times_M E \to A \times_M E^*$$

that, in fact, is precisely d_{Ff}.

[Note: Explicated, WFf is the composition

$$VA \approx A \times_M E \xrightarrow{d_{Ff}} A \times_M E^* \approx (VA)^*.]$$

Consider now

$$TFf: TA \to TE^*.$$

Taking into account the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{Ff} & E^* \\ \rho \downarrow & & \downarrow \pi^* \\ M & = & M, \end{array}$$

we see that

$$\mathrm{Ker}\ TFf \subset \mathrm{Ker}\ T\rho = VA.$$

So

$$\mathrm{Ker}\ TFf = \mathrm{Ker}\ VFf$$

or still,

$$\mathrm{Ker}\ TFf = \mathrm{Ker}\ WFf.$$

12.8 <u>LEMMA</u> Ff is a local diffeomorphism iff WFf is an isomorphism.

12.9 <u>EXAMPLE</u> Let $L \in C^{\infty}(TM)$ be a lagrangian -- then

$$FL:TM \to T^*M$$

while

$$F^2L:TM \to \text{Hom}(TM,T^*M).$$

And, in view of 12.8, L is nondegenerate iff WFL is an isomorphism (cf. 8.2 and 8.5).

12.10 <u>EXAMPLE</u> Let $L \in C^{\infty}(TM)$ be a lagrangian. Consider its energy $E_L = \Delta L - L$ -- then

$$FE_L:TM \to T^*M$$

and we have

$$FE_L(x,X) = F^2L(x,X_x)(x,X_x) \qquad (X_x \in T_xM).$$

[Note: F^2L sends

$$T_xM \text{ to } \text{Hom}(T_xM,T_x^*M),$$

so

$$F^2L(x,X_x):T_xM \to T_x^*M.$$

We shall terminate this section with a definition that could have been made at the beginning. Thus let

$$\zeta:A \to A'$$

be a morphism of fibered manifolds over id_M -- then ζ is said to be an <u>affine bundle morphism</u> if $\exists$ a vector bundle morphism

$$\bar{\zeta}:E \rightarrow E'$$

such that $\forall\ x \in M\ \&\ \forall\ a_x \in A_x,\ \forall\ e_x \in E_x,$

$$\zeta_x(r_x(a_x,e_x)) = r'_x(\zeta_x(a_x),\bar{\zeta}_x(e_x))$$

or still,

$$\zeta_x(a_x + e_x) = \zeta_x(a_x) + \bar{\zeta}_x(e_x).$$

[Note: One calls $\bar{\zeta}$ the <u>linear part</u> of ζ.]

§13. *STRUCTURAL FORMALITIES*

Let M be a connected C^∞ manifold of dimension n, $\pi:E \to M$ a fibration. Let $\Phi:N \to M$ be a C^∞ map — then a <u>section of E along Φ</u> is a C^∞ map $\sigma:N \to E$ such that $\pi \circ \sigma = \Phi$.

13.1 <u>EXAMPLE</u> Suppose that

$$\begin{cases} ((A,M,\rho),r) \\ ((A',M,\rho'),r') \end{cases}$$

are affine bundles modelled on vector bundles

$$\begin{cases} \pi:E \to M \\ \pi':E' \to M \end{cases}$$

respectively. Let

$$\zeta:A \to A'$$

be a morphism of fibered manifolds over id_M — then there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{F\zeta} & \mathrm{Hom}(E,E') \\ \rho\downarrow & & \downarrow \\ M & = & M \end{array}$$

which can be read as saying that $F\zeta$ is a section of $\mathrm{Hom}(E,E')$ along ρ.

13.2 <u>LEMMA</u> The set of sections of E along Φ can be identified with the set of sections of the fibration $N \times_M E \xrightarrow{pr_1} N$ (cf. 11.5).

PROOF Given σ, define

$$\zeta \in \sec(N \times_M E \xrightarrow{pr_1} N)$$

by

$$\zeta(y) = (y, \sigma(y))$$

and vice-versa.

13.3 EXAMPLE Take $E = TM$, $N = TM$, $\Phi = \pi_M$ and consider

$$\begin{array}{ccc} TM \times_M TM & \xrightarrow{pr_2} & TM \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ TM & \xrightarrow[\pi_M]{} & M \end{array} .$$

Then

$$\sec(TM \times_M TM \xrightarrow{pr_1} TM)$$

is in a one-to-one correspondence with the set of fiber preserving C^∞ functions $TM \rightarrow TM$. On the other hand (cf. §5), there is an exact sequence

$$0 \rightarrow TM \times_M TM \xrightarrow{\mu} TTM \xrightarrow{\nu} TM \times_M TM \rightarrow 0$$

and the identification

$$\sec(TM \times_M TM \xrightarrow{pr_1} TM) \longleftrightarrow V(TM)$$

is implemented by sending a section ζ to $\mu \circ \zeta$:

$$TM \xrightarrow{\zeta} TM \times_M TM \xrightarrow{\mu} TTM.$$

Here

$$\left[\begin{array}{l} \pi_{TM} \circ \mu \circ \zeta = pr_1 \circ \zeta = id_{TM} \\ \\ T\pi_M \circ \mu \circ \zeta = pr_2 \circ \nu \circ \mu \circ \zeta = 0. \end{array}\right.$$

In particular: If ζ corresponds to $id_{TM}:TM \to TM$, then

$$\mu \circ \zeta = \Delta.$$

[Note: The zero map $TM \to TM$ sends (x,X_x) to $(x,0)$. And, spelled out, $pr_2 \circ \nu \circ \mu \circ \zeta$ is the composition

$$(x,X_x) \xrightarrow{\zeta} ((x,X_x),(x,Y_x)) \xrightarrow{\nu \circ \mu} ((x,0),(x,0)) \xrightarrow{pr_2} (x,0).]$$

13.4 EXAMPLE Consider the pullback square

$$\begin{array}{ccc} E \times_M T^*M & \xrightarrow{pr_2} & T^*M \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi_M^*} \\ E & \xrightarrow[\pi]{} & M \end{array}$$

and the canonical injection

$$\begin{tikzcd} E \times_M T^*M \arrow[r, "i^*"] \arrow[d, "pr_1"'] & T^*E \arrow[d, "\pi_E^*"] \\ E \arrow[r, equal] & E \end{tikzcd}$$

Given

$$\zeta \in \sec(E \times_M T^*M \xrightarrow{pr_1} E),$$

put

$$\alpha_\zeta = i^* \circ \zeta.$$

Then

$$\begin{aligned} \pi_E^* \circ \alpha_\zeta &= \pi_E^* \circ i^* \circ \zeta \\ &= pr_1 \circ \zeta \\ &= id_E. \end{aligned}$$

I.e.: $\alpha_\zeta \in \Lambda^1 E$. Moreover, α_ζ annihilates the sections of VE. In general, any $\alpha \in \Lambda^1 E$ with this property is termed <u>horizontal</u> (cf. 6.14). The upshot, therefore, is that the horizontal 1-forms on E can be identified with the sections of $E \times_M T^*M \xrightarrow{pr_1} E$ or still, with the fiber preserving C^∞ functions $E \to T^*M$ (cf. 13.2). Specialize and take $E = T^*M$ -- then the horizontal 1-form on T^*M associated with $id_{T^*M}:T^*M \to T^*M$ is Θ (the fundamental 1-form on T^*M).

A <u>vector field along Φ</u> is a section of TM along Φ, i.e., is a C^∞ map $X:N \to TM$

such that $\pi_M \circ X = \Phi$. Write $\mathcal{D}^1(M;N;\Phi)$ for the set of such (thus $\mathcal{D}^1(M) = \mathcal{D}^1(M;M;\mathrm{id}_M)$) — then $\mathcal{D}^1(M;N;\Phi)$ is a module over $C^\infty(N)$.

13.5 <u>LEMMA</u> If $X:M \to TM$ is a vector field on M, then

$$X \circ \Phi \in \mathcal{D}^1(M;N;\Phi).$$

<u>PROOF</u> In fact,

$$\pi_M \circ X \circ \Phi = \mathrm{id}_M \circ \Phi = \Phi.$$

13.6 <u>LEMMA</u> If $Y:N \to TN$ is a vector field on N, then

$$T\Phi \circ Y \in \mathcal{D}^1(M;N;\Phi).$$

<u>PROOF</u> There is a commutative diagram

$$\begin{array}{ccc} TN & \xrightarrow{T\Phi} & TM \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ N & \xrightarrow[\Phi]{} & M \end{array},$$

so

$$\begin{aligned} &\pi_M \circ T\Phi \circ Y \\ &\qquad = \Phi \circ \pi_N \circ Y = \Phi \circ \mathrm{id}_N = \Phi. \end{aligned}$$

Each $X \in \mathcal{D}^1(M;N;\Phi)$ determines an arrow

$$D_X:C^\infty(M) \to C^\infty(N)$$

via the prescription

$$D_X f|_y = df_{\Phi(y)}(X(y)) \qquad (y \in N)$$

with the property that

$$D_X(f_1 f_2) = (f_1 \circ \Phi)D_X f_2 + (f_2 \circ \Phi)D_X f_1.$$

E.g.: Take $N = TM$ and let $\Phi = \pi_M$ — then

$$\mathcal{D}^1(M;TM;\pi_M)$$

is simply the set of fiber preserving C^∞ functions $TM \to TM$. In particular:

$$\mathrm{id}_{TM} \in \mathcal{D}^1(M;TM;\pi_M).$$

And in this case the associated arrow

$$D_{\mathrm{id}_{TM}}:C^\infty(M) \to C^\infty(TM)$$

sends f to $\widehat{df}$ (cf. 8.19). Agreeing to write f^T in place of $\widehat{df}$, $\forall\ X \in \mathcal{D}^1(M)$,

$$X^T f^T = (Xf)^T.$$

<u>N.B.</u> Put $D^T = D_{\mathrm{id}_{TM}}$ — then locally,

$$D^T f = v^i\left(\frac{\partial}{\partial q^i}(f \circ \pi_M)\right) \quad (f \in C^\infty(M)).$$

13.7 <u>EXAMPLE</u> Given a fiber preserving C^∞ function $F:TM \to TM$, let

$$\mathcal{D}^1_F(TM) = \{X \in \mathcal{D}^1(TM):T\pi_M \circ X = F\} \quad \text{(cf. 13.6)}.$$

Then

$$\mathcal{D}^1_{\mathrm{id}_{TM}}(TM) = \mathcal{SO}(TM).$$

Let

$$i_{21}:T^2M \to TTM$$

be the injection -- then

$$i_{21} \in \mathcal{D}^1(TM;T^2M;\pi^{21}) \quad \text{(cf. 11.6)},$$

from which an arrow

$$D_{i_{21}}:C^\infty(TM) \to C^\infty(T^2M).$$

N.B. Put $D_{21} = D_{i_{21}}$ -- then locally,

$$D_{21}f = v^i(\frac{\partial}{\partial q^i} f \circ \pi^{21}) + a^i(\frac{\partial}{\partial v^i} f \circ \pi^{21}) \quad (f \in C^\infty(TM)).$$

13.8 EXAMPLE Let $f \in C^\infty(TM)$ -- then there is a commutative diagram

$$\begin{array}{ccc} T^2M & \xrightarrow{D_{21}f} & TM \times \underline{R} \\ {\scriptstyle \pi^{21}}\downarrow & & \downarrow{\scriptstyle pr_1} \\ TM & = & TM \end{array}.$$

Recalling now that

$$((T^2M,TM,\pi^{21}),r_V)$$

is an affine bundle modelled on

$$(VTM, TM, \pi_V) \quad (\text{cf. } 12.2),$$

the definitions imply that $D_{21}f$ is an affine bundle morphism whose linear part

$$\overline{D_{21}f}: VTM \to TM \times \underline{R}$$

is $\hat{df}|VTM$.

[Note:

$$f \in C^\infty(TM) \Rightarrow df \in \Lambda^1 TM$$

$$\Rightarrow \hat{df} \in C^\infty(TTM) \quad (\text{cf. } 8.19).]$$

Let $s_{TM}: TTM \to TTM$ be the canonical involution — then

$$\pi_{TM} \circ s_{TM} = T\pi_M,$$

thus

$$s_{TM} \in \mathcal{D}^1(TM; TTM; T\pi_M).$$

Local coordinates in TTM are

$$(q^i, v^i, dq^i, dv^i).$$

To render matters more transparent, let $\dot{q}^i = dq^i$, $\dot{v}^i = dv^i$ — then

$$\begin{cases} \pi_{TM}(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, v^i) \\ T\pi_M(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, \dot{q}^i) \end{cases}$$

and

$$s_{TM}(q^i,v^i,\dot{q}^i,\dot{v}^i) = (q^i,\dot{q}^i,v^i,\dot{v}^i).$$

E.g.: Let $f \in C^\infty(M)$ — then locally,

$$D^T(D^Tf)$$

$$= \dot{q}^i(\frac{\partial}{\partial q^i} f^T \circ \pi_{TM}) + \dot{v}^i(\frac{\partial}{\partial v^i} f^T \circ \pi_{TM})$$

$$= \dot{q}^i \frac{\partial}{\partial q^i}((v^j \circ \pi_{TM})(\frac{\partial}{\partial q^j}(f \circ \pi_M) \circ \pi_{TM})$$

$$+ \dot{v}^i \frac{\partial}{\partial v^i}((v^j \circ \pi_{TM})(\frac{\partial}{\partial q^j}(f \circ \pi_M) \circ \pi_{TM})$$

$$= \dot{q}^i v^j (\frac{\partial^2}{\partial q^i \partial q^j}(f \circ \pi_M) \circ \pi_{TM})$$

$$+ \dot{v}^i (\frac{\partial}{\partial q^i}(f \circ \pi_M) \circ \pi_{TM}).$$

Therefore

$$D^T(D^Tf) \circ s_{TM} = D^T(D^Tf).$$

13.9 <u>LEMMA</u> Locally,

$$D_s : C^\infty(TM) \to C^\infty(TTM) \quad (s = s_{TM})$$

is given by

$$D_s f = v^i(\frac{\partial}{\partial q^i} f \circ T\pi_M) + \dot{v}^i(\frac{\partial}{\partial \dot{q}^i} f \circ T\pi_M) \quad (f \in C^\infty(TM)).$$

A <u>1-form along Φ</u> is a section of T^*M along Φ, i.e., is a C^∞ map $\alpha:N \to T^*M$ such that $\pi_M^* \circ \alpha = \Phi$. Write $\mathcal{D}_1(M;N;\Phi)$ for the set of such (thus $\mathcal{D}_1(M) = \mathcal{D}_1(M;M;\mathrm{id}_M)$ -- then $\mathcal{D}_1(M;N;\Phi)$ is a module over $C^\infty(N)$.

<u>N.B.</u> There is a canonical pairing

$$\mathcal{D}^1(M;N;\Phi) \times \mathcal{D}_1(M;N;\Phi) \to C^\infty(N),$$

viz.

$$(X,\alpha) \to \langle X,\alpha\rangle \ (= \alpha(X)),$$

where

$$\langle X,\alpha\rangle\Big|_y = \langle X(y),\alpha(y)\rangle.$$

13.10 <u>EXAMPLE</u> The elements of

$$\mathcal{D}_1(M;TM;\pi_M)$$

are the fiber preserving C^∞ functions $F:TM \to T^*M$. They correspond one-to-one with the elements of $h\Lambda^1 TM$ (cf. 13.4), say $\alpha \to F_\alpha$.

[Note: Each $\alpha \in h\Lambda^1 TM$ gives rise to a C^∞ function $\hat{\alpha}:TM \to \underline{R}$. Indeed, at each point $(x,X_x) \in TM$ $(X_x \in T_xM)$, $\alpha_{(x,X_x)}$ is the pullback under the tangent map of a unique element $\lambda_x \in T_x^*M$, thus the prescription is

$$\hat{\alpha}(x,X_x) = \lambda_x(X_x) \quad \text{(cf. 8.19)}.$$

In terms of the pairing

$$\mathcal{D}^1(M;TM;\pi_M) \times \mathcal{D}_1(M;TM;\pi_M) \to C^\infty(TM),$$

we have

$$\langle id_{TM}, F_\alpha \rangle = \hat{\alpha}.$$

Therefore $\hat{\alpha} = 0$ iff α annihilates the elements of $SO(TM)$.]

Let $X \in \mathcal{D}^1(M;N;\Phi)$ -- then the arrow

$$D_X : C^\infty(M) \to C^\infty(N)$$

can be extended to a degree preserving map

$$D_X : \Lambda^* M \to \Lambda^* N$$

such that

$$D_X(\alpha \wedge \beta) = D_X \alpha \wedge \Phi^* \beta + \Phi^* \alpha \wedge D_X \beta$$

and

$$D_X \circ d_M = d_N \circ D_X,$$

where d_M and d_N are the exterior derivative operators in M and N.

To accomplish this, we shall appeal to the following standard generality.

13.11 <u>LEMMA</u> Let $X \in \mathcal{D}^1(M;N;\Phi)$ -- then $\forall\, y_0 \in N$, $\exists$ neighborhoods I_0 of 0 in $\underline{R}$ and V_{y_0} in N and a C^∞ map

$$G: I_0 \times V_{y_0} \to M$$

such that $\forall\, y \in V_{y_0}$,

$$\left[\begin{array}{l} G(0,y) = \Phi(y) \\ \\ X(y) = \frac{d}{dt} G(t,y) \Big|_{t=0} . \end{array}\right.$$

Put

$$G_t = G(t,—),$$

thus $\forall\ t \in I_0$,

$$G_t : V_{y_0} \rightarrow M.$$

So, given $\alpha \in \Lambda^p M$, $\{G_t^* \alpha\}$ is a one parameter family of elements of $\Lambda^p V_{y_0}$. Moreover,

$$\frac{d}{dt} (G_t^* \alpha(y)) \Big|_{t=0} = \lim_{t \rightarrow 0} \frac{1}{t} (G_t^* \alpha(y) - \Phi^* \alpha(y)) \quad (y \in V_{y_0})$$

exists and is independent of the choice of G. Denote it by $D_X \alpha(y)$ -- then these local considerations can be reformulated globally and lead to

$$D_X : \Lambda^* M \rightarrow \Lambda^* N$$

with the stated properties.

13.12 <u>REMARK</u> Take $N = M$, $\Phi = id_M$ -- then D_X is the Lie derivative

$$L_X : \Lambda^* M \rightarrow \Lambda^* M.$$

13.13 <u>LEMMA</u> Suppose that $\Phi' : N' \rightarrow N$ is a C^∞ map. Let $X \in \mathcal{D}^1(M;N;\Phi)$ -- then

$$X \circ \Phi' \in \mathcal{D}^1(M;N';\Phi \circ \Phi')$$

and

$$D_{X \circ \Phi'} = (\Phi')^* \circ D_X.$$

Define

$$\iota_X : \Lambda^*M \to \Lambda^*N$$

by

$$\iota_X f = 0 \quad (f \in C^\infty(M))$$

and for $\alpha \in \Lambda^p M$,

$$\iota_X \alpha \big|_y (Y_1, \ldots, Y_{p-1})$$

$$= \alpha \big|_{\Phi(y)} (X(y), \Phi_* \big|_y (Y_1), \ldots, \Phi_* \big|_y (Y_{p-1})),$$

where $Y_1, \ldots, Y_{p-1} \in T_y N$.

[Note: This is the interior product in the present setting (cf. 3.7).]

13.4 <u>LEMMA</u> We have

$$D_X = \iota_X \circ d_M + d_N \circ \iota_X.$$

Let us consider in more detail the situation when $N = TM$ and $\Phi = \pi_M$. Take $X = id_{TM}$ and write D^T in place of $D_{id_{TM}}$. Therefore

$$D^T : \Lambda^*M \to \Lambda^*TM$$

and, of course,

$$D^T f = f^T \quad (f \in C^\infty(M)).$$

Given $\alpha \in \Lambda^1 M$, put

$$\alpha^T = D^T \alpha.$$

Then $\forall\ X \in \mathcal{D}^1(M)$,

$$\begin{cases} \alpha^T(X^T) = \alpha(X)^T \\ \alpha^T(X^V) = \alpha(X) \circ \pi_M. \end{cases}$$

Locally,

$$\alpha = a_i dx^i$$

$\Rightarrow$

$$\alpha^T = v^j \left(\frac{\partial}{\partial q^j}(a_i \circ \pi_M)\right) dq^i + (a_i \circ \pi_M) dv^i.$$

And when $\alpha = df$ $(f \in C^\infty(M))$,

$$(d_M f)^T = d_{TM} f^T.$$

<u>N.B.</u> Write ι_T in place of $\iota_{\mathrm{id}_{TM}}$ -- then

$$\iota_T \alpha = \hat{\alpha} \quad \text{(cf. 8.19)}.$$

One can also apply the theory to

$$i_{21} \in \mathcal{D}^1(TM; T^2M; \pi^{21}),$$

leading thereby to

$$D_{21}:\Lambda^*TM \to \Lambda^*T^2M.$$

Accordingly (cf. 13.14),

$$D_{21} = \iota_{21} \circ d_{TM} + d_{T^2M} \circ \iota_{21}.$$

Here

$$\iota_{21} = \iota_{i_{21}}.$$

The <u>differential of Lagrange</u> is, by definition, the map

$$C^\infty(TM) \to \Lambda^1T^2M$$

that sends L to δL, where

$$\delta L = D_{21}\theta_L - (\pi^{21})^*dL.$$

[Note: Thanks to 8.13,

$$\delta L = \iota_{21}d\theta_L + (\pi^{21})^*dE_L.]$$

Recall now that the triple

$$(T^2M, M, \pi^1)$$

is a fibered manifold (cf. 11.6). Relative to this structure, δL is horizontal, hence determines a fiber preserving C^∞ function

$$F_{\delta L}:T^2M \to T^*M$$

such that

$$\delta L = F^*_{\delta L}\Theta \quad \text{(cf. 13.4)}.$$

Agreeing to regard $F_{\delta L}$ as a section of the fibration $T^2M \times_M T^*M \xrightarrow{pr_1} T^2M$

(cf. 13.2), write

$$\begin{aligned} &T^2M \times_M T^*M \\ &\approx T^2M \times_{TM} (TM \times_M T^*M) \\ &\approx T^2M \times_{TM} (VTM)^* \end{aligned}$$

to get an arrow

$$\begin{tikzcd} T^2M \arrow[r, "vF_{\delta L}"] \arrow[d, "\pi^{21}"'] & (VTM)^* \arrow[d] \\ TM \arrow[r, equal] & TM. \end{tikzcd}$$

13.15 <u>LEMMA</u> $vF_{\delta L}$ is an affine bundle morphism whose linear part

$$\overline{vF_{\delta L}}: VTM \to (VTM)^*$$

is WFL.

13.16 <u>RAPPEL</u> Fix $\Gamma \in \mathcal{SO}(TM)$ -- then Γ is said to admit a lagrangian L if

$$L_\Gamma \theta_L = dL.$$

Since $\Gamma: TM \to T^2M$, for a given L, it makes sense to form $\Gamma^*\delta L$.

13.17 <u>LEMMA</u> We have

$$\Gamma^*\delta L = L_\Gamma \theta_L - dL.$$

<u>PROOF</u> Obviously,

$$\Gamma^*((\pi^{21})^* dL) = (\pi^{21} \circ \Gamma)^* dL$$

$$= dL.$$

On the other hand,

$$D_{i_{21}} \circ \Gamma = \Gamma^* \circ D_{21} \quad (\text{cf. } 13.13).$$

But

$$i_{21} \circ \Gamma \in \mathcal{D}^1(TM;TM;\pi^{21} \circ \Gamma)$$

or still,

$$i_{21} \circ \Gamma \in \mathcal{D}^1(TM;TM;\mathrm{id}_{TM}).$$

Therefore (cf. 13.12)

$$D_{i_{21}} \circ \Gamma = L_{i_{21}} \circ \Gamma$$

$$\equiv L_{\Gamma}.$$

Consequently, Γ admits L iff

$$\Gamma^* \delta L = 0.$$

13.18 <u>REMARK</u> Locally,

$$\delta L = (D_{21} \frac{\partial L}{\partial v^i} - (\pi^{21})^* \frac{\partial L}{\partial q^i}) dq^i.$$

§14. *THE EVOLUTION OPERATOR*

Let M be a connected C^∞ manifold of dimension n -- then the theory developed in §13 provides us with an arrow

$$D^T:\Lambda^*T^*M \rightarrow \Lambda^*TT^*M.$$

In particular: Denoting by Θ_M the fundamental 1-form on T^*M,

$$D^T\Theta_M \equiv \Theta_M^T \in \Lambda^1TT^*M,$$

so

$$d\Theta_M^T \in \Lambda^2TT^*M.$$

14.1 <u>LEMMA</u> The pair $(TT^*M,d\Theta_M^T)$ is a symplectic manifold.

Various systems of local coordinates are going to figure in what follows, so it's best to draw up a list of them at the beginning.

TTM: Local coordinates are

$$(q^i,v^i,\dot{q}^i,\dot{v}^i).$$

TT*M: Local coordinates are

$$(q^i,p_i,\dot{q}^i,\dot{p}_i).$$

T*TM: Local coordinates are

$$(q^i,v^i,p_i,u_i).$$

T*T*M: Local coordinates are

$$(q^i, p_i, r_i, s^i).$$

The transpose of the injection

$$VTM \rightarrow TTM$$

is the projection

$$T^*TM \rightarrow (VTM)^*.$$

But

$$VTM \approx TM \times_M TM$$

$$\Rightarrow$$

$$(VTM)^* \approx TM \times_M T^*M.$$

This said, denote by pr_{T^*M} the arrow

$$T^*TM \rightarrow (VTM)^*$$

$$\approx TM \times_M T^*M \xrightarrow{pr_2} T^*M$$

of composition.

14.2 <u>LEMMA</u> There exists a unique diffeomorphism

$$\Psi: TT^*M \rightarrow T^*TM$$

such that

$$\pi^*_{TM} \circ \Psi = T\pi^*_M \text{ and } pr_{T^*M} \circ \Psi = \pi_{T^*M},$$

i.e., such that

$$\begin{array}{ccc} TT^*M & \xrightarrow{\Psi} & T^*TM \\ {\scriptstyle T\pi^*_M}\downarrow & & \downarrow{\scriptstyle \pi^*_{TM}} \\ TM & = & TM \end{array}$$

and

$$\begin{array}{ccc} TT^*M & \xrightarrow{\Psi} & T^*TM \\ {\scriptstyle \pi_{T^*M}} \downarrow & & \downarrow {\scriptstyle pr_{T^*M}} \\ T^*M & = & T^*M \end{array}$$

commute.

PROOF Locally,

$$\begin{cases} T\pi_M^*(q^i,p_i,\dot{q}^i,\dot{p}_i) = (q^i,\dot{q}^i) \\ \\ \pi_{TM}^*(q^i,v^i,p_i,u_i) = (q^i,v^i) \end{cases}$$

and

$$\begin{cases} \pi_{T^*M}(q^i,p_i,\dot{q}^i,\dot{p}_i) = (q^i,p_i) \\ \\ pr_{T^*M}(q^i,v^i,p_i,u_i) = (q^i,u_i). \end{cases}$$

So locally,

$$\Psi(q^i,p_i,\dot{q}^i,\dot{p}_i) = (q^i,\dot{q}^i,\dot{p}_i,p_i)..$$

Finish "par recollement...".

N.B. In the notation of §13, the relation

$$\pi_{TM}^* \circ \Psi = T\pi_M^*$$

translates to

$$\Psi \in \mathcal{D}_1(TM;TT^*M;T\pi_M^*).$$

14.3 <u>LEMMA</u> Let Θ_{TM} be the fundamental 1-form on T*TM -- then

$$\Psi^*\Theta_{TM} = \Theta_M^T.$$

<u>PROOF</u> Locally,

$$\Theta_M^T = \dot{p}_i dq^i + p_i d\dot{q}^i,$$

while

$$\begin{aligned}\Psi^*\Theta_{TM} &= \Psi^*(p_i dq^i + u_i dv^i)\\ &= (p_i \circ \Psi)d(q^i \circ \Psi) + (u_i \circ \Psi)d(v^i \circ \Psi)\\ &= \dot{p}_i dq^i + p_i d\dot{q}^i.\end{aligned}$$

<u>N.B.</u> Therefore

$$\Psi:(TT^*M, d\Theta_M^T) \to (T^*TM, d\Theta_{TM})$$

is a canonical transformation.

[Note: Let $\Omega_M = d\Theta_M$ (the fundamental 2-form on T*M) -- then

$$\begin{aligned}d\Theta_M^T &= d_{TT^*M}D^T\Theta_M\\ &= D^T d_{T^*M}\Theta_M\\ &= D^T d\Theta_M\\ &= D^T\Omega_M\\ &\equiv \Omega_M^T.\end{aligned}$$

So

$$\Psi:(TT^*M,\Omega_M^T) \to (T^*TM,\Omega_{TM}),$$

where, of course, $\Omega_{TM} = d\Theta_{TM}$ is the fundamental 2-form on T^*TM.]

Write $\Omega^\flat$ for the diffeomorphism

$$TT^*M \to T^*T^*M$$

induced by $-\Omega_M$, thus locally,

$$\Omega^\flat(q^i,p_i,\dot{q}^i,\dot{p}_i) = (q^i,p_i,-\dot{p}_i,\dot{q}^i).$$

14.4 <u>LEMMA</u> We have

$$\pi^*_{T^*M} \circ \Omega^\flat = \pi_{T^*M},$$

i.e., the diagram

$$\begin{array}{ccc} TT^*M & \xrightarrow{\Omega^\flat} & T^*T^*M \\ {\scriptstyle \pi_{T^*M}}\downarrow & & \downarrow{\scriptstyle \pi^*_{T^*M}} \\ T^*M & = & T^*M \end{array}$$

commutes.

<u>PROOF</u> Locally,

$$\left[\begin{array}{l} \pi_{T^*M}(q^i,p_i,\dot{q}^i,\dot{p}_i) = (q^i,p_i) \\ \\ \pi^*_{T^*M}(q^i,p_i,r_i,s^i) = (q^i,p_i). \end{array}\right.$$

[Note: Therefore

$$\Omega^{\flat} \in \mathcal{D}_1(T^*M;TT^*M;\pi_{T^*M}) \qquad \text{(cf. §13).]}$$

The transpose of the injection

$$VT^*M \to TT^*M$$

is the projection

$$T^*T^*M \to (VT^*M)^*.$$

But

$$VT^*M \approx T^*M \times_M T^*M$$

$$\Rightarrow$$

$$(VT^*M)^* \approx T^*M \times_M TM.$$

This said, denote by pr_M the arrow

$$T^*T^*M \to (VT^*M)^*$$

$$\approx T^*M \times_M TM \xrightarrow{pr_2} TM$$

of composition.

14.5 <u>LEMMA</u> We have

$$pr_M \circ \Omega^{\flat} = T\pi_M^*,$$

i.e., the diagram

$$\begin{tikzcd} TT^*M \arrow[r, "\Omega^{\flat}"] \arrow[d, "T\pi_M^*"'] & T^*T^*M \arrow[d, "pr_M"] \\ TM \arrow[r, equal] & TM \end{tikzcd}$$

commutes.

<u>PROOF</u> Locally,

$$\begin{cases} T\pi_M^*(q^i,p_i,\dot{q}^i,\dot{p}_i) = (q^i,\dot{q}^i) \\ \\ pr_M(q^i,p_i,r_i,s^i) = (q^i,s^i). \end{cases}$$

Consider

$$(\Omega^\flat)^*\Theta_{T^*M}.$$

Here, Θ_{T^*M} is the fundamental 1-form on T^*T^*M, thus locally,

$$\Theta_{T^*M} = r_i dq^i + s^i dp_i,$$

hence

$$\begin{aligned} (\Omega^\flat)^*\Theta_{T^*M} &= (\Omega^\flat)^*(r_i dq^i + s^i dp_i) \\ &= (r_i \circ \Omega^\flat)d(q^i \circ \Omega^\flat) + (s^i \circ \Omega^\flat)d(p_i \circ \Omega^\flat) \\ &= -\dot{p}_i dq^i + \dot{q}^i dp_i \quad (\neq \Theta_M^T). \end{aligned}$$

Therefore

$$\begin{aligned} &-d(-\dot{p}_i dq^i + \dot{q}^i dp_i) \\ &\quad = d\dot{p}_i \wedge dq^i - d\dot{q}^i \wedge dp_i \\ &\quad = d\dot{p}_i \wedge dq^i + dp_i \wedge d\dot{q}^i. \end{aligned}$$

And this implies that

$$\Omega_M^T = -\, d(\Omega^\flat)^* \Theta_{T^*M}.$$

14.6 <u>REMARK</u> Define

$$\Lambda_M : TT^*M \to \underline{R}$$

by the rule

$$\Lambda_M(V) = \langle T\pi_M^*(V), \pi_{T^*M}(V)\rangle \quad (V \in TT^*M).$$

Locally,

$$\Lambda_M(q^i, p_i, \dot{q}^i, \dot{p}_i) = \dot{q}^i p_i.$$

But then

$$\Theta_M^T + (\Omega^\flat)^* \Theta_{T^*M}$$

$$= \dot{p}_i dq^i + p_i d\dot{q}^i - \dot{p}_i dq^i + \dot{q}^i dp_i$$

$$= p_i d\dot{q}^i + \dot{q}^i dp_i$$

$$= d(\dot{q}^i p_i)$$

$$= d\Lambda_M$$

=>

$$d(\Theta_M^T + (\Omega^\flat)^* \Theta_{T^*M}) = 0$$

=>

$$\Omega_M^T = -\, d(\Omega^\flat)^* \Theta_{T^*M}.$$

Let $L \in C^\infty(TM)$ be a lagrangian -- then

$$\begin{cases} dL:TM \to T^*TM \\ \Psi^{-1}:T^*TM \to TT^*M, \end{cases}$$

so it makes sense to form

$$K_L = \Psi^{-1} \circ dL,$$

which will be called the <u>evolution operator</u> attached to L.

14.7 <u>LEMMA</u> We have

$$T\pi_M^* \circ K_L = \mathrm{id}_{TM}.$$

<u>PROOF</u> For

$$\pi_{TM}^* \circ \Psi = T\pi_M^* \quad \text{(cf. 14.2)}$$

$$\Rightarrow$$

$$\begin{aligned} T\pi_M^* \circ K_L &= T\pi_M^* \circ \Psi^{-1} \circ dL \\ &= \pi_{TM}^* \circ dL \\ &= \mathrm{id}_{TM}. \end{aligned}$$

14.8 <u>LEMMA</u> We have

$$\pi_{T^*M} \circ K_L = FL.$$

<u>PROOF</u> First

$$\begin{cases} FL(q^i,v^i) = (q^i, \frac{\partial L}{\partial v^i}) \\[2ex] dL(q^i,v^i) = (q^i,v^i, \frac{\partial L}{\partial q^i}, \frac{\partial L}{\partial v^i}). \end{cases}$$

Next

$$\Psi : TT^*M \to T^*TM$$

sends

$$(q^i,p_i,\dot{q}^i,\dot{p}_i) \text{ to } (q^i,\dot{q}^i,\dot{p}_i,p_i),$$

thus

$$\Psi^{-1} : T^*TM \to TT^*M$$

sends

$$(q^i,v^i,p_i,u_i) \text{ to } (q^i,u_i,v^i,p_i).$$

Finally

$$\pi_{T^*M} \circ K_L(q^i,v^i)$$

$$= \pi_{T^*M} \circ \Psi^{-1}(q^i,v^i, \frac{\partial L}{\partial q^i}, \frac{\partial L}{\partial v^i})$$

$$= \pi_{T^*M}(q^i, \frac{\partial L}{\partial v^i}, v^i, \frac{\partial L}{\partial q^i})$$

$$= (q^i, \frac{\partial L}{\partial v^i})$$

$$= FL(q^i,v^i).$$

<u>N.B.</u> Therefore

$$K_L \in \mathcal{D}^1(T^*M;TM;FL).$$

14.9 <u>RAPPEL</u> In the formalism of §13, let

$$X \in \mathcal{D}^1(M;N;\Phi).$$

Then a curve $\gamma:I \to N$ is said to be an <u>integral curve</u> of X provided

$$T\Phi \circ \dot{\gamma} = X \circ \gamma,$$

i.e.,

$$\begin{array}{ccc} TN & \xrightarrow{T\Phi} & TM \\ {\scriptstyle\dot{\gamma}}\uparrow & & \uparrow{\scriptstyle X} \\ I & \xrightarrow[\gamma]{} & N \end{array}$$

commutes.

[Note:

$$\dot{\gamma} \in \mathcal{D}^1(N;I;\gamma).]$$

Accordingly, in this terminology, a curve $\gamma:I \to TM$ is an integral curve of K_L if

$$TFL \circ \dot{\gamma} = K_L \circ \gamma.$$

14.10 <u>LEMMA</u> A curve $\gamma:I \to TM$ is an integral curve of K_L iff the equations of Lagrange are satisfied along γ.

<u>PROOF</u> Working locally, let $\gamma = (q^i, v^i)$ $(\equiv (q^i(t), v^i(t)))$ -- then

$$\dot{\gamma} = (q^i, v^i, \dot{q}^i, \dot{v}^i)$$

and

$$\begin{cases} TFL \circ \dot{\gamma} = (q^i, \dfrac{\partial L}{\partial v^i}, \dot{q}^i, \dot{q}^j \dfrac{\partial^2 L}{\partial v^i \partial q^j} + \dot{v}^j \dfrac{\partial^2 L}{\partial v^i \partial v^j}) \\ \\ K_L \circ \gamma = (q^i, \dfrac{\partial L}{\partial v^i}, v^i, \dfrac{\partial L}{\partial q^i}). \end{cases}$$

Therefore

$$TFL \circ \dot{\gamma} = K_L \circ \gamma$$

iff

$$\dot{q}^i = v^i$$

and

$$\dot{q}^j \frac{\partial^2 L}{\partial v^i \partial q^j} + \dot{v}^j \frac{\partial^2 L}{\partial v^i \partial v^j} = \frac{\partial L}{\partial q^i}$$

or, restoring t,

$$\frac{d(q^i(\gamma(t)))}{dt} = v^i(\gamma(t))$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)\Bigg|_{\gamma(t)} = \frac{\partial L}{\partial q^i}\Bigg|_{\gamma(t)}.$$

14.11 <u>REMARK</u> Suppose that L is nondegenerate -- then 14.10 implies that

a curve $\gamma: I \to TM$ is an integral curve of Γ_L iff it is an integral curve of K_L.
Therefore

$$\Gamma_L \circ \gamma = \dot{\gamma}$$

$$\Rightarrow$$

$$\begin{aligned} TFL \circ \Gamma_L \circ \gamma &= TFL \circ \dot{\gamma} \\ &= K_L \circ \gamma. \end{aligned}$$

Since γ is arbitrary, it follows that

$$TFL \circ \Gamma_L = K_L.$$

Because

$$K_L \in \mathcal{D}^1(T^*M; TM; FL),$$

there is an arrow

$$D_{K_L}: C^\infty(T^*M) \to C^\infty(TM).$$

Locally, $\forall\, f \in C^\infty(T^*M)$,

$$\begin{aligned} &D_{K_L} f \\ &= v^i \frac{\partial}{\partial q^i}(f \circ FL) + \frac{\partial L}{\partial q^i} \frac{\partial}{\partial p_i}(f \circ FL). \end{aligned}$$

§15. DISTRIBUTIONS-CODISTRIBUTIONS

Let M be a connected C^∞ manifold of dimension n.

• A distribution on M is a subset Σ of TM such that $\forall\ x \in M$, $\Sigma_x = \Sigma \cap T_xM$ is a linear subspace of T_xM and we define $\rho_\Sigma : M \to \underline{R}$ by

$$\rho_\Sigma(x) = \dim \Sigma_x.$$

One calls Σ differentiable if $\forall\ x \in M$, $\forall\ V_x \in \Sigma_x$, $\exists$ a neighborhood U of x and a vector field $X \in \mathcal{D}^1(U)$ such that $X_x = V_x$ and $X_y \in \Sigma_y$ $(y \in U)$.

[Note: A differentiable distribution Σ is linear if ρ_Σ is constant. Therefore the linear distributions are precisely the vector subbundles of TM.]

• A codistribution on M is a subset Σ^* of T^*M such that $\forall\ x \in M$, $\Sigma^*_x = \Sigma^* \cap T^*_xM$ is a linear subspace of T^*_xM and we define $\rho_{\Sigma^*} : M \to \underline{R}$ by

$$\rho_{\Sigma^*}(x) = \dim \Sigma^*_x.$$

One calls Σ^* differentiable if $\forall\ x \in M$, $\forall\ \alpha_x \in \Sigma^*_x$, $\exists$ a neighborhood U of x and a 1-form $\omega \in \mathcal{D}_1(U)$ such that $\omega_x = \alpha_x$ and $\omega_y \in \Sigma^*_y$ $(y \in U)$.

[Note: A differentiable codistribution Σ^* is linear if ρ_{Σ^*} is constant. Therefore the linear codistributions are precisely the vector subbundles of T^*M.]

15.1 REMARK The underlying assumption is that we are working in the C^∞ category. However, on occasion, it is convenient to work in the C^ω category, since there certain results can be significantly strengthened.

[Note: Tacitly, M is paracompact, thus admits an analytic structure which is unique up to a C^∞ diffeomorphism.]

15.2 LEMMA If

$$\begin{cases} \Sigma \\ \Sigma^* \end{cases}$$

are differentiable, then the functions

$$\begin{cases} \rho_\Sigma \\ \rho_{\Sigma^*} \end{cases}$$

are lower semicontinuous.

15.3 EXAMPLE Take $M = \underline{R}$ and let

$$\Sigma_x = \text{span } \{\chi(x) \frac{\partial}{\partial x}\},$$

where

$$\chi(x) = \begin{cases} 0 & (x \neq 0) \\ 1 & (x = 0). \end{cases}$$

Then ρ_Σ is not lower semicontinuous, hence Σ is not differentiable.

Given a differentiable distribution Σ or a differentiable codistribution Σ^*, a point $x \in M$ is <u>regular</u> if ρ_Σ or ρ_{Σ^*} is constant in a neighborhood of x; otherwise x is <u>singular</u>.

15.4 <u>LEMMA</u> The set of regular points per Σ or Σ^* is open and dense.

15.5 <u>EXAMPLE</u> The set of regular points need not be connected. E.g.: Take $M = \underline{R}^2$ and let

$$\Sigma_{(x,y)} = \text{span } \{\frac{\partial}{\partial x}, y \frac{\partial}{\partial y}\}.$$

Then Σ is differentiable. Moreover, its set of singular points is the x-axis while its set of regular points has two connected components, namely the upper half-plane $y > 0$ and the lower half-plane $y < 0$.

15.6 <u>EXAMPLE</u> Take $M =]0,1[$ and fix $\varepsilon (0 < \varepsilon < 1)$ -- then $\exists$ a closed subset $A \subset M$ of Lebesgue measure ε such that $M - A$ is open and dense in M. Choose $f \in C^\infty(M):f^{-1}(0) = A$. Define a differentiable distribution Σ by

$$\Sigma_x = \text{span } \{f(x) \frac{\partial}{\partial x}\}.$$

Then

$$\left[\begin{array}{l} M - A = \text{set of regular points of } \Sigma \\ \quad A = \text{set of singular points of } \Sigma. \end{array}\right.$$

[Note: Let M be a nonempty open subset of $\underline{R}^n$. Suppose that Σ is an analytic distribution -- then it can be shown that the Lebesgue measure of the set of singular points of Σ is zero.

• Let Σ be a distribution on M -- then the <u>annihilator</u> Ann Σ of Σ is the codistribution on M specified by

$$(\text{Ann } \Sigma)_x = \{\alpha_x \in T_x^*M : \alpha_x(V_x) = 0 \ \forall \ V_x \in \Sigma_x\}.$$

• Let Σ^* be a codistribution on M -- then the <u>annihilator</u> Ann Σ^* of Σ^* is the distribution on M specified by

$$(\text{Ann } \Sigma^*)_x = \{V_x \in T_xM : \alpha_x(V_x) = 0 \ \forall \ \alpha_x \in \Sigma_x^*\}.$$

Obviously,

$$\text{Ann}(\text{Ann } \Sigma) = \Sigma, \ \text{Ann}(\text{Ann } \Sigma^*) = \Sigma^*.$$

<u>N.B.</u> Suppose that $\Sigma(\Sigma^*)$ is differentiable -- then $\rho_{\text{Ann } \Sigma}(\rho_{\text{Ann } \Sigma^*})$ is upper semicontinuous (cf. 15.2), so Ann Σ(Ann Σ^*) is not differentiable unless $\Sigma(\Sigma^*)$ is linear.

15.7 <u>EXAMPLE</u> Take $M = \underline{R}^2$ and define a differentiable distribution Σ by

$$\Sigma_{(x,y)} = \text{span } \{x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}\}.$$

Then

$$(\text{Ann } \Sigma)_{(x,y)} = \begin{cases} T_{(x,y)}^*M & (x = y = 0) \\ \text{span } \{dx\} & (x = 0, \ y \neq 0) \\ \text{span } \{dy\} & (x \neq 0, \ y = 0) \\ \{0\} \text{ otherwise.} & \end{cases}$$

• Let Σ be a distribution on M -- then an immersed, connected submanifold N of M is called an <u>integral manifold</u> of Σ if $T_y N = \Sigma_y \ \forall \ y \in N$.

• Let Σ^* be a codistribution on M -- then an immersed, connected submanifold N of M is called an <u>integral manifold</u> of Σ^* if $T_y N = (\text{Ann } \Sigma^*)_y \ \forall \ y \in N$.

15.8 <u>EXAMPLE</u> Assume that $X \in \mathcal{D}^1(M)$ never vanishes and let $\Sigma_x = \text{span } \{X_x\}$ $(x \in M)$ -- then the trajectories of X are integral manifolds of Σ.

A differentiable distribution Σ on M is <u>integrable</u> if $\forall \ x \in M$, there exists an integral manifold of Σ containing x.

15.9 <u>THEOREM</u> Suppose that Σ is integrable -- then $\forall \ x \in M$, there exists a unique integral manifold N of Σ containing x and which is maximal w.r.t. containment.

[Note: If N and N' are integral manifolds of Σ such that $N \cap N' \neq \emptyset$, then $N \cap N'$ is open in N and N' and the differentiable structures induced on $N \cap N'$ by those of N and N' are identical. Furthermore, $N \cup N'$ is an integral manifold of Σ in which both N and N' are open.]

15.10 <u>REMARK</u> The maximal integral manifolds of Σ form a partition of M, the <u>foliation</u> $\mathcal{F}_\Sigma$ of M determined by Σ (the N being the <u>leaves</u> of $\mathcal{F}_\Sigma$).

15.11 <u>EXAMPLE</u> Suppose that $\pi: E \rightarrow M$ is a fibration. Consider $VE \subset TE$ -- then VE is a vector subbundle of TE, hence is a linear distribution. In addition, VE is integrable and the leaves of the associated foliation of E are the connected components of the $E_x = \pi^{-1}(x)$ $(x \in M)$.

[Note: An <u>Ehresmann connection</u> for the fibration $\pi:E \to M$ is a linear distribution $H \subset TE$ such that $\forall\ e \in E$,

$$VE|_e \oplus H_e = T_eE.]$$

15.12 <u>EXAMPLE</u> Let $\alpha \in \Lambda^pM$ be a nonzero closed p-form on M -- then the <u>characteristic subspace</u> of α at a point $x \in M$ is Ker α_x, where

$$\text{Ker } \alpha_x = \{V_x \in T_xM: \iota_{V_x}\alpha_x = 0\},$$

and the <u>characteristic distribution</u> Ker α of α is the assignment

$$x \to \text{Ker } \alpha_x.$$

In general, Ker α is not differentiable. To remedy this, let $\mathcal{D}(\alpha)$ be the set of all locally defined vector fields X on M such that

$$\iota_X\alpha = 0.$$

Define a distribution $\Sigma(\alpha)$ on M by specifying that $\Sigma(\alpha)_x$ is to be the subspace of T_xM spanned by the X_x $(X \in \mathcal{D}(\alpha),\ x \in \text{Dom } X)$ -- then $\Sigma(\alpha)$ is contained in Ker α. Moreover, $\Sigma(\alpha)$ is differentiable and, in fact, integrable. Recall now that the <u>rank</u> of α_x is

$$\text{rk}_x\alpha = \dim(T_xM/\text{Ker } \alpha_x),$$

thus

$$p \leq \mathrm{rk}_x \alpha \leq n.$$

Impose the restriction that $x \rightarrow \mathrm{rk}_x \alpha$ is constant (i.e., that α be of constant rank) -- then in this situation,

$$\mathrm{Ker}\ \alpha = \Sigma(\alpha).$$

Therefore Ker α is linear or still, is a vector subbundle of TM. And the fiber dimension of Ker α is k if $n - k = \mathrm{rk}_x \alpha$ $(x \in M)$.

