BOSONIC QUANTUM FIELD THEORY

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ABSTRACT

The purpose of these notes is to provide a systematic account of that part of Quantum Field Theory in which symplectic methods play a major role.

CONTENTS

§1. *SELFADJOINT OPERATORS*

In what follows, $\mathcal{H}$ stands for a complex infinite dimensional Hilbert space, the convention on the inner product being that it is conjugate linear in the first slot and linear in the second slot.

A <u>linear operator</u> A is a linear transformation from a linear subspace $\text{Dom}(A) \subset \mathcal{H}$ into $\mathcal{H}$. If B is a linear operator with $\text{Dom}(B) \supset \text{Dom}(A)$ and if $B|\text{Dom}(A) = A$, then B is called an <u>extension</u> of A and we write $B \supset A$.

If A_1 and A_2 are linear operators, then $A_1 + A_2$ is the linear operator with

$$\left[\begin{array}{l} \text{Dom}(A_1 + A_2) = \text{Dom}(A_1) \cap \text{Dom}(A_2) \\ (A_1 + A_2)x = A_1x + A_2x \end{array}\right.$$

and

- A_1A_2 is the linear operator with

$$\left[\begin{array}{l} \text{Dom}(A_1A_2) = \{x \in \text{Dom}(A_2) : A_2x \in \text{Dom}(A_1)\} \\ (A_1A_2)x = A_1(A_2x). \end{array}\right.$$

- A_2A_1 is the linear operator with

$$\left[\begin{array}{l} \text{Dom}(A_2A_1) = \{x \in \text{Dom}(A_1) : A_1x \in \text{Dom}(A_2)\} \\ (A_2A_1)x = A_2(A_1x). \end{array}\right.$$

The <u>commutator</u> $[A_1, A_2]$ is the linear operator with

$$\left[\begin{array}{l} \text{Dom}([A_1, A_2]) = \text{Dom}(A_1A_2) \cap \text{Dom}(A_2A_1) \\ [A_1, A_2]x = A_1A_2x - A_2A_1x. \end{array}\right.$$

[Note: Even if $\mathrm{Dom}(A_1)$ and $\mathrm{Dom}(A_2)$ are dense, it is still perfectly possible that $\mathrm{Dom}(A_1 + A_2)$ or $\left[\begin{array}{l}\mathrm{Dom}(A_1A_2)\\ \\ \mathrm{Dom}(A_2A_1)\end{array}\right.$ is $\{0\}$ alone.]

A linear operator A is <u>bounded</u> if $\exists\ C > 0$:

$$||Ax|| \leq C\ ||x||\ \forall\ x \in \mathrm{Dom}(A),$$

otherwise A is <u>unbounded</u>. If

$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||},$$

then A is bounded iff $||A|| < \infty$.

[Note: Boundedness is tantamount to continuity.]

1.1 <u>EXAMPLE</u> Take $H = L^2(\underline{R})$ and let

$$(Qf)(x) = xf(x),$$

where

$$\mathrm{Dom}(Q) = \{f: \int_{\underline{R}} x^2|f(x)|^2 dx < \infty\}.$$

Then Q is unbounded. To see this, take $f = \chi_{[0,1]}$. With $f_n(x) = f(x - n)$, we have

$$\begin{aligned}||Qf_n||^2 &= \int_{\underline{R}} x^2 f^2(x - n)dx\\ &= \int_n^{n+1} x^2 f^2(x - n)dx\\ &\geq n^2||f_n||^2,\end{aligned}$$

which shows that $||Q|| = \infty$.

[Note: Q is called the position operator.]

1.2 EXAMPLE Take $H = L^2(\underline{R})$ and let

$$(Pf)(x) = -\sqrt{-1}\, f'(x),$$

where

$$\mathrm{Dom}(P) = \{f: \int_{\underline{R}} |f'(x)|^2 dx < \infty\}.$$

Here f' is the distributional derivative of f, so Dom(P) is the Sobolev space $W^{2,1}(\underline{R})$. We then claim that P is unbounded. Thus choose a sequence $\{f_n\} \subset C_c^\infty(\underline{R})$ such that

$$\mathrm{spt}\, f_n \subset [-\frac{1}{n},\frac{1}{n}],\ f_n \geq 0,\ ||f_n|| = 1.$$

Since $\int_{-1}^{1} f_n^2(x)dx = 1$, $\exists\ x_n \in [-\frac{1}{n},\frac{1}{n}]$: $\frac{2}{n} f_n^2(x_n) = 1$, hence

$$\sqrt{n/2} = f_n(x_n) = \int_{-1}^{x_n} f_n'(x)dx$$

$$\leq \sqrt{2}\ (\int_{-1}^{x_n} (f_n'(x))^2 dx)^{1/2}$$

$$\leq \sqrt{2}\ ||Pf_n||$$

and this implies that P is unbounded.

[Note: P is called the momentum operator.]