[Note: Take $M = \underline{R}^2$ and let $\alpha = xdx$ -- then α is closed and

$$\mathrm{Ker}\ \alpha \big|_{(x,y)} = \begin{cases} \{0\} \times \underline{R} & (x \neq 0) \\ \underline{R}^2 & (x = 0). \end{cases}$$

Therefore Ker α is not differentiable (cf. 15.2). On the other hand, if X is a vector field defined on a connected open subset of $\underline{R}^2$, then $X \in \mathcal{D}(\alpha)$ iff X has the form $g \frac{\partial}{\partial y}$, g a differentiable function. So $\Sigma(\alpha)$ is generated by $\frac{\partial}{\partial y}$, hence $\Sigma(\alpha)$ is strictly contained in Ker α.]

15.13 <u>REMARK</u> Let $L \in C^\infty(TM)$ be a lagrangian. To be in agreement with 15.12, assume that ω_L has constant rank, thus Ker ω_L is a vector subbundle of TTM. But in §8, we put

$$\mathrm{Ker}\ \omega_L = \{X \in \mathcal{D}^1(TM) : \iota_X \omega_L = 0\}.$$

This, of course, is an abuse of notation in that the sections of the bundle are being denoted by the same symbol as the bundle itself. However, no real confusion should arise from this practice.

If Σ is an integrable distribution, then a function $f \in C^\infty(M)$ is a <u>first integral</u> for Σ provided the restriction of f to each leaf $N \in \mathcal{F}_\Sigma$ is constant.

<u>N.B.</u> There may be no nontrivial first integrals. E.g.: If Σ has a leaf which is dense in M, then the only first integrals for Σ are the constants.

15.14 <u>EXAMPLE</u> Suppose that (M,ω) is a symplectic manifold. Given a linear distribution Σ, define a linear distribution $\omega^\perp\Sigma$ by

$$\omega^\perp\Sigma\big|_x = \{V_x \in T_xM: \omega_x(V_x,X_x) = 0 \ \forall \ X_x \in \Sigma_x\}.$$

In terms of

$$\omega^\flat: TM \to T^*M$$

and its inverse

$$\omega^\#: T^*M \to TM,$$

we have

$$\omega^\#(\text{Ann } \Sigma) = \omega^\perp\Sigma.$$

Assume now that Σ is integrable and let f be a first integral for Σ -- then $\omega^\# df$ is a section of $\omega^\perp\Sigma$. Thus, $\forall \ X \in \sec \Sigma$,

$$\omega(\omega^\# df, X) = \iota_{\omega^\# df}\,\omega(X)$$

$$= df(X)$$

$$= Xf$$

$$= 0,$$

the last step following from the fact that X is tangent to the leaves of $\mathcal{F}_\Sigma$.

15.15 <u>LEMMA</u> If Σ is integrable and if x is a regular point, then $\exists$ a chart $(U,\{x^1,\ldots,x^n\})$ with $x \in U$ such that

$$E_y = \text{span}\ \{\frac{\partial}{\partial x^1}\Big|_y, \ldots, \frac{\partial}{\partial x^k}\Big|_y\} \quad (y \in U).$$

[Note: Here

$$k = \rho_\Sigma(x) \quad (= \dim \Sigma_x).]$$

A differentiable distribution Σ on M is <u>involutive</u> if $\forall$ pair X,Y of vector fields defined on some open subset $U \subset M$ such that $\forall\ x \in U$, X_x & $Y_x \in \Sigma_x$, we also have

$$[X,Y]_x \in \Sigma_x.$$

15.16 <u>LEMMA</u> If Σ is integrable, then Σ is involutive.

15.17 <u>EXAMPLE</u> Take $M = \underline{R}^2$ and let

$$\Sigma_{(x,y)} = \text{span}\ \{\frac{\partial}{\partial x}, \phi(x)\,\frac{\partial}{\partial y}\},$$

where $\phi(x)$ is a C^∞ function which is 0 for $x \leq 0$ and > 0 for $x > 0$ -- then Σ is differentiable. And

$$\left[\frac{\partial}{\partial x}, \phi(x)\frac{\partial}{\partial y}\right]_{(x,y)}$$

$$= \phi'(x)\frac{\partial}{\partial y}.$$

Therefore Σ is involutive. Still, Σ is not integrable.

15.18 THEOREM (Frobenius) Suppose that Σ is linear -- then Σ is integrable iff Σ is involutive.

15.19 LEMMA A linear distribution Σ is involutive iff sec Σ is a Lie subalgebra of $\mathcal{D}^1(M)$.

15.20 EXAMPLE A presymplectic manifold is a pair (M,ω), where ω is a closed 2-form of constant rank. Consider Ker $\omega \subset TM$ (cf. 15.12) -- then Ker ω is linear and we claim that Ker ω is involutive. To see this, let $X,Y \in$ sec Ker ω -- then

$$\iota_{[X,Y]}\omega = (L_X \circ \iota_Y - \iota_Y \circ L_X)\omega$$

$$= - \iota_Y L_X \omega$$

$$= - \iota_Y(\iota_X \circ d + d \circ \iota_X)\omega$$

$$= 0,$$

so

$$[X,Y] \in \text{sec Ker } \omega.$$

Therefore Ker ω is involutive (cf. 15.19), hence integrable (cf. 15.18).

[Note: The rank of ω is necessarily even.]

15.21 THEOREM (Nagano) An analytic distribution is integrable iff it is involutive.

15.22 EXAMPLE Take $M = \underline{R}^2$ and let

$$\Sigma_{(x,y)} = \text{span } \{x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}\}.$$

Then Σ is involutive, thus is integrable (being analytic). As for the foliation F_Σ, it has 9 leaves, viz.

$$\{(0,0)\};$$

$$\begin{bmatrix} \{(x,0):x > 0\} \\ \\ \{(x,0):x < 0\} \end{bmatrix} ; \quad \begin{bmatrix} \{(0,y):y > 0\} \\ \\ \{(0,y):y < 0\} \end{bmatrix} ;$$

$$\begin{bmatrix} \{(x,y):x > 0,y > 0\} \\ \\ \{(x,y):x < 0,y < 0\} \end{bmatrix} ; \quad \begin{bmatrix} \{(x,y):x > 0,y < 0\} \\ \\ \{(x,y):x < 0,y > 0\} \end{bmatrix}.$$

15.23 LEMMA Suppose that Σ is linear of fiber dimension k -- then $\forall$ $x \in M$, $\exists$ a neighborhood U of x and linearly independent 1-forms $\omega^1,\ldots,\omega^{n-k}$ on U such that

$$\Sigma_y = \text{Ker}\ \omega^1|_y \cap \cdots \cap \text{Ker}\ \omega^{n-k}|_y \quad (y \in U).$$

[Note: Introduce

$$\begin{bmatrix} \hat{\omega}^1 : TU \to \underline{R} \\ \vdots \\ \hat{\omega}^{n-k} : TU \to \underline{R} \end{bmatrix} \qquad \text{(cf. 8.19)}.$$

Then what is being said is that $\Sigma|U$, viewed as a subset of TU, can be characterized as

$$(\hat{\omega}^1)^{-1}(0) \cap \cdots \cap (\hat{\omega}^{n-k})^{-1}(0).$$

Locally,

$$\omega^i = \sum_{j=1}^{n} a^i_j dx^j$$

$$\Rightarrow$$

$$\hat{\omega}^i = \sum_{j=1}^{n} (a^i_j \circ \pi_U) v^j.]$$

15.24 <u>REMARK</u> Σ is involutive on U iff $\exists$ 1-forms θ^i_j on U such that

$$d\omega^i = \sum_{j=1}^{n-k} \theta^i_j \wedge \omega^j \quad (i = 1,\ldots,n-k).$$

[Note: One can go further: Each $x \in U$ admits a neighborhood $U_x \subset U$ on which $\exists$ C^∞ functions C^i_j, f^j $(i,j = 1,\ldots,n-k)$ such that

$$\omega^i = \sum_{j=1}^{n-k} C^i_j df^j.]$$

If $\omega^1,\ldots,\omega^{n-k}$ are linearly independent 1-forms on M, then the prescription

$$\Sigma_x = \text{Ker}\ \omega^1|_x \cap \cdots \cap \text{Ker}\ \omega^{n-k}|_x \qquad (x \in M)$$

defines a linear distribution Σ on M of fiber dimension k.

[Note: If it is a question of a single 1-form, then the assumption is that this 1-form is nowhere vanishing.]

15.25 EXAMPLE Take $M = \underline{R}^3$ and let

$$\omega = dx + xydz.$$

Then

$$\Sigma_{(x,y,z)} = \text{span}\ \{\frac{\partial}{\partial y}, \frac{\partial}{\partial z} - xy\frac{\partial}{\partial x}\}.$$

15.26 REMARK Take $M = \underline{R}^3$ and let

$$\begin{cases} \omega^1 = dx + ydz \\ \omega^2 = dx + zdy. \end{cases}$$

Then ω^1 and ω^2 are not linearly independent. Since

$$\begin{cases} \text{Ker}\ \omega^1 = \text{span}\ \{\frac{\partial}{\partial y}, \frac{\partial}{\partial z} - y\frac{\partial}{\partial x}\} \\ \text{Ker}\ \omega^2 = \text{span}\ \{\frac{\partial}{\partial z}, \frac{\partial}{\partial y} - z\frac{\partial}{\partial x}\}, \end{cases}$$

we have

$$\left[\begin{array}{ll} \Sigma_{(x,0,0)} = \text{span } \{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\} & \\ \Sigma_{(x,0,z)} = \text{span } \{\frac{\partial}{\partial z}\} & (z \neq 0) \\ \Sigma_{(x,y,0)} = \text{span } \{\frac{\partial}{\partial y}\} & (y \neq 0) \\ \Sigma_{(x,y,z)} = \text{span } \{- z \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{z}{y} \frac{\partial}{\partial z}\} & (y \neq 0, z \neq 0). \end{array}\right.$$

So, along the x-axis ρ_Σ is not lower semicontinuous, which implies that Σ is not differentiable (cf. 15.2).

15.27 <u>LEMMA</u> Σ is integrable iff

$$d\omega^i \wedge (\omega^1 \wedge \ldots \wedge \omega^{n-k}) = 0 \quad (i = 1,\ldots,n-k).$$

E.g.: If the issue is that of (n-1) 1-forms, then

$$d\omega^i \wedge (\omega^1 \wedge \ldots \wedge \omega^{n-1}) = 0 \quad (i = 1,\ldots,n-1).$$

Therefore Σ is integrable.

15.28 <u>EXAMPLE</u> Take $M = \underline{R}^3$ and let

$$\omega = Adx + Bdy + Cdz,$$

where A,B,C are differentiable functions of x,y,z (not all vanishing simultaneously) -- then Σ is integrable iff

$$A(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}) + B(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}) + C(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}) = 0.$$

Thinking of A,B,C as the components of a vector field $\vec{F}$, the condition thus amounts to requiring that

$$\vec{F} \cdot \text{curl } \vec{F} = 0.$$

E.g.: Σ is integrable if

$$\omega = yz(y+z)dx + zx(z+x)dy + xy(x+y)dz$$

but Σ is not integrable if

$$\omega = xdy + dz.$$

15.29 <u>REMARK</u> Take $M = \underline{R}^3$ and work with 1-forms $\begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$ -- then it may very well be the case that the distributions $\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix}$ per $\begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$ individually are not integrable. Nevertheless, the distribution Σ per $\begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$ collectively must be integrable (cf. 15.27):

$$d\omega^i \wedge (\omega^1 \wedge \omega^2) = 0 \quad (i = 1,2).$$

§16. LAGRANGE MULTIPLIERS

Informally, constraints are conditions imposed on a mechanical system that restrict access to its configuration space or its velocity phase space.

So, as usual, let M be a connected C^∞ manifold of dimension n. Fix a riemannian structure g on M and let $T = \frac{1}{2} g$ -- then we shall work with the mechanical system $\mathcal{M} = (M,T,\Pi)$, where Π is horizontal.

[Note: Recall from §10 that the second order vector field $\Gamma_{\mathcal{M}}$ is characterized by the property that

$$\iota_{\Gamma_{\mathcal{M}}} \omega_T = - dT + \Pi.]$$

By a <u>system of constraints</u>, one understands a set $\omega^1,\ldots,\omega^{n-k}$ of linearly independent 1-forms on M. As will become apparent, the key point is to first study the case when k = n-1.

To this end, fix a nowhere vanishing 1-form $\omega \in \Lambda^1 M$ -- then $\hat{\omega} \in C^\infty(TM)$ (cf. 8.19) and since $\pi_M^* \omega \in h\Lambda^1 TM$, $\exists$ a unique vertical X_ω:

$$\iota_{X_\omega} \omega_T = \pi_M^* \omega \quad \text{(cf. 8.23)}.$$

<u>N.B.</u> Locally, if

$$\omega = a_i dx^i,$$

then

$$X_\omega = (W^{ij}(T)(a_j \circ \pi_M)) \frac{\partial}{\partial v^i} .$$

Here, as in §8,

$$W(T) = [W_{ij}(T)],$$

where

$$W_{ij}(T) = \frac{\partial^2 T}{\partial v^i \partial v^j} \quad (= g_{ij} \circ \pi_M)$$

and we have abbreviated

$$(W(T)^{-1})^{ij}$$

to

$$W^{ij}(T) \quad (= g^{ij} \circ \pi_M).$$

16.1 <u>LEMMA</u> Determine $X_{\hat{\omega}} \in \mathcal{D}^1(TM)$ via the prescription

$$\iota_{X_{\hat{\omega}}} \omega_T = d\hat{\omega}.$$

Then

$$SX_{\hat{\omega}} = - X_{\omega}.$$

<u>PROOF</u> From the definitions, $S^*(d\hat{\omega}) = \pi_M^*\omega$, hence

$$\begin{aligned} S^*(\iota_{X_{\hat{\omega}}} \omega_T) &= S^*(d\hat{\omega}) \\ &= \pi_M^*\omega \\ &= \iota_{X_\omega} \omega_T. \end{aligned}$$

But, on general grounds (see below), $\forall\ X \in \mathcal{D}^1(TM)$,

$$S^*(\iota_X \omega_T) + \iota_{SX} \omega_T = 0.$$

Therefore

$$\iota_{SX_{\hat{\omega}}} \omega_T = - S^*(\iota_{X_{\hat{\omega}}} \omega_T)$$

$$= - \iota_{X_\omega} \omega_T$$

=>

$$SX_{\hat{\omega}} = - X_\omega.$$

[Note: According to 6.3, $\forall\ X \in \mathcal{D}^1(TM)$,

$$\iota_X \circ \delta_S - \delta_S \circ \iota_X = \iota_{SX}.$$

So

$$\begin{aligned} \iota_{SX}\omega_T &= (\iota_X \circ \delta_S - \delta_S \circ \iota_X)\omega_T \\ &= - \delta_S \iota_X \omega_T \qquad (\text{cf. } 8.1) \\ &= - S^* \iota_X \omega_T.] \end{aligned}$$

Consequently,

$$\begin{aligned} X_\omega \hat{\omega} &= d\hat{\omega}(X_\omega) \\ &= (\iota_{X_{\hat{\omega}}} \omega_T)(X_\omega) \\ &= \omega_T(X_{\hat{\omega}}, X_\omega) \\ &= \omega_T(X_{\hat{\omega}}, - SX_{\hat{\omega}}) \\ &= \omega_T(SX_{\hat{\omega}}, X_{\hat{\omega}}). \end{aligned}$$

16.2 <u>REMARK</u> The function $X_\omega \hat{\omega}$ is never zero and, in fact, is strictly positive.

For locally,

$$X_\omega \hat{\omega} = (W^{ij}(T)(a_j \circ \pi_M)) \frac{\partial}{\partial v^i}((a_k \circ \pi_M)v^k)$$

$$= (g^{ij} \circ \pi_M)(a_j \circ \pi_M)(a_i \circ \pi_M)$$

$$= g(\omega,\omega) \circ \pi_M$$

$$> 0.$$

Let $\Sigma_\omega \subset TM$ be the linear distribution on M determined by ω -- then the assumption is that Σ_ω $(= (\hat{\omega})^{-1}(0))$ is the arena for the constrained dynamics.

[Note: The fiber dimension of Σ_ω is n-1 and Σ_ω does not have the structure of a tangent bundle.]

Given $\lambda \in C^\infty(TM)$, put

$$\Gamma_\lambda = \Gamma_M + \lambda X_\omega.$$

Then $\Gamma_\lambda \in SO(TM)$ (X_ω being vertical).

N.B. Along an interval curve γ of Γ_λ, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v^i}\right) - \frac{\partial T}{\partial q^i} = \Pi_i + \lambda(a_i \circ \pi_M) \qquad (i = 1,\ldots,n).$$

16.3 LEMMA There exists a unique $\lambda_0 \in C^\infty(TM)$ such that

$$\Gamma_{\lambda_0}\hat{\omega} = 0.$$

PROOF If

$$\Gamma_{\lambda_0}\hat{\omega} = (\Gamma_M + \lambda_0 X_\omega)(\hat{\omega})$$

$$= \Gamma_M\hat{\omega} + \lambda_0 X_\omega\hat{\omega}$$

$$= 0,$$

then

$$\lambda_0 = -\frac{\Gamma_M\hat{\omega}}{X_\omega\hat{\omega}} \quad \text{(cf. 16.2)}.$$

This particular choice of λ_0 is called the Lagrange multiplier: So we pass from

$$(M,T,\Pi) \text{ to } (M,T,\Pi,\omega)$$

and from

$$(M,T,\Pi,\omega) \text{ to } (M,T,\Pi,\omega,\lambda_0).$$

16.4 LEMMA If λ_0 is the Lagrange multiplier, then Γ_{λ_0} is tangent to Σ_ω.

[A vector field $X \in \mathcal{D}^1(TM)$ is tangent to Σ_ω iff $X\hat{\omega}\big|_{\Sigma_\omega} = 0$.]

It is now a definition that the constrained dynamics is given by the restriction of Γ_{λ_0} to Σ_ω.

Locally,

$$\Gamma_M = v^i \frac{\partial}{\partial q^i} + C_M^i \frac{\partial}{\partial v^i},$$

where

$$C^i_M = W^{ij}(T)\ \left(\frac{\partial T}{\partial q^j} - \frac{\partial^2 T}{\partial v^j \partial q^k} v^k + \Pi_j\right).$$

Put

$$|\omega|^2 = g(\omega,\omega)\ \circ\ \pi_M.$$

Then

$$\lambda_0 = -\frac{1}{|\omega|^2}\ \left(\frac{\partial(a_i\ \circ\ \pi_M)}{\partial q^j} v^i v^j + (a_k\ \circ\ \pi_M) C^k_M\right).$$

And the equations of motion are

$$\dot{q}^i = v^i,\ \dot{v}^i = C^i_M + \lambda_0 (W^{ij}(T)(a_j\ \circ\ \pi_M)).$$

16.5 <u>EXAMPLE</u> Take $M = \underline{R}^3$ and

$$g = m(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3)$$

$\Rightarrow$

$$T = \frac{m}{2}\ ((v^1)^2 + (v^2)^2 + (v^3)^2),$$

where m is a positive constant. Write

$$\Pi = \Pi_1 dq^1 + \Pi_2 dq^2 + \Pi_3 dq^3.$$

Let

$$\omega = -\ x^2 dx^1 + dx^3\ \ (\Rightarrow a_1 = -\ x^2,\ a_2 = 0,\ a_3 = 1).$$

Then

$$\Sigma_\omega \Big|_{(x^1,x^2,x^3)} = \text{span}\left\{\frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3},\ \frac{\partial}{\partial x^2}\right\}$$

and, in view of 15.28, Σ_ω is not integrable. Since

$$\omega_T = m(dv^1 \wedge dq^1 + dv^2 \wedge dq^2 + dv^3 \wedge dq^3)$$

and

$$\pi_M^* \omega = - q^2 dq^1 + dq^3,$$

it follows that

$$X_\omega = \frac{1}{m} \left(- q^2 \frac{\partial}{\partial v^1} + \frac{\partial}{\partial v^3}\right).$$

To compute the Lagrange multiplier

$$\lambda_0 = - \frac{\Gamma_M \hat{\omega}}{X_\omega \hat{\omega}},$$

note that

$$\hat{\omega} = - q^2 v^1 + v^3.$$

Using the formula for Γ_M given in 10.3, we have

$$\Gamma_M \hat{\omega} = - v^1 v^2 - q^2 \frac{\Pi_1}{m} + \frac{\Pi_3}{m}.$$

On the other hand,

$$X_\omega \hat{\omega} = \frac{1}{m} ((q^2)^2 + 1).$$

Therefore

$$\lambda_0 = \frac{mv^1 v^2 + q^2 \Pi_1 - \Pi_3}{(q^2)^2 + 1}.$$

And finally

$$\left[\begin{array}{l} \ddot{q}^1 = \dfrac{\Pi_1}{m} - q^2 \dfrac{\lambda_0}{m} \\[2ex] \ddot{q}^2 = \dfrac{\Pi_2}{m} \\[2ex] \ddot{q}^3 = \dfrac{\Pi_3}{m} + \dfrac{\lambda_0}{m}\,. \end{array}\right.$$

[Note: Take $m = 1$, $\Pi_1 = \Pi_2 = \Pi_3 = 0$, and, using the notation of the Appendix to §8, put

$$\bar{v}^1 = v^1,\ \bar{v}^2 = v^2,\ \bar{v}^3 = v^3 - q^2 v^1.$$

Then

$$\{\bar{q}^1, \bar{q}^2, \bar{q}^3, \bar{v}^1, \bar{v}^2, \bar{v}^3\}$$

is a coordinate system adapted to Σ_ω. Here

$$[f^i{}_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{bmatrix}$$

while

$$[\bar{f}^i{}_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 1 \end{bmatrix}.$$

And

$$\begin{aligned} \bullet\ \Gamma_M &= v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + v^3 \frac{\partial}{\partial q^3} \\ &= \bar{v}^1 \bar{X}_1 + \bar{v}^2 \bar{X}_2 + \bar{v}^3 \bar{X}_3 - \bar{v}^1 \bar{v}^2 \frac{\partial}{\partial \bar{v}^3} \end{aligned}$$

- $$X_\omega = -q^2 \frac{\partial}{\partial v^1} + \frac{\partial}{\partial v^3}$$

$$= -\bar{q}^2 \frac{\partial}{\partial \bar{v}^1} + ((\bar{q}^2)^2 + 1) \frac{\partial}{\partial \bar{v}^3}$$

- $$\lambda_0 = \frac{\bar{v}^1\bar{v}^2}{(\bar{q}^2)^2 + 1} .$$

Therefore

$$\Gamma_{\lambda_0} = \Gamma_M + \lambda_0 X_\omega$$

$$= \bar{v}^1\bar{X}_1 + \bar{v}^2\bar{X}_2 + \bar{v}^3\bar{X}_3 - \frac{\bar{v}^1\bar{v}^2}{(\bar{q}^2)^2 + 1} \bar{q}^2 \frac{\partial}{\partial \bar{v}^1} .$$

So the constrained dynamics is given by

$$\Gamma_{\lambda_0}\Big| \Sigma_\omega = \bar{v}^1\bar{X}_1 + \bar{v}^2\bar{X}_2 - \frac{\bar{v}^1\bar{v}^2}{(\bar{q}^2)^2 + 1} \bar{q}^2 \frac{\partial}{\partial \bar{v}^1} .]$$

16.6 LEMMA We have

$$L_{X_\omega} \theta_T = \pi_M^* \omega.$$

PROOF By definition,

$$\iota_{X_\omega} \omega_T = \pi_M^* \omega.$$

Now expand the LHS:

$$\iota_{X_\omega} \omega_T = \iota_{X_\omega} d\theta_T$$

$$= (L_{X_\omega} - d \circ \iota_{X_\omega})\theta_T.$$

But θ_T is horizontal while X_ω is vertical, hence $\theta_T(X_\omega) = 0$ (cf. 6.14). Therefore

$$\iota_{X_\omega}\omega_T = L_{X_\omega}\theta_T.$$

16.7 LEMMA We have

$$L_{\Gamma_{\lambda_0}}\theta_T = dT + \Pi + \lambda_0\pi_M^*\omega.$$

PROOF Write

$$L_{\Gamma_{\lambda_0}}\theta_T = L_{\Gamma_M + \lambda_0 X_\omega}\theta_T$$

$$= L_{\Gamma_M}\theta_T + \lambda_0 L_{X_\omega}\theta_T$$

$$= L_{\Gamma_M}\theta_T + \lambda_0\pi_M^*\omega \quad \text{(cf. 16.6)}.$$

Because Γ_M is second order,

$$\iota_{\Gamma_M}\theta_T = \Delta T \quad \text{(cf. 8.13)}$$

$$= 2T.$$

Therefore

$$L_{\Gamma_M}\theta_T = (\iota_{\Gamma_M} \circ d + d \circ \iota_{\Gamma_M})\theta_T$$

$$= \iota_{\Gamma_M}\omega_T + d(2T)$$

$$= - dT + \Pi + 2dT$$

$$= dT + \Pi.$$

16.8 LEMMA Suppose that $f \in C^{\infty}_{\Gamma_M}(TM)$. Define $X_f \in \mathcal{D}^1(TM)$ by

$$\iota_{X_f}\omega_T = df.$$

Then

$$\Gamma_{\lambda_0}(f) = -\lambda_0\pi_M^*\omega(X_f).$$

PROOF First

$$\begin{aligned}\Gamma_{\lambda_0}(f) &= \iota_{\Gamma_{\lambda_0}} df \\ &= \iota_{\Gamma_{\lambda_0}}\iota_{X_f}\omega_T \\ &= -\iota_{X_f}\iota_{\Gamma_{\lambda_0}}\omega_T.\end{aligned}$$

And

$$\begin{aligned}\iota_{\Gamma_{\lambda_0}}\omega_T &= \iota_{\Gamma_{\lambda_0}} d\theta_T \\ &= (L_{\Gamma_{\lambda_0}} - d \circ \iota_{\Gamma_{\lambda_0}})\theta_T \\ &= L_{\Gamma_{\lambda_0}}\theta_T - d(2T) \quad \text{(cf. 8.13)} \\ &= dT + \Pi + \lambda_0\pi_M^*\omega - 2dT \quad \text{(cf. 16.7)} \\ &= -dT + \Pi + \lambda_0\pi_M^*\omega.\end{aligned}$$

But

$$0 = \Gamma_M f$$

$$= \iota_{\Gamma_M} df$$

$$= \iota_{\Gamma_M} \iota_{X_f} \omega_T$$

$$= - \iota_{X_f} \iota_{\Gamma_M} \omega_T$$

$$= - \iota_{X_f} (- dT + \Pi).$$

Therefore

$$\Gamma_{\lambda_0}(f) = - \iota_{X_f} (- dT + \Pi + \lambda_0 \pi_M^* \omega)$$

$$= - \iota_{X_f} \lambda_0 \pi_M^* \omega$$

$$= - \lambda_0 \pi_M^* \omega (X_f).$$

It is thus a corollary that

$$\pi_M^* \omega (X_f) = 0 \Rightarrow f \in C_{\Gamma_{\lambda_0}}^\infty (TM).$$

16.9 <u>REMARK</u> Take $\Pi = 0$ and let $f = E_T$ -- then $E_T \in C_{\Gamma_T}^\infty (TM)$ (cf. 8.10). Here $X_{E_T} = - \Gamma_T$ $(\iota_{\Gamma_T} \omega_T = - dE_T)$ and from the above

$$\Gamma_{\lambda_0} E_T = - \lambda_0 \pi_M^* \omega (- \Gamma_T)$$

$$= \lambda_0 \hat{\omega},$$

so $E_T|\Sigma_\omega$ is a first integral for $\Gamma_{\lambda_0}|\Sigma_\omega$.

Proceeding to the general case, let $\omega^1,\ldots,\omega^{n-k}$ be a set of linearly independent 1-forms on M -- then the prescription

$$\Sigma_x = \operatorname{Ker} \omega^1|_x \cap \ldots \cap \operatorname{Ker} \omega^{n-k}|_x \qquad (x \in M)$$

defines a linear distribution Σ ($= \bigcap_{\mu=1}^{n-k} \Sigma_{\omega^\mu}$) of fiber dimension k. Write X_μ in place of X_{ω^μ}, thus

$$\iota_{X_\mu}\omega_T = \pi_M^*\omega^\mu \qquad (\mu = 1,\ldots,n-k).$$

Given $\lambda^1,\ldots,\lambda^{n-k} \in C^\infty(TM)$, put

$$\Gamma_{\underline{\lambda}} = \Gamma_M + \lambda^\mu X_\mu.$$

16.10 <u>LEMMA</u> The matrix $[M_\mu{}^\nu]$ defined by

$$M_\mu{}^\nu = X_\mu \hat{\omega}^\nu$$

is nonsingular (and symmetric).

[In fact,

$$X_\mu \hat{\omega}^\nu = g(\omega^\mu,\omega^\nu) \circ \pi_M \qquad \text{(cf. 16.2).]}$$

16.11 <u>LEMMA</u> There exists a unique (n-k)-tuple $\underline{\lambda}_0 = (\lambda_0^1,\ldots,\lambda_0^{n-k})$ ($\lambda_0^\mu \in C^\infty(TM)$, $\mu=1,\ldots,n-k$) such that

$$\Gamma_{\underline{\lambda}_0}\hat{\omega}^\nu = 0 \qquad (\nu = 1,\ldots,n - k).$$

PROOF If

$$\Gamma_{\underline{\lambda}_0}\hat{\omega}^\nu = (\Gamma_M + \lambda_0^\mu X_\mu)(\hat{\omega}^\nu)$$

$$= \Gamma_M\hat{\omega}^\nu + \lambda_0^\mu X_\mu\hat{\omega}^\nu$$

$$= 0,$$

then

$$\lambda_0^\mu = - M^\mu{}_\nu \Gamma_M\hat{\omega}^\nu,$$

where the matrix $[M^\mu{}_\nu]$ is the inverse of the matrix $[M_\mu{}^\nu]$.

We shall call $\underline{\lambda}_0$ the Lagrange multiplier. So, by construction, $\Gamma_{\underline{\lambda}_0}$ is tangent to Σ (cf. 16.4) and the agreement is that the constrained dynamics is given by $\Gamma_{\underline{\lambda}_0}|\Sigma$.

N.B. The equations of motion are $\ddot{q}^i = v^i$, $\dot{v}^i = C_M^i + \lambda_0^\mu\ (W^{ij}(T)(a^\mu{}_j \circ \pi_M))$.

16.12 EXAMPLE Take $M = \underline{R}^2 \times \underline{S}^1 \times \underline{S}^1$ and

$$g = m(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + I_3\ dx^3 \otimes dx^3 + I_4\ dx^4 \otimes dx^4$$

=>

$$T = \frac{m}{2}((v^1)^2 + (v^2)^2) + \frac{1}{2} I_3\ (v^3)^2 + \frac{1}{2} I_4\ (v^4)^2,$$

where m, I_3, I_4 are positive constants and, to keep things simple, assume that $\Pi = 0$. Let

$$\begin{cases} \omega^1 = dx^1 - (R\cos x^3)dx^4 \\ \omega^2 = dx^2 - (R\sin x^3)dx^4 \end{cases} \qquad (R > 0).$$

Then ω^1, ω^2 are linearly independent 1-forms on M and

$$\Sigma\Big|_{(x^1,x^2,x^3,x^4)}$$

$$= \text{span}\ \{R\cos x^3 \frac{\partial}{\partial x^1} + R\sin x^3 \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^3}\}.$$

So Σ is actually analytic but it is not involutive, hence is not integrable (cf. 15.18). Here

$$\omega_T = m(dv^1 \wedge dq^1 + dv^2 \wedge dq^2) + I_3(dv^3 \wedge dq^3) + I_4(dv^4 \wedge dq^4).$$

And

$$\begin{cases} X_1 = \frac{1}{m}\frac{\partial}{\partial v^1} - \frac{R}{I_4}\cos q^3 \frac{\partial}{\partial v^4} \\ X_2 = \frac{1}{m}\frac{\partial}{\partial v^2} - \frac{R}{I_4}\sin q^3 \frac{\partial}{\partial v^4}. \end{cases}$$

These relations and the fact that

$$\begin{cases} \hat{\omega}^1 = v^1 - (R\cos q^3)v^4 \\ \hat{\omega}^2 = v^2 - (R\sin q^3)v^4 \end{cases}$$

then lead to

$$\begin{cases} \lambda_0^1 = -\ (mR \sin q^3)v^3v^4 \\ \\ \lambda_0^2 = (mR \cos q^3)v^3v^4. \end{cases}$$

Therefore

$$\Gamma_{\underline{\lambda}_0} = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + v^3 \frac{\partial}{\partial q^3} + v^4 \frac{\partial}{\partial q^4}$$

$$- (R \sin q^3)v^3v^4 \frac{\partial}{\partial v^1} + (R \cos q^3)v^3v^4 \frac{\partial}{\partial v^2},$$

from which:

$$\ddot{q}^1 = -\ (R \sin q^3)\dot{q}^3\dot{q}^4,\ \ddot{q}^2 = (R \cos q^3)\dot{q}^3\dot{q}^4,$$

$$\ddot{q}^3 = 0,\ \ddot{q}^4 = 0$$

or still, subject to the initial conditions q_0^i, v_0^i $(i = 1,2,3,4)$,

$$\begin{cases} q^1(t) = R \dfrac{v_0^4}{v_0^3} \sin(v_0^3 t + q_0^3) + A_1 t + B_1 \\ \\ q^2(t) = -\ R \dfrac{v_0^4}{v_0^3} \cos(v_0^3 t + q_0^3) + A_2 t + B_2 \end{cases}$$

and

$$\begin{cases} q^3(t) = v_0^3 t + q_0^3 \\ \\ q^4(t) = v_0^4 t + q_0^4, \end{cases}$$

A_1, A_2, B_1, B_2 being constants. But

$$\hat{\omega}^1, \hat{\omega}^2 \in C^\infty_{\Gamma_{\underline{\lambda}_0}}(TM) \qquad \text{(cf. 16.11)},$$

thus are constant on the trajectories of $\Gamma_{\underline{\lambda}_0}$ (cf. 1.1). Indeed,

$$\begin{aligned}\hat{\omega}^1(q(t),v(t)) &= Rv_0^4 \cos(v_0^3 t + q_0^3) + A_1 - R\cos(v_0^3 t + q_0^3)v_0^4 \\ &= A_1\end{aligned}$$

and

$$\begin{aligned}\hat{\omega}^2(q(t),v(t)) &= Rv_0^4 \sin(v_0^3 t + q_0^3) + A_2 - R\sin(v_0^3 t + q_0^3)v_0^4 \\ &= A_2.\end{aligned}$$

So

$$A_1 = A_2 = 0$$

if the initial conditions lie in $\Sigma = (\hat{\omega}^1)^{-1}(0) \cap (\hat{\omega}^2)^{-1}(0)$.

[Note: The mechanical system represented by the preceding data is the vertical disc of radius R and of uniformly distributed mass m that rolls without slipping on a horizontal plane (I_3 and I_4 being the appropriate moments of inertia).]

Suppose again that $\omega \in \Lambda^1 M$ is a nowhere vanishing 1-form -- then in general, Σ_ω is not integrable.

16.13 <u>RAPPEL</u> Σ_ω is integrable iff the 3-form $d\omega\wedge\omega$ vanishes:

$$d\omega\wedge\omega = 0 \qquad \text{(cf. 15.27)}.$$

16.14 <u>REMARK</u> An <u>integrating factor</u> for ω is a nowhere vanishing $\phi \in C^\infty(M)$ such that $d(\phi\omega) = 0$. If ω admits an integrating factor ϕ, then Σ_ω is integrable.

Proof:

$$d(\phi\omega) = 0 \Rightarrow d\phi\wedge\omega + \phi\wedge d\omega = 0$$

$$\Rightarrow \phi\wedge d\omega\wedge\omega = 0 \Rightarrow d\omega\wedge\omega = 0.$$

Conversely, the assumption that Σ_ω is integrable implies that locally ω admits an integrating factor ϕ (cf. 15.24), hence locally

$$\phi\omega = df \ (\exists f) \Rightarrow \omega = \frac{1}{\phi}\, df.$$

If $\omega = df$ ($f \in C^\infty(M)$, $df_x \neq 0 \ \forall\, x \in M$), then Σ_{df} ($= (\hat{df})^{-1}(0)$) is integrable (cf. 16.13).

Set

$$\bar{M} = f^{-1}(0).$$

Then $\bar{M}$ is a submanifold of M and, in obvious notation, there is an induced mechanical system $\bar{\mathcal{M}} = (\bar{M},\bar{T},\bar{\Pi})$.

[Note: $\bar{M}$ is not necessarily connected but this point causes no difficulties.]

16.15 <u>LEMMA</u> The vector field Γ_{λ_0} is tangent to $T\bar{M}$ and

$$\Gamma_{\bar{\mathcal{M}}} = \Gamma_{\lambda_0}\big|T\bar{M}.$$

Here is a corollary. Assume that $\Pi = 0$ — then

$$\Gamma_{\lambda_0} = \Gamma_T + \lambda_0 X_{\hat{df}}.$$

Therefore

$$\Gamma_{\bar{T}} = \Gamma_{\lambda_0}|T\bar{M}.$$

[Note: The integral curves of $\Gamma_{\bar{T}}$ are in a one-to-one correspondence with the geodesics of $(\bar{M},\bar{g})$ (cf. 10.6). Bear in mind too that an integral curve of Γ_{λ_0} that passes through a point of $T\bar{M}$ is contained in $T\bar{M}$.]

16.16 EXAMPLE Take $M = \underline{R}^3 - \{0\}$,

$$\begin{cases} g = m(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3) \quad (m > 0) \\ f = (x^1)^2 + (x^2)^2 + (x^3)^2 - R^2 \quad (R > 0) \end{cases}$$

and suppose that $\Pi = 0$ — then

$$\begin{cases} X_{\hat{df}} = \dfrac{2q^i}{m}\dfrac{\partial}{\partial v^i} \quad (\hat{df} = 2q^i v^i) \\ \lambda_0 = -\dfrac{m}{2|q|^2}|v|^2 \quad \text{(notation as in 9.21).} \end{cases}$$

Therefore

$$\Gamma_{\lambda_0} = v^i \frac{\partial}{\partial q^i} - \frac{|v|^2}{|q|^2} q^i \frac{\partial}{\partial v^i}$$

$$\Rightarrow \quad \Gamma_{\bar{T}} = v^i \frac{\partial}{\partial q^i} - \frac{|v|^2}{R^2} q^i \frac{\partial}{\partial v^i}.$$

And on $f^{-1}(0)$,

$$\ddot{x}^i(t) + \frac{|\dot{x}(t)|^2}{R^2} x^i(t) = 0 \quad (i = 1,2,3).$$

In anticipation of the developments to come, we shall shift our point of view and fix a nondegenerate lagrangian L. Let $\omega^1,\ldots,\omega^{n-k}$ be a system of constraints -- then $\exists$ a unique vertical X_μ:

$$\iota_{X_\mu}\omega_L = \pi_M^*\omega^\mu \quad (\mu = 1,\ldots,n-k) \quad (\text{cf. } 8.23).$$

Given $\lambda^1,\ldots,\lambda^{n-k} \in C^\infty(TM)$, put

$$\Gamma_{\underline{\lambda}} = \Gamma_L + \lambda^\mu X_\mu.$$

Then the crux is the validity of 16.10 which, in general, will fail.

[Note: Locally,

$$X_\mu \hat{\omega}^\nu = (W(L)^{-1})^{k\ell} \frac{\partial\hat{\omega}^\mu}{\partial v^k} \frac{\partial\hat{\omega}^\nu}{\partial v^\ell}.]$$

16.17 <u>EXAMPLE</u> Take $M = \underline{R}^3$ and define $L:T\underline{R}^3 \to \underline{R}$ by

$$L(q^1,q^2,q^3,v^1,v^2,v^3)$$

$$= \frac{1}{2}((v^1)^2 + (v^2)^2 - (v^3)^2).$$

Then L is nondegenerate and

$$\omega_L = dv^1\wedge dq^1 + dv^2\wedge dq^2 - dv^3\wedge dq^3.$$

Letting

$$\omega = dx^2 + dx^3,$$

we have

$$X_\omega = \frac{\partial}{\partial v^2} - \frac{\partial}{\partial v^3} .$$

But

$$X_\omega \hat{\omega} = (\frac{\partial}{\partial v^2} - \frac{\partial}{\partial v^3}) \ (v^2 + v^3)$$

$$= 1 - 1 = 0.$$

[Note: L is the "T" per the semiriemannian structure

$$g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 - dx^3 \otimes dx^3$$

on $\underline{R}^3$.]

Call

$$(L, \{\omega^1, \ldots, \omega^{n-k}\})$$

regular if the matrix

$$[X_\mu \hat{\omega}^\nu]$$

is nonsingular; otherwise, call

$$(L, \{\omega^1, \ldots, \omega^{n-k}\})$$

irregular.

N.B. If

$$L = T - V \circ \pi_M,$$

where g is riemannian, then

$$(L, \{\omega^1, \ldots, \omega^{n-k}\})$$

is regular.

The upshot, therefore, is that in the presence of regularity one can determine the Lagrange multiplier $\underline{\lambda}_0$ and proceed as before.

In the irregular situation, matters are not straightforward and there may be no resolution at all. For sake of argument, let us assume that it is a question of a single constraint ω and consider the equation of tangency:

$$\Gamma_L\hat{\omega} + \lambda_0 X_\omega\hat{\omega} = 0.$$

If $X_\omega\hat{\omega}$ is never zero, then

$$\lambda_0 = -\frac{\Gamma_L\hat{\omega}}{X_\omega\hat{\omega}}$$

and we are in business. Suppose that $X_\omega\hat{\omega} \equiv 0$. If $\Gamma_L\hat{\omega} = 0$ on Σ_ω, then the dynamics is undetermined, i.e., $\forall\ \lambda$,

$$\Gamma_L\hat{\omega} + \lambda X_\omega\hat{\omega} = 0.$$

However, if $X_\omega\hat{\omega} \equiv 0$ and $\Gamma_L\hat{\omega} \not\equiv 0$ on Σ_ω, then $\forall\ \lambda$,

$$\Gamma_L\hat{\omega} + \lambda X_\omega\hat{\omega} = 0$$

on

$$\Sigma_\omega^1 = (\Gamma_L\hat{\omega})^{-1}(0) \cap \Sigma_\omega$$

and we are led to the secondary equation of tangency

$$\Gamma_L\Gamma_L\hat{\omega} + \lambda_0^1 X_\omega\Gamma_L\hat{\omega} = 0$$

whose solution is

$$\lambda_0^1 = -\frac{\Gamma_L\Gamma_L\hat{\omega}}{X_\omega\Gamma_L\hat{\omega}}$$

provided $X_\omega \Gamma_L \hat{\omega}$ is never zero. But this may fail. In that event, if $\Gamma_L \Gamma_L \hat{\omega} = 0$ on Σ_ω^1 as well, then the dynamics is undetermined. Still, it might happen that $\Gamma_L \Gamma_L \hat{\omega} \not\equiv 0$ on Σ_ω^1 and when this is so, one can pass to $\Sigma_\omega^2 \subset \Sigma_\omega^1 \ldots$.

16.18 <u>EXAMPLE</u> In the setup of 16.17, $X_\omega \hat{\omega} = 0$ and

$$\Gamma_L \hat{\omega} = (v^1 \frac{\partial}{\partial v^1} + v^2 \frac{\partial}{\partial v^2} + v^3 \frac{\partial}{\partial v^3}) \ (v^2 + v^3)$$

$$= 0,$$

so the dynamics is undetermined. Now modify L by appending the term $\frac{1}{2} (q^1)^2$ and change ω to $dx^1 + dx^3$ -- then

$$X_\omega = \frac{\partial}{\partial v^1} - \frac{\partial}{\partial v^3}$$

=>

$$X_\omega \hat{\omega} = (\frac{\partial}{\partial v^1} - \frac{\partial}{\partial v^3}) \ (v^1 + v^3)$$

$$= 1 - 1 = 0.$$

And

$$\Gamma_L \hat{\omega} = (v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + v^3 \frac{\partial}{\partial q^3} + q^1 \frac{\partial}{\partial v^1}) \ (v^1 + v^3)$$

$$= q^1.$$

Therefore $\Gamma_L \hat{\omega} \not\equiv 0$ on

$$\Sigma_\omega = \{(q^1,q^2,q^3,v^1,v^2,v^3): v^1 + v^3 = 0\}$$

and

$$\Sigma_\omega^1 = \{(q^1,q^2,q^3,v^1,v^2,v^3): q^1 = 0,\ v^1 + v^3 = 0\}.$$

But

$$X_\omega \Gamma_L \hat{\omega} = X_\omega q^1 = 0$$

while

$$\Gamma_L \Gamma_L \hat{\omega} = \Gamma_L q^1 = v^1,$$

so the next step in the procedure outlined above is to pass to

$$\Sigma_\omega^2 = \{(q^1,q^2,q^3,v^1,v^2,v^3): q^1 = 0,\ v^1 = 0,\ v^3 = 0\}.$$

Since

$$X_\omega \Gamma_L \Gamma_L \hat{\omega} = X_\omega v^1 = 1,$$

the algorithim stabilizes at Σ_ω^2, the Lagrange multiplier being

$$\lambda_0^2 = - \frac{\Gamma_L v^1}{X_\omega v^1} = - q^1$$

and

$$(\Gamma_L - q^1 X_\omega) \,|\, \Sigma_\omega^2$$

realizes the dynamics on Σ_ω^2.

By an <u>affine system of constraints</u> we shall understand a system of constraints $\omega^1,\ldots,\omega^{n-k}$ together with functions $\phi^1,\ldots,\phi^{n-k} \in C^\infty(M)$. Put

$$\Phi^\mu = \hat{\omega}^\mu + \phi^\mu \circ \pi_M \quad (\mu = 1,\ldots,n-k)$$

and set

$$C = \bigcap_{\mu=1}^{n-k} (\Phi^{\mu})^{-1} (0).$$

Assuming that

$$(L,\{\omega^1,\ldots,\omega^{n-k}\})$$

is regular, 16.11 then implies that there exists a unique (n-k)-tuple $\underline{\lambda}_0 = (\lambda_0^1,\ldots,\lambda_0^{n-k})$ $(\lambda_0^{\mu} \in C^{\infty}(TM),\ \mu = 1,\ldots,n-k)$ such that

$$\Gamma_{\underline{\lambda}_0} \Phi^{\nu} = 0 \qquad (\nu = 1,\ldots,n-k).$$

And again the agreement is that the constrained dynamics is given by $\Gamma_{\underline{\lambda}_0}|C$.

[Note: As regards the Lagrange multiplier $\underline{\lambda}_0$, we have

$$\Gamma_{\underline{\lambda}_0} \Phi^{\nu} = \Gamma_L \Phi^{\nu} + \lambda_0^{\mu} X_{\mu} \Phi^{\nu}$$

$$= \Gamma_L \Phi^{\nu} + \lambda_0^{\mu} X_{\mu} \hat{\omega}^{\nu}.$$

Here

$$X_{\mu}(\phi^{\nu} \circ \pi_M) = 0,$$

X_{μ} being vertical.]