Let A be a densely defined linear operator. Denote by Dom(A*) the set of all vectors $y \in H$ for which $\exists$ a vector $y^* \in H$ such that $< y,Ax > = < y^*,x >$ $\forall\ x \in \mathrm{Dom}(A)$ -- then the assignment $y \to y^*$ defines a linear operator A*, the adjoint of A.

[Note: If A is bounded and Dom(A) = H, then Dom(A*) = H and $||A|| = ||A^*||$.]

1.3 REMARK The domain of A* need not be dense.

[For instance, take $H = L^2(\underline{R})$ and fix a bounded measurable function ϕ_0 such that $\phi_0 \notin L^2(\underline{R})$. Let $f_0 \in L^2(\underline{R})$ be of norm 1 and put

$$Af = \langle \phi_0, f \rangle f_0,$$

where

$$\mathrm{Dom}(A) = \{f : \int_{\underline{R}} |f(x)\phi_0(x)| < \infty\}.$$

Suppose now that $g \in \mathrm{Dom}(A^*)$ — then $\forall\ f \in \mathrm{Dom}(A)$,

$$\langle A^*g, f \rangle = \langle g, Af \rangle \Rightarrow \overline{\langle A^*g, f \rangle} = \overline{\langle g, Af \rangle}$$

$$\Rightarrow$$

$$\langle f, A^*g \rangle = \langle Af, g \rangle$$

$$= \langle \langle \phi_0, f \rangle f_0, g \rangle$$

$$= \overline{\langle \phi_0, f \rangle} \langle f_0, g \rangle$$

$$= \langle f, \phi_0 \rangle \langle f_0, g \rangle$$

$$= \langle f, \langle f_0, g \rangle \phi_0 \rangle,$$

so

$$A^*g = \langle f_0, g \rangle \phi_0.$$

Since $\phi_0 \notin L^2(\underline{R})$, $< f_0, g > = 0$, thus any $g \in \text{Dom}(A^*)$ is orthogonal to f_0. Therefore $\text{Dom}(A^*)$ is not dense.]

[Note: One can even construct examples in which $\text{Dom}(A^*) = \{0\}$.]

A linear operator A is said to be <u>closed</u> if its graph

$$\Gamma_A = \{(x, Ax) : x \in \text{Dom}(A)\}$$

is a closed subset of $\mathcal{H} \times \mathcal{H}$.

[Note: A closed linear operator whose domain is all of $\mathcal{H}$ is bounded (closed graph theorem).]

1.4 <u>LEMMA</u> Let A be a densely defined linear operator -- then A^* is closed.

A linear operator A is said to <u>admit closure</u> if it has a closed extension.

[Note: When this is so, there is a smallest closed extension, the <u>closure</u> $\bar{A}$ of A, and

$$\Gamma_{\bar{A}} = \bar{\Gamma}_A .]$$

1.5 <u>LEMMA</u> Let A be a densely defined linear operator -- then A admits closure iff $\text{Dom}(A^*)$ is dense.

1.6 <u>LEMMA</u> Let A be a densely defined linear operator. Assume: A admits closure -- then $\bar{A} = A^{**}$ and $\bar{A}^* = A^*$.

A densely defined linear operator A is said to be <u>symmetric</u> if $A \subset A^*$, i.e., if

$$< y,Ax > = < Ay,x > \forall\ x,y \in \mathrm{Dom}(A).$$

[Note: A symmetric operator A whose domain is all of $\mathcal{H}$ is necessarily bounded. In fact, $A \subset A^* \Rightarrow A = A^*$, so A is closed (cf. 1.4), thus bounded.]

1.7 <u>REMARK</u> A symmetric operator A admits closure (cf. 1.5: $\mathrm{Dom}(A^*) \supset \mathrm{Dom}(A)$ is dense). But A^* is always closed (cf. 1.4), therefore $A \subset \bar{A} = A^{**} \subset A^*$ (cf. 1.6).

A densely defined linear operator A is said to be <u>selfadjoint</u> if A is symmetric and $A = A^*$.

1.8 <u>CRITERION</u> Let A be a symmetric operator -- then A is selfadjoint iff the range of $A \pm \sqrt{-1}$ is all of $\mathcal{H}$.

1.9 <u>EXAMPLE</u> Take $\mathcal{H} = L^2(\underline{R})$ -- then the position operator Q is selfadjoint. For Q is obviously symmetric. Moreover, given any $f \in L^2(\underline{R})$, we have

$$f = (x \pm \sqrt{-1}) \frac{f}{(x \pm \sqrt{-1})}$$

and

$$\frac{f}{(x \pm \sqrt{-1})} \in \mathrm{Dom}(Q).$$

1.10 <u>LEMMA</u> If $A:\mathrm{Dom}(A) \to \mathcal{H}$ is selfadjoint and if $U:\mathcal{H} \to \mathcal{H}$ is unitary, then $UAU^{-1}:U\text{Dom}(A) \to \mathcal{H}$ is selfadjoint.