16.19 REMARK Consider the case when $\Phi = \omega + \phi$ -- then

$$\Gamma_{\lambda_0} E_L = \lambda_0 \hat{\omega} \qquad \text{(cf. 16.9)}.$$

And, on C,

$$\lambda_0 \hat{\omega} = - \lambda_0 (\phi \circ \pi_M)$$

which, in general, is nonzero.

16.20 LEMMA Suppose that

$$(L, \{\omega^1, \ldots, \omega^{n-k}\})$$

is regular -- then along an integral curve γ of $\Gamma_{\underline{\lambda}_0}$, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = \sum_{\mu=1}^{n-k} \lambda_0^\mu \frac{\partial \Phi^\mu}{\partial v^i} \quad (i = 1, \ldots, n).$$

[This is an immediate consequence of the definitions.]

16.21 EXAMPLE Take

$$M = \underline{R}^2 \times]0,2\pi[\times]0,\pi[\times]0,2\pi[$$

and define $L:TM \to \underline{R}$ by

$$L(q^1,q^2,q^3,q^4,q^5,v^1,v^2,v^3,v^4,v^5)$$

$$= \frac{m}{2}((v^1)^2 + (v^2)^2)$$

$$+ \frac{I}{2}((v^3)^2 + (v^4)^2 + (v^5)^2 + 2v^3v^5 \cos q^4),$$

where $m > 0$, $I > 0$ -- then L is nondegenerate (see the Appendix, A.24). Given $R > 0$, $\Omega_0 \neq 0$, let

$$\begin{cases} \omega^1 = dx^1 - (R \sin x^5)dx^4 + (R \sin x^4 \cos x^5)dx^3 \\ \omega^2 = dx^2 + (R \cos x^5)dx^4 + (R \sin x^4 \sin x^5)dx^3 \end{cases}$$

and

$$\begin{cases} \phi^1 = \Omega_0 x^2 \\ \phi^2 = - \Omega_0 x^1. \end{cases}$$

Put

$$C = (\Phi^1)^{-1}(0) \cap (\Phi^2)^{-1}(0).$$

Then

$$C \Big|_{(x^1,x^2,x^3,x^4,x^5)}$$

is an affine subspace of

$$T_{(x^1,x^2,x^3,x^4,x^5)}M,$$

viz.

$$\text{span}\{(R \sin x^5)\frac{\partial}{\partial x^1} - (R \cos x^5)\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4},$$

$$- (R \sin x^4 \cos x^5)\frac{\partial}{\partial x^1} - (R \sin x^4 \sin x^5)\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^5}\}$$

$$+ (- \Omega_0 x^2 \frac{\partial}{\partial x^1} + \Omega_0 x^1 \frac{\partial}{\partial x^2}).$$

Since

$$(L, \{\omega^1, \omega^2\})$$

is regular, the Lagrange multiplier $\underline{\lambda}_0 = (\lambda_0^1, \lambda_0^2)$ exists, from which $\Gamma_{\underline{\lambda}_0}|C$. On general grounds,

$$\ddot{q}^i = C^i + \lambda_0^\mu (W^{ij}(L)(a^\mu{}_j \circ \pi_M) \qquad (i = 1,2,3,4,5).$$

Here

$$\begin{cases} a^1{}_1 = 1,\ a^1{}_2 = 0,\ a^1{}_3 = R \sin x^4 \cos x^5,\ a^1{}_4 = - R \sin x^5,\ a^1{}_5 = 0 \\ a^2{}_1 = 0,\ a^2{}_2 = 1,\ a^2{}_3 = R \sin x^4 \sin x^5,\ a^2{}_4 = R \cos x^5,\ a^2{}_5 = 0. \end{cases}$$

Accordingly,

$$\begin{aligned} \ddot{q}^1 = C^1 &+ \lambda_0^1 (W^{11}(L)(a^1{}_1 \circ \pi_M) \\ &\qquad + W^{13}(L)(a^1{}_3 \circ \pi_M) + W^{14}(L)(a^1{}_4 \circ \pi_M)) \\ &+ \lambda_0^2 (W^{12}(L)(a^2{}_2 \circ \pi_M) + W^{13}(L)(a^2{}_3 \circ \pi_M) + W^{14}(L)(a^2{}_4 \circ \pi_M)) \\ &= 0 + \lambda_0^1 (\frac{1}{m} + 0 + 0) + \lambda_0^2 (0 + 0 + 0) \\ &= \frac{\lambda_0^1}{m}. \end{aligned}$$

And likewise

$$\ddot{q}^2 = \frac{\lambda_0^2}{m}.$$

One can also explicate $\ddot{q}^3, \ddot{q}^4, \ddot{q}^5$ but the final formulas are on the complicated side, hence will be omitted (they will not be necessary in what follows). It remains to compute λ_0^1, λ_0^2. This can be done mechanistically by feeding the data into the machine and grinding it out. However, to shorten the discussion, we shall confine our attention just to C and employ an artifice. Consider an integral curve γ of $\Gamma_{\underline{\lambda}_0}$ lying in C (recall that $\Gamma_{\underline{\lambda}_0}$ is, by construction, tangent to C).

$$\bullet \left[\begin{array}{l} \dfrac{\partial L}{\partial v^3} = Iv^3 + Iv^5 \cos q^4 \\ \\ \dfrac{\partial L}{\partial q^3} = 0 \end{array}\right.$$

=>

$$\frac{d}{dt} (Iv^3 + Iv^5 \cos q^4) = \lambda_0^1 \frac{\partial \Phi^1}{\partial v^3} + \lambda_0^2 \frac{\partial \Phi^2}{\partial v^3}$$

$$= R \sin q^4 (\lambda_0^1 \cos q^5 + \lambda_0^2 \sin q^5).$$

$$\bullet \left[\begin{array}{l} \dfrac{\partial L}{\partial v^4} = Iv^4 \\ \\ \dfrac{\partial L}{\partial q^4} = - Iv^3 v^5 \sin q^4 \end{array}\right.$$

=>

$$\frac{d}{dt} Iv^4 + Iv^3 v^5 \sin q^4 = \lambda_0^1 \frac{\partial \Phi^1}{\partial v^4} + \lambda_0^2 \frac{\partial \Phi^2}{\partial v^4}$$

$$= \lambda_0^1 (- R \sin q^5) + \lambda_0^2 (R \cos q^5).$$

$$\bullet \left[\begin{array}{l} \dfrac{\partial L}{\partial v^5} = Iv^5 + Iv^3 \cos q^4 \\ \\ \dfrac{\partial L}{\partial q^5} = 0 \end{array}\right.$$

$\Rightarrow$

$$\frac{d}{dt}(Iv^5 + Iv^3 \cos q^4) = \lambda_0^1 \frac{\partial \Phi^1}{\partial v^5} + \lambda_0^2 \frac{\partial \Phi^2}{\partial v^5}$$

$$= \lambda_0^1 (0) + \lambda_0^2 (0)$$

$$= 0.$$

So

$$\cos q^5 (I\dot{v}^4 + Iv^3 v^5 \sin q^4)$$

$$= \lambda_0^1 (- R \sin q^5 \cos q^5) + R\lambda_0^2 (\cos q^5)^2$$

and

$$\frac{\sin q^5}{\sin q^4} \frac{d}{dt}(Iv^3 + Iv^5 \cos q^4)$$

$$= \lambda_0^1 (R \sin q^5 \cos q^5) + R\lambda_0^2 (\sin q^5)^2.$$

Now add these equations to get

$$R\lambda_0^2 = \cos q^5 (I\dot{v}^4 + Iv^3 v^5 \sin q^4)$$

$$+ \frac{\sin q^5}{\sin q^4} \frac{d}{dt}(Iv^3 + Iv^5 \cos q^4)$$

or still,

$$R\lambda_0^2 = I(\dot{v}^4 \cos q^5 + v^3 v^5 \sin q^4 \cos q^5$$

$$+ \dot{v}^3 \frac{\sin q^5}{\sin q^4} + \dot{v}^5 \sin q^5 \frac{\cos q^4}{\sin q^4} - v^4 v^5 \sin q^5).$$

But

$$\dot{v}^5 + \dot{v}^3 \cos q^4 - v^3 v^4 \sin q^4 = 0$$

=>

$$\dot{v}^5 \sin q^5 \frac{\cos q^4}{\sin q^4}$$

$$= (v^3 v^4 \sin q^4 - \dot{v}^3 \cos q^4) \sin q^5 \frac{\cos q^4}{\sin q^4}$$

=>

$$\dot{v}^3 \frac{\sin q^5}{\sin q^4} - \dot{v}^3 (\cos q^4)^2 \frac{\sin q^5}{\sin q^4}$$

$$= \frac{\dot{v}^3}{\sin q^4} \sin q^5 (1 - (\cos q^4)^2)$$

$$= \dot{v}^3 \sin q^4 \sin q^5.$$

Therefore

$$R\lambda_0^2 = I(\dot{v}^4 \cos q^5 + v^3 v^5 \sin q^4 \cos q^5$$

$$+ \dot{v}^3 \sin q^4 \sin q^5 + v^3 v^4 \cos q^4 \sin q^5 - v^4 v^5 \sin q^5).$$

On C,

$$v^2 + (R \cos q^5) v^4 + (R \sin q^4 \sin q^5) v^3 - \Omega_0 q^1 = 0.$$

Thus along γ,

$$(- R \sin q^5) v^4 v^5 + (R \cos q^5) \dot{v}^4$$

$$+ (R \cos q^4 \sin q^5) v^3 v^4 + (R \sin q^4 \cos q^5) v^3 v^5 + (R \sin q^4 \sin q^5) \dot{v}^3$$

$$= -\dot{v}^2 + \Omega_0 v^1.$$

And then

$$R\lambda_0^2 = \frac{I}{R}\,(-\dot{v}^2 + \Omega_0 v^1)$$

$$= \frac{I}{R}\,(-\ddot{q}^2 + \Omega_0 \dot{q}^1)$$

$$= \frac{I}{R}\,(-\frac{\lambda_0^2}{m} + \Omega_0 \dot{q}^1).$$

I.e.:

$$\lambda_0^2 = \frac{\frac{I}{R}}{R + \frac{I}{mR}}\,\Omega_0 \dot{q}^1$$

$$= \frac{mI}{I + mR^2}\,\Omega_0 \dot{q}^1$$

=>

$$\ddot{q}^2 = \frac{\lambda_0^2}{m} = \frac{I}{I + mR^2}\,\Omega_0 \dot{q}^1.$$

Analogously,

$$\ddot{q}^1 = \frac{\lambda_0^1}{m} = -\frac{I}{I + mR^2}\,\Omega_0 \dot{q}^2.$$

[Note: A corollary is that

$$E_L|C \notin C^\infty_{\Gamma_{\underline{\lambda}_0}|C}(C)$$

or still,

$$(\Gamma_{\underline{\lambda}_0}|C)\,(E_L|C) = \Gamma_{\underline{\lambda}_0} E_L|C$$

$$\neq 0.$$

In fact,

$$\Gamma_{\underline{\lambda}_0} E_L = \lambda_0^1 \hat{\omega}^1 + \lambda_0^2 \hat{\omega}^2 \quad \text{(cf. 16.19)}$$

$$\Rightarrow$$

$$\begin{aligned}\Gamma_{\underline{\lambda}_0} E_L | C &= (\lambda_0^1 \hat{\omega}^1 + \lambda_0^2 \hat{\omega}^2) | C \\ &= - (\lambda_0^1 \Omega_0 q^2 - \lambda_0^2 \Omega_0 q^1) | C \\ &= - \frac{mI}{I + mR^2} \Omega_0^2 (- v^2 q^2 - v^1 q^1) | C \\ &= \frac{mI}{I + mR^2} \Omega_0^2 (q^1 v^1 + q^2 v^2) | C.\end{aligned}$$

Turning to the physics that realizes the above setup, consider a homogeneous ball of radius R and mass m which rolls without slipping on a horizontal plate that rotates with constant angular velocity $\Omega_0 \neq 0$ about a vertical axis through one of its points -- then $M = \underline{R}^2 \times \underline{SO}(3)$. Fix a reference frame with origin the center of rotation of the plate and vertical axis the rotation axis of the plate. Let (x^1, x^2) denote the point of contact of the ball and the plate and let (x^3, x^4, x^5) be a chart on $\underline{SO}(3)$ per the 3-1-3 system of Euler angles (see the Appendix) -- then $L, \omega^1, \omega^2, \phi^1, \phi^2$ are as above (the potential energy corresponding to the gravitational force is constant, so there is no loss of generality in setting it equal to zero). Spelled out in traditional notation, the lagrangian is

$$\frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{I}{2} (\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \cos \theta),$$

the constraint equations expressing the condition of rolling without slipping are

$$\begin{cases} \dot{x} - R\,\dot{\theta}\sin\psi + R\,\dot{\phi}\sin\theta\cos\psi + \Omega_0 y = 0 \\ \dot{y} + R\,\dot{\theta}\cos\psi + R\,\dot{\phi}\sin\theta\sin\psi - \Omega_0 x = 0, \end{cases}$$

and

$$\begin{cases} \ddot{x} + \dfrac{I}{I + mR^2}\,\Omega_0\dot{y} = 0 \\ \ddot{y} - \dfrac{I}{I + mR^2}\,\Omega_0\dot{x} = 0. \end{cases}$$

But $I = \frac{2}{5} mR^2$ (see the Appendix, A.13), hence

$$\begin{cases} \ddot{x} + \frac{2}{7}\,\Omega_0\dot{y} = 0 \\ \ddot{y} - \frac{2}{7}\,\Omega_0\dot{x} = 0. \end{cases}$$

It is then an elementary matter to determine the motion:

$$\begin{bmatrix} x(t) \\ \\ y(t) \end{bmatrix} = \frac{7}{2}\frac{1}{\Omega_0}\begin{bmatrix} \sin(\frac{2}{7}\,\Omega_0 t) & \cos(\frac{2}{7}\,\Omega_0 t) \\ \\ -\cos(\frac{2}{7}\,\Omega_0 t) & \sin(\frac{2}{7}\,\Omega_0 t) \end{bmatrix}\begin{bmatrix} \dot{x}(0) \\ \\ \dot{y}(0) \end{bmatrix}$$

$$+ \begin{bmatrix} x(0) - \frac{7}{2}\frac{1}{\Omega_0}\,\dot{y}(0) \\ \\ y(0) + \frac{7}{2}\frac{1}{\Omega_0}\,\dot{x}(0) \end{bmatrix}.$$

Therefore the orbit of the point of contact of the ball is a circle on the plate.]

16.22 <u>REMARK</u> If we take $\Omega_0 = 0$ in the above, then the constraints are linear rather than affine. Consider

$$\begin{bmatrix} \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\ \dot{\phi} \cos \theta + \dot{\psi} \end{bmatrix}.$$

It has already been pointed out that

$$\frac{d}{dt} (\dot{\phi} \cos \theta + \dot{\psi}) = 0.$$

Next, from the preceding analysis,

$$R\lambda_0^2 = I \frac{d}{dt} (\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi).$$

But

$$\Omega_0 = 0 \Rightarrow \lambda_0^2 = 0$$

$$\Rightarrow \frac{d}{dt} (\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi) = 0.$$

Ditto

$$\frac{d}{dt} (\dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi) = 0.$$

The upshot, therefore, is that the ball rolls at constant speed in a straight line and its body angular velocity $\Omega(t)$ is constant in time (however, $\Omega(t)$ is not necessarily horizontal since $\dot{\phi} \cos \theta + \dot{\psi}$, while constant, is typically nonzero). Moreover, in this situation, $E_L|C$ is a first integral for $\Gamma_{\underline{\lambda}_0}|C$.

§17. *LIE ALGEBROIDS*

Let $\pi:E \to M$ be a vector bundle of fiber dimension k.

- Assume: sec E is a Lie algebra with bracket $[\ ,\ ]_E$.
- Assume: $\rho:E \to TM$ is a vector bundle morphism over M, i.e.,

$$\begin{array}{ccc} E & \xrightarrow{\rho} & TM \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_M} \\ M & = & M \end{array} .$$

Then the triple $(E,[\ ,]_E,\rho)$ is called a Lie algebroid over M if $\forall\ f \in C^\infty(M)$, $\forall\ s_1,s_2 \in \sec E$,

$$[s_1,fs_2]_E = f[s_1,s_2]_E + ((\rho \circ s_1)f)s_2.$$

[Note: ρ is referred to as the anchor map.]

N.B. The arrow

$$\begin{cases} \sec E \to \sec TM\ (= \mathcal{D}^1(M)) \\ \quad s \to \rho \circ s \end{cases}$$

is a homomorphism of Lie algebras: $\forall\ s_1,s_2 \in \sec E$,

$$\rho \circ [s_1,s_2]_E = [\rho \circ s_1,\rho \circ s_2],$$

where the bracket on the RHS is the usual commutator of vector fields. In fact, $\forall\ f \in C^\infty(M)$, $\forall\ s_1,s_2,s_3 \in \sec E$,

$$[[s_1,s_2]_E,fs_3]_E = f[[s_1,s_2]_E,s_3]_E + ((\rho \circ [s_1,s_2]_E)f)s_3.$$

On the other hand,

$$[[s_1,s_2]_E,fs_3]_E$$

$$= - [[s_2,fs_3]_E,s_1]_E,s_1]_E - [[fs_3,s_1]_E,s_2]_E$$

$$= - [[s_2,fs_3]_E,s_1]_E + [[s_1,fs_3]_E,s_2]_E$$

$$= - [f[s_2,s_3]_E + ((\rho \circ s_2)f)s_3,s_1]_E$$

$$+ [f[s_1,s_3]_E + ((\rho \circ s_1)f)s_3,s_2]_E$$

$$= [s_1,f[s_2,s_3]_E]_E + [s_1,((\rho \circ s_2)f)s_3]_E$$

$$- [s_2,f[s_1,s_3]_E]_E - [s_2,((\rho \circ s_1)f)s_3]_E$$

$$= f[s_1,[s_2,s_3]_E]_E + ((\rho \circ s_1)f)[s_2,s_3]_E$$

$$+ ((\rho \circ s_2)f)[s_1,s_3]_E + ((\rho \circ s_1)(\rho \circ s_2)f)s_3$$

$$- f[s_2,[s_1,s_3]_E]_E - ((\rho \circ s_2)f)[s_1,s_3]_E$$

$$- ((\rho \circ s_1)f)[s_2,s_3]_E - ((\rho \circ s_2)(\rho \circ s_1)f)s_3$$

$$= f([s_1,[s_2,s_3]_E]_E - [s_2,[s_1,s_3]_E]_E)$$

$$+ ((\rho \circ s_1)(\rho \circ s_2)f)s_3 - ((\rho \circ s_2)(\rho \circ s_1)f)s_3$$

$$= f[[s_1,s_2]_E,s_3]_E + ([\rho \circ s_1,\rho \circ s_2]f)s_3.$$

Therefore

$$\rho \circ [s_1,s_2]_E = [\rho \circ s_1, \rho \circ s_2].$$

17.1 <u>EXAMPLE</u> Every finite dimensional Lie algebra $\underline{g}$ "is" a Lie algebroid over a single point.

17.2 <u>EXAMPLE</u> The triple

$$(TM, [\ ,\ ], \mathrm{id}_{TM})$$

is a Lie algebroid: $\forall\ f \in C^\infty(M)$, $\forall\ X,Y \in \mathcal{D}^1(M)$,

$$[X,fY] = f[X,Y] + (Xf)Y.$$

[Note: If $\Sigma \subset TM$ is an integrable linear distribution, then Σ is involutive (cf. 15.18), hence can be viewed as a Lie algebroid in the obvious way.]

Other examples will be given later on.

17.3 <u>RAPPEL</u> $\Lambda^0 E = C^\infty(M)$ and $\Lambda^p E$ $(p \geq 1)$ is the set of multilinear maps

$$\omega: \sec E \times \cdots \times \sec E \to C^\infty(M)$$

which are skewsymmetric if $p > 1$.

[Note: Take $E = TM$ -- then $\sec E = \mathcal{D}^1(M)$ and in this context, $\Lambda^p E$ is what one normally calls $\Lambda^p M$, thus the symbol $\Lambda^p E$ is <u>not</u> $\Lambda^p TM$ (as it is usually understood).]

17.4 <u>LEMMA</u> Suppose that $(E,[\ ,\ ]_E,\rho)$ is a Lie algebroid over M. Define

$$d_E:\Lambda^p E \to \Lambda^{p+1} E$$

by

$$d_E\omega(s_0,\ldots,s_p)$$

$$= \sum_{i=0}^{p} (-1)^i (\rho \circ s_i)\omega(s_0,\ldots,\hat{s}_i,\ldots,s_p)$$

$$+ \sum_{i<j} (-1)^{i+j}\omega([s_i,s_j]_E,s_0,\ldots,\hat{s}_i,\ldots,\hat{s}_j,\ldots,s_p).$$

Then

$$d_E^2 = 0.$$

[Note: In the case of a Lie algebra $\underline{g}$, $d_{\underline{g}}$ is the Chevalley-Eilenberg differential and in the case of a tangent bundle TM, d_{TM} is the exterior derivative.]

<u>N.B.</u> As regards the wedge product,

$$d_E(\omega_1\wedge\omega_2) = d_E\omega_1\wedge\omega_2 + (-1)^{p_1}\omega_1\wedge d_E\omega_2 \quad (\omega_1 \in \Lambda^{p_1}E,\omega_2 \in \Lambda^* E).$$

17.5 <u>EXAMPLE</u> Consider the arrow

$$TTM \xrightarrow{\mu \circ \nu} TTM \quad (\text{cf. §5}).$$

Then

$$\pi_{TM} \circ \mu \circ \nu = pr_1 \circ \nu = \pi_{TM}.$$

So $\mu \circ \nu$ is a vector bundle morphism over TM. Next, given $X,Y \in \mathcal{D}^1(TM)$, put

$$[X,Y]_S = [SX,Y] + [X,SY] - S[X,Y].$$

Equipped with this bracket, $\mathcal{D}^1(TM)$ is a Lie algebra and $\forall\ f \in C^\infty(TM)$,

$$[X,fY]_S = f[X,Y]_S + ((SX)f)Y.$$

Therefore the triple

$$(TTM,[\ ,\ ]_S,\mu \circ \nu)$$

is a Lie algebroid over TM. And, by definition,

$$d_{TTM}\omega(X_0,\ldots,X_p)$$

$$= \sum_{i=0}^{p} (-1)^i (SX_i)\omega(X_0,\ldots,\hat{X}_i,\ldots,X_p)$$

$$+ \sum_{i<j} (-1)^{i+j}\omega([X_i,X_j]_S,X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_p).$$

I.e.:

$$d_{TTM}\omega = d_S\omega \quad (\text{cf. } \S 6).$$

Let $s \in \sec E$ -- then the <u>Lie derivative w.r.t. s</u> is the operator

$$L_s:\Lambda^p E \to \Lambda^p E$$

given by

$$L_s = \iota_s \circ d_E + d_E \circ \iota_s.$$

E.g.: Take $p = 0$ — then $\Lambda^0 E = C^\infty(M)$, $\iota_s \Lambda^0 E = 0$, and $\forall\ f \in C^\infty(M)$,

$$L_s f = \iota_s d_E f = (d_E f)(s) = (\rho \circ s)f = L_{(\rho \circ s)} f.$$

17.6 <u>LEMMA</u> $\forall\ s \in \sec E$,

$$L_s \circ d_E = d_E \circ L_s.$$

Moreover, $\forall\ s_1, s_2 \in \sec E$,

$$\begin{cases} L_{s_1} \circ L_{s_2} - L_{s_2} \circ L_{s_1} = L_{[s_1,s_2]_E} \\ \\ L_{s_1} \circ \iota_{s_2} - \iota_{s_2} \circ L_{s_1} = \iota_{[s_1,s_2]_E}. \end{cases}$$

And $\forall\ \omega_1, \omega_2 \in \Lambda^* E$,

$$L_s(\omega_1 \wedge \omega_2) = L_s\omega_1 \wedge \omega_2 + \omega_1 \wedge L_s\omega_2.$$

Suppose that

$$\begin{cases} (E, [\ ,\ ]_E, \rho) \text{ is a Lie algebroid over } M \\ \\ (E', [\ ,\ ]_{E'}, \rho') \text{ is a Lie algebroid over } M'. \end{cases}$$

Then a vector bundle morphism

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

is said to be a <u>Lie algebroid morphism</u> if $\forall\ p$, $\forall\ \omega' \in \Lambda^p E'$,

$$(F,f)^*(d_{E'}\omega') = d_E((F,f)^*\omega').$$

[Note: For $p \geq 1$,

$$((F,f)^*\omega')_x(e_1,\ldots,e_p)$$

$$= \omega'_{f(x)}(Fe_1,\ldots,Fe_p) \quad (x \in M \text{ and } e_1,\ldots,e_p \in E_x),$$

while for $p = 0$,

$$(F,f)^*f' = f' \circ f \quad (f' \in C^\infty(M')).]$$

<u>N.B.</u> If the vector bundle morphism

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

is a Lie algebroid morphism, then the diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho'} \\ TM & \xrightarrow[Tf]{} & TM' \end{array}$$

commutes.

17.7 <u>EXAMPLE</u> If $f:M \rightarrow M'$ is a C^∞ function, then there is a vector bundle morphism

$$
\begin{tikzcd}
TM \arrow[r, "Tf"] \arrow[d, "\pi_M"'] & TM' \arrow[d, "\pi_{M'}"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}
$$

which is, in fact, a Lie algebroid morphism.

17.8 EXAMPLE In the notation of the Appendix, the arrows

$$
\left[\begin{array}{l} T\underline{SO}(3) \to \underline{so}(3) \\ \\ (A,X) \longrightarrow A^{-1}X \end{array}\right. , \quad \left[\begin{array}{l} T\underline{SO}(3) \to \underline{so}(3) \\ \\ (A,X) \longrightarrow XA^{-1} \end{array}\right.
$$

are morphisms of Lie algebroids.

17.9 REMARK Matters simplify if $M = M'$, $f = \mathrm{id}_M$. For then the pair (F,id_M) is a Lie algebroid morphism iff

$$F[s_1,s_2]_E = [Fs_1,Fs_2]_{E'} \quad (s_1,s_2 \in \sec E)$$

and

$$\rho' \circ Fs = \rho \circ s \quad (s \in \sec E).$$

17.10 LEMMA If

$$
\begin{tikzcd}
E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
E' \arrow[r, "F'"] \arrow[d, "\pi'"'] & E'' \arrow[d, "\pi''"] \\
M' \arrow[r, "f'"'] & M''
\end{tikzcd}
$$

are Lie algebroid morphisms, then the composition

$$\begin{array}{ccc} E & \xrightarrow{F''} & E'' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi''} \\ M & \xrightarrow[f'']{} & M'' \end{array} \qquad (f'' = f' \circ f,\ F'' = F' \circ F)$$

is a Lie algebroid morphism.

[Note: This justifies the term "Lie algebroid morphism" in that there is a category whose objects are the Lie algebroids.]

Suppose that $(E,[\ ,\ ]_E,\rho)$ is a Lie algebroid over M and let $\Phi:M' \to M$ be a fibration. Form the pullback square

$$\begin{array}{ccc} TM' \times_{TM} E & \xrightarrow{pr_2} & E \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle\rho} \\ TM' & \xrightarrow[T\Phi]{} & TM \end{array}$$

and put

$$E' = TM' \times_{TM} E.$$

Then the points in E' are the pairs

$$((x',X'_{x'}),e) \quad (X'_{x'} \in T_{x'}M', e \in E)$$

such that

$$d\Phi_{x'}(X'_{x'}) = \rho(e).$$

[Note: It is automatic that

$$\Phi(x') = \pi(e).]$$

17.11 <u>LEMMA</u> E' is a vector bundle over M' (via $\pi' = \pi_{M'} \circ pr_1$).

<u>PROOF</u> Given $x' \in M'$,

$$(\pi')^{-1}(x') = E'_{x'}$$

is a vector subspace of $T_{x'}M' \times E_{\Phi(x')}$ of dimension

$$k + n' - \dim(d\Phi_{x'}(T_{x'}M') + \rho(E_{\Phi(x')}))$$

$$= k + n' - n.$$

The claim now is that this data gives rise to a Lie algebroid $(E',[\ ,\ ]_{E'},\rho')$ over M'. Of course the definition of ρ' is immediate, viz. take $\rho' = pr_1$. However, it is not so obvious just how to define $[\ ,\ ]_{E'}$, which requires some preparation.

17.12 <u>RAPPEL</u> Suppose that

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

is a vector bundle morphism. Form the pullback square

$$\begin{tikzcd} M \times_{M'} E' \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M'. \end{tikzcd}$$

Then there is an arrow

$$E \xrightarrow{\zeta} M \times_{M'} E'$$

and a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\zeta} & M \times_{M'} E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle pr_1} \\ M & = & M \end{array}.$$

Denote by ζ^* the induced map

$$\sec E \to \sec M \times_{M'} E'$$

of $C^\infty(M)$-modules:

$$\zeta^* s = \zeta \circ s \quad (s \in \sec E).$$

But

$$C^\infty(M) \otimes_{C^\infty(M')} \sec E' \approx \sec M \times_{M'} E',$$

where

$$\phi \otimes s' \to \phi \bar{s}'$$

and

$$\bar{s}'(x) = (x, s'(f(x))) \quad (x \in M).$$

So, modulo this identification, given $s \in \sec E$, we can write

$$\zeta^* s = \sum_i (\phi_i \otimes s_i')$$

or still,

$$F \circ s = \sum_i \phi_i (s_i' \circ f).$$

Consider anew the commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{pr_2} & E \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \rho} \\ TM' & \xrightarrow[T\Phi]{} & TM \end{array} .$$

There are pullback squares

$$\begin{array}{ccc} M' \times_M E & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow[\Phi]{} & M \end{array} , \qquad \begin{array}{ccc} M' \times_M TM & \longrightarrow & TM \\ \downarrow & & \downarrow{\scriptstyle \pi_M} \\ M' & \xrightarrow[\Phi]{} & M \end{array}$$

and arrows

$$\begin{cases} M' \times_M E \to M' \times_M TM \\ \\ TM' \to M' \times_M TM . \end{cases}$$

Now form the pullback square

$$\begin{array}{ccc} ? & \longrightarrow & M' \times_M E \\ \downarrow & & \downarrow \\ TM' & \longrightarrow & M' \times_M TM \end{array}$$

in the category of vector bundles over M' -- then

$$? = TM' \times_{M' \times_M TM} M' \times_M E$$

$$\approx \quad TM' \times_{TM} E = E'.$$

Accordingly, the sections s' of E' are pairs (X',σ), where

$$\begin{cases} X' \in \sec TM' \\ \sigma' \in \sec M' \times_M E \end{cases}$$

subject to the coincidence

$$\begin{cases} M' \xrightarrow{X'} TM' \longrightarrow M' \times_M TM \\ M' \xrightarrow{\sigma'} M' \times_M E \to M' \times_M TM . \end{cases}$$

N.B. The elements of $\sec M' \times_M E$ can be regarded as the sections of E along Φ (cf. 13.2), thus we can write

$$\sigma' = \sum_i \phi_i' (s_i \circ \Phi),$$

where $\phi_i' \in C^\infty(M')$ and $s_i \in \sec E$.

Finally, define

$$[\ ,\ ]_{E'}: \sec E' \times \sec E' \to \sec E'$$

by

$$\begin{aligned} &[s_1',s_2']_{E'} \\ &\quad = [(X_1',\sigma_1'),(X_2',\sigma_2')]_{E'} \\ &\quad = [(X_1', \sum_{i_1} \phi_{i_1}' (s_{i_1} \circ \Phi)),(X_2', \sum_{i_2} \phi_{i_2}' (s_{i_2} \circ \Phi))]_{E'} \\ &\quad = ([X_1',X_2'],W), \end{aligned}$$

W being

$$\sum_{i_1,i_2} \phi'_{i_1}\phi'_{i_2}([s_{i_1},s_{i_2}]_E \circ \Phi)$$

$$+ \sum_{i_2} X'_1(\phi'_{i_2})(s_{i_2} \circ \Phi) - \sum_{i_1} X'_2(\phi'_{i_1})(s_{i_1} \circ \Phi).$$

One can show that $[\ ,\ ]_{E'}$ is welldefined. Granted this, it is then easy to check that $(E',[\ ,\ ]_{E'},\rho')$ is a Lie algebroid over M'.

17.13 LEMMA The vector bundle morphism

$$\begin{tikzcd} E' \arrow[r, "pr_2"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "\Phi"'] & M \end{tikzcd}$$

is a Lie algebroid morphism.

[Note:

$$\rho \circ pr_2 = T\Phi \circ pr_1$$

=>

$$\pi_M \circ \rho \circ pr_2 = \pi_M \circ T\Phi \circ pr_1$$

=>

$$\pi \circ pr_2 = \Phi \circ \pi_{M'} \circ pr_1$$

$$= \Phi \circ \pi'.]$$

An important special case of the foregoing generalities arises when we take $M' = E$, $\Phi = \pi$:

$$\begin{tikzcd} TE \times_{TM} E \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & E \arrow[d, "\rho"] \\ TE \arrow[r, "T\pi"'] & TM . \end{tikzcd}$$

Put

$$LE = TE \times_{TM} E$$

and write ρ_E in place of pr_1 -- then LE is called the <u>prolongation of E</u> and $(LE,[\ ,\ ]_{LE},\rho_E)$ is a Lie algebroid over E:

$$\begin{tikzcd} LE \arrow[r, "pr_2"] \arrow[d, "\rho_E"'] & E \arrow[r, "\pi"] \arrow[d, "\rho"] & M \arrow[d, equal] \\ TE \arrow[r, "T\pi"'] \arrow[d, "\pi_E"'] & TM \arrow[r, "\pi_M"'] & M \arrow[d, equal] \\ E \arrow[rr, "\pi"'] & & M . \end{tikzcd}$$

[Note: The fiber dimension of LE is

$$k + (k + n) - n = 2k \quad (\text{cf. } 17.11),$$

k being the fiber dimension of E.]

<u>N.B.</u> The points in LE are the pairs

$$((e,X_e),p) \quad (X_e \in T_eE, p \in E)$$

such that

$$d\pi_e(X_e) = \rho(p)$$

with $\pi(e) = \pi(p)$.

17.14 EXAMPLE Let $E = TM$ -- then $LTM = TTM$ and the Lie algebroid structure of the theory is precisely that of 17.2, i.e.,

$$(TTM, [\ ,\ ], id_{TTM}).$$

Suppose that the vector bundle morphism

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

is a Lie algebroid morphism. Define

$$LF:LE \rightarrow LE'$$

by

$$LF(((e,X_e),p)) = ((Fe,dF_e(X_e)),Fp).$$

17.15 LEMMA The vector bundle morphism

$$\begin{tikzcd} LE \arrow[r, "LF"] \arrow[d, "\pi_E \circ \rho_E"'] & LE' \arrow[d, "\pi_{E'} \circ \rho_{E'}"] \\ E \arrow[r, "F"'] & E' \end{tikzcd}$$

is a Lie algebroid morphism.

Coming back to

$$\begin{array}{ccc} LE & \xrightarrow{pr_2} & E \\ {\scriptstyle \rho_E}\downarrow & & \downarrow{\scriptstyle \rho} \\ TE & \xrightarrow[T\pi]{} & TM, \end{array}$$

call the elements of Ker pr_2 <u>vertical</u> and denote the set of such by VLE -- then VLE is a vector subbundle of LE and its points have the form

$$((e,X_e),0),$$

where X_e is a vertical vector tangent to E at e.

Given $e,p \in E$ with $\pi(e) = \pi(p)$, denote by $X^V_{e,p} \in T_eE$ the vector tangent to the curve $e + tp$ at $t = 0$ -- then it is clear that

$$((e,X^V_{e,p}),0) \in VLE.$$

This said, define

$$\Xi^V : E \times_M E \to VLE$$

by

$$\Xi^V(e,p) = ((e,X^V_{e,p}),0).$$

Then Ξ^V is an isomorphism of vector bundles over E:

$$\begin{array}{ccc} E \times_M E & \xrightarrow{\Xi^V} & VLE \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi_E \circ \rho_E} \\ E & = & E. \end{array}$$

17.16 <u>EXAMPLE</u> Put

$$\Delta_E(e) = \Xi^V(e,e) \qquad (e \in E).$$

Then

$$\Delta_E \in \sec VLE.$$

[Note:

$$\rho_E \circ \Delta_E \in \sec VE$$

is the dilation vector field Δ on E (cf. 4.2). In detail: Identify VE with $E \times_M E$ (cf. §5) -- then Δ corresponds to the section $p \to (p,p)$ of $E \times_M E$.]

17.17 <u>LEMMA</u> If $s_1, s_2 \in \sec VLE$, then

$$[s_1, s_2]_{LE} \in \sec VLE.$$

We shall now extend the operations

$$\left[\begin{array}{c} \sec TM \to \sec TTM \\ X \to X^V, \end{array}\right. \qquad \left[\begin{array}{c} \sec TM \to \sec TTM \\ X \to X^T \end{array}\right.$$

to operations

$$\left[\begin{array}{c} \sec E \to \sec LE \\ s \to s^V, \end{array}\right. \qquad \left[\begin{array}{c} \sec E \to \sec LE \\ s \to s^T. \end{array}\right.$$

17.18 <u>RAPPEL</u> Every $\omega \in \Lambda^1 E$ determines a C^∞ function $\hat{\omega}: E \to \underline{R}$.

[Note: Given $f \in C^\infty(M)$, put

$$f^T = d_E^\wedge f.$$

Then

$$f^T(e) = \rho(e)f \quad (e \in E).]$$

Let $s \in \sec E$ -- then its <u>vertical lift</u> is the section s^V of LE defined by the prescription

$$s^V(e) = \Xi^V(e,s(\pi(e))) \quad (e \in E).$$

17.19 <u>LEMMA</u> $\forall\ f \in C^\infty(M)$,

$$(fs)^V = (f \circ \pi)s^V \text{ and } (\rho_E \circ s^V)(f \circ \pi) = 0.$$

17.20 <u>LEMMA</u> $\forall\ \omega \in \Lambda^1 E$,

$$(\rho_E \circ s^V)\hat{\omega} = \iota_s\omega \circ \pi.$$

17.21 <u>RAPPEL</u> Let $s_{TM}: TTM \to TTM$ be the canonical involution -- then $\forall\ X \in \mathcal{D}^1(TM)$,

$$X^T = s_{TM} \circ TX \quad (\text{cf. §4}).$$

Fix a point

$$((e,X_e),p) \in LE.$$

Then

$$\pi_E \circ \rho_E(((e,X_e),p)) = \pi_E(e,X_e) = e.$$

I.e.:

$$((e,X_e),p) \in (LE)_e.$$

17.22 <u>LEMMA</u> Put $x = \pi(e)$ $(= \pi(p))$ — then $\exists$ a unique tangent vector $V_p \in T_pE$ such that

1. $\forall\ f \in C^\infty(M)$,

$$V_p(f \circ \pi) = f^T(e).$$

2. $\forall\ \omega \in \Lambda^1 E$,

$$V_p\hat{\omega} = X_e\hat{\omega} + (d_E\omega)\big|_x(e,p).$$

<u>PROOF</u> V_p is determined by its action on the $f \circ \pi$ and the $\hat{\omega}$ provided that the conditions are compatible. First

$$\begin{aligned} V_p(\widehat{f\omega}) &= V_p((f \circ \pi)\hat{\omega}) \\ &= V_p(f \circ \pi)\hat{\omega}(p) + (f \circ \pi)(p)(V_p\hat{\omega}) \\ &= f^T(e)\hat{\omega}(p) + f(x)(V_p\hat{\omega}). \end{aligned}$$

Now compare this with

$$V_p(\widehat{f\omega}) = X_e(\widehat{f\omega}) + d_E(f\omega)\big|_x(e,p)$$

$$= X_e(f \circ \pi)\hat{\omega}(e) + (f \circ \pi)(e)X_e\hat{\omega}$$

$$+ (d_E f \wedge \omega)\big|_x(e,p) + f(x)(d_E\omega)\big|_x(e,p)$$

$$= f(x)(X_e\hat{\omega} + (d_E\omega)\big|_x(e,p))$$

$$+ X_e(f \circ \pi)\hat{\omega}(e) + f^T(e)\hat{\omega}(p) - (\rho(p)f)\hat{\omega}(e)$$

$$= f(x)(V_p\hat{\omega}) + f^T(e)\hat{\omega}(p).$$

[Note: Here we have used the fact that $X_e(f \circ \pi) = d\pi_e(X_e)f = \rho(p)f$.]

<u>N.B.</u> $\forall\, f \in C^\infty(M)$,

$$d\pi_p(V_p)f = V_p(f \circ \pi) = f^T(e) = \rho(e)f.$$

17.23 <u>LEMMA</u> Define

$$s_E: LE \rightarrow LE$$

by

$$s_E((e,X_e),p) = ((p,V_p),e).$$

Then

$$s_E \circ s_E = \mathrm{id}_{LE} \text{ and } \mathrm{pr}_2 \circ s_E = \pi_E \circ \rho_E.$$

[Both points are immediate. Incidentally, s_E is smooth (argue locally (cf. infra)).]

We shall call s_E the <u>canonical involution</u> associated with the Lie algebroid E.

[Note: If $E = TM$, then s_{TM} is the canonical involution on TTM (cf. 17.21).]

17.24 <u>REMARK</u> The vector bundle $T\pi:TE \to TM$ can be equipped with a Lie algebroid structure in which the anchor map is $s_{TM} \circ T\rho$. Proceeding, one can then construct a Lie algebroid structure on the vector bundle $TE \times_{TM} E \xrightarrow{pr_2} E$. On the other hand, s_E is a vector bundle morphism

$$\begin{array}{ccc}
LE & \xrightarrow{s_E} & TE \times_{TM} E \\
{\scriptstyle \pi_E \circ \rho_E}\downarrow & & \downarrow{\scriptstyle pr_2} \\
E & = & E
\end{array}$$

that, in fact, is a Lie algebroid morphism.

Let $s \in \sec E$ -- then

$$s:M \to E \Rightarrow Ts:TM \to TE$$

$$\Rightarrow Ts \circ \rho:E \to TE.$$

Abuse the notation and regard $Ts \circ \rho$ as an element of

$$\sec(TE \times_{TM} E \xrightarrow{pr_2} E).$$

Put

$$s^T = s_E \circ Ts \circ \rho.$$

Then

$$\pi_E \circ \rho_E \circ s^T = \pi_E \circ \rho_E \circ s_E \circ Ts \circ \rho$$

$$= pr_2 \circ Ts \circ \rho$$

$$= id_E.$$

Therefore

$$s^T : E \to LE$$

is a section of LE, the <u>lift</u> of s.

[Note: We have

$$pr_2 \circ s^T = pr_2 \circ s_E \circ Ts \circ \rho$$

$$= \pi_E \circ \rho_E \circ Ts \circ \rho$$

$$= \pi_E \circ Ts \circ \rho$$

$$= s \circ \pi.]$$

17.25 <u>LEMMA</u> $\forall\ f \in C^\infty(M)$,

$$(fs)^T = (f \circ \pi)s^T + f^T s^V$$

and

$$(\rho_E \circ s^T)(f \circ \pi) = L_s f \circ \pi \ (= ((\rho \circ s)f) \circ \pi).$$

17.26 <u>LEMMA</u> $\forall\ \omega \in \Lambda^1 E$,

$$(\rho_E \circ s^T)\hat{\omega} = L_s\hat{\omega}.$$

17.27 <u>REMARK</u> Viewed as a map $s^T:E \to LE$,

$$s^T = (\rho_E \circ s^T, s \circ \pi),$$

where $\rho_E \circ s^T \in \mathcal{D}^1(E)$ is characterized by its action on the $f \circ \pi$ and the $\hat{\omega}$. To confirm compatibility, write

$$\begin{aligned}(\rho_E \circ s^T)(\widehat{f\omega}) &= (\rho_E \circ s^T)((f \circ \pi)\hat{\omega}) \\ &= (\rho_E \circ s^T)(f \circ \pi)\hat{\omega} + (f \circ \pi)(\rho_E \circ s^T)\hat{\omega} \\ &= (L_s f \circ \pi)\hat{\omega} + (f \circ \pi)L_s\hat{\omega}\end{aligned}$$

or

$$\begin{aligned}(\rho_E \circ s^T)(\widehat{f\omega}) &= (L_s(f\omega))^\wedge \\ &= ((L_s f)\omega)^\wedge + (f(L_s\omega))^\wedge \\ &= (L_s f \circ \pi)\hat{\omega} + (f \circ \pi)L_s\hat{\omega}.\end{aligned}$$

17.28 <u>LEMMA</u> $\forall\ f \in C^\infty(M)$,

$$\begin{cases} (\rho_E \circ s^V)f^T = ((\rho \circ s)f) \circ \pi \\ (\rho_E \circ s^T)f^T = ((\rho \circ s)f)^T. \end{cases}$$

Let $s_1, s_2 \in \sec E$ -- then

$$\left[\begin{array}{l} [s_1^V, s_2^V]_{LE} = 0 \\ \\ [s_1^V, s_2^T]_{LE} = [s_1, s_2]_E^V \\ \\ [s_1^T, s_2^T]_{LE} = [s_1, s_2]_E^T. \end{array}\right.$$

[Note: We have

$$[s_1^T, s_2^V]_{LE} = - [s_2^V, s_1^T]_{LE}$$

$$= - [s_2, s_1]_E^V$$

$$= [s_1, s_2]_E^V.]$$

17.29 <u>EXAMPLE</u> To run a reality check, let $f \in C^\infty(M)$ -- then

$$[s_1^T, (fs_2)^T]_{LE} = [s_1, fs_2]_E^T$$

$$= (f[s_1, s_2]_E + ((\rho \circ s_1)f)s_2)^T$$

$$= (f \circ \pi)[s_1, s_2]_E^T + f^T[s_1, s_2]_E^V + (((\rho \circ s_1)f) \circ \pi)s_2^T + ((\rho \circ s_1)f)^T s_2^V.$$

On the other hand,

$$[s_1^T, (fs_2)^T]_{LE} = [s_1^T, (f \circ \pi)s_2^T + f^T s_2^V]_{LE}$$

$$= [s_1^T,(f \circ \pi)s_2^T]_{LE} + [s_1^T,f^T s_2^V]_{LE}$$

$$= (f \circ \pi)[s_1^T,s_2^T]_{LE} + ((\rho_E \circ s_1^T)(f \circ \pi))s_2^T + f^T[s_1^T,s_2^V]_{LE} + ((\rho_E \circ s_1^T)f^T)s_2^V$$

$$= (f \circ \pi)[s_1,s_2]_E^T + (((\rho \circ s_1)f) \circ \pi)s_2^T + f^T[s_1,s_2]_E^V + ((\rho \circ s_1)f)^T s_2^V.$$

17.30 RAPPEL Let $X \in \mathcal{D}^1(M)$ -- then

$$\begin{cases} [\Delta,X^V] = - X^V & \text{(cf. 4.6)} \\ [\Delta,X^T] = 0 & \text{(cf. 4.4).} \end{cases}$$

N.B. $\forall f \in C^\infty(M)$,

$$(\rho_E \circ \Delta_E)\ (f \circ \pi) = 0,$$

and $\forall \omega \in \Lambda^1 E$,

$$(\rho_E \circ \Delta_E)\hat{\omega} = \hat{\omega}.$$

17.31 LEMMA Let $s \in \sec E$ -- then

$$\begin{cases} [\Delta_E,s^V]_{LE} = - s^V \\ [\Delta_E,s^T]_{LE} = 0. \end{cases}$$

PROOF To check the first point, note that $[\Delta_E,s^V]_{LE}$ is vertical (cf. 17.17), hence it suffices to show that

$$(\rho_E \circ [\Delta_E, s^V]_{LE})\hat{\omega} = -(\rho_E \circ s^V)\hat{\omega}$$

for all $\omega \in \Lambda^1 E$. But

$$(\rho_E \circ [\Delta_E, s^V]_{LE})\hat{\omega} = [\rho_E \circ \Delta_E, \rho_E \circ s^V]\hat{\omega}$$

$$= (\rho_E \circ \Delta_E)(\rho_E \circ s^V)\hat{\omega} - (\rho_E \circ s^V)(\rho_E \circ \Delta_E)\hat{\omega}$$

$$= (\rho_E \circ \Delta_E)(\iota_s \omega \circ \pi) - (\rho_E \circ s^V)\hat{\omega} \quad \text{(cf. 17.20)}$$

$$= -(\rho_E \circ s^V)\hat{\omega}.$$

Turning to the second point, $\forall\ f \in C^\infty(M)$,

$$(\rho_E \circ [\Delta_E, s^T]_{LE})\ (f \circ \pi)$$

$$= (\rho_E \circ \Delta_E)(\rho_E \circ s^T)(f \circ \pi) - (\rho_E \circ s^T)(\rho_E \circ \Delta_E)(f \circ \pi)$$

$$= (\rho_E \circ \Delta_E)(L_s f \circ \pi) \quad \text{(cf. 17.25)}$$

$$= 0$$

and $\forall\ \omega \in \Lambda^1 E$,

$$(\rho_E \circ [\Delta_E, s^T]_{LE})\hat{\omega}$$

$$= (\rho_E \circ \Delta_E)(\rho_E \circ s^T)\hat{\omega} - (\rho_E \circ s^T)(\rho_E \circ \Delta_E)\hat{\omega}$$

$$= (\rho_E \circ \Delta_E)\widehat{L_s \omega} - (\rho_E \circ s^T)\hat{\omega} \quad \text{(cf. 17.26)}$$

$$= L_{\hat{s}}\omega - L_{\hat{s}}\omega$$

$$= 0.$$

Let S stand for the composition of the arrow

$$\begin{cases} LE \longrightarrow E \times_M E \\ ((e,X_e),p) \rightarrow (e,p) \end{cases}$$

with Ξ^V -- then S is called the vertical morphism:

$$\begin{array}{ccc} LE & \xrightarrow{S} & LE \\ {\scriptstyle \pi_E \circ \rho_E}\downarrow & & \downarrow{\scriptstyle \pi_E \circ \rho_E} \\ E & = & E. \end{array}$$

[Note: $\forall$ s $\in$ sec E,

$$\begin{cases} S \circ s^T = s^V \\ S \circ s^V = 0.] \end{cases}$$

17.32 LEMMA $S^2 = 0$ and

$$\text{Ker } S = \text{Im } S,$$

the vertical subbundle VLE of LE.

17.33 RAPPEL $\Gamma \in \mathcal{D}^1(TM)$ is second order provided $\Gamma TM \subset T^2M$ or still, if $T\pi_M \circ \Gamma = \text{id}_{TM}$.

Put

$$\mathrm{Adm}(E) = \{((e,X_e),p) \in LE : e = p\}.$$

Let $\Gamma \in \sec LE$ -- then Γ is <u>second order</u> provided $\Gamma E \subset \mathrm{Adm}(E)$ or still, if $pr_2 \circ \Gamma = id_E$.

17.34 <u>LEMMA</u> Let $\Gamma \in \sec LE$ -- then Γ is second order iff $S \circ \Gamma = \Delta_E$ (cf. 5.8).

<u>PROOF</u> Suppose that $\Gamma E \subset \mathrm{Adm}(E)$ -- then $\forall\ e \in E$,

$$\Gamma(e) = ((e,X_e),e)$$

$$\Rightarrow$$

$$S(\Gamma(e)) = \Xi^V(e,e) = \Delta_E(e).$$

Conversely, if

$$\Gamma(e) = ((e,X_e),p),$$

then

$$S(\Gamma(e)) = \Xi^V(e,p)$$

$$= ((e,X_{e,p}^V),0).$$

But

$$S \circ \Gamma = \Delta_E$$

$$\Rightarrow$$

$$((e,X_{e,p}^V),0) = ((e,X_{e,e}^V),0)$$

$$\Rightarrow$$

$$X_{e,p}^V = X_{e,e}^V \Rightarrow e = p.$$

Therefore

$$\Gamma(e) \in \mathrm{Adm}(E).$$

A Lie algebroid $(E,[\ ,\ ]_E,\rho)$ over M can be localized to any nonempty open subset $U \subset M$, the claim being that the bracket

$$[\ ,\ ]_E : \sec E \times \sec E \to \sec E$$

induces a Lie algebroid structure on $\pi^{-1}(U)$. To see this, it is enough to prove that if $s_1, s_2 \in \sec E$ and if $s_2|U = 0$, then $[s_1,s_2]_E|U = 0$. Thus let $x_0 \in U$ and choose $f \in C^\infty(M): f(x_0) = 0$ & $f(M-U) = 1$ — then

$$[s_1,s_2]_E(x_0) = [s_1,fs_2]_E(x_0)$$

$$= f(x_0)[s_1,s_2]_E(x_0) + ((\rho \circ s_1)f)\Big|_{x_0} s_2(x_0)$$

$$= 0.$$

Work now with local coordinates $\{x^i,y^\alpha\}$ in $\pi^{-1}(U)$ determined by local coordinates x^i $(i = 1,\ldots,n)$ in U and a frame $e_\alpha(\alpha = 1,\ldots,k)$ for E over U -- then from the definitions

$$\rho \circ e_\alpha = \rho_\alpha^i \frac{\partial}{\partial x^i} \text{ and } [e_\alpha,e_\beta]_E = C_{\alpha\beta}^\gamma e_\gamma.$$

Here

$$\rho_\alpha^j \frac{\partial \rho_\beta^i}{\partial x^j} - \rho_\beta^j \frac{\partial \rho_\alpha^i}{\partial x^j} = \rho_\gamma^i C_{\alpha\beta}^\gamma$$

and

$$\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + \rho^i_\beta \frac{\partial C^\nu_{\gamma\alpha}}{\partial x^i} + \rho^i_\gamma \frac{\partial C^\nu_{\alpha\beta}}{\partial x^i}$$

$$+ C^\mu_{\beta\gamma} C^\nu_{\alpha\mu} + C^\mu_{\gamma\alpha} C^\nu_{\beta\mu} + C^\mu_{\alpha\beta} C^\nu_{\gamma\mu} = 0.$$

[Note: The ρ^i_α and the $C^\gamma_{\alpha\beta}$ are C^∞ functions on U. Of course an x^i, when viewed as a function on $\pi^{-1}(U)$, should really be denoted by $x^i \circ \pi$ ... .]

17.35 EXAMPLE If $E = \underline{g}$ (cf. 17.1), then $\rho^i_\alpha = 0$ and the $C^\gamma_{\alpha\beta}$ are the structure constants of the Lie algebra.

17.36 EXAMPLE If E = TM (cf. 17.2), if the x^i are the q^i, and if the y^α are the v^i, then $\rho^i_j = \delta^i_j$, $C^k_{ij} = 0$.

[Note: Make the replacements

$$\begin{bmatrix} M \rightarrow TM \\ TM \rightarrow TTM. \end{bmatrix}$$

Then in the notation of the Appendix to §8, the set

$$\{\bar{X}_1, \ldots, \bar{X}_n, \frac{\partial}{\partial \bar{v}^1}, \ldots, \frac{\partial}{\partial \bar{v}^n}\}$$

is a basis for

$$\mathcal{D}^1((\pi_M)^{-1}U).$$

And

$$[\bar{X}_i, \bar{X}_j] = \gamma^k_{ij} \bar{X}_k.]$$

17.37 <u>REMARK</u> Let $\{e^\alpha\}$ be the frame dual to $\{e_\alpha\}$ -- then $\forall\ f \in C^\infty(M)$,

$$d_E f = \frac{\partial f}{\partial x^i}\, \rho^i_\alpha e^\alpha,$$

hence

$$f^T(x^i, y^\alpha) = (\frac{\partial f}{\partial x^i} \circ \pi)(\rho^i_\alpha \circ \pi) y^\alpha.$$

Starting with the e_α, put

$$X_\alpha = e^T_\alpha + (C^\gamma_{\alpha\beta} \circ \pi) y^\beta e^V_\alpha \text{ and } Y_\alpha = e^V_\alpha.$$

Then $\{X_\alpha, Y_\alpha\}$ is a frame for LE over $\pi^{-1}(U)$.

[Note: Let

$$U_{LE} = (\pi_E \circ \rho_E)^{-1}\, (\pi^{-1}(U)).$$

Then

$$\begin{cases} X_\alpha \in \sec(U_{LE} \to \pi^{-1}(U)) \\ Y_\alpha \in \sec(U_{LE} \to \pi^{-1}(U)). \end{cases}$$

And

$$\begin{cases} SX_\alpha = Y_\alpha \\ SY_\alpha = 0.] \end{cases}$$

17.38 <u>EXAMPLE</u> Locally,

$$\Delta_E = y^\alpha Y_\alpha \text{ and } \rho_E \circ \Delta_E = y^\alpha \frac{\partial}{\partial y^\alpha}\ .$$

17.39 <u>LEMMA</u> We have

$$[X_\alpha, X_\beta]_{LE} = (C^\gamma_{\alpha\beta} \circ \pi) X_\gamma$$

and

$$\begin{cases} [X_\alpha, Y_\beta]_{LE} = 0 \\ [Y_\alpha, Y_\beta]_{LE} = 0. \end{cases}$$

17.40 <u>LEMMA</u> We have

$$(\rho_E \circ X_\alpha) = (\rho^i_\alpha \circ \pi) \frac{\partial}{\partial x^i}, \ \rho_E \circ Y_\alpha = \frac{\partial}{\partial y^\alpha}.$$

<u>N.B.</u> If $\{X^\alpha, Y^\alpha\}$ is the frame dual to $\{X_\alpha, Y_\alpha\}$, then

$$d_{LE}\phi = (\rho^i_\alpha \circ \pi) \frac{\partial \phi}{\partial x^i} X^\alpha + \frac{\partial \phi}{\partial y^\alpha} Y^\alpha \quad (\phi \in C^\infty(\pi^{-1}(U))).$$

In particular:

$$\begin{cases} d_{LE}x^i = (\rho^i_\alpha \circ \pi) X^\alpha \\ d_{LE}y^\alpha = Y^\alpha. \end{cases}$$

Furthermore

$$d_{LE}X^\alpha = -\frac{1}{2} (C^\alpha_{\beta\gamma} \circ \pi) X^\beta \wedge X^\gamma$$

while

$$d_{LE}Y^\alpha = 0.$$

Suppose that $\Gamma \in \sec LE$ is second order -- then locally,

$$\Gamma = y^\alpha X_\alpha + C^\alpha V_\alpha$$

and

$$\rho_E \circ \Gamma = (\rho^i_\alpha \circ \pi) y^\alpha \frac{\partial}{\partial x^i} + C^\alpha \frac{\partial}{\partial y^\alpha} .$$

[Note: An integral curve γ of $\rho_E \circ \Gamma$ is a solution to

$$\frac{dx^i}{dt} = (\rho^i_\beta \circ \pi) y^\beta, \ \frac{dy^\alpha}{dt} = C^\alpha .]$$

Suppose that C is a vector subbundle of E -- then the restriction $\pi|C:C \rightarrow M$ is a fibration. So we can form the pullback square

$$\begin{tikzcd} TC \times_{TM} E \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & E \arrow[d, "\rho"] \\ TC \arrow[r, "T(\pi|C)"'] & TM \end{tikzcd}$$

and put

$$L_C E = TC \times_{TM} E$$

to get a Lie algebroid $(L_C E, [\ ,\ ]_{L_C E}, \rho_C)$ over C.

[Note: Here C plays the role of M' and $\pi|C$ plays the role of Φ.]

There is another pullback square that can be formed, namely

$$\begin{array}{ccc}
TC \times_{TM} C & \xrightarrow{pr_2} & C \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \rho|C} \\
TC & \xrightarrow[T(\pi|C)]{} & TM\ .
\end{array}$$

Put

$$LC = TC \times_{TM} C.$$

Then, in general, the vector bundle $LC \rightarrow C$ is not a Lie algebroid (but it will be if C is a Lie subalgebroid of E, i.e., if sec C is closed per $[\ ,\ ]_E$).

<u>N.B.</u> LC is a vector subbundle of $L_C E$.

17.41 <u>EXAMPLE</u> Take $E = TM$ and write Σ in place of C -- then $L_\Sigma E = T\Sigma$ and $L\Sigma$ is a linear distribution on Σ. E.g.: Let $\omega^1,\ldots,\omega^{n-k}$ be a system of constraints and

$$\Sigma = \bigcap_{\mu=1}^{n-k} \Sigma_{\omega^\mu} \qquad \text{(cf. §16)},$$

where

$$\Sigma_{\omega^\mu} = (\hat{\omega}^\mu)^{-1}(0).$$

Set

$$\Sigma^* = \bigcap_{\mu=1}^{n-k} \operatorname{Ker} \pi_M^*(\omega^\mu).$$

Then Σ^* is a linear distribution on TM and

$$L\Sigma = \Sigma^* \cap T\Sigma.$$

Suppose that $\Gamma \in SO(TM)$, thus

$$\forall\ \mu,\ (\pi_M^* \omega^\mu)\ (\Gamma) = \hat{\omega}^\mu.$$

So, if Γ is tangent to Σ, then

$$\Gamma|\Sigma \in \text{sec}\ L\Sigma.$$

APPENDIX

Suppose that

$$\begin{tikzcd} M_1' \arrow[r, "\Psi'"] \arrow[d, "\Phi_1"'] & M_2' \arrow[d, "\Phi_2"] \\ M_1 \arrow[r, "\Psi"'] & M_2 \end{tikzcd}$$

is a morphism of fibered manifolds. Let

$$\left[\begin{array}{l} (E_1, [\ ,\ ]_{E_1}, \rho_1) \text{ be a Lie algebroid over } M_1 \\ \\ (E_2, [\ ,\ ]_{E_2}, \rho_2) \text{ be a Lie algebroid over } M_2 \end{array}\right.$$

and let

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\Psi"'] & M_2 \end{tikzcd}$$

be a vector bundle morphism such that $T\Psi \circ \rho_1 = \rho_2 \circ F$. Form

$$\begin{tikzcd} E_1' \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & E_1 \arrow[d, "\rho_1"] \\ TM_1' \arrow[r, "T\Phi_1"'] & TM_1 \end{tikzcd} , \qquad \begin{tikzcd} E_2' \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & E_2 \arrow[d, "\rho_2"] \\ TM_2' \arrow[r, "T\Phi_2"'] & TM_2 \end{tikzcd}$$

and let

$$F':E_1' \to E_2'$$

be the arrow that sends

$$((x',X_{x'}'),e) \text{ to } ((\Psi'(x'),d\Psi_{x'}'(X_{x'}')),F(e)).$$

Then F' determines a vector bundle morphism

$$\begin{tikzcd} E_1' \arrow[r, "F'"] \arrow[d, "\pi_1'"'] & E_2' \arrow[d, "\pi_2'"] \\ M_1' \arrow[r, "\Psi'"'] & M_2' \end{tikzcd}$$

such that $T\Psi' \circ \rho_1' = \rho_2' \circ F'$. Moreover, F' is a Lie algebroid morphism iff F is a Lie algebroid morphism.

[Note: This construction is "functorial" w.r.t. composition.]

§18. *LAGRANGIAN FORMALISM*

It is straightforward to extend the considerations of §8 to an arbitrary Lie algebroid $(E, [\ ,\ ]_E, \rho)$ over M, bearing in mind that

$$\left[\begin{array}{l} E \leftrightarrow TM \\ \\ LE \leftrightarrow TTM. \end{array}\right.$$

First, we shall agree that a <u>lagrangian</u> is simply any element $L \in C^\infty(E)$.

[Note: Local coordinates in E are the x^i and the y^α, hence it makes sense to take the partial derivatives of L w.r.t. the x^i and the y^α.]

18.1 <u>RAPPEL</u> If $E = TM$, then

$$\theta_L = d_S L$$

or still,

$$\theta_L = S^*(dL)$$

or still,

$$\theta_L = S^*(d_{TTM} L).$$

[Note: Spelled out,

$$d_{TM} \leftrightarrow d \text{ per } \Lambda^* M$$

and

$$d_{TTM} \leftrightarrow d \text{ per } \Lambda^* TM.]$$

<u>N.B.</u> The vertical morphism $S:LE \to LE$ induces a map

$$\sec LE \to \sec LE,$$

hence operates by duality on $\Lambda^* LE$, thus there is an arrow

$$S^*:\Lambda^* LE \to \Lambda^* LE.$$

In particular:

$$\begin{bmatrix} S \circ X_\alpha = Y_\alpha \\ \\ S \circ Y_\alpha = 0 \end{bmatrix} \Rightarrow \begin{bmatrix} S^* X^\alpha = 0 \\ \\ S^* Y^\alpha = X^\alpha. \end{bmatrix}$$

Given L, put

$$\theta_L = S^*(d_{LE}L).$$

18.2 <u>LEMMA</u> Locally,

$$\theta_L = \frac{\partial L}{\partial y^\alpha} X^\alpha.$$

[On general grounds,

$$d_{LE}L = (\rho_\alpha^i \circ \pi) \frac{\partial L}{\partial x^i} X^\alpha + \frac{\partial L}{\partial y^\alpha} Y^\alpha.]$$

Given L, put

$$\omega_L = d_{LE}\theta_L.$$

18.3 <u>LEMMA</u> Locally,

$$\omega_L = -\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} X^\alpha \wedge Y^\beta$$

$$+ ((\rho^i_\alpha \circ \pi) \frac{\partial^2 L}{\partial x^i \partial y^\beta} - \frac{1}{2} (C^\gamma_{\alpha\beta} \circ \pi) \frac{\partial L}{\partial y^\gamma}) X^\alpha \wedge X^\beta .$$

PROOF For

$$d_{LE}\theta_L = (d_{LE} \frac{\partial L}{\partial y^\beta}) \wedge X^\beta + \frac{\partial L}{\partial y^\gamma} \wedge d_{LE} X^\gamma$$

$$= (\rho^i_\alpha \circ \pi) \frac{\partial^2 L}{\partial x^i \partial y^\beta} X^\alpha \wedge X^\beta + \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} Y^\alpha \wedge X^\beta$$

$$+ \frac{\partial L}{\partial y^\gamma} (- \frac{1}{2} (C^\gamma_{\alpha\beta} \circ \pi)) X^\alpha \wedge X^\beta .$$

Given L, put

$$E_L = (\rho_E \circ \Delta_E) L - L (\equiv L_{\Delta_E} L - L).$$

Then E_L is the energy function attached to L.

[Note: Locally,

$$E_L = \frac{\partial L}{\partial y^\alpha} y^\alpha - L.]$$

18.4 LEMMA We have

$$\iota_{\Delta_E} \omega_L = S^*(d_{LE} E_L).$$

PROOF Locally,

$$\Delta_E = y^\alpha y_\alpha \quad \text{(cf. 17.38).}$$

Therefore

$$\left[\begin{array}{l} \iota_{\Delta_E} X^\beta = X^\beta(\Delta_E) = 0 \\ \\ \iota_{\Delta_E} y^\beta = y^\beta(\Delta_E) = y^\beta. \end{array}\right.$$

Consequently,

$$\iota_{\Delta_E}\omega_L = -\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}(\iota_{\Delta_E} X^\alpha \wedge y^\beta - X^\alpha \wedge \iota_{\Delta_E} y^\beta)$$

$$+ (\ldots)(\iota_{\Delta_E} X^\alpha \wedge X^\beta - X^\alpha \wedge \iota_{\Delta_E} X^\beta)$$

$$= \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} y^\beta X^\alpha.$$

On the other hand,

$$S^*(d_{LE}E_L) = \frac{\partial E_L}{\partial y^\alpha} X^\alpha$$

$$= \frac{\partial}{\partial y^\alpha}\left(\frac{\partial L}{\partial y^\beta} y^\beta - L\right) X^\alpha$$

$$= \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} y^\beta + \frac{\partial L}{\partial y^\alpha} - \frac{\partial L}{\partial y^\alpha}\right) X^\alpha$$

$$= \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} X^\alpha.$$

L is said to be <u>nondegenerate</u> if ω_L is symplectic; otherwise, L is said to be <u>degenerate</u>. The analog of 8.5 is valid: L is nondegenerate iff for all coordinate systems $\{x^i, y^\alpha\}$,

$$\det \left[\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \right] \neq 0.$$

18.5 <u>EXAMPLE</u> Define a lagrangian $L:E \to \underline{R}$ by

$$L(e) = \frac{1}{2} G(e,e) - (V \circ \pi)(e) \qquad (e \in E),$$

where $G:E \times_M E \to \underline{R}$ is a bundle metric on E and V is a C^∞ function on M -- then L is nondegenerate.

Let

$$D_L = \{X \in \sec LE: \iota_X \omega_L = - d_{LE} E_L\}.$$

Then L is said to admit <u>global dynamics</u> if D_L is nonempty.

18.6 <u>LEMMA</u> Let $X \in D_L$ -- then $L_X \omega_L = 0$.

<u>PROOF</u> One has only to write

$$\begin{aligned} L_X \omega_L &= (\iota_X \circ d_{LE} + d_{LE} \circ \iota_X)\omega_L \\ &= 0 + d_{LE}(- d_{LE} E_L) \\ &= 0. \end{aligned}$$

[Note: Recall that

$$d_{LE}^2 = 0.]$$

18.7 <u>REMARK</u> Let $X \in D_L$ -- then

$$\begin{aligned} L_X E_L &= \iota_X d_{LE} E_L \\ &= - \iota_X \iota_X \omega_L \\ &= 0. \end{aligned}$$

But

$$L_X E_L = (\rho_E \circ X) E_L.$$

Therefore E_L is a first integral for $\rho_E \circ X$ (cf. 8.10).

18.8 <u>LEMMA</u> $\forall\ X \in \sec LE$,

$$\iota_{S \circ X} \omega_L = - S^*(\iota_X \omega_L).$$

18.9 <u>LEMMA</u> If L is nondegenerate, then L admits global dynamics: $\exists$ a (unique) $\Gamma_L \in \sec LE$ such that

$$\iota_{\Gamma_L} \omega_L = - d_{LE} E_L.$$

And Γ_L is second order.

<u>PROOF</u> The existence (and uniqueness) of Γ_L is implied by the assumption that ω_L is symplectic. To establish that Γ_L is second order, write

$$\iota_{\Delta_E}\omega_L = S^*(d_{LE}E_L) \quad (\text{cf. } 18.4)$$

$$= - S^*(\iota_{\Gamma_L}\omega_L)$$

$$= \iota_{S \circ \Gamma_L}\omega_L \quad (\text{cf. } 18.8).$$

But then

$$S \circ \Gamma_L = \Delta_E,$$

so Γ_L is second order (cf. 17.34).

[Note: Locally,

$$\Gamma_L = y^\alpha X_\alpha + C^\alpha \mathcal{V}_\alpha.$$

And $\forall\ \alpha$,

$$(\rho^i_\beta \circ \pi) y^\beta \frac{\partial^2 L}{\partial x^i \partial y^\alpha} + C^\beta \frac{\partial^2 L}{\partial y^\beta \partial y^\alpha}$$

$$= (\rho^i_\alpha \circ \pi) \frac{\partial L}{\partial x^i} - (C^\gamma_{\alpha\beta} \circ \pi) y^\beta \frac{\partial L}{\partial y^\gamma}$$

or still,

$$L_{\Gamma_L}\left(\frac{\partial L}{\partial y^\alpha}\right) = (\rho_E \circ \Gamma_L) \frac{\partial L}{\partial y^\alpha}$$

$$= (\rho^i_\alpha \circ \pi) \frac{\partial L}{\partial x^i} - (C^\gamma_{\alpha\beta} \circ \pi) y^\beta \frac{\partial L}{\partial y^\gamma}.]$$

18.10 <u>REMARK</u> Along an integral curve γ of $\rho_E \circ \Gamma_L$, we have

$$\frac{dx^i}{dt} = (\rho^i_\beta \circ \pi)y^\beta, \ \frac{dy^\alpha}{dt} = C^\alpha.$$

Therefore

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \frac{\partial^2 L}{\partial x^i \partial y^\alpha}\frac{dx^i}{dt} + \frac{\partial^2 L}{\partial y^\beta \partial y^\alpha}\frac{dy^\beta}{dt}$$

$$= \frac{\partial^2 L}{\partial x^i \partial y^\alpha}(\rho^i_\beta \circ \pi)y^\beta + \frac{\partial^2 L}{\partial y^\beta \partial y^\alpha}C^\beta.$$

I.e.:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = (\rho^i_\alpha \circ \pi)\frac{\partial L}{\partial x^i} - (C^\gamma_{\alpha\beta} \circ \pi)y^\beta\frac{\partial L}{\partial y^\gamma},$$

which will be termed the <u>equations of Lagrange</u>.]

18.11 <u>EXAMPLE</u> Let $\underline{g}$ be a finite dimensional Lie algebra. Fix a basis e_α for $\underline{g}$ ($\alpha = 1,\ldots,k$) ($k = \dim \underline{g}$) -- then

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}e_\gamma$$

and the equations of Lagrange are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = -C^\gamma_{\alpha\beta}y^\beta\frac{\partial L}{\partial y^\gamma}.$$

E.g.: Take $\underline{g} = \underline{R}^3$ and

$$\left[\begin{array}{l} e_1 = (1,0,0) \\ \\ e_2 = (0,1,0) \\ \\ e_3 = (0,0,1). \end{array}\right.$$

Then

$$[e_\alpha, e_\beta] = e_\alpha \times e_\beta = \sum_{\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma} e_\gamma$$

and in vector notation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y}\right) = \frac{\partial L}{\partial y} \times y.$$

To illustrate, let

$$L(y) = L(y^1, y^2, y^3) = \frac{1}{2}\left(I_1 (y^1)^2 + I_2 (y^2)^2 + I_3 (y^3)^2\right),$$

where $I_1 > 0$, $I_2 > 0$, $I_3 > 0$ — then the equations of Lagrange become

$$\left[\begin{array}{l} \dot{y}^1 = \dfrac{(I_2 - I_3)}{I_1} y^2 y^3 \\[2ex] \dot{y}^2 = \dfrac{(I_3 - I_1)}{I_2} y^3 y^1 \\[2ex] \dot{y}^3 = \dfrac{(I_1 - I_2)}{I_3} y^1 y^2. \end{array}\right.$$

So, from the Lie algebroid viewpoint, the "Euler equations" of the Appendix are instances of the equations of Lagrange.

APPENDIX

Suppose that $(E, [\ ,\ ]_E, \rho)$ is a Lie algebroid over M. Let $\pi' : E' \to M$ be a vector bundle -- then an <u>E-connection</u> on E' is a map

$$\left[\begin{array}{l} \nabla : \sec E \times \sec E' \to \sec E' \\ \qquad (s, s') \to \nabla_s s' \end{array}\right.$$

such that

1. $\nabla_{s_1+s_2} s' = \nabla_{s_1} s' + \nabla_{s_2} s'$;

2. $\nabla_s(s_1' + s_2') = \nabla_s s_1' + \nabla_s s_2'$;

3. $\nabla_{fs} s' = f\nabla_s s'$;

4. $\nabla_s(fs') = ((\rho \circ s)f)s' + f\nabla_s s'$.

A.1 <u>REMARK</u> The choice

$$(E,[\ ,\ ]_E,\rho) = (TM,[\ ,\ ],\mathrm{id}_{TM})$$

leads to the usual notion of a connection in a vector bundle.

In what follows, we shall take $E' = E$ and use the term "connection on E".

So let ∇ be a connection on E -- then locally, the <u>connection coefficients</u> of ∇ are the C^∞ functions $\Gamma_{\alpha\beta}^\gamma$ on U defined by

$$\nabla_{e_\alpha} e_\beta = \Gamma_{\alpha\beta}^\gamma e_\gamma.$$

Accordingly, if

$$\left[\begin{array}{l} s = s^\alpha e_\alpha \\ \\ t = t^\beta e_\beta \end{array}\right. \qquad (s^\alpha, t^\beta \in C^\infty(U)),$$

then

$$\nabla_s t = s^\alpha \nabla_{e_\alpha}(t^\beta e_\beta)$$

$$= s^\alpha(((\rho \circ e_\alpha)t^\beta)e_\beta + t^\beta \nabla_{e_\alpha} e_\beta)$$

$$= s^\alpha(\rho^i_\alpha \frac{\partial t^\beta}{\partial x^i} e_\beta + t^\beta \Gamma^\gamma_{\alpha\beta} e_\gamma)$$

$$= s^\alpha(\rho^i_\alpha \frac{\partial t^\gamma}{\partial x^i} + t^\beta \Gamma^\gamma_{\alpha\beta})e_\gamma.$$

Assume now that $G: E \times_M E \to \underline{R}$ is a bundle metric on E.

A.2 LEMMA There exists a unique connection ∇^G on E such that

$$\nabla^G_{s_1} s_2 - \nabla^G_{s_2} s_1 = [s_1, s_2]_E$$

and

$$(\rho \circ s_1)(G(s_2,s_3)) = G(\nabla^G_{s_1} s_2, s_3) + G(s_2, \nabla^G_{s_1} s_3).$$

PROOF ∇^G is determined by the formula

$$2G(\nabla^G_{s_1} s_2, s_3) = (\rho \circ s_1)G(s_2,s_3) + (\rho \circ s_2)G(s_1,s_3) - (\rho \circ s_3)G(s_1,s_2)$$

$$+ G(s_1,[s_3,s_2]_E) + G(s_2,[s_3,s_1]_E) + G([s_1,s_2]_E,s_3).$$

N.B. ∇^G is called the metric connection attached to G.

Locally,

$$G = G_{\alpha\beta} e^\alpha \otimes e^\beta$$

and the connection coefficients of ∇^G are given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} G^{\alpha\nu}([\nu,\beta;\gamma] + [\nu,\gamma;\beta] + [\beta,\gamma;\nu]),$$

where

$$[\alpha,\beta;\gamma] = \frac{\partial G_{\alpha\beta}}{\partial x^i}\rho^i_\gamma + C^\mu_{\alpha\beta}G_{\mu\gamma}.$$

A.3 <u>LEMMA</u> Put

$$L_G(e) = \frac{1}{2}G(e,e) \quad (e \in E) \quad (\text{cf. } 18.5).$$

Write Γ_G in place of Γ_{L_G} (cf. 18.9) -- then locally,

$$\Gamma_G = y^\gamma X_\gamma - (\Gamma^\gamma_{\alpha\beta} \circ \pi)y^\alpha y^\beta Y_\gamma \quad (\text{cf. } 10.6).$$

Given $V \in C^\infty(M)$, its gradient $\text{grad}_G V$ is the section of E characterized by

$$G(\text{grad}_G V, s) = d_E V(s) \quad (s \in \sec E).$$

Locally,

$$\text{grad}_G V = (G^{\alpha\beta}\rho^i_\beta \frac{\partial V}{\partial x^i})e_\alpha.$$

A.4 <u>LEMMA</u> Put

$$L_{G,V}(e) = \frac{1}{2}G(e,e) - (V \circ \pi)(e) \quad (e \in E) \quad (\text{cf. } 18.5).$$

Write $\Gamma_{G,V}$ in place of $\Gamma_{L_{G,V}}$ (cf. 18.9) -- then

$$\Gamma_{G,V} = \Gamma_G - (\text{grad}_G V)^V \quad (\text{cf. } 10.8).$$

§19. *CONSTRAINT THEORY*

To set the stage, let us recall the following points.

19.1 <u>RAPPEL</u> Suppose given C^∞ functions

$$\Phi^\mu : TM \to \underline{R} \quad (\mu = 1,\ldots,n-k).$$

Then the Φ^μ combine to give a map

$$\Phi : TM \to \underline{R}^{n-k}.$$

Consider the level set $\Phi^{-1}(0)$. Assume: $\forall\ p \in \Phi^{-1}(0)$, $\Phi_*|_p$ has rank $n - k$ -- then $\Phi^{-1}(0)$ is a closed submanifold of TM.

[Note: The assumption is equivalent to demanding that $\forall\ p \in \Phi^{-1}(0)$, the 1-forms

$$d\Phi^1|_p, \ldots, d\Phi^{n-k}|_p$$

are linearly independent or still, that

$$d\Phi^1 \wedge \cdots \wedge d\Phi^{n-k} \neq 0$$

on $\Phi^{-1}(0)$.]

19.2 <u>EXAMPLE</u> Take $M = \underline{R}$ and let $\Phi(q,v) = v$ -- then $\Phi^{-1}(0) = \{(q,v) : v = 0\}$ satisfies the above conditions. On the other hand, the alternative descriptions of the q-axis given by

$$\Phi(q,v) = v^2 \text{ or } \Phi(q,v) = \sqrt{|v|}$$

are not admissible.

19.3 EXAMPLE Take $M = \underline{R}^4$ and define $\Phi:TM = \underline{R}^4 \times \underline{R}^4$ by

$$\Phi(q^1,q^2,q^3,q^4,v^1,v^2,v^3,v^4)$$

$$= v^1v^4 - v^2v^3 = \det \begin{bmatrix} v^1 & v^2 \\ v^3 & v^4 \end{bmatrix}.$$

Then the level set $\Phi^{-1}(0)$ is not a submanifold of TM.

[Note: Removing the zero section from $\Phi^{-1}(0)$ gives rise to a submanifold of TM. Physically, it is a question of two point masses A and B forced to move in a plane with parallel velocities. The lagrangian is

$$\frac{1}{2} m_A((v^1)^2 + (v^2)^2) + \frac{1}{2} m_B((v^3)^2 + (v^4)^2)$$

and Φ represents the constraints on the velocities. Elimination of the zero section imposes the additional restriction that the velocities cannot be simultaneously zero.]

A constraint is a submanifold $C \subset TM$ such that $\pi_M|C$ is a fibration. E.g.: C might be a vector or affine subbundle of TM.

In the applications, however, one is ordinarily handed C^∞ functions

$$\Phi^\mu:TM \to \underline{R} \quad (\mu = 1,\ldots,n-k)$$

satisfying the conditions of 19.1 and then one takes

$$C = \Phi^{-1}(0),$$

the data being such that $\pi_M|C$ is a fibration. So, in the sequel, this will be our standing assumption.

19.4 <u>REMARK</u> Suppose given an affine system of constraints

$$\Phi^{\mu} = \hat{\omega}^{\mu} + \phi^{\mu} \circ \pi_M \quad (\mu = 1,\ldots,n-k).$$

Then

$$C = \Phi^{-1}(0)$$

is a constraint. To see this, work locally -- then the rank of

$$\begin{bmatrix} \frac{\partial\Phi^1}{\partial v^1} & \cdots & \frac{\partial\Phi^1}{\partial v^n} \\ \vdots & & \vdots \\ \frac{\partial\Phi^{n-k}}{\partial v^1} & \cdots & \frac{\partial\Phi^{n-k}}{\partial v^n} \end{bmatrix}$$

equals the rank of

$$\begin{bmatrix} a^1{}_1 & \cdots & a^1{}_n \\ \vdots & & \vdots \\ a^{n-k}{}_1 & \cdots & a^{n-k}{}_n \end{bmatrix}.$$

But the rank of the latter is precisely $n - k$ (recall that the set $\omega^1,\ldots,\omega^{n-k}$ is linearly independent).

[Note:

$$\omega^{\mu} = a^{\mu}{}_i dx^i$$

$$\Rightarrow$$

$$\hat{\omega}^\mu = (a^\mu{}_i \circ \pi_M) v^i$$

=>

$$\frac{\partial \Phi^\mu}{\partial v^i} = \frac{\partial \hat{\omega}^\mu}{\partial v^i} + \frac{\partial (\phi^\mu \circ \pi_M)}{\partial v^i}$$

$$= \frac{\partial \hat{\omega}^\mu}{\partial v^i}$$

$$= a^\mu{}_i \circ \pi_M.]$$

19.5 <u>LEMMA</u> Given a point $(x, V_x) \in C$, $\exists$ an open interval I containing the origin and a curve $\gamma: I \to M$ such that $\dot{\gamma}(0) = V_x$ and $(\gamma(t), \dot{\gamma}(t)) \in C$ $(t \in I)$.

<u>PROOF</u> Since $\pi_M | C$ is a fibration, hence is a submersion, $\exists$ an open set $U \subset M$ containing x and a local section $X: U \to C$ such that $X(x) = (x, V_x)$. This said, choose an integral curve $\gamma: I \to M$ for X such that $\dot{\gamma}(0) = V_x$ and $\gamma(t) \in U$ $(t \in I)$.

Fix a nondegenerate lagrangian L. Define $X_\mu \in \mathcal{D}^1(TM)$ by the requirement that

$$\iota_{X_\mu} \omega_L = S^*(d\Phi^\mu) \quad (\mu = 1, \ldots, n-k).$$

Then X_μ is necessarily vertical (cf. 8.23). Given $\lambda^1, \ldots, \lambda^{n-k} \in C^\infty(TM)$, put

$$\Gamma_{\underline{\lambda}} = \Gamma_L + \lambda^\mu X_\mu.$$

Impose the condition of tangency

$$0 = \Gamma_{\underline{\lambda}}(\Phi^\nu)$$

$$= \Gamma_L(\Phi^\nu) + \lambda^\mu X_\mu(\Phi^\nu).$$

Call

$$(L, \{\Phi^1, \ldots, \Phi^{n-k}\})$$

<u>regular</u> if the matrix

$$[X_\mu \Phi^\nu]$$

is nonsingular; otherwise, call

$$(L, \{\Phi^1, \ldots, \Phi^{n-k}\})$$

<u>irregular</u>.

So, in the regular situation, one can determine the Lagrange multiplier $\underline{\lambda}_0$ and the dictum is that the constrained dynamics is given by $\Gamma_{\underline{\lambda}_0}|C$.

<u>N.B.</u> Locally,

$$X_\mu \Phi^\nu = (W(L)^{-1})^{k\ell} \frac{\partial \Phi^\mu}{\partial v^k} \frac{\partial \Phi^\nu}{\partial v^\ell}.$$

Therefore

$$(L, \{\Phi^1, \ldots, \Phi^{n-k}\})$$

is regular if

$$L = T - V \circ \pi_M,$$

where g is riemannian.

19.6 <u>EXAMPLE</u> Take $M = \underline{R}^3$ and put

$$|v| = ((v^1)^2 + (v^2)^2 + (v^3)^2)^{1/2}.$$

Let

$$L = \frac{m}{2} (|v|^2) - mgq^3 \quad (m > 0,\ g > 0).$$

Then

$$E_L = \frac{m}{2}\,(|v|^2) + mgq^3$$

and

$$\left[\begin{aligned} \omega_L &= m(dv^1 \wedge dq^1 + dv^2 \wedge dq^2 + dv^3 \wedge dq^3) \\ \Gamma_L &= v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + v^3 \frac{\partial}{\partial q^3} - g \frac{\partial}{\partial v^3}\,. \end{aligned}\right.$$

Take

$$\Phi = |v|^2 - R \quad (R > 0).$$

Then

$$S^*(d\Phi) = 2v^1 \frac{\partial}{\partial q^1} + 2v^2 \frac{\partial}{\partial q^2} + 2v^3 \frac{\partial}{\partial q^3}\,.$$

Define X_Φ by

$$\iota_{X_\Phi}\omega_L = S^*(d\Phi).$$

Then

$$X_\Phi = \frac{2}{m}\,(v^1 \frac{\partial}{\partial v^1} + v^2 \frac{\partial}{\partial v^2} + v^3 \frac{\partial}{\partial v^3}).$$

To compute the Lagrange multiplier

$$\lambda_0 = -\frac{\Gamma_L\Phi}{X_\Phi\Phi},$$

note that

$$\Gamma_L\Phi = -\,2gv^3$$

and

$$X_\Phi\Phi = \frac{4}{m}\,|v|^2.$$

Therefore

$$\lambda_0 = \frac{mgv^3}{2|v|^2} .$$

So

$$\Gamma_{\lambda_0}|C = (\Gamma_L + \lambda_0 X_\Phi)|C$$

$$= v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + v^3 \frac{\partial}{\partial q^3}$$

$$+ \frac{g}{R} v^3 v^1 \frac{\partial}{\partial v^1} + \frac{g}{R} v^3 v^2 \frac{\partial}{\partial v^2} + (\frac{gv^3}{R} - g) \frac{\partial}{\partial v^3} .$$

19.7 <u>EXAMPLE</u> Take $M = \underline{R}^4$ and consider the setup of 19.3 -- then

$$\omega_L = m_A(dv^1 \wedge dq^1 + dv^2 \wedge dq^2) + m_B(dv^3 \wedge dq^3 + dv^4 \wedge dq^4)$$

while

$$S^*(d\Phi) = v^4 dq^1 - v^3 dq^2 - v^2 dq^3 + v^1 dq^4.$$

Therefore

$$X_\Phi = \frac{1}{m_A} (v^4 \frac{\partial}{\partial v^1} - v^3 \frac{\partial}{\partial v^2}) + \frac{1}{m_B} (- v^2 \frac{\partial}{\partial v^3} + v^1 \frac{\partial}{\partial v^4}).$$

Determine λ_0 per

$$\lambda_0 = - \frac{\Gamma_L \Phi}{X_\Phi \Phi} .$$

Since

$$\Gamma_L = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + v^3 \frac{\partial}{\partial q^3} + v^4 \frac{\partial}{\partial q^4},$$

it is clear that $\Gamma_L \Phi = 0$. Thus the upshot is that the motion is the free motion of the point masses A and B subject to parallel initial velocities.

[Note: Strictly speaking, the analysis is formal since $\Phi^{-1}(0)$ is not a submanifold of TM. However, matters are correct provided we stay away from the zero section. In this connection, observe that

$$X_\Phi \Phi = \frac{1}{m_A} ((v^3)^2 + (v^4)^2) + \frac{1}{m_B} ((v^1)^2 + (v^2)^2).]$$

A constraint C is said to be <u>homogeneous</u> if Δ is tangent to C.

19.8 <u>LEMMA</u> C is homogeneous iff

$$\Delta\Phi^\mu|_C = 0 \quad (\mu = 1,\ldots,n-k)$$

or still, iff

$$v^i \frac{\partial\Phi^\mu}{\partial v^i}\Big|_C = 0 \quad (\mu = 1,\ldots,n-k).$$

19.9 <u>EXAMPLE</u> If each Φ^μ is homogeneous of degree $r(\mu) \geq 0$ in the velocities, i.e., if

$$\Phi^\mu(x, tX_x) = t^{r(\mu)} \Phi^\mu(x, X_x) \quad (0 \leq t \leq 1),$$

then C is homogeneous. Indeed,

$$\Phi^\mu(q^1,\ldots,q^n,tv^1,\ldots,tv^n)$$

$$= t^{r(\mu)} \Phi^\mu(q^1,\ldots,q^n,v^1,\ldots,v^n)$$

=>

$$v^i \frac{\partial \Phi^\mu}{\partial v^i} = r(\mu)\Phi^\mu$$

=>

$$v^i \frac{\partial \Phi^\mu}{\partial v^i}\Big|_C = r(\mu)\Phi^\mu\Big|_C = 0.$$

E.g.: The linear distribution Σ defined by a system of constraints $\omega^1,\ldots,\omega^{n-k}$ is homogeneous.

19.10 LEMMA Suppose that C is homogeneous -- then $E_L|C$ is a first integral for $\Gamma_{\underline{\lambda}_0}|C$:

$$E_L|C \in C^\infty_{\Gamma_{\underline{\lambda}_0}|C}(C).$$

PROOF In fact,

$$\begin{aligned}
\Gamma_{\underline{\lambda}_0} E_L &= (\Gamma_L + \lambda_0^\mu X_\mu)E_L \\
&= \lambda_0^\mu X_\mu E_L \\
&= \lambda_0^\mu dE_L(X_\mu) \\
&= -\lambda_0^\mu \iota_{\Gamma_L}\omega_L(X_\mu) \\
&= -\lambda_0^\mu \omega_L(\Gamma_L, X_\mu) \\
&= \lambda_0^\mu \omega_L(X_\mu, \Gamma_L)
\end{aligned}$$

$$= \lambda_0^\mu \iota_{X_\mu} \omega_L(\Gamma_L)$$

$$= \lambda_0^\mu S^*(d\Phi^\mu)(\Gamma_L)$$

$$= \lambda_0^\mu d\Phi^\mu(S\Gamma_L)$$

$$= \lambda_0^\mu d\Phi^\mu(\Delta)$$

$$= \lambda_0^\mu \Delta\Phi^\mu.$$

19.11 EXAMPLE In the notation of 19.6,

$$\Phi = |v|^2 - R \ (R > 0)$$

is not homogeneous. Here

$$E_L|C = \frac{m}{2} R + mgq^3$$

and

$$(\Gamma_{\lambda_0}|C)(E_L|C) = mgv^3 \neq 0.$$

Suppose now that $(E,[\ ,\ ]_E,\rho)$ is a Lie algebroid over M -- then in this context, a constraint is a submanifold $C \subset E$ such that $\pi|C$ is a fibration.

[Note: The constraint is linear if C is a vector subbundle of E.]

N.B. Consider the pullback square

$$\begin{array}{ccc} TC \times_{TM} E & \xrightarrow{pr_2} & E \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \rho} \\ TC & \xrightarrow[T(\pi|C)]{} & TM. \end{array}$$

Put

$$L_C E = TC \times_{TM} E.$$

Then

$$(L_C E, [\ ,\ ]_{L_C E}, \rho_C)$$

is a Lie algebroid over C, the <u>prolongation of C over E</u>.

[Note: Needless to say, $L_E E = LE$.]

In line with the earlier theory, we shall assume henceforth that $\exists$ C^∞ functions

$$\Phi^\mu : E \to \underline{R} \quad (\mu = 1,\ldots,K)$$

such that

$$C = \bigcap_{\mu=1}^{K} (\Phi^\mu)^{-1}(0) \quad \text{(cf. 19.1)}.$$

[Note: The fiber dimension of C is

$$r = \dim C - \dim M = \dim C - n.$$

And

$$\begin{aligned} K &= \dim E - \dim C \\ &= (n + k) - (n + r) \\ &= k - r, \end{aligned}$$

k the fiber dimension of E (as in §17). To run a reality check, take $E = TM$, thus in this case $k = n$. On the other hand, the codimension of $C \subset TM$ is, by our notational agreements, $n - k$... . Therefore

$$\dim C = 2n - (n - k)$$

$$= n + k$$

$$\Rightarrow$$

$$r = (n + k) - n = k$$

$$\Rightarrow$$

$$K = n - k.]$$

Fix a nondegenerate lagrangian L. Define $X_\mu \in \sec LE$ by the requirement that

$$\iota_{X_\mu} \omega_L = S^*(d_{LE}\Phi^\mu) \qquad (\mu = 1,\ldots,K).$$

19.12 <u>LEMMA</u> X_μ is vertical, i.e.,

$$X_\mu \in \sec VLE.$$

Locally,

$$X_\mu = (W(L)^{-1})^{\alpha\beta} \frac{\partial \Phi^\mu}{\partial y^\alpha} V_\beta.$$

[Note: $W(L)^{-1}$ is the inverse of

$$W(L) = [W_{\alpha\beta}(L)],$$

where

$$W_{\alpha\beta}(L) = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}.]$$

N.B. Locally,

$$S^*(d_{LE}\Phi^{\mu}) = \frac{\partial\Phi^{\mu}}{\partial y^{\alpha}} \chi^{\alpha}.$$

19.13 LEMMA Let $s \in \sec LE$. Suppose that

$$(\rho_E \circ s)\Phi^{\mu} = 0 \quad (\mu = 1,\ldots,K).$$

Then

$$s|C \in \sec L_C E.$$

[Note: Recall that

$$\rho_E \circ s \in \mathcal{D}^1(E).]$$

Given $\lambda^1,\ldots,\lambda^K \in C^{\infty}(E)$, put

$$\Gamma_{\underline{\lambda}} = \Gamma_L + \lambda^{\mu}X_{\mu}.$$

In view of 19.13, to force

$$\Gamma_{\underline{\lambda}}|C \in \sec L_C E,$$

it suffices to demand that

$$(\rho_E \circ \Gamma_{\underline{\lambda}})\Phi^{\nu} = 0 \quad (\nu = 1,\ldots,K)$$

or still,

$$(\rho_E \circ \Gamma_L)\Phi^{\nu} + \lambda^{\mu}(\rho_E \circ X_{\mu})\Phi^{\nu} = 0 \quad (\nu = 1,\ldots,K).$$

Call

$$(L,\{\Phi^1,\ldots,\Phi^K\})$$

<u>regular</u> if the matrix

$$[(\rho_E \circ X_\mu)\Phi^\nu]$$

is nonsingular; otherwise, call

$$(L,\{\Phi^1,\ldots,\Phi^K\})$$

<u>irregular</u>.