1.11 EXAMPLE Take $\mathcal{H} = L^2(\underline{R})$ -- then the momentum operator P is selfadjoint. Indeed, $P = U_F^{-1}QU_F$, where $U_F:L^2(\underline{R}) \to L^2(\underline{R})$ is the unitary operator provided by the Plancherel theorem.

[Note: On $\mathcal{S}(\underline{R})$,

$$\begin{cases} U_F f = \hat{f} \\ ||f||^2 = ||\hat{f}||^2, \end{cases}$$

where

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\sqrt{-1}\lambda x}\, dx.]$$

1.12 REMARK There are analogs of Q and P when $L^2(\underline{R})$ is replaced by $L^2(\underline{R}^n)$.

Q_j: Let

$$(Q_j f)(x) = x_j f(x),$$

where

$$\text{Dom}(Q_j) = \{f: \int_{\underline{R}^n} x_j^2 |f(x)|^2\, dx < \infty\}.$$

Then Q_j is selfadjoint (cf. 1.8).

[Note: Q_j is the j^{th} position operator $(j = 1,\ldots,n)$.]

P_j: Let $U_F:L^2(\underline{R}^n) \to L^2(\underline{R}^n)$ be the unitary operator provided by the Plancherel theorem -- then, by definition,

$$P_j = U_F^{-1} Q_j U_F,$$

where

$$\mathrm{Dom}(P_j) = U_F^{-1}\mathrm{Dom}(Q_j).$$

Since Q_j is selfadjoint and U_F is unitary, P_j is selfadjoint (cf. 1.10). And, $\forall\, f \in S(\underline{R}^n)$,

$$\begin{aligned}(P_j f)(x) &= (U_F^{-1} Q_j \hat{f})(x) \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} (Q_j \hat{f})(\lambda) e^{\sqrt{-1}\, x\cdot\lambda}\, d\lambda \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} \lambda_j \hat{f}(\lambda) e^{\sqrt{-1}\, x\cdot\lambda}\, d\lambda \\ &= -\frac{\sqrt{-1}}{(2\pi)^{n/2}} \frac{\partial}{\partial x_j} \int_{\underline{R}^n} \hat{f}(\lambda) e^{\sqrt{-1}\, x\cdot\lambda}\, d\lambda \\ &= -\sqrt{-1}\, \frac{\partial}{\partial x_j} f(x).\end{aligned}$$

[Note: P_j is the j^{th} <u>momentum operator</u> $(j = 1,\ldots,n)$.]

A densely defined linear operator A is said to be <u>essentially selfadjoint</u> if A is symmetric and $\bar{A}$ is selfadjoint. For example, if D is a dense linear proper subspace of $\mathcal{H}$, then its identity map is not selfadjoint but it is essentially selfadjoint.

[Note: A symmetric operator A is essentially selfadjoint iff the range of $A \pm \sqrt{-1}$ is dense in $\mathcal{H}$ (observe that $\overline{\mathrm{Ran}(A \pm \sqrt{-1})} = \mathrm{Ran}(\bar{A} \pm \sqrt{-1})$ and apply 1.8).]

1.13 EXAMPLE Take H separable and let $\{e_n\}$ be an orthonormal basis. Given a sequence $r = \{r_n\}$ of real numbers, define a linear operator A_r on the linear span of the e_n by $A_r e_n = r_n e_n$ — then A_r is symmetric (but A_r is bounded iff r is bounded). The adjoint A_r^* of A_r has for its domain

$$\{x = \sum_n c_n e_n \in H: \sum_n |c_n r_n|^2 < \infty\},$$

with

$$A_r^* x = \sum_n c_n r_n e_n.$$

Therefore A_r is not selfadjoint. On the other hand, $\bar{A}_r = A_r^*$, hence $\bar{A}_r^* = A_r^{**} = \bar{A}_r$, so $\bar{A}_r$ is selfadjoint, i.e., A_r is essentially selfadjoint.

1.14 LEMMA If A is essentially selfadjoint and if $B \supset A$ is symmetric, then B is essentially selfadjoint and $\bar{A} = \bar{B}$.

[Note: In particular, an essentially selfadjoint operator admits a unique selfadjoint extension.]

A symmetric operator need not be essentially selfadjoint (in fact, a symmetric operator need not have any selfadjoint extensions whatsoever). Suppose, however, that A is symmetric and $D \subset \mathrm{Dom}(A)$ is a dense linear subspace such that $A|D$ is essentially selfadjoint -- then A is essentially selfadjoint and $\bar{A} = \overline{A|D}$ (cf. 1.14).