So, when

$$(L,\{\Phi^1,\ldots,\Phi^K\})$$

is regular, one can find the Lagrange multiplier $\underline{\lambda}_0$, thence

$$\Gamma_{\underline{\lambda}_0}|C \in \sec L_C E.$$

<u>N.B.</u> Locally,

$$\begin{aligned}(\rho_E \circ X_\mu)\Phi^\nu &= (W(L)^{-1})^{\alpha\beta} \frac{\partial\Phi^\mu}{\partial y^\alpha} (\rho_E \circ Y_\beta)\Phi^\nu \\ &= (W(L)^{-1})^{\alpha\beta} \frac{\partial\Phi^\mu}{\partial y^\alpha} \frac{\partial\Phi^\nu}{\partial y^\beta} \qquad \text{(cf. 17.40)}.\end{aligned}$$

Therefore

$$(L,\{\Phi^1,\ldots,\Phi^K\})$$

is regular if

$$L = \frac{1}{2} G - V \circ \pi,$$

where $G:E \times_M E \to \underline{R}$ is a bundle metric on E and V is a C^∞ function on M.

19.14 <u>EXAMPLE</u> Keep to the assumptions and notation of 18.11. Define

$I_0:\underline{R}^3 \to \underline{R}^3$ by

$$\begin{cases} I_0e_1 = I_1e_1 \\ I_0e_2 = I_2e_2 \\ I_0e_3 = I_3e_3. \end{cases}$$

Then

$$L(y) = \frac{1}{2} \langle I_0y, y\rangle \qquad (y \in \underline{R}^3).$$

And Γ_L is the Euler vector field $\Gamma_0:\underline{R}^3 \to \underline{R}^3$, thus

$$\Gamma_0 y = I_0^{-1}(I_0 y \times y) \quad (y \in \underline{R}^3) \quad \text{(see the Appendix, A.16)}.$$

Fix a unit vector $U \in \underline{R}^3$. Let $\Phi:\underline{R}^3 \to \underline{R}$ be the function $y \to \langle y,U\rangle$ and take

$$C = \Phi^{-1}(0).$$

Then

$$W(L) = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

=>

$$X_\Phi = I_0^{-1}(U)$$

=>

$$X_\Phi \Phi = \frac{U^1}{I_1}\frac{\partial}{\partial y^1}\Phi + \frac{U^2}{I_2}\frac{\partial}{\partial y^2}\Phi + \frac{U^3}{I_3}\frac{\partial}{\partial y^3}\Phi$$

$$= \frac{(U^1)^2}{I_1} + \frac{(U^2)^2}{I_2} + \frac{(U^3)^2}{I_3}$$

$$= \langle U, I_0^{-1} U \rangle.$$

Therefore

$$\lambda_0(y) = - \frac{\Gamma_0 \Phi}{X_\Phi \Phi}(y)$$

$$= - \frac{\langle I_0 y \times y, I_0^{-1} U \rangle}{\langle U, I_0^{-1} U \rangle} .$$

19.15 REMARK If C is linear and, in addition, is a Lie subalgebroid of E, then

$$\Gamma_{\underline{\lambda}_0} | C \in \sec LC$$

and

$$\Gamma_{L|C} = \Gamma_{\underline{\lambda}_0} | C.$$

19.16 LEMMA If $\rho_E \circ \Delta_E$ is tangent to C, then

$$(\rho_C \circ \Gamma_{\underline{\lambda}_0} | C)(E_L | C) = 0 \quad \text{(cf. 19.10)}.$$

[Note: The tangency assumption is always met by a linear C.]

19.17 EXAMPLE To check the validity of 19.16 in the setting of 19.14, note that

$$\left(y^1 \frac{\partial}{\partial y^1} + y^2 \frac{\partial}{\partial y^2} + y^3 \frac{\partial}{\partial y^3}\right) \langle y, U \rangle$$

$$= y^1U^1 + y^2U^2 + y^3U^3$$

$$= \langle y, U \rangle = \Phi(y).$$

Of course, one can also proceed directly, bearing in mind that here $E_L = L$, hence

$$\Gamma_L E_L = 0.$$

On the other hand,

$$X_\Phi L = \Phi.$$

§20. *CHAPLYGIN SYSTEMS*

Suppose that $\pi:E \to M$ is a fibration (cf. §11) -- then an Ehresmann connection is a linear distribution $H \subset TE$ such that $\forall\ e \in E$,

$$VE|_e \oplus H_e = T_eE \quad \text{(cf. 15.11)}.$$

[Note: Let k be the fiber dimension of E, thus $\dim E = n + k$. Since

$$VE|_e = T_e(E_{\pi(e)}),$$

it follows that

$$\dim H_e = \dim T_eE - \dim VE|_e$$

$$= n + k - k = n.$$

Therefore

$$\dim H = 2n + k.]$$

Associated with H are vertical and horizontal projections

$$\begin{cases} \underline{v}:\mathcal{D}^1(E) \to \sec VE \\ \underline{h}:\mathcal{D}^1(E) \to \sec H \end{cases}$$

and its <u>curvature</u> is the map

$$R:\mathcal{D}^1(E) \times \mathcal{D}^1(E) \to \mathcal{D}^1(E)$$

defined by

$$R(X,Y)$$

$$= [\underline{h}X,\underline{h}Y] - \underline{h}[\underline{h}X,Y] - \underline{h}[X,\underline{h}Y] + \underline{h}[X,Y].$$

20.1 LEMMA $\forall$ $X,Y \in \mathcal{D}^1(E)$,

$$R(\underline{h}X,\underline{h}Y) = \underline{v}([\underline{h}X,\underline{h}Y])$$

and

$$\begin{array}{|l} R(\underline{h}X,\underline{v}Y) = 0 = R(\underline{v}X,\underline{h}Y) \\ \\ R(\underline{v}X,\underline{v}Y) = 0. \end{array}$$

Therefore

$$R(X,Y) = R(\underline{h}X + \underline{v}X,\underline{h}Y + \underline{v}Y)$$

$$= R(\underline{h}X,\underline{h}Y) + R(\underline{h}X,\underline{v}Y) + R(\underline{v}X,\underline{h}Y) + R(\underline{v}X,\underline{v}Y)$$

$$= R(\underline{h}X,\underline{h}Y)$$

$$= \underline{v}([\underline{h}X,\underline{h}Y]).$$

20.2 LEMMA H is integrable (or still, involutive (cf. 15.18)) iff $R = 0$.

PROOF Suppose that $R = 0$ -- then $\forall$ $X,Y \in \mathcal{D}^1(E)$,

$$[\underline{h}X,\underline{h}Y] = \underline{h}[\underline{h}X,Y] + \underline{h}[X,\underline{h}Y] - \underline{h}[X,Y]$$

$$= \underline{h}([\underline{h}X,Y] + [X,\underline{h}Y] - [X,Y])$$

$$\in \sec H.$$

Therefore H is involutive (cf. 15.19). Conversely,

$$R(X,Y) = \underline{v}([\underline{h}X,\underline{h}Y])$$

$$= 0$$

if H is involutive.

20.3 RAPPEL Because $\pi:E \to M$ is a fibration, hence a submersion, each point in E admits a neighborhood U on which $\exists$ local coordinates

$$\{x^1,\ldots,x^n,y^1,\ldots,y^k\}$$

such that

$$(\pi|U)(x^i,y^\alpha) = (x^i).$$

Denote by X^h the horizontal lift of an $X \in \mathcal{D}^1(M)$, thus

$$X^h|_e = (T_e\pi|H_e)^{-1}X|_{\pi(e)}.$$

[Note: Bear in mind that

$$T\pi|H:H \to TM$$

is a fiberwise isomorphism.]

The distribution H is locally spanned by the vector fields

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - A_i^\alpha \frac{\partial}{\partial y^\alpha} \qquad (1 \le i \le n),$$

where $A_i^\alpha \in C^\infty(U)$, i.e.,

$$H_e = \mathrm{span}\left\{\left(\frac{\partial}{\partial x^1}\right)^h\Big|_e,\ldots,\left(\frac{\partial}{\partial x^n}\right)^h\Big|_e\right\} \quad (e \in U).$$

N.B. The set

$$\left\{\left(\frac{\partial}{\partial x^i}\right)^h, \frac{\partial}{\partial y^\alpha}\right\}$$

is a basis for $\mathcal{D}^1(U)$.

20.4 <u>REMARK</u> The A_i^α are called the <u>connection components</u> of the Ehresmann connection H. E.g.: Take $E = TM$ and let $\Gamma \in SO(TM)$ -- then as we have seen in §5, one may attach to Γ an Ehresmann connection H, where

$$A_i^j = -\frac{1}{2}\frac{\partial C^j}{\partial v^i}$$

if

$$\Gamma = v^i \frac{\partial}{\partial q^i} + C^i \frac{\partial}{\partial v^i}.$$

Put

$$\omega^\alpha = A_i^\alpha dx^i + dy^\alpha \quad (1 \leq \alpha \leq k).$$

20.5 <u>LEMMA</u> The 1-forms $\omega^1,\ldots,\omega^k$ on U are linearly independent and

$$H_e = \text{Ker } \omega^1|_e \cap \ldots \cap \text{Ker } \omega^k|_e \quad (e \in U).$$

[Note: This is 15.23 in the present setting (the dimension of E is $n + k$ and the fiber dimension of H is n, so the "$n - k$" there is $n + k - n = k$ here.]

<u>N.B.</u> Denote the velocity coordinates by v^i $(i = 1,\ldots,n)$ and u^α $(\alpha = 1,\ldots,k)$. Put

$$\Phi^\alpha = A_i^\alpha v^i + u^\alpha \quad (\text{a.k.a. } \hat{\omega}^\alpha).$$

Then the Φ^α combine to give a map

$$\Phi: TE \rightarrow \underline{R}$$

and locally,

$$H = \Phi^{-1}(0).$$

[Note: To be completely precise, $H|U$ is a vector subbundle of TU ($\equiv TE|U$) and what we are saying is that

$$H|U = \Phi^{-1}(0).$$

Also, in the definition of Φ^α, there is an abuse of notation in that

$$A_i^\alpha \circ \pi_E$$

has been abbreviated to A_i^α.]

Write

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = R_{ij}^\alpha \frac{\partial}{\partial y^\alpha} .$$

20.6 <u>LEMMA</u> We have

$$R_{ij}^\alpha = \frac{\partial A_i^\alpha}{\partial x^j} - \frac{\partial A_j^\alpha}{\partial x^i} + A_i^\beta \frac{\partial A_j^\alpha}{\partial y^\beta} - A_j^\beta \frac{\partial A_i^\alpha}{\partial y^\beta} .$$

Fix a nondegenerate lagrangian L (per TE, not TM). Working locally, define a vector field $X_\alpha \in \mathcal{D}^1(TU)$ by the requirement that

$$\iota_{X_\alpha} \omega_L = \pi_U^* \omega^\alpha \quad (\alpha = 1,\ldots,k).$$

20.7 <u>LEMMA</u> $\exists$ one and only one distribution Σ_L on TE which is locally generated by the X_α.

Since H is a vector subbundle of TE, it can play the role of a constraint (but H is not necessarily the zero set of a C^∞ function). This said, let us term the pair (L,H) <u>regular</u> if locally,

$$(L,\{\Phi^1,\ldots,\Phi^k\})$$

is regular, i.e., if the matrix

$$[X_\alpha \Phi^\beta]$$

is nonsingular.

20.8 <u>LEMMA</u> Suppose that (L,H) is regular -- then $\forall\, x \in H$,

$$T_x H \cap \Sigma_L|_x = 0.$$

<u>PROOF</u> Let $X_x \in T_x H \cap \Sigma_L|_x$ -- then

$$X_x = \sum_\alpha \lambda^\alpha X_\alpha|_x \qquad (\lambda^\alpha \in \underline{R})$$

=>

$$\sum_\alpha \lambda^\alpha (X_\alpha \Phi^\beta)\Big|_x = 0 \qquad (\beta = 1,\ldots,k)$$

=>

$$\lambda^1 = 0,\ldots,\lambda^k = 0$$

=>

$$X_x = 0.$$

Put

$$\Sigma_{(L,H)} = \Sigma_L | H.$$

Then from the above,

$$TTE|H = TH \oplus \Sigma_{(L,H)},$$

so there are projections P and Q given pointwise by

$$\begin{bmatrix} P_x : T_x TE \to T_x H \\ \\ Q_x : T_x TE \to \Sigma_{(L,H)}|x \end{bmatrix} \quad (x \in H).$$

The fundamental stipulation is now:

$$\Gamma_{(L,H)} \equiv P(\Gamma_L | H)$$

represents the constrained dynamics.

[Note:

$$\Gamma_L | H \in \sec\ (TTE|H)$$

=>

$$P(\Gamma_L | H) \in \sec\ TH.$$

I.e.:

$$P(\Gamma_L | H) \in \mathcal{D}^1(H).]$$

20.9 <u>REMARK</u> Working locally, define the Lagrange multiplier $\underline{\lambda}_0$ in the evident manner and form

$$\Gamma_{\underline{\lambda}_0} = \Gamma_L | TU + \lambda^\alpha X_\alpha .$$

Explicating the relation

$$\Gamma_{(L,H)} = \Gamma_L | H - Q(\Gamma_L | H)$$

then gives

$$\Gamma_{\underline{\lambda}_0} | (H|U) = \Gamma_{(L,H)} | (H|U) .$$

Furthermore, along an integral curve γ of $\Gamma_{\underline{\lambda}_0}$, we have

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial v^i}\right) - \dfrac{\partial L}{\partial x^i} = \sum\limits_{\alpha=1}^{k} \lambda_0^\alpha \dfrac{\partial \Phi^\alpha}{\partial v^i} \\[2em] \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial u^\alpha}\right) - \dfrac{\partial L}{\partial y^\alpha} = \sum\limits_{\beta=1}^{k} \lambda_0^\beta \dfrac{\partial \Phi^\beta}{\partial u^\alpha} \end{cases}$$

or still,

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial v^i}\right) - \dfrac{\partial L}{\partial x^i} = \sum\limits_{\alpha=1}^{k} \lambda_0^\alpha A_i^\alpha \\[2em] \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial u^\alpha}\right) - \dfrac{\partial L}{\partial y^\alpha} = \lambda_0^\alpha . \end{cases}$$

To reflect the presence of the connection, call L <u>H-invariant</u> if $\forall\, x \in M$ & $\forall\, X_x \in T_x M$,

$$L(e_1, (X_x)^h\big|_{e_1}) = L(e_2, (X_x)^h\big|_{e_2}),$$

where

$$\pi(e_1) = x = \pi(e_2).$$

[Note: If L is H-invariant, then

$$L(x^i, y^\alpha, v^i, -A_i^\alpha v^i)$$

is independent of y^α. Therefore

$$\frac{\partial L}{\partial y^\alpha} = \frac{\partial L}{\partial u^\beta} v^i \frac{\partial A_i^\beta}{\partial y^\alpha} .]$$

20.10 EXAMPLE Take $E = \underline{R}^2 \times \underline{S}^1$, $M = \underline{R}^2$, and let

$$\pi(x^1, x^2, \theta) = (x^1, x^2) \qquad (\theta = y^1).$$

Put

$$L = \frac{1}{2}((v^1)^2 + (v^2)^2) + \frac{1}{2}(v^2) \qquad (v = u^1).$$

Define the Ehresmann connection H by

$$H_{(x^1, x^2, \theta)} = \text{span}\{\frac{\partial}{\partial x^1} - \sin\theta \frac{\partial}{\partial \theta}, \frac{\partial}{\partial x^2} + \cos\theta \frac{\partial}{\partial \theta}\}.$$

Then L is not H-invariant.

If L is H-invariant, then L induces a lagrangian $\bar{L} \in C^\infty(TM)$ via the prescription

$$\bar{L}(x, X_x) = L(e, (X_x)^h|_e) \quad (\pi(e) = x).$$

[Note: Locally,

$$\bar{L}(x^i, v^i) = L(x^i, y^\alpha, v^i, -A_i^\alpha v^i).]$$

20.11 LEMMA Suppose that L is H-invariant -- then (L,H) is regular iff $\bar{L}$ is nondegenerate.

PROOF Let $W = W(L)^{-1}$ (recall that by assumption, L is nondegenerate) -- then

$$W = \begin{bmatrix} W^{ij} & W^{i\beta} \\ & \\ W^{\alpha j} & W^{\alpha\beta} \end{bmatrix}$$

and we have

$$X_\alpha \Phi^\beta = W^{ij} A_i^\alpha A_j^\beta + W^{i\beta} A_i^\alpha + W^{\alpha j} A_j^\beta + W^{\alpha\beta}$$

or still,

$$[X_\alpha \Phi^\beta] = \begin{bmatrix} A & 0 \\ & \\ 0 & I_{k \times k} \end{bmatrix} W \begin{bmatrix} A & 0 \\ & \\ 0 & I_{k \times k} \end{bmatrix}^T,$$

where

$$A_{\alpha i} = A_i^\alpha.$$

On the other hand,

$$\frac{\partial^2 \bar{L}}{\partial v^i \partial v^j}$$

$$= \frac{\partial^2 L}{\partial v^i \partial v^j} - A_i^\alpha \frac{\partial^2 L}{\partial u^\alpha \partial v^j} - A_j^\beta \frac{\partial^2 L}{\partial u^\beta \partial v^i} + A_i^\alpha A_j^\beta \frac{\partial^2 L}{\partial u^\alpha \partial u^\beta}$$

or still,

$$W(\bar{L}) = \begin{bmatrix} I_{n \times n} & 0 \\ & \\ 0 & -A^T \end{bmatrix} W^{-1} \begin{bmatrix} I_{n \times n} & 0 \\ & \\ 0 & -A^T \end{bmatrix}^T,$$

$W(L)$ being W^{-1}. Combining these facts with some elementary matrix theory then leads to the desired conclusion.

Assume henceforth that L is H-invariant and (L,H) is regular. Let

$$\gamma(t) = (x^i(t), y^\alpha(t), v^i(t), u^\alpha(t))$$

be an integral curve for

$$\Gamma_{\underline{\lambda}_0} | (H|U) = \Gamma_{(L,H)} | (H|U).$$

Pass to

$$\bar{L}(x^i(t), v^i(t))$$

and consider

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial v^i}\right) - \frac{\partial \bar{L}}{\partial x^i}$$

taken along

$$\bar{\gamma}(t) = (x^i(t), v^i(t)).$$

1. $$\frac{\partial \bar{L}}{\partial x^i} = \frac{\partial L}{\partial x^i} + \frac{\partial L}{\partial u^\alpha} \frac{\partial}{\partial x^i}(-A_j^\alpha v^j)$$

$$= \frac{\partial L}{\partial x^i} + \frac{\partial L}{\partial u^\alpha}\left(-\frac{\partial A_j^\alpha}{\partial x^i} v^j\right).$$

2. $$\frac{\partial \bar{L}}{\partial v^i} = \frac{\partial L}{\partial v^i} + \frac{\partial L}{\partial u^\alpha} \frac{\partial}{\partial v^i} (- A_j^\alpha v^j)$$

$$= \frac{\partial L}{\partial v^i} + \frac{\partial L}{\partial u^\alpha} (- A_i^\alpha).$$

3. $$\frac{d}{dt} (\frac{\partial \bar{L}}{\partial v^i}) = \frac{d}{dt} (\frac{\partial L}{\partial v^i}) + \frac{d}{dt} (\frac{\partial L}{\partial u^\alpha} (- A_i^\alpha))$$

=>

$$\frac{d}{dt} (\frac{\partial \bar{L}}{\partial v^i}) - \frac{\partial \bar{L}}{\partial x^i}$$

$$= \frac{d}{dt} (\frac{\partial L}{\partial v^i}) - \frac{\partial L}{\partial x^i}$$

$$+ \frac{d}{dt} (\frac{\partial L}{\partial u^\alpha}) (- A_i^\alpha) + \frac{\partial L}{\partial u^\alpha} \frac{d}{dt} (- A_i^\alpha) + \frac{\partial L}{\partial u^\alpha} v^j \frac{\partial A_j^\alpha}{\partial x^i} .$$

4. $$\frac{d}{dt} (\frac{\partial L}{\partial v^i}) - \frac{\partial L}{\partial x^i} = \sum_{\alpha=1}^{k} \lambda_0^\alpha A_i^\alpha.$$

5. $$\frac{d}{dt} (\frac{\partial L}{\partial u^\alpha}) - \frac{\partial L}{\partial y^\alpha} = \lambda_0^\alpha.$$

6. $$\frac{\partial L}{\partial y^\alpha} = \frac{\partial L}{\partial u^\beta} v^j \frac{\partial A_j^\beta}{\partial y^\alpha}$$

=>

$$\frac{d}{dt} (\frac{\partial \bar{L}}{\partial v^i}) - \frac{\partial \bar{L}}{\partial x^i}$$

$$= \sum_{\alpha=1}^{k} \lambda_0^\alpha A_i^\alpha - \sum_{\alpha=1}^{k} \lambda_0^\alpha A_i^\alpha + \frac{\partial L}{\partial y^\alpha} (- A_i^\alpha)$$

$$+ \frac{\partial L}{\partial u^\alpha} \frac{d}{dt} (- A_i^\alpha) + \frac{\partial L}{\partial u^\alpha} v^j \frac{\partial A_j^\alpha}{\partial x^i}$$

$$= \frac{\partial L}{\partial u^\beta} v^j \frac{\partial A_j^\beta}{\partial y^\alpha} (- A_i^\alpha)$$

$$+ \frac{\partial L}{\partial u^\alpha} \frac{d}{dt} (- A_i^\alpha) + \frac{\partial L}{\partial u^\alpha} v^j \frac{\partial A_j^\alpha}{\partial x^i} .$$

7. $\frac{d}{dt} x^j(t) = v^j(t) \equiv v^j.$

8. $\frac{d}{dt} y^\beta(t) = u^\beta(t) \equiv - v^j A_j^\beta.$

9. $\frac{\partial L}{\partial u^\alpha} \frac{d}{dt} (- A_i^\alpha)$

$$= \frac{\partial L}{\partial u^\alpha} \frac{d}{dt} (- A_i^\alpha(x^j(t), y^\beta(t)))$$

$$= \frac{\partial L}{\partial u^\alpha} (- \frac{\partial A_i^\alpha}{\partial x^j} \frac{d}{dt} x^j(t) - \frac{\partial A_i^\alpha}{\partial y^\beta} \frac{d}{dt} y^\beta(t))$$

$$= \frac{\partial L}{\partial u^\alpha} (v^j (- \frac{\partial A_i^\alpha}{\partial x^j}) + v^j A_j^\beta \frac{\partial A_i^\alpha}{\partial y^\beta})$$

=>

$$\frac{d}{dt} (\frac{\partial \bar{L}}{\partial v^i}) - \frac{\partial \bar{L}}{\partial x^i}$$

$$= \frac{\partial L}{\partial u^\alpha} (- A_i^\beta) \frac{\partial A_j^\alpha}{\partial y^\beta}$$

$$+ \frac{\partial L}{\partial u^\alpha} v^j \left(- \frac{\partial A_i^\alpha}{\partial x^j} + A_j^\beta \frac{\partial A_i^\alpha}{\partial y^\beta}\right)$$

$$+ \frac{\partial L}{\partial u^\alpha} v^j \frac{\partial A_j^\alpha}{\partial x^i}$$

$$= \frac{\partial L}{\partial u^\alpha} v^j \left(\frac{\partial A_j^\alpha}{\partial x^i} - \frac{\partial A_i^\alpha}{\partial x^j} + A_j^\beta \frac{\partial A_i^\alpha}{\partial y^\beta} - A_i^\beta \frac{\partial A_j^\alpha}{\partial y^\beta}\right)$$

$$= - \frac{\partial L}{\partial u^\alpha} v^j R_{ij}^\alpha \quad \text{(cf. 20.6)}.$$

This sets the stage for reduction theory which, however, we are not going to delve into. Let's just say: Under certain circumstances, the vector field $\Gamma_{(L,H)}$ is $T\pi$-projectable onto a second order vector field $\bar{\Gamma}_{(L,H)} \in \mathcal{D}^1(TM)$ such that

$$\iota_{\bar{\Gamma}_{(L,H)}} \omega_{\bar{L}} = - dE_{\bar{L}} + \Pi_{(L,H)},$$

where $\Pi_{(L,H)}$ is a horizontal 1-form on TM given locally by

$$- \frac{\partial L}{\partial u^\alpha} v^j R_{ij}^\alpha dq^i,$$

a potentially ambiguous expression.

20.12 <u>REMARK</u> It can be shown that

$$\iota_{\bar{\Gamma}_{(L,H)}} \Pi_{(L,H)} = 0.$$

Consequently,

$$\bar{\Gamma}_{(L,H)} E_{\bar{L}}$$

$$= \langle \bar{\Gamma}_{(L,H)}, dE_{\bar{L}} \rangle$$

$$= \langle \bar{\Gamma}_{(L,H)}, -\iota_{\bar{\Gamma}_{(L,H)}} \omega_{\bar{L}} + \Pi_{(L,H)} \rangle$$

$$= -\omega_{\bar{L}}(\bar{\Gamma}_{(L,H)}, \bar{\Gamma}_{(L,H)}) + \iota_{\bar{\Gamma}_{(L,H)}} \Pi(L,H)$$

$$= 0.$$

So $E_{\bar{L}}$ is a first integral for $\bar{\Gamma}_{(L,H)}$ (cf. 8.10).

<u>N.B.</u> In the language of §10, the triple $\mathcal{M} = (M, \bar{L}, \Pi_{(L,H)})$ is a nondegenerate mechanical system, $\Pi_{(L,H)}$ being the (external) force field.

20.13 <u>REMARK</u> If $\Pi_{(L,H)}$ is not identically zero, then $\Pi_{(L,H)}$ is not closed (in which case our mechanical system is not conservative). To see this, let

$$\Gamma = \bar{\Gamma}_{(L,H)}$$

and write

$$\Pi_{(L,H)} = a_i dq^i \quad (a_i = -\frac{\partial L}{\partial u^\alpha} v^j R^\alpha_{ij}).$$

Then

$$d\Pi_{(L,H)} = 0$$

=>

$$L_\Gamma \Pi_{(L,H)} = (\iota_\Gamma \circ d + d \circ \iota_\Gamma) \Pi_{(L,H)}$$

$$= 0$$

=>

$$0 = (L_\Gamma a_i)dq^i + a_i(L_\Gamma dq^i)$$

$$= (L_\Gamma a_i)dq^i + a_i(dL_\Gamma q^i)$$

$$= (L_\Gamma a_i)dq^i + a_i dv^i \quad (\Gamma \in SO(TM))$$

=>

$$a_i \equiv 0 \Rightarrow \Pi_{(L,H)} \equiv 0.$$

[Note: If H is integrable, then $\Pi_{(L,H)}$ is identically zero (cf. 20.2) (but the converse is false (cf. 20.15)).]

20.14 <u>EXAMPLE</u> Take $E = \underline{R}^3$, $M = \underline{R}^2$ and let

$$\pi(x^1,x^2,y^1) = (x^1,x^2).$$

Then

$$H|_{(x^1,x^2,y^1)} = \text{span}\{\frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^2}\}$$

is an Ehresmann connection. Here

$$\omega^1 = - x^2 dx^1 + dy^1$$

=>

$$A_1^1 = - x^2, \ A_2^1 = 0$$

=>

$$\left[\begin{array}{l} R^1_{11} = 0,\ R^1_{21} = 1 \\ \\ R^1_{12} = -1,\ R^1_{22} = 0 \end{array}\right. \qquad \text{(cf. 20.6)}.$$

Let

$$L = \frac{1}{2}\,((v^1)^2 + (v^2)^2 + (u^1)^2).$$

Then L is H-invariant and (L,H) is regular. To compute $\Pi_{(L,H)}$, note that

$$\left[\begin{array}{l} \frac{\partial L}{\partial u^1}\,(v^1R^1_{11} + v^2R^1_{12})dq^1 = -\,u^1v^2dq^1 = -\,q^2v^1v^2dq^1 \\ \\ \frac{\partial L}{\partial u^2}\,(v^1R^1_{21} + v^2R^1_{22})dq^2 = u^1v^1dq^2 = q^2(v^1)^2dq^2 \end{array}\right.$$

=>

$$\Pi_{(L,H)} = q^2v^1v^2dq^1 - q^2(v^1)^2dq^2.$$

In addition,

$$\bar{L} = \frac{1}{2}\,(((q^2)^2 + 1)(v^1)^2 + (v^2)^2).$$

But, as has been seen in 16.5,

$$\Gamma_{\lambda_0} = v^1\,\frac{\partial}{\partial q^1} + v^2\,\frac{\partial}{\partial q^2} + v^3\,\frac{\partial}{\partial q^3}$$

$$+\,\frac{v^1v^2}{(q^2)^2 + 1}\,(-\,q^2\,\frac{\partial}{\partial v^1} + \frac{\partial}{\partial v^3}),$$

so

$$\Gamma_{(L,H)} = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + q^2 v^1 \frac{\partial}{\partial q^3}$$

$$- \frac{v^1 v^2}{(q^2)^2 + 1} \; q^2 \frac{\partial}{\partial v^1} ,$$

from which

$$\bar{\Gamma}_{(L,H)} = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} - \frac{v^1 v^2}{(q^2)^2 + 1} q^2 \frac{\partial}{\partial v^1} .$$

To check that

$$\iota_{\bar{\Gamma}(L,H)} \omega_{\bar{L}} = - dE_{\bar{L}} + \Pi_{(L,H)} ,$$

it suffices to check that

$$L_{\bar{\Gamma}(L,H)} \theta_{\bar{L}} = d\bar{L} + \Pi_{(L,H)} \qquad \text{(cf. 8.14)}.$$

To this end, write

$$\theta_{\bar{L}} = \frac{\partial \bar{L}}{\partial v^1} dq^1 + \frac{\partial \bar{L}}{\partial v^2} dq^2$$

$$= ((q^2)^2 + 1) v^1 dq^1 + v^2 dq^2 .$$

Then

$$L_{\bar{\Gamma}(L,H)} \theta_{\bar{L}}$$

$$= L_{\bar{\Gamma}(L,H)} (((q^2)^2 + 1) v^1) \wedge dq^1$$

$$+ ((q^2)^2 + 1)v^1 \wedge L_{\bar{\Gamma}(L,H)} dq^1$$

$$+ L_{\bar{\Gamma}(L,H)} v^2 \wedge dq^2 + v^2 \wedge L_{\bar{\Gamma}(L,H)} dq^2$$

$$= q^2 v^1 v^2 dq^1 + ((q^2)^2 + 1)v^1 dv^1 + v^2 dv^2.$$

On the other hand,

$$d\bar{L} + \Pi_{(L,H)}$$

$$= q^2 (v^1)^2 dq^2 + ((q^2)^2 + 1)v^1 dv^1 + v^2 dv^2$$

$$+ q^2 v^1 v^2 dq^1 - q^2 (v^1)^2 dq^2$$

$$= q^2 v^1 v^2 dq^1 + ((q^2)^2 + 1)v^1 dv^1 + v^2 dv^2$$

$$= L_{\bar{\Gamma}(L,H)} \theta_{\bar{L}}.$$

[Note: $E_{\bar{L}}$ is a first integral for $\bar{\Gamma}(L,H)$ (cf. 20.12). Proof:

$$\bar{\Gamma}_{(L,H)} (\frac{1}{2} (((q^2)^2 + 1) (v^1)^2 + (v^2)^2))$$

$$= v^2 q^2 (v^1)^2 - \frac{v^1 v^2}{((q^2)^2 + 1)} q^2 ((q^2)^2 + 1) v^1$$

$$= q^2 (v^1)^2 v^2 - q^2 (v^1)^2 v^2$$

$$= 0.$$

Another first integral for $\bar{\Gamma}_{(L,H)}$ is the function

$$((q^2)^2 + 1)^{1/2} v^1.$$

Proof:

$$\bar{\Gamma}_{(L,H)}(((q^2)^2 + 1)^{1/2} v^1)$$

$$= v^1 v^2 \frac{q^2}{((q^2)^2 + 1)^{1/2}} - \frac{v^1 v^2}{((q^2)^2 + 1)} q^2 ((q^2)^2 + 1)^{1/2}$$

$$= 0.]$$

20.15 <u>EXAMPLE</u> Take $E = \underline{S}^1 \times \underline{S}^1 \times \underline{R}^2$, $M = \underline{S}^1 \times \underline{S}^1$ and let

$$\pi(\theta^1,\theta^2,y^1,y^2) = (\theta^1,\theta^2).$$

Then the distribution Σ figuring in 16.12 is an Ehresmann connection, call it H:

$$H\Big|_{(\theta^1,\theta^2,y^1,y^2)}$$

$$= \text{span}\{R \cos \theta^1 \frac{\partial}{\partial y^1} + R \sin \theta^1 \frac{\partial}{\partial y^2} + \frac{\partial}{\partial \theta^2}, \frac{\partial}{\partial \theta^1}\}.$$

Here

$$\begin{cases} \omega^1 = -(R \cos \theta^1) d\theta^2 + dy^1 \\ \omega^2 = -(R \sin \theta^1) d\theta^2 + dy^2 \end{cases}$$

$\Rightarrow$

$$\begin{cases} A^1_1 = 0,\ A^1_2 = -R \cos \theta^1 \\ A^2_1 = 0,\ A^2_2 = -R \sin \theta^1 \end{cases}$$

=>

$$\left[\begin{array}{l} R^1_{11} = 0,\ R^1_{12} = -\ R \sin \theta^1,\ R^1_{21} = R \sin \theta^1,\ R^1_{22} = 0 \\ \\ R^2_{11} = 0,\ R^2_{12} = R \cos \theta^1,\ R^2_{21} = -\ R \cos \theta^1,\ R^2_{22} = 0 \end{array}\right. \qquad \text{(cf. 20.6).}$$

Let

$$L = \frac{1}{2} I_1 (v^1)^2 + \frac{1}{2} I_2 (v^2)^2 + \frac{m}{2} ((u^1)^2 + (u^2)^2),$$

where I_1, I_2, and m are positive constants -- then L is H-invariant and (L,H) is regular. And, from the definitions,

$$\bar{L} = \frac{1}{2} (I_1 (v^1)^2 + (mR^2 + I_2)(v^2)^2).$$

However, in this situation,

$$\Pi_{(L,H)} = 0.$$

E.g.: The coefficient of dq^1 is the negative of

$$\frac{\partial L}{\partial u^1} (v^1 R^1_{11} + v^2 R^1_{12}) + \frac{\partial L}{\partial u^2} (v^1 R^2_{11} + v^2 R^2_{12})$$

$$= mu^1 (v^2 (-\ R \sin \theta^1)) + mu^2 (v^2 (R \cos \theta^1))$$

$$= m(R \cos \theta^1) v^2 (v^2 (-\ R \sin \theta^1))$$

$$+\ m(R \sin \theta^1) v^2 (v^2 (R \cos \theta^1))$$

$$= 0.$$

[Note: H is not involutive, hence is not integrable (cf. 15.18).]

A <u>Chaplygin system</u> has two ingredients.

- A principal bundle $\pi:E \to M$ with structure group G and a principal connection H.
- A nondegenerate lagrangian $L \in C^{\infty}(TE)$ that is G-invariant for the lifted action of G on TE and for which (L,H) is regular.

It is then a fundamental point that this data realizes all the assumptions of the preceding setup.

[Note: The dynamics on H can be reconstructed from the dynamics on TM via the horizontal lift operation.]

§21. DEPENDENCE ON TIME

Let M be a connected C^∞ manifold of dimension n. Put

$$\begin{cases} J^0M = \underline{R} \times M \\ J^1M = \underline{R} \times TM \\ J^2M = \underline{R} \times T^2M. \end{cases}$$

Then J^1M is called the evolution space of a time-dependent (a.k.a. non-autonomous) mechanical system whose configuration space is M.

21.1 EXAMPLE Consider the motion of a plane pendulum whose length $\ell(t) > 0$ is a function of time -- then

$$M = \underline{S}^1 \Rightarrow J^1M = \underline{R} \times (\underline{S}^1 \times \underline{R})$$

and its motion is governed by the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta - \frac{2}{\ell}\frac{d\ell}{dt}\frac{d\theta}{dt},$$

where $\theta = \theta(t)$ is the angle made by the pendulum with the vertical and g is the gravitational acceleration (cf. 21.19).

Local coordinates in J^1M are (t,q^i,v^i) and there is a canonical inclusion

$$J^1M \to TJ^0M,$$

viz.

$$(t,q^i,v^i) \to (t,q^i,1,v^i).$$

Local coordinates in J^2M are (t,q^i,v^i,a^i) (cf. 11.6) and there is a canonical inclusion

$$J^2M \to \underline{R} \times TTM,$$

viz.

$$(t,q^i,v^i,a^i) \to (t,q^i,v^i,v^i,a^i).$$

Since $\underline{R} \times TTM$ can be embedded in TJ^1M, it makes sense to write

$$J^2M \subset TJ^1M.$$

This being the case, let $\Gamma \in \mathcal{D}^1(J^1M)$ — then Γ is said to be <u>second order</u> provided $\Gamma J^1M \subset J^2M$.

21.2 <u>LEMMA</u> Let $\Gamma \in \mathcal{D}^1(J^1M)$ — then Γ is second order iff locally,

$$\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial q^i} + C^i \frac{\partial}{\partial v^i},$$

where

$$C^i = C^i(t,q^i,v^i).$$

The vertical morphism

$$S:\mathcal{D}^1(TM) \to \mathcal{D}^1(TM)$$

and the dilation vector field

$$\Delta \in \mathcal{D}^1(TM)$$

can be regarded as living on J^1M. Agreeing to denote these extensions by the same symbols, define

$$S_{dt} \in \mathcal{D}^1_1(J^1M)$$

by

$$S_{dt} = S - \Delta \otimes dt.$$

Then locally,

$$S_{dt} = \frac{\partial}{\partial v^i} \otimes (dq^i - v^i dt).$$

<u>N.B.</u> Viewing S_{dt} as an element of

$$\mathrm{Hom}_{C^\infty(J^1M)}(\mathcal{D}^1(J^1M), \mathcal{D}^1(J^1M)),$$

we have

$$S_{dt}(\frac{\partial}{\partial t}) = - v^i \frac{\partial}{\partial v^i},\ S_{dt}(\frac{\partial}{\partial q^i}) = \frac{\partial}{\partial v^i},\ S_{dt}(\frac{\partial}{\partial v^i}) = 0.$$

The triple (J^1M, J^0M, π^{10}) is a fibered manifold, from which

$$V^{10}J^1M \subset TJ^1M \quad (\text{cf. §11}).$$

21.3 <u>LEMMA</u> $S_{dt}^2 = 0$, hence

$$\mathrm{Im}\ S_{dt} \subset \mathrm{Ker}\ S_{dt}.$$

Moreover,

$$\mathrm{Im}\ S_{dt} = \sec V^{10}J^1M \equiv \mathcal{V}^{10}(J^1M).$$

[Note: The containment

$$\text{Im } S_{dt} \subset \text{Ker } S_{dt}$$

is proper.]

21.4 <u>REMARK</u> It can be shown that $\forall\ X,Y \in \mathcal{D}^1(J^1M)$,

$$[S_{dt}X, S_{dt}Y] - S_{dt}[S_{dt}X, Y] - S_{dt}[X, S_{dt}Y]$$

$$= (\iota_X dt) S_{dt}Y - (\iota_Y dt) S_{dt}X \quad \text{(cf. 5.9)}.$$

21.5 <u>LEMMA</u> Let $\Gamma \in \mathcal{D}^1(J^1M)$ -- then Γ is second order iff $S\Gamma = \Delta$ and $S_{dt}\Gamma = 0$.

<u>PROOF</u> The necessity is obvious. To see the sufficiency, work locally and write

$$\Gamma = \tau \frac{\partial}{\partial t} + A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial v^i}.$$

Then

$$S\Gamma = \Delta \Rightarrow A^i = v^i \quad (1 \le i \le n)$$

=>

$$0 = S_{dt}\Gamma = (1 - \tau) v^i \frac{\partial}{\partial v^i}$$

=>

$$(1 - \tau) v^i = 0 \quad (1 \le i \le n)$$

$$\Rightarrow \tau = 1.$$

21.6 <u>LEMMA</u> Let $\Gamma \in \mathcal{D}^1(J^1M)$ -- then Γ is second order iff $dt(\Gamma) = 1$ and $S_{dt}\Gamma = 0$.

21.7 <u>LEMMA</u> Suppose that $\Gamma \in \mathcal{D}^1(J^1M)$ is second order -- then $\forall$ π^{10}-vertical X,

$$S_{dt}([X,\Gamma]) = X.$$

An element $L \in C^\infty(J^1M)$ is, by definition, a (time-dependent) <u>lagrangian</u>. This said, put

$$\begin{cases} \Theta_L = S_{dt}^*(dL) + Ldt \\ \Omega_L = d\Theta_L. \end{cases}$$

21.8 <u>LEMMA</u> Locally,

$$\Theta_L = \frac{\partial L}{\partial v^i}(dq^i - v^i dt) + Ldt.$$

[One has only to note that

$$S_{dt}^*(dt) = 0,\ S_{dt}^*(dq^i) = 0,\ S_{dt}^*(dv^i) = dq^i - v^i dt.]$$

<u>N.B.</u> On general grounds (cf. 13.4), the horizontal 1-forms $\alpha \in \Lambda^1 J^1 M$ per the fibration $\pi^{10}: J^1M \to J^0M$ are characterized by the property that they annihilate the sections of $V^{10}J^1M$. Locally, these are the $\alpha \in \Lambda^1 J^1 M$ that can be written in the form

$$\alpha = adt + a_i dq^i,$$

where

$$\begin{cases} a = a(t,q^1,\ldots,q^n,v^1,\ldots,v^n) \\ a_i = a_i(t,q^1,\ldots,q^n,v^1,\ldots,v^n). \end{cases}$$

In particular: Θ_L is π^{10}-horizontal.

21.9 LEMMA Locally,

$$\Omega_L = \frac{\partial^2 L}{\partial q^i \partial v^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dv^i \wedge dq^j$$

$$+ \frac{\partial^2 L}{\partial v^i \partial v^j} v^i dt \wedge dv^j$$

$$+ \left(\frac{\partial^2 L}{\partial v^i \partial t} + \frac{\partial^2 L}{\partial q^i \partial v^j} v^j - \frac{\partial L}{\partial q^i}\right) dt \wedge dq^i.$$

Therefore

$$dt \wedge \Omega_L^n = \pm n! \det \left[\frac{\partial^2 L}{\partial v^i \partial v^j} \right] dt \wedge dv^1 \wedge \ldots \wedge dv^n \wedge dq^1 \wedge \ldots \wedge dq^n.$$

Motivated by this, call L <u>nondegenerate</u> if $dt \wedge \Omega_L^n$ is a volume form; otherwise, call L <u>degenerate</u>.

21.10 LEMMA L is nondegenerate iff for all coordinate systems $\{t,q^1,\ldots,q^n,v^1,\ldots,v^n\}$,

$$\det \left[\frac{\partial^2 L}{\partial v^i \partial v^j} \right] \neq 0$$

everywhere (cf. 8.5).

21.11 EXAMPLE Take $M = \underline{R}$ and let

$$L = \frac{1}{2} v^2 - \frac{1}{2} \omega^2(t) q^2.$$

Then L is nondegenerate.

[Note: This lagrangian is that of the time dependent harmonic oscillator.]

21.12 EXAMPLE Take $M = \underline{R}^2$ and let

$$L = \frac{1}{2} (v^1 + tv^2)^2.$$

Then

$$\det \begin{bmatrix} \dfrac{\partial^2 L}{\partial v^i \partial v^j} \end{bmatrix} = \det \begin{bmatrix} 1 & t \\ t & t^2 \end{bmatrix} = 0,$$

so L is degenerate.

21.13 RAPPEL Suppose that N is a connected $(2n+1)$-dimensional manifold -- then a cosymplectic structure on N is a pair (η,Ω), where $\eta \in \Lambda^1 N$ is a closed 1-form on N and $\Omega \in \Lambda^2 N$ is a closed 2-form on N such that $\eta \wedge \Omega^n \neq 0$.

[Note: It follows that the rank of Ω is $2n$.]

Accordingly, a nondegenerate lagrangian L determines a cosymplectic structure (dt, Ω_L) on $J^1 M = \underline{R} \times TM$.

21.14 LEMMA Suppose that (η,Ω) is a cosymplectic structure on N -- then there exists a unique vector field $X_{\eta,\Omega} \in \mathcal{D}^1(N)$:

$$\begin{cases} \iota_{X_{\eta,\Omega}} \Omega = 0 \\ \iota_{X_{\eta,\Omega}} \eta = 1. \end{cases}$$

PROOF The arrow

$$b_{\eta,\Omega}:\mathcal{D}^1(N) \to \mathcal{D}_1(N)$$

that sends X to

$$\iota_X\Omega + \eta(X)\eta$$

is an isomorphism. Put

$$X_{\eta,\Omega} = (b_{\eta,\Omega})^{-1}(\eta),$$

thus

$$\iota_{X_{\eta,\Omega}}\Omega + \eta(X_{\eta,\Omega})\eta = \eta.$$

To check that $X_{\eta,\Omega}$ has the stated properties, observe that

$$\iota_{X_{\eta,\Omega}}\iota_{X_{\eta,\Omega}}\Omega + \eta(X_{\eta,\Omega})\eta(X_{\eta,\Omega}) = \eta(X_{\eta,\Omega}).$$

I.e.:

$$\eta(X_{\eta,\Omega})^2 = \eta(X_{\eta,\Omega})$$

$$\Rightarrow$$

$$\eta(X_{\eta,\Omega}) \equiv 0 \text{ or } \eta(X_{\eta,\Omega}) \equiv 1.$$

The first possibility would imply that

$$\iota_{X_{\eta,\Omega}} \Omega = \eta.$$

But then

$$\eta\wedge\Omega^n \neq 0$$

$$\Rightarrow$$

$$\iota_{X_{\eta,\Omega}} \Omega\wedge\Omega^n \neq 0.$$

On the other hand,

$$\Omega\wedge\Omega^n = 0$$

$$\Rightarrow$$

$$\iota_{X_{\eta,\Omega}} \Omega\wedge\Omega^n + \Omega\wedge\iota_{X_{\eta,\Omega}} \Omega^n = 0$$

$$\Rightarrow$$

$$\iota_{X_{\eta,\Omega}} \Omega\wedge\Omega^n + \iota_{X_{\eta,\Omega}} \Omega^n\wedge\Omega = 0$$

$$\Rightarrow$$

$$(n + 1)\,\iota_{X_{\eta,\Omega}} \Omega\wedge\Omega^n = 0,$$

a contradiction. Therefore

$$\eta(X_{\eta,\Omega}) = 1$$

$$\Rightarrow$$

$$\iota_{X_{\eta,\Omega}} \Omega + \eta = \eta$$

$$\Rightarrow$$

$$\iota_{X_{\eta,\Omega}} \Omega = 0.$$

[Note: $X_{\eta,\Omega}$ is called the <u>Reeb vector field</u> attached to (η,Ω).]

21.15 <u>EXAMPLE</u> Let Ω be the fundamental 2-form on T^*M. Form the product $\underline{R} \times T^*M$ and let $\pi^*:\underline{R} \times T^*M \to T^*M$ be the projection -- then the pair $(dt,\pi^*\Omega)$ is a cosymplectic structure on $\underline{R} \times T^*M$ and its Reeb vector field is $\frac{\partial}{\partial t}$.

Given a nondegenerate lagrangian L, set

$$\Gamma_L = X_{dt,\Omega_L}.$$

Then

$$\left[\begin{array}{l} \iota_{\Gamma_L}\Omega_L = 0 \\ \\ \iota_{\Gamma_L} dt = 1. \end{array}\right.$$

21.16 <u>REMARK</u> Suppose that $L:TM \to \underline{R}$ is a nondegenerate lagrangian. Define $\tilde{L}:J^1M \to \underline{R}$ by $\tilde{L} = L \circ \pi$, where $\pi:\underline{R} \times TM \to TM$ is the projection — then $\tilde{L}$ is nondegenerate and

$$\Omega_{\tilde{L}} = -\pi^*\omega_L + dt\wedge\pi^*(dE_L).$$

Furthermore,

$$\Gamma_{\tilde{L}} = \frac{\partial}{\partial t} + \Gamma_L.$$

[Note: Recall that $\iota_{\Gamma_L}\omega_L = -dE_L$ and $\Gamma_L E_L = 0$.]

21.17 <u>LEMMA</u> Γ_L is second order.

<u>PROOF</u> To apply 21.2, write

$$\Gamma_L = \tau \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + C^i \frac{\partial}{\partial v^i} .$$

Then

$$1 = \iota_{\Gamma_L} dt = \tau .$$

As for the X^i, use the fact that $\iota_{\Gamma_L} \Omega_L = 0$ and 21.9 to conclude:

$$X^i \frac{\partial^2 L}{\partial v^i \partial v^j} = v^i \frac{\partial^2 L}{\partial v^i \partial v^j} .$$

But L is nondegenerate, so

$$X^i = v^i .$$

Let

$$\gamma(s) = (t(s), q^1(s), \ldots, q^n(s), v^1(s), \ldots, v^n(s))$$

be an integral curve of Γ_L -- then

$$\frac{d}{ds} t(s) = 1.$$

Because of this, we can and will choose the evolution parameter s to be the "time" t.

[Note: Time reparametrization is thus a form of "gauge fixing".]

21.18 LEMMA If

$$\gamma(t) = (t, q^1(t), \ldots, q^n(t), v^1(t), \ldots, v^n(t))$$

is an integral curve of Γ_L, then

$$\frac{d}{dt} q^i(t) = v^i(t), \frac{d^2}{dt^2} q^i(t) = C^i$$

and along γ, the equations of Lagrange

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0 \qquad (i = 1, \ldots, n)$$

are in force.

[Manipulation of the relation $\iota_{\Gamma_L} \Omega_L = 0$ gives

$$\frac{\partial^2 L}{\partial t \partial v^i} + v^j \frac{\partial^2 L}{\partial v^i \partial q^j} + C^j \frac{\partial^2 L}{\partial v^i \partial v^j} - \frac{\partial L}{\partial q^i} = 0 \qquad (i = 1, \ldots, n).]$$

21.19 EXAMPLE Take $M = \underline{S}^1$ and consider the setup of 21.1. Let

$$L(t, \theta, v) = \frac{1}{2} m\ell^2 v^2 + mg\ell \cos \theta \quad (\theta = q).$$

Explicating the equations of Lagrange then leads to the differential equation stated there.

Given any $L \in C^\infty(J^1 M)$, its energy is the function

$$E_L = \Delta L - L.$$

N.B. We have

$$\Theta_L = S^*(dL) - E_L dt.$$

21.20 LEMMA Suppose that L is nondegenerate — then

$$\Gamma_L E_L = -\frac{\partial L}{\partial t}.$$

PROOF For

$$\begin{aligned}
\Gamma_L E_L &= \iota_{\Gamma_L} dE_L \\
&= -\iota_{\Gamma_L} d\iota_{\partial/\partial t}\Theta_L \\
&= -\iota_{\Gamma_L}(L_{\partial/\partial t} - \iota_{\partial/\partial t}d)\Theta_L \\
&= \iota_{\Gamma_L}\iota_{\partial/\partial t}d\Theta_L - \iota_{\Gamma_L}L_{\partial/\partial t}\Theta_L \\
&= \iota_{\Gamma_L}\iota_{\partial/\partial t}\Omega_L - \iota_{\Gamma_L}L_{\partial/\partial t}\Theta_L \\
&= -\iota_{\partial/\partial t}\iota_{\Gamma_L}\Omega_L - \iota_{\Gamma_L}L_{\partial/\partial t}\Theta_L \\
&= -\iota_{\Gamma_L}L_{\partial/\partial t}\Theta_L \\
&= \iota_{[\partial/\partial t,\Gamma_L]}\Theta_L - L_{\partial/\partial t}\iota_{\Gamma_L}\Theta_L \\
&= -L_{\partial/\partial t}\iota_{\Gamma_L}\Theta_L
\end{aligned}$$

$$= - L_{\partial/\partial t} \iota_{\Gamma_L} (S^*_{dt}(dL) + L dt)$$

$$= - L_{\partial/\partial t} \iota_{\Gamma_L} S^*_{dt}(dL) - L_{\partial/\partial t} L \iota_{\Gamma_L} dt$$

$$= - L_{\partial/\partial t} \iota_{S_{dt}\Gamma_L} dL - \frac{\partial L}{\partial t}$$

$$= - L_{\partial/\partial t} \iota_0 dL - \frac{\partial L}{\partial t} \quad \text{(cf. 21.5)}$$

$$= - \frac{\partial L}{\partial t} .$$

21.21 REMARK Maintaining the assumption that L is nondegenerate, let $\gamma(t)$ be an integral curve of Γ_L and consider $E_L|_{\gamma(t)}$ -- then

$$\frac{dE}{dt} = \frac{d}{dt} (v^i \frac{\partial L}{\partial v^i} - L)$$

$$= \frac{dv^i}{dt} \frac{\partial L}{\partial v^i} + v^i \frac{d}{dt} (\frac{\partial L}{\partial v^i}) - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial q^i} \frac{dq^i}{dt} - \frac{\partial L}{\partial v^i} \frac{dv^i}{dt}$$

$$= v^i (\frac{d}{dt}(\frac{\partial L}{\partial v^i}) - \frac{\partial L}{\partial q^i}) - \frac{\partial L}{\partial t}$$

$$= - \frac{\partial L}{\partial t} \text{ (cf. 21.18).}$$

It is not difficult to extend constraint theory to the time-dependent case but I shall not stop to run through the formalities. However, there is one point to be made, namely that in general the constraints will depend on time. To illustrate, consider a particle of mass m moving in the plane and subject to the constraint

$$v^1 - tv^2 - C = 0 \quad (C \in \underline{R}).$$

This constraint is affine in the velocities and the 1-form

$$\omega = dq^1 - tdq^2$$

defines a time-dependent vector subbundle of $T\underline{R}^2 = \underline{R}^4$.

[Note: Refer back to 16.21 but assume that the horizontal plate rotates with nonconstant angular velocity $\Omega(t)$ -- then the vector field

$$-\Omega(t)x^2 \frac{\partial}{\partial x^1} + \Omega(t)x^1 \frac{\partial}{\partial x^2}$$

now depends on time. Still, the analysis given there goes through without essential change.]

There is one final topic that demands consideration, viz. the notion of fiber derivative. So let $L \in C^\infty(J^1M)$ be an arbitrary lagrangian. Since Θ_L is π^{10}-horizontal, it determines a fiber preserving C^∞ function

$$\widehat{FL}: J^1M \rightarrow T^*J^0M$$

over J^0M, i.e., the diagram

$$\begin{array}{ccc} J^1M & \xrightarrow{\widehat{FL}} & T^*J^0M \\ {\scriptstyle \pi^{10}} \downarrow & & \downarrow {\scriptstyle \pi^*_{J^0M}} \\ J^0M & = & J^0M \end{array}$$

commutes.

Locally,

$$\hat{FL}(t,q^i,v^i) = (t,q^i,-E_L, \frac{\partial L}{\partial v^i}).$$

N.B. If Θ is the fundamental 1-form on T^*J^0M, then

$$\Theta_L = (\hat{FL})^*\Theta.$$

We have

$$\begin{aligned} T^*J^0M &= T^*(\underline{R} \times M) \\ &\approx T^*\underline{R} \times T^*M \\ &\xrightarrow{pr_{\underline{R}}} \underline{R} \times T^*M, \end{aligned}$$

where

$$pr_{\underline{R}} = \pi^*_{\underline{R}} \times id_{T^*M}.$$

The fiber derivative FL of L is then the composition

$$pr_{\underline{R}} \circ \hat{FL}.$$

Therefore

$$FL:\underline{R} \times TM \to \underline{R} \times T^*M$$

and there is a commutative diagram

$$\begin{array}{ccc} \underline{R} \times TM & \xrightarrow{FL} & \underline{R} \times T^*M \\ \| & & \downarrow{\scriptstyle id_{\underline{R}} \times \pi^*_M} \\ \underline{R} \times TM & \xrightarrow[\pi^{10}]{} & \underline{R} \times M . \end{array}$$

Locally,

$$FL(t,q^i,v^i) = (t,q^i, \frac{\partial L}{\partial v^i}).$$

21.22 <u>LEMMA</u> The pair (dt,Ω_L) is a cosymplectic structure on J^1M iff FL is a local diffeomorphism.

The central conclusion of this § is that the time-dependent theory is more or less parallel to the time-independent theory. But there is one important difference: If L_1 and L_2 are nondegenerate and if $\Omega_{L_1} = \Omega_{L_2}$, then $\Gamma_{L_1} = \Gamma_{L_2}$, the analog of this in the autonomous setting being false.

21.23 <u>EXAMPLE</u> Take $M = \underline{R}$ and let

$$\begin{cases} L_1(q,v) = \dfrac{v^2}{2} \\ \\ L_2(q,v) = \dfrac{v^2}{2} + q. \end{cases}$$

Then both L_1 and L_2 are nondegenerate with

$$\left.\begin{matrix} \omega_{L_1} \\ \\ \omega_{L_2} \end{matrix}\right\} = dv\wedge dq.$$

However

$$\begin{cases} \Gamma_{L_1} = v\,\dfrac{\partial}{\partial q} \\ \\ \Gamma_{L_2} = v\,\dfrac{\partial}{\partial q} + \dfrac{\partial}{\partial v}\,. \end{cases}$$

§22. DEGENERATE LAGRANGIANS

Up until now, the focus has been on nondegenerate lagrangians but, for the applications, it is definitely necessary to consider degenerate lagrangians as well (a case in point being general relativity, albeit this is an infinite dimensional setting).

Suppose, therefore, that $L \in C^\infty(TM)$ is degenerate -- then ω_L is no longer of maximal rank and, in general, is not of constant rank.

22.1 <u>EXAMPLE</u> Take $M = \underline{R}$ and let

$$L(q,v) = v^3.$$

Then

$$\omega_L = \frac{\partial^2 L}{\partial q^i \partial v^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dv^i \wedge dq^j$$

$$= 6v dv \wedge dq,$$

so ω_L is not of constant rank.

Henceforth, our standing assumption will be that the rank of ω_L is constant, thus the pair (TM, ω_L) is a presymplectic manifold (cf. 15.20).

<u>N.B.</u> Recall the convention of 15.13: Ker ω_L has two meanings, dictated by context.

Let

$$D_L = \{X \in \mathcal{D}^1(TM) : \iota_X \omega_L = - dE_L\}.$$

Then in the terminology of §8, L is said to admit <u>global dynamics</u> if D_L is nonempty.

22.2 <u>LEMMA</u> If L admits global dynamics and if $\iota_X \omega_L = - dE_L$ is a particular solution, then the general solution has the form X + Z, where $Z \in \operatorname{Ker} \omega_L$.

While a given lagrangian might not admit global dynamics, there still might be a subset of TM on which the relation

$$\iota_X \omega_L = - dE_L$$

does obtain.

22.3 <u>EXAMPLE</u> Take $M = \underline{R}^3$ and let

$$L(q^1,q^2,q^3,v^1,v^2,v^3) = v^1 v^3 + \frac{1}{2}\,((q^2)^2 q^3).$$

Then

$$\begin{aligned}\omega_L &= \frac{\partial^2 L}{\partial q^i \partial v^j}\, dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j}\, dv^i \wedge dq^j \\ &= dv^1 \wedge dq^3 + dv^3 \wedge dq^1.\end{aligned}$$

So

$$\left[\begin{array}{l} \omega_L^2 \neq 0 \\ \\ \omega_L^3 = 0 \end{array}\right. \implies \operatorname{rank} \omega_L = 4.$$

And Ker ω_L is generated by $\dfrac{\partial}{\partial q^2}$ and $\dfrac{\partial}{\partial v^2}$. Next

$$\iota_X \omega_L = B^3 dq^1 + B^1 dq^3 - A^3 dv^1 - A^1 dv^3$$

if

$$X = A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial v^i} .$$

On the other hand,

$$- dE_L = q^2 q^3 dq^2 + \frac{(q^2)^2}{2} dq^3 - v^3 dv^1 - v^1 dv^3 .$$

Therefore

$$\iota_X \omega_L \neq - dE_L$$

unless $q^2 q^3 = 0$, in which case

$$\begin{cases} A^1 = v^1 \\ \\ A^3 = v^3 \end{cases} , \quad \begin{cases} B^1 = \frac{(q^2)^2}{2} \\ \\ B^3 = 0. \end{cases}$$

The general solution on $q^2 q^3 = 0$ is thus

$$v^1 \frac{\partial}{\partial q^1} + v^3 \frac{\partial}{\partial q^3} + \frac{(q^2)^2}{2} \frac{\partial}{\partial v^1} + A^2 \frac{\partial}{\partial q^2} + B^2 \frac{\partial}{\partial v^2} ,$$

where A^2, B^2 are arbitrary C^∞ functions.

[Note: The condition $q^2 q^3 = 0$ does not, strictly speaking, define a submanifold of TM.]

Put

$$\mathrm{Ker}^V \omega_L = \mathrm{Ker}\, \omega_L \cap V(TM) .$$

22.4 <u>LEMMA</u> We have

$$S(\mathrm{Ker}\ \omega_L) \subset \mathrm{Ker}^V\ \omega_L.$$

<u>PROOF</u> Let $Z \in \mathrm{Ker}\ \omega_L$ — then

$$\iota_Z\omega_L = 0.$$

But

$$\iota_{SZ}\omega_L = -\ \iota_Z\omega_L \circ S \quad \text{(see the note appended to 16.1).}$$

Therefore

$$\iota_{SZ}\omega_L = 0 \Rightarrow SZ \in \mathrm{Ker}\ \omega_L.$$

And

$$SZ \in V(TM).$$

Terminology: L is

$$\begin{cases} \text{Type I} & \text{if } S(\mathrm{Ker}\ \omega_L) = \mathrm{Ker}^V\ \omega_L \\ \text{Type II} & \text{if } S(\mathrm{Ker}\ \omega_L) \neq \mathrm{Ker}^V\ \omega_L. \end{cases}$$

22.5 <u>EXAMPLE</u> Take $M = \underline{R}^2$ and let

$$L(q^1,q^2,v^1,v^2) = \frac{1}{2}(v^1)^2 e^{q^2}.$$

Then

$$\omega_L = \frac{\partial^2 L}{\partial q^i \partial v^j}\, dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j}\, dv^i \wedge dq^j$$

$$= v^1 e^{q^2} dq^2 \wedge dq^1 + e^{q^2} dv^1 \wedge dq^1.$$

So

$$\left[\begin{array}{l} \omega_L \neq 0 \\ \\ \omega_L^2 = 0 \end{array}\right. \quad \Rightarrow \text{rank } \omega_L = 2.$$

To determine Ker ω_L, write

$$X = A^1 \frac{\partial}{\partial q^1} + A^2 \frac{\partial}{\partial q^2} + B^1 \frac{\partial}{\partial v^1} + B^2 \frac{\partial}{\partial v^2}$$

and set $\iota_X \omega_L$ equal to zero, hence

$$\left[\begin{array}{l} A^1 e^{q^2} = 0 \Rightarrow A^1 = 0 \\ \\ (A^2 v^1 + B^1) e^{q^2} = 0 \Rightarrow B^1 = - A^2 v^1 \end{array}\right.$$

$\Rightarrow$

$$X = A^2 \left(\frac{\partial}{\partial q^2} - v^1 \frac{\partial}{\partial v^1}\right) + B^2 \frac{\partial}{\partial v^2} .$$

Therefore Ker ω_L is generated by

$$\frac{\partial}{\partial q^2} - v^1 \frac{\partial}{\partial v^1} \text{ and } \frac{\partial}{\partial v^2} .$$

And here

$$\text{Ker}^V \omega_L = \{f \frac{\partial}{\partial v^2} : f \in C^\infty(\underline{R}^4)\}$$

$$= S(\text{Ker } \omega_L),$$

meaning that L is Type I. Still, L does not admit global dynamics.

22.6 LEMMA If L admits global dynamics and is Type I, then $\exists$ a $\Gamma \in \mathcal{D}^1(TM)$ of second order such that

$$\iota_\Gamma \omega_L = - dE_L.$$

PROOF Choose $X \in \mathcal{D}^1(TM)$:

$$\iota_X \omega_L = - dE_L.$$

Then

$$\begin{aligned} \iota_{SX - \Delta}\omega_L &= - \iota_X\omega_L \circ S - \iota_\Delta\omega_L \\ &= dE_L \circ S - dE_L \circ S \quad \text{(cf. 8.7)} \\ &= 0 \end{aligned}$$

$\Rightarrow$

$$SX - \Delta \in \text{Ker}^V \omega_L$$

$\Rightarrow$

$$SX - \Delta = SY \quad (\exists\ Y \in \text{Ker}\ \omega_L)$$

$\Rightarrow$

$$\begin{aligned} \iota_{X - Y}\omega_L &= \iota_X\omega_L \\ &= - dE_L. \end{aligned}$$

And

$$S(X - Y) = \Delta$$

$\Rightarrow$

$$\Gamma = X - Y \in \mathcal{SO}(TM) \quad \text{(cf. 5.8)}.$$

22.7 <u>EXAMPLE</u> Let $g \in \mathcal{D}_2^0(M)$ be symmetric. Assign to each $x \in M$ the subspace

$$K_x = \{X_x \in T_xM : g_x(X_x, Y_x) = 0 \ \forall \ Y_x \in T_xM\}.$$

Then g is said to be a <u>degenerate metric</u> if $\exists \ d > 0$ such that $\forall \ x \in M$, $\dim K_x = d$ and the bilinear form induced by g_x on T_xM/K_x is positive definite. It has been shown by Crampin that there exists a linear connection ∇ with zero torsion such that $\nabla g = 0$ iff $L_Z g = 0$ for all $Z \in K = \bigcup_{x \in M} K_x$ (the null distribution attached to g). This condition implies that K is integrable. In fact, if $Y, Z \in K$, then for any X,

$$0 = (L_Y g)(Z,X) = Yg(Z,X) - g([Y,Z],X) - g(Z,[Y,X])$$

$$\Rightarrow$$

$$g([Y,Z],X) = Yg(Z,X) - g(Z,[Y,X])$$

$$= 0.$$

On the other hand, K may be integrable even when this condition is not satisfied. For example, let $M = \underline{R}^2$ and put $g = \phi(q^1) dq^2 \otimes dq^2$ with $\phi > 0$ — then K is spanned by $\partial/\partial q^1$, hence is integrable, but $L_{\partial/\partial q^1} g \neq 0$ unless ϕ is a constant. Take now for $L \in C^\infty(TM)$ the function

$$(x, X_x) \to \frac{1}{2} g_x(X_x, X_x) \qquad (X_x \in T_xM).$$

Then it turns out that L is Type I iff K is integrable and when this is so, L admits global dynamics iff $L_Z g = 0 \ \forall \ Z \in K$.

22.8 <u>EXAMPLE</u> Let $\omega \in \Lambda^1 M$ and put $L = \hat{\omega}$ (cf. 8.19) -- then

$$\begin{cases} \theta_L = \pi_M^* \omega \\ \omega_L = \pi_M^* d\omega. \end{cases}$$

Furthermore, in suggestive notation,

$$\omega_L(X,\_\_) = d\omega((\pi_M)_* X,\_\_),$$

which implies that

$$\operatorname{Ker} \omega_L \supset V(TM).$$

Accordingly, if $d\omega$ is nondegenerate, then

$$\operatorname{Ker} \omega_L = V(TM)$$

and L is Type II. For instance, take $M = \underline{R}^2$ and consider

$$L((q^1,q^2),(v^1,v^2)) = \frac{1}{2}(q^2 v^1 - q^1 v^2).$$

Let

$$\omega = \frac{1}{2}(q^2 dq^1 - q^1 dq^2).$$

Then $L = \hat{\omega}$. Since $d\omega = dq^2 \wedge dq^1$ is nondegenerate, $\operatorname{Ker} \omega_L$ is generated by $\frac{\partial}{\partial v^1}$ and $\frac{\partial}{\partial v^2}$.

22.9 <u>LEMMA</u> We have

$$\operatorname{Ker}^V \omega_L = \operatorname{Ker} FL_*.$$

It remains to consider the time-dependent situation. So suppose that $L \in C^\infty(J^1M)$ is degenerate, hence $dt \wedge \Omega_L^n$ is not a volume form. Given $t \in \underline{R}$, let

$$L_t = L|\{t\} \times TM.$$

Then in what follows it will be assumed that $\exists$ $r: 0 < r < n$ (= dim M), where $\forall$ $t \in \underline{R}$,

$$\text{rank } \omega_{L_t} = 2r.$$

Therefore

$$dt \wedge \Omega_L^r \neq 0, \ dt \wedge \Omega_L^{r+1} = 0, \ \Omega_L^{2r+2} = 0$$

$\Rightarrow$

$$2r \leq \text{rank } \Omega_L \leq 2r + 2.$$

N.B. While convenient, this assumption is certainly not automatic: Take $M = \underline{R}$ and consider

$$L(t,q,v) = t \frac{v^2}{2}.$$

22.10 EXAMPLE Take $M = \underline{R}^2$ and let

$$L = \frac{1}{2} (v^1 + tv^2)^2.$$

Then L is degenerate (cf. 21.12). We have

$$\omega_{L_t} = \frac{\partial^2 L_t}{\partial q^i \partial v^j} dq^i \wedge dq^j + \frac{\partial^2 L_t}{\partial v^i \partial v^j} dv^i \wedge dq^j$$

$$= dv^1 \wedge dq^1 + tdv^1 \wedge dq^2 + tdv^2 \wedge dq^1 + t^2 dv^2 \wedge dq^2.$$

So $\forall$ t,

$$\left.\begin{array}{l} \omega_{L_t} \neq 0 \\ \\ \omega_{L_t}^2 = 0 \end{array}\right] \Rightarrow \text{rank } \omega_{L_t} = 2.$$

Now use 21.9 to get

$$\Omega_L = t dv^2 \wedge dq^1 + v^2 dt \wedge dq^1 + t dv^1 \wedge dq^2$$

$$+ t^2 dv^2 \wedge dq^2 + (v^1 + 2tv^2) dt \wedge dq^2$$

$$+ (v^1 + tv^2) dt \wedge dv^1 + dv^1 \wedge dq^1 + t(v^1 + tv^2) dt \wedge dv^2.$$

Therefore

$$2 \leq \text{rank } \Omega_L \leq 4.$$

Let $C = f^{-1}(0)$, where

$$f(t,q^1,q^2,v^1,v^2) = v^1 + tv^2.$$

Then

$$C = \{x = (t,q^1,q^2,v^1,v^2) : \text{rank}(\Omega_L)_x = 2\}.$$

Motivated by 21.14 (and subsequent discussion), let

$$D_L = \{X \in \mathcal{D}^1(J^1M) : \iota_X \Omega_L = 0,\ \iota_X dt = 1\}.$$

Then L is said to admit <u>global dynamics</u> if D_L is nonempty.

22.11 <u>LEMMA</u> L admits global dynamics iff Ω_L has constant rank 2r.

This is a consequence of 22.12 and 22.14 infra.

22.12 <u>LEMMA</u> Fix $x \in J^1M$ -- then $\text{rank}(\Omega_L)_x = 2r$ iff $\exists\ X_x \in T_xJ^1M$ such that $\iota_{X_x}(\Omega_L)_x = 0$, $\iota_{X_x}(dt)_x = 1$.

<u>PROOF</u> If $\text{rank}(\Omega_L)_x = 2r$, then $\exists$ a linearly independent set

$$\{e^1,\ldots,e^r,e^{r+2},\ldots,e^{2r}\} \subset T_x^*J^1M$$

such that

$$(\Omega_L)_x = \sum_{i=1}^{r} e^i \wedge e^{r+i}.$$

But $(dt)_x \wedge (\Omega_L)_x \neq 0$, thus

$$\{(dt)_x,e^1,\ldots,e^r,e^{r+1},\ldots,e^{2r}\} \subset T_x^*J^1M$$

is also linearly independent. Complete it to a basis

$$\{(dt)_x,e^1,\ldots,e^r,e^{r+1},\ldots,e^{2r},f^1,\ldots,f^{2n-2r}\}$$

for $T_x^*J^1M$ and pass to the dual basis

$$\{X_x,e_1,\ldots,e_r,e_{r+1},\ldots,e_{2r},f_1,\ldots,f_{2n-2r}\}$$

for T_xJ^1M -- then

$$(dt)_x(X_x) = 1 \text{ and } e^i(X_x) = e^{r+i}(X_x) = f^j(X_x) = 0$$

$\Rightarrow$

$$\left[\begin{array}{l} \iota_{X_x}(\Omega_L)_x = 0 \\ \\ \iota_{X_x}(dt)_x = 1. \end{array}\right.$$

Conversely, if X_x has the stated properties, then

$$0 = \iota_{X_x}((dt)_x \wedge (\Omega_L)_x^{r+1})$$

$$= (\Omega_L)_x^{r+1}$$

$$\Rightarrow$$

$$\operatorname{rank}(\Omega_L)_x = 2r.$$

22.13 <u>RAPPEL</u> Suppose that N is a connected (2n+1)-dimensional manifold -- then a <u>precosymplectic structure</u> on N of rank 2r is a pair(η,Ω), where $\eta \in \Lambda^1 N$ is a closed 1-form on N and $\Omega \in \Lambda^2 N$ is a closed 2-form on N of constant rank 2r such that $\eta \wedge \Omega^r \neq 0$.

22.14 <u>LEMMA</u> If (η,Ω) is a precosymplectic structure on N of rank 2r, then there exists a vector field $X \in \mathcal{D}^1(N)$:

$$\begin{cases} \iota_X \Omega = 0 \\ \iota_X \eta = 1. \end{cases}$$

<u>PROOF</u> By a variation on a wellknown theme, each $y \in N$ admits a neighborhood U_y with local coordinates $\{(t,q^i,p_i,u^s\}$ $(1 \leq i \leq r, 1 \leq s \leq 2n-2r)$ such that

$$\Omega = dp_i \wedge dq^i, \ \eta = dt.$$

Therefore

$$\begin{cases} \iota_{\partial/\partial t} \Omega = 0 \\ \iota_{\partial/\partial t} \eta = 1. \end{cases}$$

Pass from this point via a partition of unity... .

[Note: In general, X is far from unique.]

22.15 <u>EXAMPLE</u> Take $M = \underline{R}^3$ and let

$$L(t,q^1,q^2,q^3,v^1,v^2,v^3) = \frac{1}{2}(v^1)^2 - v^2q^3 - V(t,q^1,q^2,q^3),$$

where $V:\underline{R} \times \underline{R}^3 \to \underline{R}$ is C^∞ -- then it is clear that L is degenerate. Moreover,

$$\omega_{L_t} = \frac{\partial^2 L_t}{\partial q^i \partial v^j} dq^i \wedge dq^j + \frac{\partial^2 L_t}{\partial v^i \partial v^j} dv^i \wedge dq^j$$

$$= dq^2 \wedge dq^3 + dv^1 \wedge dq^1$$

=>

$$\left[\begin{array}{l} \omega_{L_t}^2 \neq 0 \\ \\ \omega_{L_t}^3 = 0 \end{array}\right. \quad \Rightarrow \text{rank } \omega_{L_t} = 4.$$

Next (cf. 21.9)

$$\Omega_L = v^1 dt \wedge dv^1 + dq^2 \wedge dq^3 + dv^1 \wedge dq^1$$

$$+ \frac{\partial V}{\partial q^1} dt \wedge dq^1 + \frac{\partial V}{\partial q^2} dt \wedge dq^2 + \frac{\partial V}{\partial q^3} dt \wedge dq^3.$$

So

$$\text{rank } \Omega_L = 4.$$

Therefore L admits global dynamics (cf. 22.11), the general solution being

$$X = \frac{\partial}{\partial t} + v^1 \frac{\partial}{\partial q^1} - \frac{\partial V}{\partial q^3}\frac{\partial}{\partial q^2} + \frac{\partial V}{\partial q^2}\frac{\partial}{\partial q^3}$$

$$- \frac{\partial V}{\partial q^1}\frac{\partial}{\partial v^1} + B^2 \frac{\partial}{\partial v^2} + B^3 \frac{\partial}{\partial v^3} .$$

Here B^2, B^3 are arbitrary C^∞ functions on $J^1\underline{R}^3$.

22.16 <u>REMARK</u> The lagrangian

$$L = \frac{1}{2}(v^1 + tv^2)^2$$

figuring in 22.10 does not admit global dynamics. However, if matters are limited to the submanifold $C = f^{-1}(0)$, then

$$TC = \{\frac{\partial}{\partial t} - v^2 \frac{\partial}{\partial v^1}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial v^2} - t\frac{\partial}{\partial v^1}\}$$

and the general solution is

$$X = \frac{\partial}{\partial t} - At\frac{\partial}{\partial q^1} + A\frac{\partial}{\partial q^2} - (v^2 + Bt)\frac{\partial}{\partial v^1} + B\frac{\partial}{\partial v^2} ,$$

where A,B are C^∞ functions on C.

Put

$$K_L = \text{Ker } dt \cap \text{Ker } \Omega_L$$

and then set

$$K_L^v = K_L \cap V^{10}(J^1M) .$$

22.17 LEMMA We have

$$S_{dt}(\mathcal{K}_L) \subset \mathcal{K}_L^V.$$

22.18 LEMMA We have

$$S_{dt}(\text{Ker } \Omega_L) \subset \text{Ker } \Omega_L \cap \mathcal{V}^{10}(TM).$$

N.B.

$$22.18 \Rightarrow 22.17.$$

For

$$X \in \mathcal{K}_L \Rightarrow X \in \text{Ker } \Omega_L$$

$$\Rightarrow S_{dt}X \in \text{Ker } \Omega_L \cap \mathcal{V}^{10}(J^1M).$$

On the other hand,

$$S_{dt}^*(dt) = 0 \Rightarrow dt(\text{Im } S_{dt}) = 0.$$

The proof of 22.18 hinges on an auxilliary result.

22.19 LEMMA $\forall\ X,Y \in \mathcal{D}^1(J^1M)$ and $\forall\ \omega \in \mathcal{D}_1(J^1M)$,

$$(d(\omega \circ S_{dt}) + \omega\wedge dt)(S_{dt}X,Y) + (d(\omega \circ S_{dt}) + \omega\wedge dt)(X,S_{dt}Y)$$

$$= d\omega(S_{dt}X,S_{dt}Y).$$

PROOF Since $S_{dt}^2 = 0$ (cf. 21.3),

$$\begin{cases} d(\omega \circ S_{dt})(S_{dt}X,Y) = S_{dt}X(\omega(S_{dt}Y)) - \omega(S_{dt}[S_{dt}X,Y]) \\ d(\omega \circ S_{dt})(X,S_{dt}Y) = - S_{dt}Y(\omega(S_{dt}X)) - \omega(S_{dt}[X,S_{dt}Y]) \end{cases}$$

$\Rightarrow$

$$d(\omega \circ S_{dt})(S_{dt}X,Y) + d(\omega \circ S_{dt})(X,S_{dt}Y)$$

$$= S_{st}X(\omega(S_{dt}Y)) - S_{dt}Y(\omega(S_{dt}X))$$

$$- \omega(S_{dt}[S_{dt}X,Y]) - \omega(S_{dt}[X,S_{dt}Y])$$

$$= d\omega(S_{dt}X,S_{dt}Y) + \omega([S_{dt}X,S_{dt}Y])$$

$$- \omega(S_{dt}[S_{dt}X,Y]) - \omega(S_{dt}[X,S_{dt}Y])$$

$$= d\omega(S_{dt}X,S_{dt}Y) + \omega((\iota_X dt)S_{dt}Y - (\iota_Y dt)S_{dt}X) \quad \text{(cf. 21.4)}.$$

But

$$\begin{cases} (\omega\wedge dt)(S_{dt}X,Y) = \omega(S_{dt}X)\,\iota_Y dt - \omega(Y)\,dt(S_{dt}X) \\ (\omega\wedge dt)(X,S_{dt}Y) = \omega(X)\,dt(S_{dt}Y) - \omega(S_{dt}Y)\,\iota_X dt \end{cases}$$

or still,

$$\begin{cases} (\omega\wedge dt)(S_{dt}X,Y) = \omega(S_{dt}X)\,\iota_Y dt \\ (\omega\wedge dt)(X,S_{dt}Y) = - \omega(S_{dt}Y)\,\iota_X dt \end{cases}$$

$\Rightarrow$

$$(\omega\wedge dt)(S_{dt}X,Y) + (\omega\wedge dt)(X,S_{dt}Y)$$

$$= \omega((\iota_Y dt)S_{dt}X - (\iota_X dt)S_{dt}Y).$$

In 22.19, let $\omega = dL$ -- then

$$\Theta_L = dL \circ S_{dt} + Ldt$$

$$\Rightarrow$$

$$\Omega_L = d\Theta_L = d(dL \circ S_{dt}) + dL \wedge dt.$$

So, $\forall\ X,Y \in \mathcal{D}^1(J^1M)$,

$$\begin{aligned}\Omega_L(S_{dt}X,Y) + \Omega_L(X,S_{dt}Y) &= d(dL)(S_{dt}X,S_{dt}Y)\\ &= 0.\end{aligned}$$

Accordingly,

$$X \in \text{Ker } \Omega_L \Rightarrow \Omega_L(X,S_{dt}Y) = 0$$

$$\Rightarrow \Omega_L(S_{dt}X,Y) = 0 \Rightarrow S_{dt}X \in \text{Ker } \Omega_L,$$

thereby establishing 22.18.

Terminology: L is

$$\begin{cases} \text{Type I if } S_{dt}(K_L) = K_L^V \\ \text{Type II if } S_{dt}(K_L) \neq K_L^V. \end{cases}$$

22.20 <u>LEMMA</u> If L admits global dynamics and is Type I, then $\exists$ a $\Gamma \in \mathcal{D}^1(J^1M)$ of second order such that

$$\begin{cases} \iota_\Gamma \Omega_L = 0 \\ \iota_\Gamma dt = 1. \end{cases}$$

<u>PROOF</u> Choose $X \in \mathcal{D}^1(J^1M)$:

$$\begin{cases} \iota_X \Omega_L = 0 \\ \iota_X dt = 1. \end{cases}$$

Then

$$S_{dt}X = Y \in \mathcal{K}_L^V.$$

Choose $Z \in \mathcal{K}_L$:

$$S_{dt}Z = Y$$

and let $\Gamma = X - Z$ -- then

$$\begin{cases} \iota_\Gamma \Omega_L = \iota_X \Omega_L - \iota_Z \Omega_L = 0 \\ \iota_\Gamma dt = \iota_X dt - \iota_Z dt = 1. \end{cases}$$

Finally

$$\begin{aligned} S_{dt}\Gamma &= S_{dt}X - S_{dt}Z \\ &= Y - Y \\ &= 0. \end{aligned}$$

Therefore Γ is second order (cf. 21.6).

22.21 <u>REMARK</u> The lagrangian introduced in 22.15 admits global dynamics but there are no second order solutions, thus L is not Type I.

22.22 LEMMA We have

$$\mathrm{Ker}\ \Omega_L \cap V^{10}(J^1M) = \mathrm{Ker}\ FL_* = \mathrm{Ker}\ \widehat{FL}_*.$$

APPENDIX

Fix a lagrangian $L \in C^\infty(TM)$ and put

$$\begin{cases} W(L) = [W_{ij}(L)] \\ T(L) = [T_{ij}(L)], \end{cases}$$

where

$$\begin{cases} W_{ij}(L) = \dfrac{\partial^2 L}{\partial v^i \partial v^j} \\ T_{ij}(L) = \dfrac{\partial^2 L}{\partial v^i \partial q^j} - \dfrac{\partial^2 L}{\partial q^i \partial v^j}\ . \end{cases}$$

Let

$$X = A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial v^i}\ .$$

Then in abbreviated notation, the differential equations that govern the relation

$$\iota_X \omega_L = -\, dE_L$$

are

$$\begin{bmatrix} T(L) & W(L) \\ -\,W(L) & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -\,d_q E_L \\ -\,W(L)v \end{bmatrix}$$

or still,

$$\begin{cases} T(L)A + W(L)B = -d_q E_L \\ W(L)(A - v) = 0. \end{cases}$$

Therefore

$$A = v + \xi (W(L)\xi = 0),$$

so

$$W(L)B = -T(L)(v + \xi) - d_q E_L$$

$$= -T(L)v - d_q E_L - T(L)\xi$$

$$= \Xi - T(L)\xi.$$

Here

$$\Xi = -T(L)v - d_q E_L$$

=>

$$\Xi_i = \frac{\partial^2 L}{\partial q^i \partial v^j} v^j - \frac{\partial^2 L}{\partial v^i \partial q^j} v^j - \frac{\partial^2 L}{\partial q^i \partial v^j} v^j + \frac{\partial L}{\partial q^i}$$

$$= \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial v^i \partial q^j} v^j.$$

An integral curve γ for

$$X = (v^i + \xi^i)\frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial v^i}$$

is determined by the differential equations

$$\begin{cases} \dot{q}^i = \dfrac{dq^i(\gamma(t))}{dt} = v^i(\gamma(t)) + \xi^i(\gamma(t)) \\ \dfrac{dv^i(\gamma(t))}{dt} = B^i(\gamma(t)). \end{cases}$$

But

$$W(L)B = \Xi - T(L)\xi$$

=>

$$\frac{\partial^2 L}{\partial v^i \partial v^j} B^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial v^i \partial q^j} v^j + \left(\frac{\partial^2 L}{\partial q^i \partial v^j} - \frac{\partial^2 L}{\partial v^i \partial q^j}\right) \xi^j$$

=>

$$\frac{\partial^2 L}{\partial v^i \partial v^j} (\ddot{q}^j - \dot{\xi}^j) = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial v^i \partial q^j} (\dot{q}^j - \xi^j) + \left(\frac{\partial^2 L}{\partial q^i \partial v^j} - \frac{\partial^2 L}{\partial v^i \partial q^j}\right) \xi^j$$

$$= \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial v^i \partial q^j} \dot{q}^j + \frac{\partial^2 L}{\partial q^i \partial v^j} \xi^j$$

=>

$$\frac{\partial^2 L}{\partial v^i \partial v^j} \ddot{q}^j + \frac{\partial^2 L}{\partial v^i \partial q^j} \dot{q}^j - \frac{\partial L}{\partial q^i}$$

$$= \frac{\partial^2 L}{\partial v^i \partial v^j} \dot{\xi}^j + \frac{\partial^2 L}{\partial q^i \partial v^j} \xi^j.$$

These relations are thus a generalization of the equations of Lagrange (to which they reduce when $\xi = 0$).

A.1 <u>REMARK</u> It is to be emphasized that this analysis is predicated on the assumption that L admits global dynamics:

$$\iota_X \omega_L = - dE_L \quad (\exists\ X \in \mathcal{D}^1(TM)).$$

A.2 <u>EXAMPLE</u> Take $M = \underline{R}$ and let $L(q,v) = q$ -- then

$$W(L) = \begin{bmatrix} 0 & 0 \\ & \\ 0 & 0 \end{bmatrix}.$$

And

$$\omega_L = 0,\ E_L = -\ q,$$

so $\nexists$ X: $\iota_X \omega_L = -\ dE_L$. In addition, the preceding differential equation reduces to "1 = 0".

A.3 EXAMPLE Take $M = \underline{R}$ and let $L(q,v) = v$ — then

$$W(L) = \begin{bmatrix} 0 & 0 \\ & \\ 0 & 0 \end{bmatrix}.$$

And

$$\omega_L = 0,\ E_L = 0,$$

so $\forall$ X: $\iota_X \omega_L = -\ dE_L$. In addition, the preceding differential equation reduces to "0 = 0".

There are similar results in the time-dependent case but I shall leave their explication to the reader.

§23. PASSAGE TO THE COTANGENT BUNDLE

Let M be a connected C^∞ manifold of dimension n. Suppose that $L \in C^\infty(TM)$ is degenerate.

23.1 <u>ASSUMPTION</u> For some $k < n$, FL is of constant rank $n + k$, $\Sigma = FL(TM)$ is a closed submanifold of T^*M of dimension $n + k$, and $\forall\ \sigma \in \Sigma$, the fiber $(FL)^{-1}(\sigma)$ is connected.

[Note: $\forall\ \sigma \in \Sigma$,

$$\begin{aligned}\dim(FL)^{-1}(\sigma) &= \dim TM - \dim \Sigma \\ &= 2n - (n + k) \\ &= n - k.]\end{aligned}$$

<u>N.B.</u> The matrix

$$W(L) = [W_{ij}(L)],$$

where

$$W_{ij}(L) = \frac{\partial^2 L}{\partial v^i \partial v^j},$$

has constant rank k.

[Note: A $2n \times 2n$ matrix of the form

$$\begin{bmatrix} I_n & 0 \\ A & X \end{bmatrix}$$

is row equivalent to

$$\begin{bmatrix} I_n & 0 \\ 0 & X \end{bmatrix}.]$$

For motivation, recall the following standard fact.

23.2 RAPPEL Let M', M'' be C^∞ manifolds; let $f:M' \to M''$ be a C^∞ map of constant rank r -- then each point of M' admits a neighborhood U such that $f(U)$ is an r-dimensional submanifold of M'' and the restriction $U \to f(U)$ is a submersion with connected fibers.

Since $FL:TM \to \Sigma$ is a fibration, the kernel of

$$TFL:TTM \to T\Sigma$$

determines a vector subbundle $V_L TM$ of TTM (cf. §11). Viewed as a linear distribution, $V_L TM$ is integrable and the leaves of the associated foliation of TM are the $(FL)^{-1}(\sigma)$ $(\sigma \in \Sigma)$ (cf. 15.11):

$$TM = \coprod_{\sigma \in \Sigma} (FL)^{-1}(\sigma).$$

N.B. The fiber dimension of $\operatorname{Ker}^V \omega_L$ is $n - k$ (cf. 22.9).

We claim now that ω_L has constant rank, thus the machinery developed in §22 is applicable. To this end, let Ω be the fundamental 2-form on T^*M and put

$$\Omega_\Sigma = i_\Sigma^* \Omega \quad (i_\Sigma : \Sigma \to T^*M).$$

23.3 LEMMA The rank of Ω_Σ is constant and, in fact,

$$\operatorname{rank} \Omega_\Sigma = k + \ell,$$

where $k \leq \ell \leq n$ $(k < n)$.

[Note: The pair (Σ,Ω_Σ) is a presymplectic manifold (cf. 15.20) and the fiber dimension of Ker Ω_Σ is

$$(n + k) - (k + \ell) = n - \ell.]$$

Therefore

$$\operatorname{rank} \omega_L = k + \ell.$$

<u>N.B.</u> The fiber dimension of Ker ω_L is

$$2n - (k + \ell) = (n - k) + (n - \ell).$$

23.4 <u>REMARK</u> L is Type I iff

$$(n - k) + (n - \ell) = 2(n - k),$$

i.e., iff $\ell = k$.

23.5 <u>RAPPEL</u> Let (V,Ω) be a symplectic vector space of dimension 2n. Given a subspace $W \subset V$, its symplectic complement $W^\perp$ is

$$\{v \in V : \Omega(v,W) = 0\}$$

and

$$\dim W + \dim W^\perp = 2n.$$

Denote by Ω_W the restriction of Ω to $W \times W$ — then

$$\operatorname{Ker} \Omega_W = \{w \in W : \iota_w \Omega_W = 0\} = W \cap W^\perp,$$

so (W,Ω_W) is a symplectic vector space iff $W \cap W^{\perp} = \{0\}$.

Given $\sigma \in \Sigma$, regard $T_\sigma\Sigma$ as a subspace of $T_\sigma T^*M$ -- then

$$(T_\sigma\Sigma)^{\perp} = \{X_\sigma \in T_\sigma T^*M : \Omega_\sigma(X_\sigma, T_\sigma\Sigma) = 0\}.$$

Following Dirac, Σ is said to be <u>first class</u> if $\forall\ \sigma \in \Sigma$,

$$(T_\sigma\Sigma)^{\perp} \subset T_\sigma\Sigma$$

or <u>second class</u> if $\forall\ \sigma \in \Sigma$,

$$T_\sigma\Sigma \cap (T_\sigma\Sigma)^{\perp} = \{0\}.$$

23.6 <u>LEMMA</u> Σ is first class iff $\ell = k$.

<u>PROOF</u> To begin with,

$$(n - k) + \dim \Sigma = (n - k) + (n + k) = 2n$$

$$\Rightarrow$$

$$(n - k) + \dim T_\sigma\Sigma = 2n$$

$$\Rightarrow$$

$$(n - k) = \dim(T_\sigma\Sigma)^{\perp}.$$

But

$$(n - k) + (n - \ell) = (n - k) + \dim(T_\sigma\Sigma \cap (T_\sigma\Sigma)^{\perp}).$$

Therefore $\ell = k$

$$\Leftrightarrow (n - k) + (n - \ell) = 2(n - k)$$

$$\Leftrightarrow (n - k) + \dim(T_\sigma\Sigma \cap (T_\sigma\Sigma)^{\perp})$$

$$\Leftrightarrow \dim(T_\sigma\Sigma)^{\perp} = \dim(T_\sigma\Sigma \cap (T_\sigma\Sigma)^{\perp})$$

$$\Leftrightarrow (T_\sigma\Sigma)^{\perp} \subset T_\sigma\Sigma.$$

23.7 <u>LEMMA</u> Σ is second class iff $\ell = n$.

[Note: When this is the case, the pair (Σ, Ω_Σ) is a symplectic manifold.]

23.8 <u>REMARK</u> Because k is less than n, Σ cannot be simultaneously first and second class.

[Note: In general, Σ is neither but rather is of "mixed type".]

The $F \in C^\infty(TM)$ which are constant on the $(FL)^{-1}(\sigma)$ are annihilated by the $X \in \text{Ker}^V \omega_L$ and conversely. Denote by $C_L^\infty(TM)$ the set of such -- then $C^\infty(\Sigma) \approx C_L^\infty(TM)$ via

$$f \rightarrow (FL)^*f \quad (= f \circ FL).$$

23.9 <u>LEMMA</u> The energy $E_L = \Delta L - L$ lies in $C_L^\infty(TM)$, hence

$$E_L = (FL)^*H_\Sigma,$$

where $H_\Sigma \in C^\infty(\Sigma)$.

<u>PROOF</u> Working locally, take an $X \in \text{Ker}^V \omega_L$ and write

$$X = A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial v^i}.$$

Then

$$A^i = 0, \ \frac{\partial^2 L}{\partial v^i \partial v^j} B^j = 0.$$

Therefore

$$XE_L = \sum_i \sum_j B^j \frac{\partial}{\partial v^j} (v^i \frac{\partial L}{\partial v^i}) - \sum_i B^i \frac{\partial L}{\partial v^i}$$

$$= \sum_i \sum_j B^j \left(\frac{\partial v^i}{\partial v^j} \frac{\partial L}{\partial v^i} - v^i \frac{\partial^2 L}{\partial v^i \partial v^j}\right) - \sum_i B^i \frac{\partial L}{\partial v^i}$$

$$= \sum_i B^i \frac{\partial L}{\partial v^i} - \sum_i v^i \sum_j B^j \frac{\partial^2 L}{\partial v^i \partial v^j} - \sum_i B^i \frac{\partial L}{\partial v^i}$$

$$= 0.$$

[Note: H_Σ is the <u>hamiltonian</u> attached to L.]