1.15 EXAMPLE Take $H = L^2(\underline{R}^n)$ and let

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

where

$$\text{Dom}(\Delta) = \{f : \Delta f \in L^2(\underline{R}^n)\}.$$

Here Δf is understood in the sense of distributions, hence $\text{Dom}(\Delta)$ is the Sobolev space $W^{2,2}(\underline{R}^n)$. There are then two points:

1. Δ is selfadjoint.

2. $\Delta | C_c^\infty(\underline{R}^n)$ is essentially selfadjoint.

Using the Fourier transform, the first follows from the fact that multiplication by $|x|^2$ is selfadjoint on

$$\{f : \int_{\underline{R}^n} |x|^4 |f(x)|^2 \, dx < \infty\}.$$

As for the second, since $\Delta | C_c^\infty(\underline{R}^n)$ is symmetric, it suffices to show that

$$(\Delta | C_c^\infty(\underline{R}^n))^* = \Delta.$$

Indeed, this gives

$$\overline{\Delta | C_c^\infty(\underline{R}^n)} = (\Delta | C_c^\infty(\underline{R}^n))^{**} = \Delta^* = \Delta.$$

Let g be in the domain of $(\Delta | C_c^\infty(\underline{R}^n))^*$ -- then $\forall \ f \in C_c^\infty(\underline{R}^n)$,

$$< g, \Delta f > = < (\Delta | C_c^\infty(\underline{R}^n))^* g, f >,$$

thus $\Delta g \in L^2(\underline{R}^n)$ in the sense of distributions, so $g \in \text{Dom}(\Delta)$ and $(\Delta | C_c^\infty(\underline{R}^n))^* g = \Delta g$. Therefore

$$(\Delta | C_c^\infty(\underline{R}^n))^* \subset \Delta.$$

The reverse containment is equally clear.

[Note: It is a corollary that $\Delta|S(\underline{R}^n)$ is essentially selfadjoint (in 1.14, let $A = \Delta|C_c^\infty(\underline{R}^n)$ and $B = \Delta|S(\underline{R}^n)$).]

1.16 TABLE

A symmetric $\longrightarrow$	$A \subset \bar{A} = A^{**} \subset A^*$
A symmetric and closed $\longrightarrow$	$A = \bar{A} = A^{**} \subset A^*$
A essentially selfadjoint $\longrightarrow$	$A \subset \bar{A} = A^{**} = A^*$
A selfadjoint $\longrightarrow$	$A = \bar{A} = A^{**} = A^*$

[Note: Suppose that A is symmetric -- then

A essentially selfadjoint <=> A^* symmetric.]

Let A be a densely defined linear operator -- then a C^∞ vector for A is any element of $\bigcap_{k=1}^{\infty} \text{Dom}(A^k)$.

1.17 REMARK If A is selfadjoint, then spectral theory implies that its set of C^∞ vectors is dense but if A is merely symmetric, then $\text{Dom}(A^2)$ can be $\{0\}$, hence in this case, the only analytic vector is the zero vector.

Let x be a C^∞ vector for A -- then x is said to be analytic if the power series

$$\sum_{k=0}^{\infty} \frac{||A^k x||}{k!} t^k$$

has a positive radius of convergence.

[Note: The set of analytic vectors for A is a linear subspace of Dom(A).]

1.18 THEOREM (Nelson) If A is symmetric and if Dom(A) contains a dense set of analytic vectors, then A is essentially selfadjoint.

1.19 EXAMPLE (Annihilation and Creation) Take H separable. Fix an orthonormal basis $\{e_n : n \geq 0\}$ for H and let D be the set of $x \in H$:

$$\#\{n: < e_n, x > \neq 0\} < \infty.$$

Define linear operators $\underline{a}$ and $\underline{c}$ on D by

$$\underline{a}x = < e_1, x > e_0 + \sqrt{2} < e_2, x > e_1 + \sqrt{3} < e_3, x > e_2 + \cdots .$$

and

$$\underline{c}x = < e_0, x > e_1 + \sqrt{2} < e_1, x > e_2 + \sqrt{3} < e_2, x > e_3 + \cdots.$$

Then:

1. $\underline{a}D \subset D$, $\underline{c}D \subset D$.
2. $\underline{a}e_0 = 0$ & $\underline{a}e_n = \sqrt{n}\, e_{n-1}$ $(n \geq 1)$.
3. $\underline{c}e_n = \sqrt{n+1}\, e_{n+1}$ $(n \geq 0)$.
4. $e_n = \dfrac{\underline{c}^n}{\sqrt{n!}} e_0$ $(n \geq 1)$.
5. $[\underline{a}, \underline{c}] = I$.
6. $< \underline{c}y, x > = < y, \underline{a}x > \forall\, x, y \in D$.