23.10 <u>CORRESPONDENCE PRINCIPLE</u> To each $X_L \in \mathcal{D}^1(TM)$ such that

$$\iota_{X_L} \omega_L = - dE_L$$

there corresponds an $X_\Sigma \in \mathcal{D}^1(\Sigma)$ such that

$$\iota_{X_\Sigma} \Omega_\Sigma = - dH_\Sigma$$

with

$$X_L F \Big|_{(x,X_x)} = X_\Sigma f \Big|_{FL(x,X_x)} \quad (F = (FL)^* f).$$

Conversely, to each $X_\Sigma \in \mathcal{D}^1(\Sigma)$ such that

$$\iota_{X_\Sigma} \Omega_\Sigma = - dH_\Sigma$$

there corresponds an $X_L \in \mathcal{D}^1(TM)$ such that

$$\iota_{X_L} \omega_L = - dE_L$$

with

$$X_L F \Big|_{(x,X_x)} = X_\Sigma f \Big|_{FL(x,X_x)} \quad (F = (FL)^* f).$$

[Note: As a corollary, L admits global dynamics iff H_Σ admits global dynamics (in the obvious sense).]

To proceed further, it will be convenient to assume that $\exists\ \Phi_\mu \in C^\infty(T^*M)$ $(\mu = k + 1,\ldots,n)$ such that

$$\Sigma = \bigcap_\mu (\Phi_\mu)^{-1}(0)$$

with

$$\bigwedge_\mu d\Phi_\mu \neq 0$$

on Σ.

[Note: Bear in mind that $\dim \Sigma = n + k = 2n - (n - k)$.]

23.11 <u>EXAMPLE</u> Take $M = \underline{R}^n$ and let $L = 0$ -- then $k = 0$ and Σ consists of those points

$$(q^1,\ldots,q^n,p_1,\ldots,p_n) \in \underline{R}^{2n}$$

such that

$$p_i = 0 \quad (i = 1,\ldots,n),$$

so

$$\Phi_1 = p_1,\ldots,\Phi_n = p_n.$$

And here, of course, $H_\Sigma = 0$.

23.12 <u>EXAMPLE</u> Take $M = \underline{R}^n$ and let

$$L(q^1,\ldots,q^n,v^1,\ldots,v^n) = -\sum_{i=1}^n \frac{1}{2}(q^i)^2.$$

Then $k = 0$ and Σ is the same as in 23.11 but this time

$$H_\Sigma(q^1,\ldots,q^n) = \sum_{i=1}^{n} \frac{1}{2} (q^i)^2.$$

23.13 EXAMPLE Take $M = \underline{R}^2$ and let

$$L(q^1,q^2,v^1,v^2) = \frac{1}{2} (v^1)^2 e^{q^2}.$$

Then $k = 1$ and

$$FL(q^1,q^2,v^1,v^2) = (q^1,q^2,v^1 e^{q^2},0)$$

$\Rightarrow$

$$\Sigma = \{(q^1,q^2,p_1,p_2) \in \underline{R}^4 : p_2 = 0\}.$$

Furthermore

$$\Omega_\Sigma = i_\Sigma^*(dp_1 \wedge dq^1 + dp_2 \wedge dq^2) = dp_1 \wedge dq^1$$

$$\Rightarrow \text{rank } \Omega_\Sigma = 2 \Rightarrow \ell = 1.$$

So L is Type I (cf. 23.4). Finally

$$H_\Sigma(q^1,q^2,p_1) = \frac{1}{2} (p_1)^2 e^{-q^2}.$$

Indeed

$$\begin{aligned} & H_\Sigma \circ FL(q^1,q^2,v^1,v^2) \\ & = H_\Sigma(q^1,q^2,v^1 e^{q^2}) \\ & = \frac{1}{2} (v^1 e^{q^2})^2 e^{-q^2} \\ & = \frac{1}{2} (v^1)^2 e^{q^2} \end{aligned}$$

$$= E_L(q^1,q^2,v^1,v^2).$$

[Note: L does not admit global dynamics (cf. 22.5), thus 23.10 is not applicable.]

Any $f \in C^\infty(T^*M)$ such that

$$\Sigma \subset f^{-1}(0)$$

is called a constraint.

[Note: The Φ_μ are called primary constraints.]

N.B. A vector field $X \in \mathcal{D}^1(T^*M)$ is tangent to Σ iff $Xf|\Sigma = 0$ for all constraints f.

[Note: $\forall\ \sigma \in \Sigma$, $T_\sigma\Sigma$ consists of those $X_\sigma \in T_\sigma T^*M$ such that $X_\sigma f = 0$ for all constraints f.]

23.14 LEMMA Let f be a constraint -- then $\exists\ C^\infty$ functions f^μ such that

$$f = \sum_\mu f^\mu \Phi_\mu.$$

PROOF Given a point $\sigma \in \Sigma$, choose a coordinate system $\{\phi,\psi\}$ valid in a neighborhood U_σ of σ having the Φ_μ as its first coordinates. By hypothesis,

$$f(0,\psi) = 0$$

=>

$$f(\phi,\psi) = \int_0^1 \frac{d}{dt} f(t\phi,\psi)dt$$

$$= \sum_\mu f_\sigma^\mu \Phi_\mu,$$

where

$$f^{\mu}_{\sigma} = \int_0^1 f,_{\mu} \, (t\phi,\psi)\,dt.$$

To extend this to all of T*M, let U_{μ} be the set where $\Phi_{\mu} \neq 0$ and fix a C^{∞} partition of unity $\{\zeta_{\mu},\zeta_{\sigma}\}$ subordinate to the open covering

$$(\bigcup_{\mu} U_{\mu}) \cup (\bigcup_{\sigma} U_{\sigma}).$$

Put

$$f^{\mu} = f\,\frac{\zeta_{\mu}}{\Phi_{\mu}} + \sum_{\sigma} f^{\mu}_{\sigma}\zeta_{\sigma}.$$

Then

$$\begin{aligned}
f &= f(\sum_{\mu} \zeta_{\mu} + \sum_{\sigma} \zeta_{\sigma}) \\
&= \sum_{\mu} f\zeta_{\mu} + \sum_{\sigma} f\zeta_{\sigma} \\
&= \sum_{\mu} f\,\frac{\zeta_{\mu}}{\Phi_{\mu}}\,\Phi_{\mu} + \sum_{\sigma}\sum_{\mu} f^{\mu}_{\sigma}\zeta_{\sigma}\Phi_{\mu} \\
&= \sum_{\mu} f^{\mu}\Phi_{\mu}.
\end{aligned}$$

23.15 <u>RAPPEL</u> There are two arrows

$$\left[\begin{array}{l}
\Omega^{\flat}:\mathcal{D}^1(T^*M) \rightarrow \mathcal{D}_1(T^*M) \\
\\
\Omega^{\#}:\mathcal{D}_1(T^*M) \rightarrow \mathcal{D}^1(T^*M)
\end{array}\right.$$

that are mutually inverse, the <u>hamiltonian vector fields</u> being those elements of the form $X_f = - \Omega^{\#}df$ $(f \in C^{\infty}(T^*M))$.

[Note: The explanation for the minus sign is this. If in canonical local coordinates

$$df = \sum_i \left(\frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial p_i} dp_i\right),$$

then

$$- \Omega^{\#}df = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}\right).$$

Therefore, along an integral curve of X_f, we have

$$\begin{cases} \dot{q}^i = \dfrac{dq^i}{dt} = \dfrac{\partial f}{\partial p_i} \\ \\ \dot{p}_i = \dfrac{dp_i}{dt} = - \dfrac{\partial f}{\partial q^i}, \end{cases}$$

the equations of Hamilton.]

23.16 <u>LEMMA</u> Put $X_\mu = X_{\Phi_\mu}$ $(\mu = k + 1,\ldots,n)$ -- then $\forall\ \sigma \in \Sigma$, the span of the $X_\mu|_\sigma$ is $(T_\sigma\Sigma)^{\perp}$.

[Note: If f is a constraint, then

$$X_f|\Sigma \in (T\Sigma)^{\perp} = \bigcup_{\sigma\in\Sigma} (T_\sigma\Sigma)^{\perp}.]$$

The issue of whether L admits global dynamics can be shifted to the issue of whether H_Σ admits global dynamics (cf. 23.10). And for the latter there is a criterion.

23.17 <u>THEOREM</u> The equation

$$\iota_{X_\Sigma}\Omega_\Sigma = - dH_\Sigma$$

has a solution X_Σ iff $\exists$ an extension $H \in C^\infty(T^*M)$ of H_Σ with the property that

$$X_H\big|_\sigma \in T_\sigma\Sigma \ \forall \ \sigma \in \Sigma.$$

<u>PROOF</u> Under the assumption that such an extension exists, put $X_\Sigma = X_H\big|_\Sigma$ -- then $\forall \ X \in \mathcal{D}^1(\Sigma)$,

$$\begin{aligned}\iota_{X_\Sigma}\Omega_\Sigma(X) &= \Omega(X_\Sigma, X)\\ &= \Omega(X_H|\Sigma, X)\\ &= - d(H|\Sigma)(X)\\ &= - dH_\Sigma(X).\end{aligned}$$

Turning to the converse, let H be any extension of H_Σ -- then $\forall \ \sigma \in \Sigma \ \& \ \forall \ X \in T_\sigma\Sigma$,

$$\begin{aligned}&\Omega_\sigma(X_\Sigma\big|_\sigma - X_H\big|_\sigma, X)\\ &\quad = - dH_\Sigma\big|_\sigma(X) + dH\big|_\sigma(X)\\ &\quad = 0\end{aligned}$$

$\Rightarrow$

$$X_\Sigma\big|_\sigma - X_H\big|_\sigma \in (T_\sigma\Sigma)^\perp.$$

So, $\exists\ \Lambda^\mu \in C^\infty(T^*M)$ such that on Σ,

$$X_\Sigma - X_H = \Lambda^\mu X_\mu \quad \text{(cf. 23.16)}.$$

But $H + \Lambda^\mu \Phi_\mu$ is also an extension of H_Σ and on Σ

$$\begin{aligned} & d(H + \Lambda^\mu \Phi_\mu) \\ & \quad = dH + (d\Lambda^\mu)\Phi_\mu + \Lambda^\mu(d\Phi_\mu) \\ & \quad = dH + \Lambda^\mu(d\Phi_\mu) \\ & \quad = -\Omega^\flat(X_H) - \Lambda^\mu\Omega^\flat(X_\mu) \\ & \quad = -\Omega^\flat(X_H + \Lambda^\mu X_\mu) \\ & \quad = -\Omega^\flat(X_\Sigma). \end{aligned}$$

Therefore the hamiltonian vector field corresponding to $H + \Lambda^\mu \Phi_\mu$ is tangent to Σ.

23.18 <u>EXAMPLE</u> Take $M = \underline{R}^2$ and let

$$L(q^1,q^2,v^1,v^2) = \frac{1}{2}(v^1 + v^2)^2 - V(q^1 + q^2) \quad (V \in C^\infty(\underline{R})).$$

Then $k = 1$ and

$$FL(q^1,q^2,v^1,v^2) = (q^1,q^2,v^1 + v^2,v^1 + v^2)$$

$$\Rightarrow$$

$$\Sigma = \{(q^1,q^2,p_1,p_2) \in \underline{R}^4 : p_1 - p_2 = 0\},$$

so $\exists$ one primary constraint, viz.

$$\Phi(q^1,q^2,p_1,p_2) = p_1 - p_2,$$

thus

$$\Omega_\Sigma = i^*_\Sigma(dp_1 \wedge dq^1 + dp_2 \wedge dq^2)$$

$$= dp_1 \wedge dq^1 + dp_1 \wedge dq^2.$$

Consequently, if

$$X = f\,\frac{\partial}{\partial p_1} + A^1\,\frac{\partial}{\partial q^1} + A^2\,\frac{\partial}{\partial q^2}\,,$$

then

$$\iota_X \Omega_\Sigma = f dq^1 + f dq^2 - (A^1 + A^2) dp_1.$$

Therefore Ker Ω_Σ is spanned by $\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}$. Noting that

$$H_\Sigma(q^1,q^2,p_1,p_2) = \frac{1}{2}\,(p_1)^2 + V(q^1 + q^2),$$

consider the equation

$$\iota_{X_\Sigma} \Omega_\Sigma = - dH_\Sigma$$

$$= - p_1 dp_1 - V'(q^1 + q^2) dq^1 - V'(q^1 + q^2) dq^2.$$

Then a particular solution is

$$X_\Sigma = - V'(q^1 + q^2)\,\frac{\partial}{\partial p_1} + p_1\,\frac{\partial}{\partial q^1}$$

and the general solution is

$$X_\Sigma + F\left(\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}\right),$$

where F is some C^∞ function.

[Note: Since $\ell = k = 1$, Σ is first class (cf. 23.6). It is also clear that H_Σ can be extended to an H whose hamiltonian vector field X_H is tangent to Σ.]

23.19 <u>RAPPEL</u> The <u>Poisson bracket</u> is the bilinear function

$$\{\ ,\ \}: C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$$

defined by the rule

$$\{f,g\} = X_f g \ (= - X_g f) = \Omega(X_f, X_g).$$

Properties:

1. $\{f,g\} = - \{g,f\}$;

2. $\{f_1 f_2, g\} = \{f_1, g\} f_2 + f_1 \{f_2, g\}$;

3. $\{f, g_1 g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\}$;

4. $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$;

5. $X_{\{f,g\}} = [X_f, X_g]$.

In canonical local coordinates,

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i}\right).$$

Therefore

$$\{q^i, q^j\} = 0,\ \{p_i, p_j\} = 0,\ \{p_i, q^j\} = \delta_i^{\ j}.$$

[Note: Fix $H \in C^\infty(T^*M)$ and consider any C^∞ function $F(q^1, \ldots, q^n, p_1, \ldots, p_n)$ of the canonical local coordinates — then along an integral curve of X_H,

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial q^i} \dot{q}^i + \frac{\partial F}{\partial p_i} \dot{p}_i\right)$$

$$= \sum_i \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i}\right)$$

$$= \{H,F\}.$$

In particular:

$$\dot{q}^i = \{H,q^i\}, \ \dot{p}_i = \{H,p_i\}.]$$

23.20 <u>EXAMPLE</u> Suppose that

$$\iota_{X_\Sigma} \Omega_\Sigma = - dH_\Sigma \quad (X_\Sigma \in \mathcal{D}^1(\Sigma)).$$

Let $H \in C^\infty(T^*M)$ be any extension of H_Σ -- then $\exists\ \Lambda^\mu \in C^\infty(T^*M)$ such that

$$X_{H + \Lambda^\mu \Phi_\mu}$$

is tangent to Σ (cf. 23.17). Accordingly on Σ, $\forall$ constraint f

$$0 = X_{H + \Lambda^\mu \Phi_\mu} f$$

$$= \{H + \Lambda^\mu \Phi_\mu, f\}$$

$$= \{H,f\} + \{\Lambda^\mu \Phi_\mu, f\}$$

$$= \{H,f\} + \{\Lambda^\mu, f\}\Phi_\mu + \Lambda^\mu\{\Phi_\mu, f\}$$

$$= \{H,f\} + \Lambda^\mu\{\Phi_\mu, f\}.$$

Let $f \in C^\infty(T^*M)$ -- then f is said to be <u>first class</u> (w.r.t. Σ) if X_f is tangent to Σ.

23.21 <u>REMARK</u> In this terminology, one can restate 23.17: The equation

$$\iota_{X_\Sigma} \Omega_\Sigma = - dH_\Sigma$$

has a solution X_Σ iff $\exists$ an extension $H \in C^\infty(T^*M)$ of H_Σ which is first class.

23.22 <u>LEMMA</u> A function $f \in C^\infty(T^*M)$ is first class iff

$$\{f,\Phi_\mu\}|\Sigma = 0$$

for all primary constraints Φ_μ.

<u>PROOF</u> If f is first class, then X_f is tangent to Σ, so $\forall\ \mu$, $X_f\Phi_\mu|\Sigma = 0$, i.e.,

$$\{f,\Phi_\mu\}|\Sigma = 0.$$

To go the other way, take any constraint g and using 23.14, write

$$g = \sum_\mu g^\mu \Phi_\mu.$$

Then

$$X_f g = \sum_\mu (X_f g^\mu)\Phi_\mu + \sum_\mu g^\mu (X_f \Phi_\mu).$$

But

$$\Phi_\mu|\Sigma = 0 \text{ and } X_f\Phi_\mu|\Sigma = 0.$$

Therefore

$$X_f g|\Sigma = 0.$$

E.g.: If f is a constraint, then f^2 is first class. Proof: $\forall\ \mu$,

$$\{f^2,\Phi_\mu\} = 2f\{f,\Phi_\mu\}$$

$$\Rightarrow$$

$$\{f^2,\Phi_\mu\}|\Sigma = 0.$$

N.B. Σ is first class iff each of the primary constraints Φ_μ is first class or still, iff $\forall\ \mu',\mu''$:

$$\{\Phi_{\mu'},\Phi_{\mu''}\}|\Sigma = 0 \quad \text{(cf. 23.16)}.$$

23.23 EXAMPLE In the setup of 23.20, H is first class provided Σ is first class. To see this, take $f = \Phi_{\mu_0}$ -- then as there,

$$0 = \{H,\Phi_{\mu_0}\}|\Sigma + \Lambda^\mu\{\Phi_\mu,\Phi_{\mu_0}\}|\Sigma$$

$$= \{H,\Phi_{\mu_0}\}|\Sigma.$$

Finish by quoting 23.22.

23.24 REMARK It can be shown that a necessary and sufficient condition that the hamiltonian vector field $X_f \in \mathcal{D}^1(T^*M)$ be the projection through the fiber derivative FL of a vector field $\bar{X}_f \in \mathcal{D}^1(TM)$ is that f be first class.

[Note: There is then a commutative diagram

$$\begin{tikzcd} TTM \arrow[r, "TFL"] & TT^*M \\ TM \arrow[u, "\bar{X}_f"] \arrow[r, "FL"'] & T^*M, \arrow[u, "X_f"'] \end{tikzcd}$$

where $\bar{X}_f$ is unique up to an element of Ker TFL (= Ker FL_*). By means of a careful analysis, matters can be arranged so that

$$\bar{X}_f((FL)^*g) = (FL)^*\{f,g\} \qquad (g \in C^\infty(T^*M))$$

and

$$[\bar{X}_{f_1}, \bar{X}_{f_2}] = \bar{X}_{\{f_1,f_2\}},$$

the second point making sense since $\{f_1,f_2\}$ is again first class (cf. 23.25).

Let F_Σ be the set of functions $f \in C^\infty(T^*M)$ which are first class.

23.25 <u>LEMMA</u> F_Σ is closed under the formation of the Poisson bracket.

<u>PROOF</u> Let $f_1, f_2 \in F_\Sigma$ and fix μ -- then

$$\begin{cases} \{f_1,\Phi_\mu\}|\Sigma = 0 \\ \{f_2,\Phi_\mu\}|\Sigma = 0 \end{cases} \qquad \text{(cf. 23.22).}$$

But this simply means that

$$\begin{cases} \{f_1,\Phi_\mu\} \\ \{f_2,\Phi_\mu\} \end{cases}$$

are constraints, thus in view of 23.14

$$\begin{cases} \{f_1,\Phi_\mu\} = \Phi_1 \\ \\ \{f_2,\Phi_\mu\} = \Phi_2, \end{cases}$$

where Φ_1,Φ_2 are certain C^∞ linear combinations of the primary constraints. Now write

$$\begin{aligned} \{\{f_1,f_2\},\Phi_\mu\}|\Sigma &= \{f_1,\{f_2,\Phi_\mu\}\}|\Sigma - \{f_2,\{f_1,\Phi_\mu\}\}|\Sigma \\ &= \{f_1,\Phi_2\}|\Sigma - \{f_2,\Phi_1\}|\Sigma \\ &= 0. \end{aligned}$$

If Σ is not first class ($\Rightarrow k < \ell$ (cf. 23.6)), then it is possible to choose the primary constraints Φ_μ in such a way that

$$\Phi_{\ell + 1},\ldots,\Phi_n$$

are first class,

$$\Phi_{k + 1},\ldots,\Phi_\ell$$

then being termed <u>second class</u> primary constraints.

[Note: To arrange this, assume outright that the matrix

$$[\{\Phi_\mu,\Phi_\nu\}]$$

has constant rank $\ell - k$ on an open subset U of T^*M containing Σ and redefine the data (building in 23.27 below).]

23.26 <u>EXAMPLE</u> Take $M = \underline{R}^4$ and let

$$L(q^1,q^2,q^3,q^4,v^1,v^2,v^3,v^4)$$

$$= (q^2 + q^3)v^1 + q^4v^3 + \frac{1}{2}((q^4)^2 - 2q^2q^3 - (q^3)^2).$$

Then

$$W(L) = [0_4],$$

thus $k = 0$. Since

$$\frac{\partial L}{\partial v^1} = q^2 + q^3, \ \frac{\partial L}{\partial v^2} = 0, \ \frac{\partial L}{\partial v^3} = q^4, \ \frac{\partial L}{\partial v^4} = 0,$$

there are four primary constraints:

$$\Phi_1 = p_1 - q^2 - q^3, \ \Phi_2 = p_2, \ \Phi_3 = p_3 - q^4, \ \Phi_4 = p_4.$$

We have

$$[\{\Phi_\mu,\Phi_\nu\}] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ & & & \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

so $\Phi_1,\Phi_2,\Phi_3,\Phi_4$ are second class primary constraints. Next

$$\Omega_\Sigma = i_\Sigma^*(dp_1\wedge dq^1 + dp_2\wedge dq^2 + dp_3\wedge dq^3 + dp_4\wedge dq^4)$$

$$= dq^2\wedge dq^1 + dq^3\wedge dq^1 + dq^4\wedge dq^3,$$

which is symplectic, hence Σ is second class. Here

$$H_\Sigma(q^1,q^2,q^3,q^4,p_1,p_3)$$

$$= -\frac{1}{2}(q^2)^2 + \frac{1}{2}(p_1)^2 - \frac{1}{2}(p_3)^2.$$

Indeed

$$H_\Sigma \circ FL(q^1,q^2,q^3,q^4,v^1,v^2,v^3,v^4)$$

$$= -\frac{1}{2}(q^2)^2 + \frac{1}{2}(q^2+q^3)^2 - \frac{1}{2}(q^4)^2$$

$$= -\frac{1}{2}((q^4)^2 - 2q^2q^3 - (q^3)^2)$$

$$= E_L(q^1,q^2,q^3,q^4,v^1,v^2,v^3,v^4).$$

And the unique $X_\Sigma \in \mathcal{D}^1(\Sigma)$ such that

$$\iota_{X_\Sigma}\Omega_\Sigma = -\, dH_\Sigma$$

is

$$X_\Sigma = q^3\frac{\partial}{\partial q^1} + q^4\frac{\partial}{\partial q^2} - q^4\frac{\partial}{\partial q^3} - q^2\frac{\partial}{\partial q^4}.$$

At this point, it will be necessary to adopt an index convention, say:

$$\begin{cases} k+1 \le a,b \le \ell \\ \ell+1 \le u,v \le n. \end{cases}$$

Then

$$\begin{cases} X_a = X_{\Phi_a},\ X_b = X_{\Phi_b} \\ X_u = X_{\Phi_u},\ X_v = X_{\Phi_v}. \end{cases}$$

Put

$$[C_{ab}] = \begin{bmatrix} \{\Phi_{k+1}, \Phi_{k+1}\} & \cdots & \{\Phi_{k+1}, \Phi_{\ell}\} \\ \vdots & & \vdots \\ \{\Phi_{\ell}, \Phi_{k+1}\} & \cdots & \{\Phi_{\ell}, \Phi_{\ell}\} \end{bmatrix}.$$

23.27 <u>LEMMA</u> The matrix $[C_{ab}]$ is skewsymmetric and nonsingular on an open subset U of T*M containing Σ.

[Note: Therefore the number of second class primary constraints is even.]

For simplicity, it will be assumed in what follows that U = T*M (which is typically the case in practice) and we shall agree to write $[C^{ab}]$ for the inverse of $[C_{ab}]$.

Suppose that

$$<\mathrm{Ker}\ \Omega_{\Sigma}, - dH_{\Sigma}> = 0.$$

Then for any extension $H \in C^{\infty}(T^*M)$ of H_{Σ},

$$\{H, \Phi_v\} | \Sigma = 0 \quad (v = \ell + 1, \ldots, n).$$

Given $\Lambda^u \in C^{\infty}(T^*M)$, let

$$X = \{H, \Phi_a\} C^{ab} X_b - X_H + \Lambda^u X_u.$$

23.28 <u>LEMMA</u> X is tangent to Σ.

<u>PROOF</u> The Φ_u are first class, thus it is automatic that $\Lambda^u \Phi_u$ is tangent

to Σ, so we need only consider

$$\{H,\Phi_a\}C^{ab}X_b - X_H.$$

• $\{H,\Phi_a\}C^{ab}X_b\Phi_v|\Sigma - X_H\Phi_v|\Sigma$

$$= \{H,\Phi_a\}C^{ab}\{\Phi_b,\Phi_v\}|\Sigma - \{H,\Phi_v\}|\Sigma$$

$$= 0 \quad \text{(cf. 23.22)}.$$

• $\{H,\Phi_a\}C^{ab}X_b\Phi_{a'} - X_H\Phi_{a'}$

$$= \{H,\Phi_a\}C^{ab}C_{ba'} - \{H,\Phi_{a'}\}$$

$$= \{H,\Phi_{a'}\} = \{H,\Phi_{a'}\}$$

$$= 0.$$

Set

$$X_\Sigma = X|\Sigma.$$

Then the definitions imply that

$$\iota_{X_\Sigma}\Omega_\Sigma = - dH_\Sigma.$$

Therefore H_Σ admits global dynamics.

23.29 <u>REMARK</u> In general, the equation

$$\iota_{X_\Sigma}\Omega_\Sigma = - dH_\Sigma$$

need not be solvable on all of Σ. This sets the stage for an implementation of the constraint algorithm, the subject of the next §.

The foregoing theory can also be written in the time-dependent case. While relevant and interesting, I am nevertheless going to omit the details.

§24. THE CONSTRAINT ALGORITHM

Let M_0 be a connected C^∞ manifold of dimension n_0. Fix a closed 2-form $\omega_0 \in \Lambda^2 M_0$ of constant rank which is degenerate in the sense that

$$\text{Ker } \omega_0 = \{X_0 \in \mathcal{D}^1(M_0) : \iota_{X_0}\omega_0 = 0\}$$

is nontrivial.

[Note: The pair (M_0,ω_0) is a presymplectic manifold (cf. 15.20).]

Let $\alpha_0 \in \Lambda^1 M_0$ be a closed 1-form. Consider the equation

$$\iota_{X_0}\omega_0 = \alpha_0 \quad (X_0 \in \mathcal{D}^1(M_0)).$$

Then a solution, if there is one, is determined only up to an element of Ker ω_0.

[Note:

$$\iota_{X_0}\omega_0 = \alpha_0$$

$\Rightarrow$

$$\begin{aligned} L_{X_0}\omega_0 &= (\iota_{X_0} \circ d + d \circ \iota_{X_0})\omega_0 \\ &= d\iota_{X_0}\omega_0 \\ &= d\alpha_0 = 0.] \end{aligned}$$

24.1 <u>EXAMPLE</u> To realize this setup, take

$$\begin{bmatrix} M_0 = TM \\ \omega_0 = \omega_L \\ \alpha_0 = - dE_L, \end{bmatrix}$$

where L is a degenerate lagrangian per §22.

24.2 EXAMPLE To realize this setup, take

$$\begin{cases} M_0 = \Sigma \\ \omega_0 = \Omega_\Sigma \\ \alpha_0 = - dH_\Sigma, \end{cases}$$

where L is a degenerate lagrangian per §23.

Let $M \subset M_0$ be a submanifold, $i:M \to M_0$ the inclusion. Write

$$\begin{cases} \mathcal{D}^1(M_0;M) \text{ in place of } \mathcal{D}^1(M_0;M;i) \\ \mathcal{D}_1(M_0;M) \text{ in place of } \mathcal{D}_1(M_0;M;i) \end{cases} \quad (\text{cf. §13}).$$

Then there is a canonical pairing

$$\mathcal{D}^1(M_0;M) \times \mathcal{D}_1(M_0;M) \to C^\infty(M).$$

Let

$$\mathrm{Ker}(\omega_0|M) = \{X_0 \in \mathcal{D}^1(M_0;M) : (\omega_0|M)(X_0,X) = 0 \ \forall\ X \in \mathcal{D}^1(M)\}.$$

Denote by $(\omega_0|M)^\flat$ the map $\mathcal{D}^1(M) \to \mathcal{D}_1(M_0;M)$ which sends X to $(\omega_0|M)(X,\text{—})$.

24.3 LEMMA The range of $(\omega_0|M)^\flat$ consists of those $\alpha \in \mathcal{D}_1(M_0;M)$ such that

$$<\mathrm{Ker}(\omega_0|M),\alpha> = 0.$$

PROOF The annihilator of

$$(\omega_0|M)^\flat(\mathcal{D}^1(M))$$

is comprised of those $X_0 \in \mathcal{D}^1(M;M_0)$ with the property that

$$<X_0,(\omega_0|M)(X,\text{—})> = 0 \ \forall\ X \in \mathcal{D}^1(M)$$

or still,

$$(\omega_0|M)(X_0,X) = 0 \ \forall\ X \in \mathcal{D}^1(M).$$

I.e.:

$$\mathrm{Ann}((\omega_0|M)^\flat(\mathcal{D}^1(M))) = \mathrm{Ker}(\omega_0|M)$$

$$\Rightarrow$$

$$(\omega_0|M)^\flat(\mathcal{D}^1(M)) = \mathrm{Ann}\ \mathrm{Ann}((\omega_0|M)^\flat(\mathcal{D}^1(M)))$$

$$= \mathrm{Ann}\ \mathrm{Ker}(\omega_0|M).$$

Consider again the equation

$$\iota_{X_0}\omega_0 = \alpha_0.$$

Since $\omega_0^\flat$ is not surjective, the relation

$$<\mathrm{Ker}\ \omega_0,\alpha_0> = 0$$

need not be true, so let

$$M_1 = \{x_0 \in M_0 : <\mathrm{Ker}\ \omega_0,\alpha_0>(x_0) = 0\}.$$

We assume that M_1 is a submanifold. Put

$$\omega_1 = \omega_0|M_1,\ \alpha_1 = \alpha_0|M_1$$

and consider the equation

$$\iota_{X_1}\omega_1 = \alpha_1,$$

where now $X_1 \in \mathcal{D}^1(M_1)$. If α_1 is in the range of $\omega_1^{\flat}$, the process stops. Otherwise, let

$$M_2 = \{x_1 \in M_1 : \langle \mathrm{Ker}\ \omega_1, \alpha_1 \rangle\ (x_1) = 0\}$$

and continue on, generating thereby a chain of submanifolds

$$\cdots\ M_2 \rightarrow M_1 \rightarrow M_0.$$

If at the k^{th} stage,

$$\langle \mathrm{Ker}\ \omega_k, \alpha_k \rangle = 0$$

on all of M_k, the procedure ends since by construction $\exists\ X_k \in \mathcal{D}^1(M_k)$:

$$\iota_{X_k}\omega_k = \alpha_k.$$

M_k is called the final constraint manifold.

[Note: Conceivably, M_k could be empty or discrete, possibilities that we shall simply ignore.]

On the final constraint submanifold M_k, we have

$$\iota_{X_k}\omega_k = \alpha_k$$

for some $X_k \in \mathcal{D}^1(M_k)$. I.e.:

$$(\omega_0|M_k)(X_k, \text{—}) = \alpha_0|M_k,$$

this being an equality of elements of $\mathcal{D}_1(M_0;M_k)$. Let $i_k:M_k \to M_0$ be the inclusion -- then $\forall\ X \in \mathcal{D}^1(M_k)$,

$$(\omega_0|M_k)(X_k,X) = (\iota_{X_k}(i_k^*\omega_0))(X)$$

and

$$(\alpha_0|M_k)(X) = (i_k^*\alpha_0)(X),$$

thus

$$\iota_{X_k}(i_k^*\omega_0) = i_k^*\alpha_0.$$

[Note: In general, the set of X_k for which

$$\iota_{X_k}\omega_k = \alpha_k$$

is strictly contained in the set of X_k for which

$$\iota_{X_k}(i_k^*\omega_0) = i_k^*\alpha_0.]$$

24.4 <u>REMARK</u> If

$$Z \in \mathcal{D}^1(M_k) \cap \mathrm{Ker}(\omega_0|M_k),$$

then, as a functional on $\mathcal{D}^1(M_0;M_k)$,

$$(\omega_0|M_k)(Z,—) = 0,$$

hence

$$\iota_{X_k}\omega_k = \alpha_k$$

$$\Rightarrow$$

$$\iota_{X_k+Z}\omega_k = \alpha_k.$$

This failure of uniqueness is called <u>gauge freedom</u>.]

24.5 <u>EXAMPLE</u> Let M_0 be the submanifold of $T^*\underline{R}^4$ determined by the conditions $p_1 - q^4 = p_3 = p_4 = 0$ and take for ω_0 the pullback

$$i_0^*\Omega = i_0^* \left(\sum_{i=1}^{4} dp_i \wedge dq^i\right)$$

$$= dp_1 \wedge dq^1 + dq^4 \wedge dq^2,$$

$i_0 : M_0 \to T^*\underline{R}^4$ the inclusion -- then

$$\text{rank } \omega_0 = 4$$

and Ker ω_0 is spanned by $\frac{\partial}{\partial q^3}$. Let $\alpha_0 = - dH_0$, where

$$H_0 = \frac{1}{2}\left((p_1 - q^2)^2 + (q^3)^2\right),$$

and consider the equation

$$\iota_{X_0}\omega_0 = - dH_0 \qquad (X_0 \in \mathcal{D}^1(M_0)).$$

Using q^1, q^2, q^3, q^4, p_1 as coordinates on M_0, write

$$X_0 = f\frac{\partial}{\partial p_1} + \sum_{i=1}^{4} A^i \frac{\partial}{\partial q^i}.$$

Then

$$\left[\begin{array}{l} \iota_{X_0}(dp_1 \wedge dq^1) = f dq^1 - A^1 dp_1 \\ \iota_{X_0}(dq^4 \wedge dq^2) = A^4 dq^2 - A^2 dq^4 \end{array}\right.$$

$\Rightarrow$

$$\iota_{X_0}\omega_0 = -A^1 dp_1 + f dq^1 + A^4 dq^2 - A^2 dq^4.$$

On the other hand,

$$dH_0 = (p_1 - q^2)dp_1 + (q^2 - p_1)dq^2 + q^3 dq^3.$$

Restricting the data to $M_1 = \{q^3 = 0\}$ and comparing $\iota_{X_0}\omega_0$ with $-dH_0$, we find that

$A^1 = p_1 - q^2$, $A^2 = 0$, $A^4 = p_1 - q^2$, $f = 0$, thus

$$X_0 = (p_1 - q^2)\ (\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^4}) + A^3 \frac{\partial}{\partial q^3},$$

A^3 being undetermined. Now choose $A^3 = 0$ -- then

$$X_1 = (p_1 - q^2)\ (\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^4})$$

is tangent to M_1, so the algorithm terminates at this point.

Expanding on 23.29, if H_Σ does not admit global dynamics, then the resolution is to set the constraint algorithm into motion:

$$\Sigma \supset \Sigma', \Sigma' \supset \Sigma'', \ldots .$$

Here (cf. 24.2),

$$M_0 \leftrightarrow \Sigma$$

$$M_1 \leftrightarrow \Sigma'$$

$$M_2 \leftrightarrow \Sigma''$$

$$\vdots$$

In more detail, one supposes that there is a solution valid on some submanifold $\Sigma' \subset \Sigma$ which is described by <u>secondary constraints</u>. Such a solution need not be tangent to Σ'. One then has to pass to a submanifold $\Sigma'' \subset \Sigma'$ where the solution is tangent to Σ', Σ'' being described by <u>tertiary constraints</u>. And so forth... . For a physical system with reasonable dynamics this process terminates at a submanifold $\Sigma_0 \subset \Sigma$ described by certain constraints and on which the equation

$$\iota_{X_{\Sigma_0}} \Omega_{\Sigma_0} = - dH_{\Sigma_0}$$

can be solved (but, of course, it need not be true that $\pi_M^*(\Sigma_0) = M$).

To make matters precise, let us suppose that Σ' is a submanifold of Σ of dimension $(n + k) + (n - k')$, where $n \leq k' \leq n + (n + k)$ (thus the codimension of Σ' w.r.t. Σ is $(n + k) - ((n + k) + (n - k')) = k' - n$ and the codimension of Σ' w.r.t. T^*M is $2n - ((n + k) + (n - k')) = k' - k$). In addition, we shall impose a regularity condition, viz. that $\exists\ \chi_\tau \in C^\infty(T^*M)$ $(\tau = n + 1,\ldots,k')$ such that

$$\Sigma' = \Sigma \cap \bigcap_\tau \chi_\tau^{-1}(0)$$

with

$$\wedge_\tau d\chi_\tau|_{\sigma'} \neq 0 \ \forall \ \sigma' \in \Sigma'.$$

[Note: The χ_τ are called <u>secondary constraints</u>.]

24.6 <u>REMARK</u> Initially,

$$\Sigma' = \{\sigma \in \Sigma : \langle \text{Ker } \Omega_\Sigma, dH_\Sigma \rangle (\sigma) = 0\}$$

and, by construction,

$$\Sigma'' = \{\sigma' \in \Sigma' : \langle \mathrm{Ker}(\Omega_\Sigma|\Sigma'), \, dH_\Sigma|\Sigma'\rangle(\sigma') = 0\}.$$

To say that there are no tertiary constraints amounts to saying that $\Sigma' = \Sigma''$, thus the final constraint submanifold is Σ' itself. So, $\exists\ X_{\Sigma'} \in \mathcal{D}^1(\Sigma')$:

$$(\Omega_\Sigma|\Sigma')(X_{\Sigma'}, \text{---}) = -\, dH_\Sigma|\Sigma',$$

this being an equality of elements of $\mathcal{D}_1(\Sigma;\Sigma')$. Put

$$\Omega_{\Sigma'} = i^*_{\Sigma'}\Omega_\Sigma(i_{\Sigma'}:\Sigma' \to \Sigma).$$

Then

$$\iota_{X_{\Sigma'}}\Omega_{\Sigma'} = -\, dH_{\Sigma'},$$

where $H_{\Sigma'} = H_\Sigma|\Sigma'$ (observe that $dH_{\Sigma'} = d(H_\Sigma|\Sigma') = d(i^*_{\Sigma'}H_\Sigma) = i^*_{\Sigma'}dH_\Sigma$).

24.7 <u>EXAMPLE</u> Take $M = \underline{R}^2$ and let

$$L(q^1,q^2,v^1,v^2) = \frac{1}{2}\,(v^1)^2 + \frac{1}{2}\,(q^1)^2 q^2.$$

Then

$$W(L) = \begin{bmatrix} 1 & 0 \\ & \\ 0 & 0 \end{bmatrix},$$

thus $k = 1$. Because

$$\frac{\partial L}{\partial v^1} = v^1, \ \frac{\partial L}{\partial v^2} = 0,$$

there is one primary constraint, viz.

$$\Phi(q^1,q^2,p_1,p_2) = p_2.$$

So

$$\Sigma = \{(q^1,q^2,p_1,p_2):p_2 = 0\}.$$

And

$$\begin{cases} \Omega_\Sigma = dp_1 \wedge dq^1 \\ H_\Sigma = \frac{1}{2}(p_1)^2 - \frac{1}{2}(q^1)^2 q^2. \end{cases}$$

Given

$$X_\Sigma = f\frac{\partial}{\partial p_1} + A^1\frac{\partial}{\partial q^1} + A^2\frac{\partial}{\partial q^2} \in \mathcal{D}^1(\Sigma),$$

we have

$$\iota_{X_\Sigma}\Omega_\Sigma = \iota_{X_\Sigma}(dp_1 \wedge dq^1)$$

$$= f dq^1 - A^1 dp_1.$$

Accordingly, Ker Ω_Σ is spanned by $\frac{\partial}{\partial q^2}$. But

$$dH_\Sigma = p_1 dp_1 - q^1 q^2 dq^1 - \frac{1}{2}(q^1)^2 dq^2,$$

hence

$$\Sigma' = \{\sigma \in \Sigma : \langle \text{Ker } \Omega_\Sigma, dH_\Sigma \rangle(\sigma) = 0\}$$

$$= \{(q^1,q^2,p_1,0):q^1 = 0\}.$$

Therefore Σ' is described by the secondary constraint

$$\chi(q^1,q^2,p_1,p_2) = q^1.$$

However $\nexists\ X_{\Sigma'} \in \mathcal{D}^1(\Sigma')$:

$$(\Omega_\Sigma|\Sigma')(X_{\Sigma'}, —) = -\, dH_\Sigma|\Sigma'.$$

To proceed, it is necessary to impose the tertiary constraint $p_1 = 0$. To confirm this, let us determine Σ'' which, by definition, is the set of $\sigma' \in \Sigma'$:

$$<\mathrm{Ker}(\Omega_\Sigma|\Sigma'), dH_\Sigma|\Sigma'>(\sigma') = 0.$$

Let

$$\left[\begin{array}{l} X = F\dfrac{\partial}{\partial p_1} + A^1\dfrac{\partial}{\partial q^1} + A^2\dfrac{\partial}{\partial q^2} \in \mathcal{D}^1(\Sigma;\Sigma') \\[2em] Y = G\dfrac{\partial}{\partial p_1} + B^2\dfrac{\partial}{\partial q^2} \in \mathcal{D}^1(\Sigma'). \end{array}\right.$$

Then

$$X \in \mathrm{Ker}(\Omega_\Sigma|\Sigma')$$

iff $\forall$ Y,

$$dp_1 \wedge dq^1(X,Y) = 0$$

<=>

$$dp_1(X)dq^1(Y) - dp_1(Y)dq^1(X) = 0$$

<=>

$$F\cdot 0 - GA^1 = 0$$

=>

$$A^1 = 0.$$

Since

$$dH_\Sigma|\Sigma' = p_1 dp_1,$$

it follows that

$$\langle F \frac{\partial}{\partial p_1} + A^2 \frac{\partial}{\partial q^2},\ p_1 dp_1 \rangle = Fp_1$$

is zero for all F precisely at those σ' at which $p_1 = 0$. Moreover the dynamics on Σ'' are trivial. Indeed,

$$\iota_{\partial/\partial q^2}\Omega_\Sigma|\Sigma'' = 0 = dH_\Sigma|\Sigma''.$$

[Note: Consider the constraints of the preceding example:

$$p_2 = 0 \text{ --- primary}$$

$$q^1 = 0 \text{ --- secondary}$$

$$p_1 = 0 \text{ --- tertiary.}$$

Then

$$\{p_1,p_1\} = 0,\ \{p_2,p_2\} = 0,\ \{q^1,q^1\} = 0$$

$$\{p_1,p_2\} = 0,\ \{p_1,q^1\} = 1,\ \{p_2,q^1\} = 0.]$$

APPENDIX

There are physically reasonable lagrangians that lead to constraints beyond the tertiary level.

Thus let $M = \underline{R}^3$ and put

$$L = \frac{1}{2}\,((v^1)^2 + (v^2)^2) - \frac{1}{2}\,q^3((q^1)^2 + (q^2)^2 - 1).$$

Since

$$W(L) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has constant rank $k = 2$, it follows that $\dim \Sigma = n + k = 3 + 2 = 5$, the primary constraint being $p_3 = 0$. Therefore

$$\Omega_\Sigma = dp_1 \wedge dq^1 + dp_2 \wedge dq^2.$$

So, if

$$X_\Sigma = f_1 \frac{\partial}{\partial p_1} + f_2 \frac{\partial}{\partial p_2} + \sum_{i=1}^{3} A^i \frac{\partial}{\partial q^i},$$

then

$$\iota_{X_\Sigma} \Omega_\Sigma = f_1 dq^1 + f_2 dq^2 - A^1 dp_1 - A^2 dp_2.$$

Accordingly, Ker Ω_Σ is spanned by $\frac{\partial}{\partial q^3}$. On the other hand,

$$H_\Sigma = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2} q^3((q^1)^2 + (q^2)^2 - 1)$$

$\Rightarrow$

$$dH_\Sigma = p_1 dp_1 + p_2 dp_2 + q^1 q^3 dq^1 + q^2 q^3 dq^2 + \frac{1}{2}((q^1)^2 + (q^2)^2 - 1) dq^3.$$

Thus

$$\Sigma' = \{\sigma \in \Sigma : \langle \text{Ker } \Omega_\Sigma, dH_\Sigma \rangle (\sigma) = 0\}$$

$$= \{(q^1, q^2, q^3, p_1, p_2) : (q^1)^2 + (q^2)^2 = 1\}.$$

I.e.: Σ' is described by the secondary constraint $(q^1)^2 + (q^2)^2 = 1$ and there

$$\iota_{X_\Sigma} \Omega_\Sigma = - dH_\Sigma,$$

where

$$X_\Sigma = - q^1q^3 \frac{\partial}{\partial p_1} - q^2q^3 \frac{\partial}{\partial p_2} + p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2} \,.$$

But X_Σ is not tangent to Σ' unless we impose the tertiary constraint $p_1q^1 + p_2q^2 = 0$.

To see that this agrees with what is predicted by the theory, it is necessary to consider Σ'', the set of $\sigma' \in \Sigma'$:

$$\langle \mathrm{Ker}(\Omega_\Sigma|\Sigma'), dH_\Sigma|\Sigma'\rangle(\sigma') = 0.$$

Let

$$\begin{cases} X = F_1 \dfrac{\partial}{\partial p_1} + F_2 \dfrac{\partial}{\partial p_2} + \displaystyle\sum_{i=1}^{3} A^i \dfrac{\partial}{\partial q^i} \in \mathcal{D}^1(\Sigma;\Sigma') \\ Y = G_1 \dfrac{\partial}{\partial p_1} + G_2 \dfrac{\partial}{\partial p_2} + - q^2 \dfrac{\partial}{\partial q^1} + q^1 \dfrac{\partial}{\partial q^2} \in \mathcal{D}^1(\Sigma'). \end{cases}$$

Then

$$X \in \mathrm{Ker}(\Omega_\Sigma|\Sigma')$$

iff $\forall$ Y,

$$dp_1 \wedge dq^1(X,Y) + dp_2 \wedge dq^2(X,Y) = 0$$

$\Leftrightarrow$

$$- q^2F_1 + q^1F_2 = G_1A^1 + G_2A^2$$

$\Rightarrow$

$$\begin{cases} A^1 = 0 \\ A^2 = 0, \end{cases}$$

G_1 and G_2 being arbitrary. But

$$dH_\Sigma|\Sigma' = p_1dp_1 + p_2dp_2 + q^1q^3dq^1 + q^2q^3dq^2,$$

hence

$$\langle X, dH_\Sigma|\Sigma'\rangle = p_1F_1 + p_2F_2$$

vanishes for all $X \in \mathrm{Ker}(\Omega_\Sigma|\Sigma')$ at those σ':

$$p_1F_1 + p_2F_2 = 0$$

subject to

$$- q^2F_1 + q^1F_2 = 0.$$

The condition

$$p_1q^1 + p_2q^2 \neq 0$$

allows only the trivial solution $F_1 = F_2 = 0$, thus the tertiary constraint is $p_1q^1 + p_2q^2 = 0$.