The last property implies that $\underline{c} \subset \underline{a}^*$ and $\underline{a} \subset \underline{c}^*$. Therefore both $\underline{a}$ and $\underline{c}$ admit closure (cf. 1.5). Put $N = \underline{ca}$ — then $Ne_n = ne_n$ $(n \geq 0)$ and

$$[N,\underline{a}] = -\underline{a}, \quad [N,\underline{c}] = \underline{c}.$$

Suppose now that $r \in \underline{R}$, $z \in \underline{C}$ and consider $rN + z\underline{c} + \bar{z}\underline{a}$. It is symmetric and we claim that it is actually essentially selfadjoint. To see this, let us first show that

$$||(rN + z\underline{c} + \bar{z}\underline{a})^k e_n|| \leq (|r| + 2|z|)^k \frac{(n+k)!}{n!}$$

This is certainly true if $k = 0$. Proceeding by induction, assume that it holds for $k > 0$ and then note that

$$||(rN + z\underline{c} + \bar{z}\underline{a})^{k+1} e_n||$$

$$= ||(rN + z\underline{c} + \bar{z}\underline{a})^k (rne_n + z\sqrt{n+1}\, e_{n+1} + \bar{z}\sqrt{n}\, e_{n-1})||$$

$$\leq |r|n||(rN + z\underline{c} + \bar{z}\underline{a})^k e_n||$$

$$+ |z| \sqrt{n+1}\, ||(rN + z\underline{c} + \bar{z}\underline{a})^k e_{n+1}||$$

$$+ |z|\sqrt{n}\, ||(rN + z\underline{c} + \bar{z}\underline{a})^k e_{n-1}||$$

$$\leq (|r| + 2|z|)^k \, [|r|n \frac{(n+k)!}{n!}$$

$$+ |z| \sqrt{n+1} \frac{(n+k+1)!}{(n+1)!} + |z| \sqrt{n} \frac{(n+k-1)!}{(n-1)!}]$$

$$\leq (|r| + 2|z|)^{k+1} \frac{(n+k+1)!}{n!}$$

which completes the induction. From this it follows that the elements of D are analytic vectors for $rN + z\underline{c} + \bar{z}\underline{a}$:

$$\sum_{k=0}^{\infty} ||(rN + z\underline{c} + \bar{z}\underline{a})^k e_n|| \frac{|t|^k}{k!}$$

$$\leq \sum_{k=0}^{\infty} (|r| + 2|z|)^k \frac{(n+k)!}{k!n!} |t|^k$$

$$= (1 - |t|(|r| + 2|z|))^{-(n+1)} < \infty$$

so long as $|t|$ is sufficiently small. That $rN + z\underline{c} + \bar{z}\underline{a}$ is essentially selfadjoint is thus a consequence of Nelson's theorem. In particular: The combinations

$$\begin{cases} Q = \dfrac{1}{\sqrt{2}} (\underline{c} + \underline{a}) \\[2ex] P = \dfrac{\sqrt{-1}}{\sqrt{2}} (\underline{c} - \underline{a}) \end{cases}$$

are essentially selfadjoint.

[Note: By definition, $\underline{a}$ is the <u>annihilation operator</u>, $\underline{c}$ is the <u>creation operator</u>, and N is the <u>number operator</u> (all this being, of course, w.r.t. the given orthonormal basis).]

1.20 <u>REMARK</u> As was shown above, we have $\underline{c} \subset \underline{a}^*$ and $\underline{a} \subset \underline{c}^*$. To simplify notation, denote their respective closures by $\bar{c}$ and $\bar{a}$ (rather than $\bar{\underline{c}}$ and $\bar{\underline{a}}$) -- then $\underline{c}^* = \bar{a}$ and $\underline{a}^* = \bar{c}$. Consequently,

$$< \bar{a}x, y > = < x, \bar{c}y > \quad (x \in \text{Dom}(\bar{a}),\ y \in \text{Dom}(\bar{c})).$$

[Note: Actually,

$$\begin{cases} \text{Dom}(\bar{a}) = \bar{D} \\ \text{Dom}(\bar{c}) = \bar{D}, \end{cases}$$

where we have put

$$\bar{D} = \{x \in H\text{: } \sum_{n=0}^{\infty} n|<e_n,x>|^2 < \infty\}.$$

Since $\bar{a}$ and $\bar{c}$ are the respective closures of $\underline{a}$ and $\underline{c}$, it is clear that $\bar{D} \subset \text{Dom}(\bar{a})$ and $\bar{D} \subset \text{Dom}(\bar{c})$ with

$$\bar{a}x = <e_1,x> e_0 + \sqrt{2} <e_2,x> e_1 + \sqrt{3} <e_3,x> e_2 + \cdots$$

and

$$\bar{c}x = <e_0,x> e_1 + \sqrt{2} <e_1,x> e_2 + \sqrt{3} <e_2,x> e_3 + \cdots.$$

Turning to the reverse containments, let $x \in \text{Dom}(\bar{a})$ -- then

$$\begin{aligned} \bar{a}x &= \sum_0^{\infty} <e_n,\bar{a}x> e_n \\ &= \sum_0^{\infty} <e_n,\underline{c}^*x> e_n \\ &= \sum_0^{\infty} <\underline{c}e_n,x> e_n \\ &= \sum_0^{\infty} \sqrt{n+1} <e_{n+1},x> e_n \end{aligned}$$

and

$$\sum_0^{\infty} (n+1)|<e_{n+1},x>|^2 = \sum_1^{\infty} n|<e_n,x>|^2 < \infty.$$

Therefore $\text{Dom}(\bar{a}) \subset \bar{D}$. By the same token, $\text{Dom}(\bar{c}) \subset \bar{D}$.]

1.21 <u>LEMMA</u> Suppose that A is symmetric. Let D be a dense linear subspace of Dom(A) which contains a dense set of analytic vectors for A -- then $A|D$ is essentially selfadjoint if $AD \subset D$.

[Note: There is a subtlety here: If $x \in D$ is to be analytic for $A|D$, then first of all it must be C^∞ for $A|D$, meaning that $A^n x \in D \ \forall \ n$. But this is not automatic, thus the requirement that $AD \subset D$.]

1.22 <u>EXAMPLE</u> Take $\mathcal{H} = \ell^2(\underline{N})$, let $\{e_n\}$ be its usual orthonormal basis, and define A by $Ae_n = ne_n$ $(n \geq 1)$ -- then A is selfadjoint and

$$\text{Dom}(A) = \{x \in \mathcal{H}: \sum_{n=1}^{\infty} n^2 |<e_n, x>|^2 < \infty\}.$$

Let D be the set of all finite linear combinations of the form $\sum_{k=1}^{K} c_k e_k$, where $\sum_{k=1}^{K} c_k = 0$ (K arbitrary) -- then D is dense and its elements are analytic for A. However, $A|D$ is not essentially selfadjoint. To see this, let $y = \sum_{1}^{\infty} \frac{1}{n} e_n$ -- then $\forall \ x \in D$,

$$\begin{aligned} <y, Ax> &= <y, \sum_{k=1}^{K} kc_k e_k> \\ &= <\sum_{k=1}^{K} \frac{1}{k} e_k, \sum_{k=1}^{K} kc_k e_k> \\ &= \sum_{k=1}^{K} c_k = 0 \end{aligned}$$

=>

$$y \in \mathrm{Dom}((A|D)^*).$$

But $y \notin \mathrm{Dom}(A)$ and this implies that $A|D$ is not essentially selfadjoint. For if it were, then $\overline{A|D} = A$ (cf. 1.14) and $\overline{A|D} = (A|D)^*$ (cf. 1.16), i.e., we would have $(A|D)^* = A$, an impossibility since their domains are different ($(A|D)^*$ is, of course, an extension of A).

[Note: D is not invariant under A.]

1.23 REMARK The set of analytic vectors for a selfadjoint operator is dense (cf. 2.28) but there exist essentially selfadjoint operators whose set of analytic vectors is not dense.

[It can happen that a selfadjoint operator A has a domain of essential selfadjointness $D \subset \mathrm{Dom}(A)$ such that $D \cap \mathrm{Dom}(A^2) = \{0\}$.]

In quantum mechanics, an observable is a selfadjoint operator. But there is a difficulty: The sum of two selfadjoint operators need not be selfadjoint (or even essentially selfadjoint), hence the set of observables is not a linear space.

[Note: Recall that by assumption, $\mathcal{H}$ is infinite dimensional (if $\mathcal{H}$ is finite dimensional, then there are no problems).]

1.24 EXAMPLE Take $\mathcal{H} = L^2(\underline{R})$, let $\{q_n\}$ be an enumeration of the rationals, and put

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n!} |x - q_n|^{-1/2}.$$

Let Q_f be the multiplication operator determined by f, thus $Q_f = f\psi$, where

$$\mathrm{Dom}(Q_f) = \{\psi \in L^2(\underline{R}) : f\psi \in L^2(\underline{R})\},$$

and Q_f is selfadjoint. It is clear that f is locally integrable. However, f is not square integrable on any interval of positive length. If g is continuous and nonzero at a point x_0, then $\exists\ \varepsilon > 0: |g(x)| \geq \varepsilon$ for all x in some neighborhood of x_0, so $\int_{\underline{R}} |fg|^2\, dx = \infty$. Accordingly, $\mathrm{Dom}(Q_f)$ does not contain any nonzero continuous functions. Since a given element of Dom(P) always admits an absolutely continuous representative, it follows that $\mathrm{Dom}(P) \cap \mathrm{Dom}(Q_f) = \{0\}$. Therefore $P + Q_f$ is not selfadjoint.]

1.25 <u>REMARK</u> The uncertainty relations in quantum mechanics involve the commutator [A,B], where A and B are selfadjoint. However, some care has to be exercised: Dom([A,B]) may reduce to {0} even if B (say) is bounded.