Recall now that

$$X_\Sigma = - q^1q^3 \frac{\partial}{\partial p_1} - q^2q^3 \frac{\partial}{\partial p_2} + p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2} \in \mathcal{D}^1(\Sigma;\Sigma')$$

and put

$$X_\Sigma' = X_\Sigma|\Sigma''.$$

Then $X_\Sigma' \in \mathcal{D}^1(\Sigma';\Sigma'')$ but X_Σ' is not tangent to Σ'', thus it will be necessary to impose yet another constraint. Consider

$$A \frac{\partial}{\partial q^1} + B \frac{\partial}{\partial q^2} + C \frac{\partial}{\partial p_1} + D \frac{\partial}{\partial p_2} \in T_{\sigma''}\Sigma'.$$

To figure out the conditions on A,B,C,D which guarantee that this vector is in $T_{\sigma''}\Sigma''$, let

$$f(q^1,q^2,p_1,p_2) = p_1q^1 + p_2q^2.$$

Then

$$\left[\begin{array}{l} \dfrac{\partial f}{\partial q^1} = p_1 \ , \ \dfrac{\partial f}{\partial q^2} = p_2 \\ \\ \dfrac{\partial f}{\partial p_1} = q^1 \ , \ \dfrac{\partial f}{\partial p_2} = q^2 \end{array}\right.$$

=>

$$\nabla f \cdot (A,B,C,D)$$

$$= p_1A + p_2B + q^1C + q^2D.$$

Therefore

$$A \frac{\partial}{\partial q^1} + B \frac{\partial}{\partial q^2} + C \frac{\partial}{\partial p_1} + D \frac{\partial}{\partial p_2} \in T_{\sigma''}\Sigma''$$

iff

$$p_1A + p_2B + q^1C + q^2D = 0.$$

In our case:

$$A = p_1,\ B = p_2,\ C = -\,q^1q^3,\ D = -\,q^2q^3,$$

so the next constraint is

$$p_1^2 + p_2^2 - q^3((q^1)^2 + (q^2)^2) = 0$$

or still,

$$p_1^2 + p_2^2 = q^3.$$

Additional computation shows that there are no other constraints. Therefore the final constraint submanifold $\Sigma_0 \subset \Sigma$ is described by

$$\begin{cases} (q^1)^2 + (q^2)^2 = 1 \\ p_1 q^1 + p_2 q^2 = 0 \\ p_1^2 + p_2^2 = q^3, \end{cases}$$

hence Σ_0 is two dimensional.

We have

$$\Sigma \supset \Sigma' \supset \Sigma'' \supset \Sigma_0$$

with

$$\begin{cases} X_\Sigma \in \mathcal{D}^1(\Sigma;\Sigma') \\ X_\Sigma' \in \mathcal{D}^1(\Sigma';\Sigma''). \end{cases}$$

So, if $X_0 = X_\Sigma' | \Sigma_0$, then by construction,

$$X_0 \in \mathcal{D}^1(\Sigma_0)$$

and

$$\iota_{X_0}(\Omega_\Sigma | \Sigma_0) = - dH_\Sigma | \Sigma_0,$$

this being an equality of elements of $\mathcal{D}_1(\Sigma;\Sigma_0)$ (or $\mathcal{D}_1(\Sigma_0)$, provided the data is pulled back to Σ_0).

The integral curves of X_0 depend on two parameters θ,ω and are given by

$$\begin{cases} q^1(t) = \cos(\omega t + \theta) \\ q^2(t) = \sin(\omega t + \theta) \end{cases} \qquad q^3(t) = \omega^2,$$

$$\begin{cases} p_1(t) = -\,\omega \sin(\omega t + \theta) \\ p_2(t) = \omega \cos(\omega t + \theta). \end{cases}$$

<u>N.B.</u> In the situation at hand, there is no gauge freedom, i.e., X_0 is unique. To see this, it suffices to note that the pullback of

$$dp_1 \wedge dq^1 + dp_2 \wedge dq^2$$

to Σ_0 is nondegenerate. Thus define a map

$$f:]0,2\pi[\times \underline{R} \to \Sigma_0$$

by the prescription

$$\begin{cases} q^1 = \cos\theta,\ q^2 = \sin\theta,\ q^3 = \omega^2 \\ p_1 = -\,\omega\sin\theta,\ p_2 = \omega\cos\theta. \end{cases}$$

Then

$$d(-\,\omega\sin\theta) \wedge d\cos\theta + d(\omega\cos\theta) \wedge d\sin\theta$$

$$= (-\sin\theta\, d\omega - \omega\cos\theta\, d\theta) \wedge (-\sin\theta) d\theta$$

$$+ (\cos\theta\, d\omega - \omega\sin\theta\, d\theta) \wedge (\cos\theta) d\theta$$

$$= (\sin^2\theta + \cos^2\theta) d\omega \wedge d\theta = d\omega \wedge d\theta.$$

Turning to the physical interpretation, the above lagrangian is that of a particle of unit mass moving on a circle of radius 1 in a two dimensional plane spanned by q^1, q^2 with q^3 being the force necessary to make the particle stay on the circle.

§25. FIRST CLASS SYSTEMS

Let (M,Ω) be a symplectic manifold of dimension $2n$ (M connected).

Suppose that $C \subset M$ is a closed connected submanifold. Assume: $\exists\ \Phi_\mu \in C^\infty(M)$ $(\mu = 1,\ldots,k\ (k < n))$ such that

$$C = \bigcap_\mu (\Phi_\mu)^{-1}(0)$$

with

$$\wedge_\mu d\Phi_\mu \neq 0$$

on C.

Put

$$\omega_C = i_C^*\Omega \quad (i_C : C \to M)$$

and impose the a priori hypothesis that the rank of ω_C is constant, hence that the pair (C,ω_C) is a presymplectic manifold. Therefore $\operatorname{Ker}\omega_C$ is integrable (cf. 15.20), so there is a decomposition

$$C = \coprod_i C_i,$$

C_i a generic leaf of the associated foliation.

Next, introduce

$$(TC)^\perp \subset TM|C.$$

Then C is said to be <u>first class</u> if

$$(TC)^\perp \subset TM.$$

<u>N.B.</u> Consequently,

$$\operatorname{Ker}\omega_C = (TC)^\perp.$$

In what follows, we shall take C first class.

Let $f \in C^\infty(M)$ -- then f is said to be a <u>Dirac observable</u> if X_f is tangent to C.

[Note: As usual, X_f is the hamiltonian vector field attached to f.]

25.1 <u>REMARK</u> In the context of §23, the Dirac observables are precisely the $f \in C^\infty(T^*M)$ which are first class (w.r.t. Σ).

25.2 <u>LEMMA</u> A function $f \in C^\infty(M)$ is a Dirac observable iff $\forall\ \mu$,

$$\{f,\Phi_\mu\}|C = 0.$$

[The argument used in 23.22 is clearly applicable here as well.]

In particular: $\forall\ \mu',\mu''$,

$$\{\Phi_{\mu'},\Phi_{\mu''}\}|C = 0$$

=>

$$\{\Phi_{\mu'},\Phi_{\mu''}\} = \sum_\mu f^\mu{}_{\mu'\mu''}\Phi_\mu,$$

where

$$f^\mu{}_{\mu'\mu''} \in C^\infty(M) \qquad \text{(cf. 23.14)}.$$

Fix a positive definite quadratic form K and let

$$\underline{M} = \frac{1}{2} K^{\mu\nu}\Phi_\mu\Phi_\nu.$$

Then

$$C = \underline{M}^{-1}(0).$$

[Note:

$$d\underline{M} = \frac{1}{2}\,(K^{\mu\nu}(d\Phi_\mu)\Phi_\nu + K^{\mu\nu}\Phi_\mu(d\Phi_\nu))$$

$$\Rightarrow$$

$$d\underline{M}|C = 0.]$$

25.3 <u>LEMMA</u> $\forall\ f \in C^\infty(M)$,

$$\{f,\underline{M}\}|C = 0.$$

<u>PROOF</u> In fact,

$$\begin{aligned}\{f,\underline{M}\}|C &= (X_f\underline{M})|C \\ &= d\underline{M}(X_f)|C \\ &= 0.\end{aligned}$$

25.4 <u>LEMMA</u> Let $f \in C^\infty(M)$ -- then f is a Dirac observable iff

$$\{f,\{f,\underline{M}\}\}|C = 0.$$

<u>PROOF</u> We have

$$\begin{aligned}&\{f,\{f,\underline{M}\}\} \\ &= \frac{1}{2}\,\{f,\{f,K^{\mu\nu}\Phi_\mu\Phi_\nu\}\} \\ &= \frac{1}{2}\,\{f,K^{\mu\nu}\{f,\Phi_\mu\Phi_\nu\}\} \\ &= \frac{1}{2}\,\{f,K^{\mu\nu}(\{f,\Phi_\mu\}\Phi_\nu + \{f,\Phi_\nu\}\Phi_\mu)\}\end{aligned}$$

$$= \frac{1}{2} K^{\mu\nu}(\{f,\{f,\Phi_\mu\}\Phi_\nu\} + \{f,\{f,\Phi_\nu\}\Phi_\mu\})$$

$$= \frac{1}{2} K^{\mu\nu}(\{f,\{f,\Phi_\mu\}\}\Phi_\nu + \{f,\Phi_\nu\}\{f,\Phi_\mu\}$$

$$+ \{f,\{f,\Phi_\nu\}\}\Phi_\mu + \{f,\Phi_\mu\}\{f,\Phi_\nu\})$$

=>

$$\{f,\{f,\underline{M}\}\}|C$$

$$= (\{f,\Phi_\mu\}|C)K^{\mu\nu}(\{f,\Phi_\nu\}|C).$$

Therefore

$$\{f,\{f,\underline{M}\}\}|C = 0$$

iff

$$\{f,\Phi_1\}|C = 0,\ldots,\{f,\Phi_k\}|C = 0$$

or still,

$$\{f,\{f,\underline{M}\}\}|C = 0$$

iff f is a Dirac observable (cf. 25.2).

Let $H \in C^\infty(C)$ -- then H is said to admit <u>global dynamics</u> if $\exists\, X_H \in \mathcal{D}^1(C)$:

$$\iota_{X_H}\omega_C = - dH.$$

25.5 <u>LEMMA</u> If H admits global dynamics, then H is constant on the C_i, hence is a first integral for Ker ω_C.

PROOF Suppose that X is tangent to C_i, thus $X \in (TC)^{\perp}$ and

$$\begin{aligned} XH &= dH(X) \\ &= -\iota_{X_H}\omega_C(X) \\ &= -\omega_C(X_H,X) \\ &= \omega_C(X,X_H) \\ &= 0. \end{aligned}$$

In general, the quotient $C/\mathrm{Ker}\ \omega_C$ does not carry the structure of a C^∞ manifold. However, let us assume that it does and that the projection

$$\pi: C \to C/\mathrm{Ker}\ \omega_C$$

is a fibration.

N.B. Under these circumstances, one calls $C/\mathrm{Ker}\ \omega_C$ the reduced phase space of the theory.

Write $\tilde{C}$ for $C/\mathrm{Ker}\ \omega_C$ -- then there is a 2-form $\omega_{\tilde{C}}$ on $\tilde{C}$ such that

$$\omega_C = \pi^*\omega_{\tilde{C}}.$$

To see this, let $\tilde{X}_1, \tilde{X}_2$ be two vectors tangent to $\tilde{x} \in \tilde{C}$. Choose a point x in the leaf C_i lying over $\tilde{x}$ and let X_1, X_2 be two vectors tangent to x:

$$\left[\begin{array}{l} \tilde{X}_1 = \pi_* X_1 \\ \\ \tilde{X}_2 = \pi_* X_2. \end{array}\right.$$

Set

$$\omega_{\tilde{C}}|_{\tilde{x}}(\tilde{X}_1,\tilde{X}_2) = \omega_C|_x(X_1,X_2).$$

25.6 <u>LEMMA</u> $\omega_{\tilde{C}}$ is welldefined.

<u>PROOF</u> We have to show that the definition is independent of the choice of x and the choice of X_1,X_2. First, ω_C is constant along a leaf: $\forall\ Z \in (TC)^{\perp}$,

$$L_Z\omega_C = (\iota_Z \circ d + d \circ \iota_Z)\omega_C = 0.$$

Second, if

$$\left[\begin{array}{l} \tilde{X}_1 = \pi_* Y_1 \\ \\ \tilde{X}_2 = \pi_* Y_2, \end{array}\right.$$

then

$$\left[\begin{array}{l} Y_1 = X_1 + Z_1 \\ \\ Y_2 = X_2 + Z_2, \end{array}\right.$$

where $Z_1,Z_2 \in (TC)^{\perp}$. Therefore

$$\begin{aligned} \omega_C|_x(Y_1,Y_2) &= \omega_C|_x(X_1 + Z_1,X_2 + Z_2) \\ &= \omega_C|_x(X_1,X_2). \end{aligned}$$

25.7 <u>LEMMA</u> $\omega_{\tilde{C}}$ is symplectic.

PROOF Suppose that for some $\tilde{X}_0$:

$$\omega_{\tilde{C}}(\tilde{X}_0,\tilde{X}) = 0 \ \forall \ \tilde{X}.$$

Then

$$\omega_C|_x(X_0,X) = 0 \ \forall \ X$$

$$\Rightarrow$$

$$X_0 \in \operatorname{Ker} \omega_C|_x$$

$$\Rightarrow$$

$$\tilde{X}_0 = \pi_* X_0 = 0.$$

The function H projects to a function $\tilde{H} \in C^\infty(\tilde{C})$ (cf. 25.5). Furthermore, there exists a unique $X_{\tilde{H}} \in \mathcal{D}^1(\tilde{C})$:

$$\iota_{X_{\tilde{H}}} \omega_{\tilde{C}} = - d\tilde{H}.$$

And finally $X_{\tilde{H}}$ is the projection of any X_H:

$$\iota_{X_H} \omega_C = - dH.$$

25.8 REMARK All Dirac observables project to $\tilde{C}$.

APPENDIX: KINEMATICS OF THE FREE RIGID BODY

To establish notation, let

$$\begin{cases} \underline{SO}(3) = \{A \in \underline{GL}(3,\underline{R}) : AA^{T} = I,\ \det A = 1\} \\ \\ \underline{so}(3) = \{X \in \underline{g\ell}(3,\underline{R}) : X + X^{T} = 0\}, \end{cases}$$

the "T" standing for transpose -- then $\underline{so}(3)$ is the Lie algebra of $\underline{SO}(3)$.

A.1 RAPPEL The arrow $\underline{R}^3 \to \underline{so}(3)$ that sends

$$x = (x^1, x^2, x^3)$$

to

$$\hat{x} = \begin{bmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{bmatrix}$$

is an isomorphism of the Lie algebra $(\underline{R}^3, \times)$ with the Lie algebra $(\underline{so}(3), [\ ,\ ])$:

$$(x \times y)^{\wedge} = [\hat{x}, \hat{y}] \qquad (x, y \in \underline{R}^3).$$

It is equivariant in the sense that $\forall\ A \in \underline{SO}(3)$,

$$(Ax)^{\wedge} = A\hat{x}A^{-1} \qquad (x \in R^3).$$

[Note: Equip $\underline{so}(3)$ with the metric derived from the Killing form, thus

$$k(X,Y) = -\frac{1}{2}\operatorname{tr}(XY) \qquad (X,Y \in \underline{so}(3)).$$

Then the arrow $x \to \hat{x}$ is an isometry:

$$\langle x,y\rangle = k(\hat{x},\hat{y}) \qquad (x,y \in \underline{R}^3).]$$

The tangent bundle $T\underline{SO}(3)$ admits two trivializations, viz.

$$\begin{cases} \lambda: T\underline{SO}(3) \to \underline{SO}(3) \times \underline{so}(3) & \text{(left)} \\ \rho: T\underline{SO}(3) \to \underline{SO}(3) \times \underline{so}(3) & \text{(right)}. \end{cases}$$

To explain this, view $\underline{GL}(3,\underline{R})$ as an open subset of $\underline{R}^{3\times 3}$ -- then the tangent space of $\underline{GL}(3,\underline{R})$ at a given point is naturally isomorphic to $\underline{g\ell}(3,\underline{R})$. Since $\underline{SO}(3)$ is contained in $\underline{GL}(3,\underline{R})$, it follows that the elements of $T_A\underline{SO}(3)$ are matrix pairs (A,X). One then puts

$$\begin{cases} \lambda(A,X) = (A,A^{-1}X) \\ \rho(A,X) = (A,XA^{-1}). \end{cases}$$

[Note: To check, e.g., that $A^{-1}X \in \underline{so}(3)$, fix a curve $t \to A(t)$ such that $A(0) = A$, $A'(0) = X$ -- then

$$A(t)^{T}A(t) = I$$

$$\Rightarrow$$

$$\dot{A}(t)^{T}A(t) + A(t)^{T}\dot{A}(t) = 0$$

=>

$$X^T A + A^T X = 0$$

=>

$$\begin{aligned}(A^{-1}X)^T &= (A^T X)^T \\ &= X^T A \\ &= - A^T X \\ &= - A^{-1}X.]\end{aligned}$$

N.B. The classical terminology is that

$$\begin{cases} A^{-1}X \text{ is the } \underline{\text{body angular velocity}} \text{ per } X \\ XA^{-1} \text{ is the } \underline{\text{spatial angular velocity}} \text{ per } X. \end{cases}$$

It is also traditional to write

$$\begin{cases} \hat{\Omega} \\ \\ \hat{\omega} \end{cases} \text{for a generic} \begin{cases} \text{body} \\ \\ \text{spatial} \end{cases} \text{angular velocity}$$

at A, hence

$$\begin{cases} \hat{\Omega} = \begin{bmatrix} 0 & -\Omega^3 & \Omega^2 \\ \Omega^3 & 0 & -\Omega^1 \\ -\Omega^2 & \Omega^1 & 0 \end{bmatrix} \longleftrightarrow (\Omega^1, \Omega^2, \Omega^3) \\ \hat{\omega} = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix} \longleftrightarrow (\omega^1, \omega^2, \omega^3). \end{cases}$$

Suppose that $A:I \to \underline{SO}(3)$ is a curve -- then its lift to $T\underline{SO}(3)$ is given by

$$t \to (A(t),\dot{A}(t)).$$

Write

$$\begin{cases} \hat{\Omega}(t) = A(t)^{-1}\dot{A}(t) \\ \hat{\omega}(t) = \dot{A}(t)A(t)^{-1}. \end{cases}$$

Then

$$\begin{cases} t \to \Omega(t) \\ t \to \omega(t) \end{cases}$$

are curves in $\underline{R}^3$.

A.2 <u>EXAMPLE</u> If

$$A(t) = \begin{bmatrix} \cos\phi(t)\ \cos\theta(t) & \sin\phi(t)\ \cos\theta(t) & \sin\theta(t) \\ \cos\phi(t)\ \sin\theta(t) & \sin\phi(t)\ \sin\theta(t) & -\cos\theta(t) \\ -\sin\phi(t) & \cos\phi(t) & 0 \end{bmatrix},$$

then

$$\hat{\Omega}(t) = \begin{bmatrix} 0 & \dot{\phi}(t) & \dot{\theta}(t)\ \cos\phi(t) \\ -\dot{\phi}(t) & 0 & \dot{\theta}(t)\ \sin\phi(t) \\ -\dot{\theta}(t)\ \cos\phi(t) & -\dot{\theta}(t)\ \sin\phi(t) & 0 \end{bmatrix},$$

so

$$\Omega(t) = (-\dot{\theta}(t)\ \sin\phi(t),\ \dot{\theta}(t)\ \cos\phi(t),\ -\dot{\phi}(t)).$$

[Note: Analogously,

$$\hat{\omega}(t) = \begin{bmatrix} 0 & -\dot{\theta}(t) & \dot{\phi}(t)\cos\theta(t) \\ \dot{\theta}(t) & 0 & \dot{\phi}(t)\sin\theta(t) \\ -\dot{\phi}(t)\cos\theta(t) & -\dot{\phi}(t)\sin\theta(t) & 0 \end{bmatrix},$$

so

$$\omega(t) = (-\dot{\phi}(t)\sin\theta(t), \dot{\phi}(t)\cos\theta(t), \dot{\theta}(t)).]$$

A <u>rigid body</u> is a pair (Ξ,μ), where $\Xi \subset \underline{R}^3$ is compact and μ is a finite Borel measure on $\underline{R}^3$ with spt $\mu = \Xi$. One calls

$$\mu(\Xi) = \int_\Xi d\mu(\xi)$$

the <u>mass</u> of the body, its <u>center of mass</u> then being the point

$$\xi_C = \frac{1}{\mu(\Xi)} (\int_\Xi \xi d\mu(\xi)).$$

[Note: ξ_C is the unique point for which

$$\int_\Xi (\xi - \xi_C) d\mu(\xi) = 0.]$$

A.3 <u>EXAMPLE</u> A particle of mass m is a special case of a rigid body. Thus suppose the particle is situated at a point $\xi_0 \in \underline{R}^3$ and take $\mu = m\delta_{\xi_0}$ -- then spt $\mu = \{\xi_0\}$ and the center of mass is

$$\xi_C = m^{-1}(m\xi_0) = \xi_0.$$

The <u>inertia operator</u> of a rigid body (Ξ,μ) about a point $x_0 \in \underline{R}^3$ is the linear map

$$I_{x_0}:\underline{R}^3 \rightarrow \underline{R}^3$$

defined by

$$I_{x_0}(x) = \int_{\Xi} (\xi - x_0)\times(x \times (\xi - x_0))d\mu(\xi).$$

[Note: We have

$$(\xi - x_0)\times(x \times (\xi - x_0))$$

$$= |\xi - x_0|^2 x - \langle\xi - x_0,x\rangle (\xi - x_0).]$$

A.4 <u>EXAMPLE</u> Keeping to the setup of A.3,

$$I_{x_0}(x) = m(\xi_0 - x_0)\times(x \times (\xi_0 - x_0)).$$

Let (a^1,a^2,a^3) be the components of $a = \xi_0 - x_0$ -- then the matrix of I_{x_0} is

$$m\begin{bmatrix} (a^2)^2 + (a^3)^2 & -a^1a^2 & -a^1a^3 \\ -a^2a^1 & (a^3)^2 + (a^1)^2 & -a^2a^3 \\ -a^3a^1 & -a^3a^2 & (a^1)^2 + (a^2)^2 \end{bmatrix}$$

and its eigenvalues are

$$\{m|a|^2, m|a|^2, 0\}.$$

A.5 LEMMA I_{x_0} is symmetric, i.e., $\forall\, x_1, x_2$,

$$\langle I_{x_0}(x_1), x_2\rangle = \langle x_1, I_{x_0}(x_2)\rangle$$

and positive semidefinite, i.e., $\forall\, x$,

$$\langle I_{x_0}(x), x\rangle \geq 0.$$

PROOF First write

$$\begin{aligned}
&\langle I_{x_0}(x_1), x_2\rangle \\
&= \int_\Xi \langle (\xi - x_0)\times(x_1 \times (\xi - x_0)), x_2\rangle d\mu(\xi) \\
&= \int_\Xi \langle x_1 \times (\xi - x_0),\ x_2 \times (\xi - x_0)\rangle d\mu(\xi) \\
&= \int_\Xi \langle x_1, (\xi - x_0)\times(x_2 \times (\xi - x_0))\rangle d\mu(\xi) \\
&= \langle x_1, I_{x_0}(x_2)\rangle.
\end{aligned}$$

Then take $x_1 = x_2 = x$ to get

$$\begin{aligned}
&\langle I_{x_0}(x), x\rangle \\
&= \int_\Xi \langle x \times (\xi - x_0), x \times (\xi - x_0)\rangle d\mu(\xi) \\
&\geq 0.
\end{aligned}$$

Therefore the eigenvalues of I_{x_0} are real and nonnegative.

A.6 LEMMA If I_{x_0} has a zero eigenvalue, then the other two eigenvalues are equal.

[Note: I_{x_0} has a zero eigenvalue iff Ξ is contained in a line through x_0.]

A.7 LEMMA If I_{x_0} has two zero eigenvalues, then $\Xi = \{x_0\}$.

A.8 REMARK If there is no line through x_0 that contains the support of μ, then I_{x_0} is an isomorphism.

Take $x_0 = \xi_C$ and write I_C in place of I_{ξ_C}.

A.9 LEMMA $\forall\, x \in \underline{R}^3$,

$$I_C(x) = \int_\Xi \xi \times (x \times \xi) d\mu(\xi)$$

$$- \mu(\Xi)(\xi_C \times (x \times \xi_C)).$$

In the case of a particle ξ_0 of mass m, $\mu = m\delta_{\xi_0}$, hence

$$I_C(x) = m(\xi_0 \times (x \times \xi_0)) - m(\xi_0 \times (x \times \xi_0))$$

$$= 0.$$

A.10 REMARK Given x_0, define C by $x_0 = \xi_C + C$ -- then

$$I_{x_0}(x) = I_C(x) + \mu(\Xi)(C \times (x \times C)).$$

E.g.: Take $x_0 = 0$ — then $C = -\xi_C$, so

$$I_0(x) = \int_\Xi \xi \times (x \times \xi) d\mu(\xi)$$

$$= I_C(x) + \mu(\Xi)(-\xi_C \times (x \times -\xi_C))$$

or still,

$$I_C(x) = \int_\Xi \xi \times (x \times \xi) d\mu(\xi)$$

$$- \mu(\Xi)(\xi_C \times (x \times \xi_C)),$$

in agreement with A.9.

[Note: Bear in mind that

$$\int_\Xi (\xi - \xi_C) d\mu(\xi) = 0.]$$

Let us now consider the description of the free rotation of an isolated rigid body (Ξ,μ) about a fixed point, which we take to be the origin in $\underline{R}^3$, and, to minimize trivialities, we shall assume that I_0 is positive definite.

Define a lagrangian

$$L_0 : T\underline{SO}(3) \to \underline{R}$$

by

$$L_0(A,X) = \frac{1}{2} \langle I_0 \Omega, \Omega \rangle.$$

[Note: Recall that Ω depends on (A,X) via the prescription

$$A^{-1}X = \hat{\Omega}.]$$

Explicated,

$$\frac{1}{2}\langle I_0\Omega,\Omega\rangle = \frac{1}{2}\int_\Xi \langle \xi \times (\Omega \times \xi),\Omega\rangle d\mu(\xi)$$

$$= \frac{1}{2}\int_\Xi |\Omega \times \xi|^2 d\mu(\xi)$$

or still,

$$\frac{1}{2}\langle I_0\Omega,\Omega\rangle = \frac{1}{2}\langle I_C\Omega,\Omega\rangle$$

$$+ \frac{1}{2}\mu(\Xi)\langle\Omega \times \xi_C,\Omega \times \xi_C\rangle.$$

N.B. $\underline{SO}(3)$ operates to the left on $T\underline{SO}(3)$ and relative to this action, L_0 is invariant.

A.11 REMARK Define an inner product $\langle,\rangle_0$ on $\underline{R}^3$ by

$$\langle x,y\rangle_0 = \int_\Xi \langle x \times \xi, y \times \xi\rangle d\mu(\xi).$$

Transfer it to $\underline{so}(3)$, viewed as the tangent space to the identity of $\underline{SO}(3)$, thence by left translation to the tangent space at an arbitrary point of $\underline{SO}(3)$. Call g_0 the left invariant riemannian structure resulting thereby -- then its "kinetic energy" is L_0, i.e., in the notation of 8.4,

$$L_0 = \frac{1}{2} g_0.$$

Consequently, L_0 is nondegenerate.

[Note: The metric connection ∇_0 associated with g_0 is left invariant, thus, on general grounds, induces a bilinear map

$$\underline{so}(3) \times \underline{so}(3) \rightarrow \underline{so}(3)$$

or still, a bilinear map

$$\underline{R}^3 \times \underline{R}^3 \rightarrow \underline{R}^3,$$

viz.

$$(x,y) \rightarrow \frac{1}{2} (x \times y) + \frac{1}{2} I_0^{-1} (x \times I_0 y + y \times I_0 x).]$$

A.12 LEMMA We have

$$I_0 = \int_{\Xi} \begin{bmatrix} (\xi^2)^2 + (\xi^3)^2 & -\xi^1\xi^2 & -\xi^1\xi^3 \\ -\xi^2\xi^1 & (\xi^3)^2 + (\xi^1)^2 & -\xi^2\xi^3 \\ -\xi^3\xi^1 & -\xi^3\xi^2 & (\xi^1)^2 + (\xi^2)^2 \end{bmatrix} d\mu(\xi).$$

A.13 EXAMPLE Take for Ξ a ball of radius R centered at the origin and suppose that μ has a spherically symmetric density: $d\mu(\xi) = \rho(|\xi|)d\xi$ -- then

$$I_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

where

$$3I = 2\int_{\Xi} \rho(|\xi|)|\xi|^2 d\xi$$

$$= 8\pi\int_0^R \rho(r)r^4 dr.$$

Therefore

$$I = \frac{8\pi}{3} \int_0^R \rho(r) r^4 dr.$$

If the mass distribution is actually homogeneous, i.e.,

$$\rho = \frac{3m}{4\pi R^3},$$

then $I = \frac{2}{5} mR^2$, hence the inner product $<,>_0$ arising from the choices $m = \frac{5}{2}$, $R = 1$ is the usual inner product on $\underline{R}^3$.

A.14 EXAMPLE Take for Ξ a cone with vertex at the origin and of height h above the $\xi^1\xi^2$-plane ($\xi^3 = h(\frac{r}{R})$ $(0 \le r \le R)$). Assume that the mass distribution is homogeneous, thus $\rho = 3m/\pi R^2 h$ and the center of mass is at $(0,0,\frac{3h}{4})$. Here, the off diagonal entries in A.12 are obviously zero, so

$$I_0 = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

and by an elementary calculation, one finds that

$$\begin{cases} I_1 = I_2 = (3/5)m(\frac{R^2}{4} + h^2) \\ I_3 = (3/10)mR^2. \end{cases}$$

Using A.9, one can then compute the matrix representing I_C, which is necessarily diagonal:

$$I_C = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

In the formula

$$\xi_C \times (x \times \xi_C)$$

$$= \langle \xi_C, \xi_C \rangle x - \langle \xi_C, x \rangle \xi_C,$$

successively insert

$$x = (1,0,0),\ (0,1,0),\ (0,0,1).$$

Then it follows that

$$\left[\begin{array}{l} \lambda_1 = I_1 - m(\frac{3h}{4})^2 = (3/20)m(R^2 + \frac{h^2}{4}) \\ \\ \lambda_2 = I_2 - m(\frac{3h}{4})^2 = (3/20)m(R^2 + \frac{h^2}{4}) \end{array}\right.$$

and

$$\lambda_3 = I_3 + (3/10)mR^2.$$

Determine $\Gamma_{L_0} \in \mathcal{D}^1(T\underline{SO}(3))$ per 8.12.

A.15 <u>THEOREM</u> Let

$$\gamma(t) = (A(t), \dot{A}(t))$$

be a curve in $\underline{TSO}(3)$. Put

$$\hat{\Omega}(t) = A(t)^{-1}\dot{A}(t).$$

Then $\gamma(t)$ is an integral curve of Γ_{L_0} iff $\Omega(t)$ satisfies Euler's equations, i.e., iff

$$I_0\dot{\Omega}(t) = I_0\Omega(t) \times \Omega(t).$$

A.16 REMARK The projection $\pi_{\underline{SO}(3)}:\underline{TSO}(3) \to \underline{SO}(3)$ of the integral curves of Γ_{L_0} are the geodesics of $(\underline{SO}(3),g_0)$ (cf. 10.6) and these are what the motion should follow. Define now the Euler vector field $\Gamma_0:\underline{R}^3 \to \underline{R}^3$ by

$$\Gamma_0\xi = I_0^{-1}(I_0\xi \times \xi) \qquad (\xi \in \underline{R}^3).$$

Then a curve $t \to \xi(t)$ is an integral curve of Γ_0 iff

$$\dot{\xi}(t) = (\Gamma_0)_{\xi(t)}$$

or still, iff

$$I_0\dot{\xi}(t) = I_0\xi(t) \times \xi(t).$$

One can thus view A.15 as providing an alternative description of the motion, which turns out to be more amenable to explicit computation.

Define a function

$$\Pi:\underline{SO}(3) \to \underline{R}^3$$

by

$$\Pi(A,X) = AI_0\Omega.$$

[Note: Π is called the <u>angular momentum</u> of the system.]

A.17 <u>LEMMA</u> Π is constant on the trajectories γ of Γ_{L_0}.

<u>PROOF</u> Consider the restriction of Π to such a γ:

$$t \to A(t)I_0\Omega(t).$$

Then

$$\begin{aligned}
&(A(t)I_0\Omega(t))^{\bullet} \\
&= \dot{A}(t)I_0\Omega(t) + A(t)I_0\dot{\Omega}(t) \\
&= \dot{A}(t)I_0\Omega(t) + A(t)(I_0\Omega(t) \times \Omega(t)) \\
&= A(t)\hat{\Omega}(t)I_0\Omega(t) + A(t)(I_0\Omega(t) \times \Omega(t)) \\
&= A(t)(\Omega(t) \times I_0\Omega(t)) + A(t)(I_0\Omega(t) \times \Omega(t)) \\
&= 0.
\end{aligned}$$

[Note: Therefore the components of Π are first integrals for Γ_{L_0} (cf. 1.1). Another first integral for Γ_{L_0} is E_{L_0} (cf. 8.10):

$$\begin{aligned}
E_{L_0}(\gamma(t)) &= L_0(\gamma(t)) \\
&= \frac{1}{2} \langle I_0\Omega(t), \Omega(t)\rangle
\end{aligned}$$

=>

$$\frac{d}{dt} \frac{1}{2} \langle I_0\Omega(t), \Omega(t)\rangle$$

$$= \frac{1}{2} \langle I_0\dot{\Omega}(t), \Omega(t)\rangle + \frac{1}{2} \langle I_0\Omega(t), \dot{\Omega}(t)\rangle$$

$$= \frac{1}{2} \langle I_0\dot{\Omega}(t), \Omega(t)\rangle + \frac{1}{2} \langle \Omega(t), I_0\dot{\Omega}(t)\rangle$$

$$= \langle I_0\dot{\Omega}(t), \Omega(t)\rangle$$

$$= \langle I_0\Omega(t) \times \Omega(t), \Omega(t)\rangle$$

$$= - \langle \Omega(t) \times I_0\Omega(t), \Omega(t)\rangle$$

$$= - \langle I_0\Omega(t), \Omega(t) \times \Omega(t)\rangle$$

$$= 0.]$$

A.18 REMARK The functions

$$\left[\begin{array}{l} \xi \to \frac{1}{2} \langle I_0\xi, \xi\rangle \\ \\ \xi \to \langle I_0\xi, I_0\xi\rangle \end{array}\right.$$

are constant on the trajectories of Γ_0, hence belong to $C^\infty_{\Gamma_0}(\underline{R}^3)$ (cf. 1.1).

Fix a positively oriented orthonormal basis $\{E_1, E_2, E_3\}$:

$$\left[\begin{array}{l} I_0E_1 = I_1E_1 \\ \\ I_0E_2 = I_2E_2 \\ \\ I_0E_3 = I_3E_3. \end{array}\right.$$

Then

$$\Omega = \Omega_1 E_1 + \Omega_2 E_2 + \Omega_3 E_3$$

$$\Rightarrow$$

$$\frac{1}{2} \langle I_0 \Omega, \Omega \rangle = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2).$$

<u>N.B.</u> In terms of this data, the Euler equations read

$$\left[\begin{array}{l} \dot{\Omega}_1 = \dfrac{(I_2 - I_3)}{I_1} \Omega_2 \Omega_3 \\[2ex] \dot{\Omega}_2 = \dfrac{(I_3 - I_1)}{I_2} \Omega_3 \Omega_1 \\[2ex] \dot{\Omega}_3 = \dfrac{(I_1 - I_2)}{I_3} \Omega_1 \Omega_2. \end{array}\right.$$

A.19 <u>EXAMPLE</u> Take for Ξ a uniform cube of side ℓ whose pivot is at the origin and whose sides are lined up along the coordinate axes in the first octant -- then $\rho = \frac{m}{\ell^3}$ and

$$I_0 = \frac{2}{3} m\ell^2 \begin{bmatrix} 1 & c & c \\ c & 1 & c \\ c & c & 1 \end{bmatrix},$$

where $c = -3/8$. The eigenvalue equation for

$$\begin{bmatrix} 1 & c & c \\ c & 1 & c \\ c & c & 1 \end{bmatrix}$$

is

$$(1 - \lambda)^3 - 3c^2(1 - \lambda) + 2c^3$$

$$= (1 - \lambda - c)^2(1 - \lambda + 2c) = 0,$$

the solutions to which are

$$I_1 = \frac{1}{4},\ I_2 = \frac{11}{8},\ I_3 = \frac{11}{8}.$$

An unnormalized eigenvector per I_1 is $(1,1,1)$, hence lies along the diagonal of the cube. On the other hand, eigenvectors per $I_2 = I_3$ constitute a subspace of dimension 2 perpendicular to the diagonal.

[Note: From the definitions,

$$\xi_C = (\frac{\ell}{2}, \frac{\ell}{2}, \frac{\ell}{2}).$$

Claim: The eigenvalues of I_C are

$$\{\frac{m\ell^2}{6}, \frac{m\ell^2}{6}, \frac{m\ell^2}{6}\}.$$

In fact, thanks to A.9,

$$I_C(\xi_C) = I_0(\xi_C) - m(\xi_C \times (\xi_C \times \xi_C))$$

$$= I_0(\xi_C)$$

$$= \frac{m\ell^2}{6}\ \xi_C.$$

Now let $\Lambda \in \{\xi_C\}^\perp$ — then

$$\xi_C \times (\Lambda \times \xi_C)$$

$$= \langle\xi_C, \xi_C\rangle\Lambda - \langle\xi_C, \Lambda\rangle\xi_C$$

$$= \frac{3\ell^2}{4}\Lambda.$$

So, applying A.9 once again,

$$I_C(\Lambda) = I_0(\Lambda) - (3/4)m\ell^2\Lambda$$

$$= (11/12)m\ell^2\Lambda - (3/4)m\ell^2\Lambda$$

$$= \frac{m\ell^2}{6}\Lambda.]$$

Put

$$\left[\begin{array}{l} 2E = I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 \\ \\ L = I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_3^2. \end{array}\right.$$

Then 2E and L are first integrals for Γ_0 (cf. A.18).

Turning to the solutions of the Euler equations, we shall consider three cases.

<u>Case 1</u>: $I_1 = I_2 = I_3$.

<u>Case 2</u>: $I_1 = I_2 \neq I_3$.

<u>Case 3</u>: $I_1 < I_2 < I_3$.

The first case is trivial: $\exists$ constants C_1, C_2, C_3 such that

$$\Omega_1 = C_1,\ \Omega_2 = C_2,\ \Omega_3 = C_3.$$

As for the second case, we have

$$\begin{cases} I_1\dot{\Omega}_1 - (I_2 - I_3)\Omega_2\Omega_3 = 0 \\ I_1\dot{\Omega}_2 - (I_3 - I_1)\Omega_3\Omega_1 = 0 \end{cases}$$

and

$$\dot{\Omega}_3 = 0.$$

So $\Omega_3 = C_3$ and matters reduce to

$$\begin{cases} \dot{\Omega}_1 - C\Omega_2 = 0 \\ \dot{\Omega}_2 + C\Omega_1 = 0, \end{cases}$$

where

$$C = \frac{(I_1 - I_3)C_3}{I_1}.$$

Eliminating Ω_2 gives

$$\ddot{\Omega}_1 + C^2\Omega_1 = 0,$$

the general solution to which is

$$\Omega_1 = K\sin(Ct + \tau)$$

for certain constants K and τ. And then

$$\Omega_2 = K\cos(Ct + \tau).$$

N.B. Here

$$2E = I_1(\Omega_1^2 + \Omega_2^2) + I_3 C_3^2$$

$$= I_1 K^2 + I_3 \left[\frac{I_1 C}{I_1 - I_3} \right]^2$$

$$= I_1 \left(K^2 + \frac{I_1 I_3}{(I_1 - I_3)^2} C^2\right)$$

and, analogously,

$$L = I_1^2 \left(K^2 + \frac{I_3^2}{(I_1 - I_3)^2} C^2\right).$$

Therefore

$$K^2 = \frac{1}{I_1(I_1 - I_3)} (L - 2I_3 E)$$

while

$$C^2 = \frac{I_1 - I_3}{I_1^2 I_3} (2I_1 E - L).$$

The third case is more complicated but doable, the details being a bit messy. Suffice it to say that explicit solutions can be given in terms of the Jacobi elliptic functions sn, cn, dn.

[Note: In $\underline{R}^3$, consider the differential equations

$$\begin{cases} \dot{x} = yz \\ \dot{y} = -xz \\ \dot{z} = -k^2 xy \qquad (0 < k < 1). \end{cases}$$

Then the triple

$$t \to (\mathrm{sn}(t;k),\ \mathrm{cn}(t;k),\ \mathrm{dn}(t;k))$$

is the solution to this system subject to the initial condition (0,1,1) (if $k = 0$, then $\mathrm{sn}(t;0) = \sin t$, $\mathrm{cn}(t;0) = \cos t$, $\mathrm{dn}(t;0) = 1$). To see where this is going, put

$$\begin{cases} c_1 = I_1^{-1},\ c_2 = I_2^{-1},\ c_3 = I_3^{-1} \\ u_1 = I_1\Omega_1,\ u_2 = I_2\Omega_2,\ u_3 = I_3\Omega_3 \end{cases}$$

and rewrite the Euler equations as

$$\begin{cases} \dot{u}_1 = -\ (c_2 - c_3)u_2u_3 \\ \dot{u}_2 = (c_1 - c_3)u_1u_3 \\ \dot{u}_3 = -\ (c_1 - c_2)u_1u_2, \end{cases}$$

the point of departure... .]

The motion of (Ξ,μ) is a geodesic w.r.t. the left invariant riemannian structure g_0. To exploit A.15, fix $A_0 \in \underline{SO}(3)$, $X_0 \in T_{A_0}\underline{SO}(3)$. Translate X_0 to $\underline{so}(3)$ and then to $\underline{R}^3$ to get Ω_0. Let $\Omega(t)$ be the solution of the Euler equations subject to the initial condition Ω_0. Pass to $\hat{\Omega}(t)$ -- then

$$\dot{A}(t) = A(t)\hat{\Omega}(t)$$

is a system of linear differential equations with time dependent coefficients, the so-called <u>reconstruction equation</u>. Solve it for $A(t)$, subject to $A(0) = A_0$, thus

$$\dot{A}(0) = A(0)\hat{\Omega}_0$$

$$= A_0(A_0^{-1}X_0)$$

$$= X_0,$$

and so

$$\gamma(t) = (A(t),\dot{A}(t))$$

is an integral curve of Γ_{L_0} passing through (A_0,X_0) at $t = 0$.

<u>N.B.</u> This is what happens in principle. What happens in practice is, however, a different matter, at least if one wants to be completely explicit. Case 3 is particularly vexsome but Case 1 is simple. For then $\hat{\Omega}(t)$ is constant in time: $\hat{\Omega}(t) = \hat{\Omega}_0 \ \forall \ t$, hence the solution is

$$A(t) = A_0 e^{t\hat{\Omega}_0}.$$

A.20 <u>RAPPEL</u> Let $\{e_1,e_2,e_3\}$ be the standard basis for $\underline{R}^3$ -- then $\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ is the standard basis for $\underline{so}(3)$.

The manifold $\underline{SO}(3)$ can be equipped with a number of charts, all derived from the notion of "Euler angle", but the subject is potentially confusing due to the variety of choices that can be made.

Given ϕ,θ,ψ, put

$$\begin{bmatrix} c_\phi = \cos\phi \\ \\ s_\phi = \sin\phi \end{bmatrix}, \quad \begin{bmatrix} c_\theta = \cos\theta \\ \\ s_\theta = \sin\theta \end{bmatrix}, \quad \begin{bmatrix} c_\psi = \cos\psi \\ \\ s_\psi = \sin\psi. \end{bmatrix}$$

Then

$$\exp(\phi\hat{e}_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix},$$

$$\exp(\theta\hat{e}_2) = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix},$$

$$\exp(\psi\hat{e}_3) = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A.21 <u>LEMMA</u> The map

$$]-\pi,\pi[\times]-\frac{\pi}{2},\frac{\pi}{2}[\times]-\pi,\pi[$$

that sends (ψ,θ,ϕ) to

$$A(\psi,\theta,\phi) = \exp(\psi\hat{e}_3)\exp(\theta\hat{e}_2)\exp(\phi\hat{e}_1)$$

is one-to-one and its image U_{321} is open.

[Note: The inverse

$$U_{321} \to \left]-\pi,\pi\right[\times \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\times \left]-\pi,\pi\right[$$

can be computed in terms of atan(x,y), the 2-argument arctangent function.]

Therefore this data defines a chart on $\underline{SO}(3)$ with local coordinates ψ,θ,ϕ. [Note: Local coordinates on $\underline{TSO}(3)$ will be denoted by ψ,θ,ϕ, v_ψ,v_θ,v_ϕ.]

Given $A \in U_{321}$, the entries of the associated triple (ψ,θ,ϕ) are called its <u>3-2-1 Euler angles</u>.

<u>N.B.</u> All told, there are 12 possible rotation sequences, namely:

1 - 2 - 1	2 - 1 - 2	3 - 1 - 3
1 - 3 - 1	2 - 3 - 2	3 - 2 - 3
1 - 2 - 3	2 - 3 - 1	3 - 1 - 2
1 - 3 - 2	2 - 1 - 3	3 - 2 - 1.

A.22 <u>REMARK</u> In the engineering literature, the 3-2-1 rotation sequence is referred to as yaw-pitch-roll.

The 3-1-3 convention is also a popular choice:

$$\left[\begin{array}{l} A \longleftrightarrow (\phi,\theta,\psi) \\ \\ A = \exp(\phi\hat{e}_3)\exp(\theta\hat{e}_1)\exp(\psi\hat{e}_3), \end{array}\right.$$

where

$$0 < \phi < 2\pi,\ 0 < \theta < \pi,\ 0 < \psi < 2\pi.$$

Consider a curve $t \to A(t)$ and pass to $\hat{\Omega}(t) = A(t)^{-1}\dot{A}(t)$. Put

$$\left[\begin{array}{l} A_\phi = \exp(\phi(t)\hat{e}_3) \\ \\ A_\theta = \exp(\theta(t)\hat{e}_1) \\ \\ A_\psi = \exp(\psi(t)\hat{e}_3). \end{array}\right.$$

Then

$$\Omega(t) = (A_\phi A_\theta A_\psi)^{-1} \frac{d}{dt} (A_\phi A_\theta A_\psi)$$

$$= A_\psi^{-1} A_\theta^{-1} A_\phi^{-1} (\dot{\phi} (\frac{d}{d\phi} A_\phi) A_\theta A_\psi$$

$$+ \dot{\theta} A_\phi (\frac{d}{d\theta} A_\theta) A_\psi + \dot{\psi} A_\phi A_\theta (\frac{d}{d\psi} A_\psi))$$

$$= \dot{\phi} A_\psi^{-1} A_\theta^{-1} A_\phi^{-1} (\frac{d}{d\phi} A_\phi) A_\theta A_\psi$$

$$+ \dot{\theta} A_\psi^{-1} A_\theta^{-1} (\frac{d}{d\theta} A_\theta) A_\psi + \dot{\psi} A_\psi^{-1} (\frac{d}{d\psi} A_\psi).$$

A.23 <u>LEMMA</u> We have

$$\Omega(t) = \begin{bmatrix} \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \dot{\phi} \cos\theta + \dot{\psi} \end{bmatrix}.$$

A.24 <u>EXAMPLE</u> Take for Ξ a uniform ball of mass m and radius R centered at the origin, hence $I = \frac{2}{5} mR^2$ (cf. A.13). Locally, in the 3-1-3 system,

$$L_0(\phi,\theta,\psi)$$

$$= \frac{1}{2} I((v_\phi \sin\theta \sin\psi + v_\theta \cos\psi)^2$$

$$+ (v_\phi \sin\theta \cos\psi - v_\theta \sin\psi)^2 + (v_\phi \cos\theta + v_\psi)^2)$$

or still,

$$L_0(\phi,\theta,\psi) = \frac{1}{2} I(v_\phi^2 + v_\theta^2 + v_\psi^2 + 2v_\phi v_\psi \cos\theta).$$

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