[Proceeding as above, take $\mathcal{H} = L^2(\underline{R})$ but this time put

$$f(x) = \sum_x 2^{-n},$$

where $\sum_x$ stands for a sum over all n such that $q_n < x$ — then $0 < f(x) < 1$ and f is discontinuous at each q_n. Let $A = P$, $B = Q_f$ (B is selfadjoint and bounded). If $g \in \mathrm{Dom}([A,B])$, then both g and fg are continuous on $\underline{R}$. Therefore f is continuous at all points x_0 at which $g(x_0) \neq 0$. But f is discontinuous at each q_n, thus $g(q_n) = 0\ \forall\ n$ and so $g \equiv 0$. I.e.: $\mathrm{Dom}([A,B]) = \{0\}$.

If A and B are selfadjoint and if Dom(A + B) is dense, then

$$(A + B)^* \supset A^* + B^* = A + B.$$

Therefore A + B is symmetric and is essentially selfadjoint iff $(A + B)^*$ is symmetric (cf. 1.16).

1.26 <u>REMARK</u> Suppose that A is an unbounded selfadjoint operator -- then it is always possible to find another selfadjoint operator B such that A + B is densely defined (thus symmetric) but has no selfadjoint extensions.

[Note: B is necessarily unbounded (see below).]

1.27 <u>THEOREM</u> (Kato-Rellich) Suppose that A is selfadjoint and B is symmetric with $\text{Dom}(A) \subset \text{Dom}(B)$. Assume: $\exists$ constants $0 \le a < 1$, $b \ge 0$ such that

$$||Bx|| \le a||Ax|| + b||x|| \quad (x \in \text{Dom}(A)).$$

Then A + B is selfadjoint.

Consequently, if A is a selfadjoint operator and if B is a bounded selfadjoint operator, then A + B is selfadjoint. Proof: In 1.26, take $a = 0$, $b = ||B||$.

1.28 <u>REMARK</u> If A is selfadjoint and unbounded and if B is selfadjoint and bounded, then AB need not be selfadjoint. Thus choose $x \in \mathcal{H} - \text{Dom}(A)$ and let B be the orthogonal projection onto $\underline{C}x$ -- then $\text{Dom}(AB) = \{\underline{C}x\}^{\perp}$, which is not dense in $\mathcal{H}$.

1.29 <u>THEOREM</u> (Wüst) Suppose that A is essentially selfadjoint and B is symmetric with $\text{Dom}(A) \subset \text{Dom}(B)$. Assume: $\exists\ b \geq 0$ such that

$$||Bx|| \leq ||Ax|| + b||x|| \quad (x \in \text{Dom}(A)).$$

Then $A + B$ is essentially selfadjoint.

[Note: If the hypothesis that "A is essentially selfadjoint" is strengthened to "A is selfadjoint", the conclusion remains the same: $A + B$ is essentially selfadjoint. E.g.: Take $B = -A$ with A unbounded -- then the sum $A - A$ is the zero operator on $\text{Dom}(A)$, which is essentially selfadjoint (its closure being the zero operator on $\overline{\text{Dom}(A)} = H$).]

A closed densely defined linear operator A is said to be <u>normal</u> if $\text{Dom}(A^*A) = \text{Dom}(AA^*)$ and there $A^*A = AA^*$.

Every selfadjoint operator is normal as is every unitary operator.

1.30 <u>REMARK</u> If A is a closed densely defined linear operator, then

$$\begin{bmatrix} A^*A \\ AA^* \end{bmatrix} \text{ are selfadjoint and}$$

$$\begin{bmatrix} \overline{A|\text{Dom}(A^*A)} = A \\ \overline{A^*|\text{Dom}(AA^*)} = A^*. \end{bmatrix}$$

1.31 <u>LEMMA</u> Suppose that A is closed and densely defined — then A is normal iff $\text{Dom}(A) = \text{Dom}(A^*)$ and there $||Ax|| = ||A^*x||$.

An easy application of this result is the fact that if A is normal, then $\forall\ z \in \underline{C}$, $z + A$ is normal.

1.32 LEMMA Suppose that A is normal — then

$$\frac{A + A^*}{2} \quad \text{and} \quad \frac{A - A^*}{2\sqrt{-1}}$$

are essentially selfadjoint on Dom(A) = Dom(A*).

[Note: Put

$$\begin{cases} \text{Re } A = \overline{\dfrac{A + A^*}{2}} \\ \\ \text{Im } A = \overline{\dfrac{A - A^*}{2\sqrt{-1}}}. \end{cases}$$

Then

$$\text{Dom}(A) = \text{Dom}(\text{Re } A) \cap \text{Dom}(\text{Im } A)$$

and there

$$A = \text{Re } A + \sqrt{-1} \text{ Im } A.]$$

Suppose that $A: H \to H$ is bounded — then A is said to be nonnegative if $\langle x, Ax \rangle \geq 0 \ \forall \ x \in H$.

[Note: A nonnegative operator is necessarily selfadjoint (H is complex).]

1.33 LEMMA If A is nonnegative, then there is a unique nonnegative operator $\sqrt{A}$ such that $(\sqrt{A})^2 = A$.

1.34 LEMMA If A is nonnegative and $B: H \to H$ is bounded, then AB = BA iff $\sqrt{A}\, B = B \sqrt{A}$.

1.35 <u>EXAMPLE</u> If A and B are nonnegative and if AB = BA, then $\sqrt{A}\,\sqrt{B} = \sqrt{B}\,\sqrt{A}$, thus

$$AB = \sqrt{A}\,\sqrt{A}\,\sqrt{B}\,\sqrt{B} = \sqrt{A}\,\sqrt{B}\,\sqrt{A}\,\sqrt{B} = (\sqrt{A}\,\sqrt{B})^2,$$

from which it follows that AB is also nonnegative.

Suppose that $A:H \to H$ is bounded — then A^*A is nonnegative, hence by 1.33 admits a unique square root and we write

$$|A| = (A^*A)^{1/2}.$$

1.36 <u>EXAMPLE</u> If

$$|A|^2 = |B|^2 + I,$$

then $|A|$ and $|B|$ commute. For $|A|^2\,|B|^2 = |B|^2\,|A|^2$, thus (cf. 1.35)

$$(|A|^2)^{1/2}\,(|B|^2)^{1/2} = (|B|^2)^{1/2}\,(|A|^2)^{1/2}$$

or still,

$$|A|\,|B| = |B|\,|A|.$$

APPENDIX

Denote by $\mathcal{B}(H)$ the set of bounded linear operators on H.

- $\underline{L}_2(H)$ is the two sided $*$-ideal in $\mathcal{B}(H)$ consisting of the Hilbert-Schmidt operators.

- $\underline{L}_1(H)$ is the two sided $*$-ideal in $\mathcal{B}(H)$ consisting of the trace class operators.

Recall that $\underline{L}_2(H)$ is a Hilbert space while $\underline{L}_1(H)$ is a Banach space. In fact, $\underline{L}_1(H) \subset \underline{L}_2(H)$ with

$$||A||_1 \geq ||A||_2 \geq ||A||.$$

[Note: By definition,

$$\begin{cases} ||A||_1 = \text{tr}(|A|) \\ ||A||_2 = (\text{tr}(|A|^2))^{1/2}. \end{cases}$$
]

<u>LEMMA</u> Let $A \in B(H)$ -- then $A \in \underline{L}_1(H)$ iff $\exists$ $B,C \in \underline{L}_2(H)$ such that $A = BC$.

[Note: Matters can always be arranged so as to ensure that

$$||A||_1 = ||B||_2 \, ||C||_2.]$$

<u>REMARK</u> Let $A \in B(H)$. Assume: A is invertible -- then

$$I = AA^{-1} \Rightarrow A \notin \underline{L}_p(H) \quad (p = 1,2).$$

[Bear in mind that H is, by hypothesis, infinite dimensional.]

In practice, it is sometimes necessary to consider two inner products on H, say $<\ ,\ >$, $<\ ,\ >'$, which we shall assume are equivalent -- then the Riesz representation theorem implies that $\exists$ a bounded linear operator $T':H \to H$ such that $\forall$ $x,y \in H$,

$$< x,y > = < x,T'y >'.$$

Observing that T' is positive and selfadjoint per $<\ ,\ >'$, put $T = (T')^{1/2}$, so that $\forall\ x,y \in \mathcal{H}$

$$< x,y > = < Tx,Ty >'.$$

[Note: T is invertible.]

<u>REMARK</u> Let $A \in \mathcal{B}(\mathcal{H})$ and denote its adjoint per $<\ ,\ >'$ by A* — then $\forall\ x,y \in \mathcal{H}$,

$$\begin{aligned} < x,Ay > &= < Tx,TAy >' \\ &= < T^2x,Ay >' \\ &= < A^*T^2x,y >' \\ &= < T^{-2}A^*T^2x,y >, \end{aligned}$$

thus the adjoint of A per $<\ ,\ >$ is $T^{-2}A^*T^2$.

E.g.: Take A = T — then the adjoint of T per $<\ ,\ >$ is $T^{-2}TT^2 = T$, i.e., T is also selfadjoint per $<\ ,\ >$.

<u>LEMMA</u> Let $A \in \mathcal{B}(\mathcal{H})$ — then $A \in \underline{L}_p(\mathcal{H})$ $(p = 1,2)$ per $<\ ,\ >$ iff $A \in \underline{L}_p(\mathcal{H})$ $(p = 1,2)$ per $<\ ,\ >'$.

[Note: Suppose that A is trace class — then

$$\text{tr}(A) = \text{tr}'(A).]$$

§2. *SPECTRAL THEORY*

Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space -- then by $\text{Pro}_{\mathcal{H}}$ we understand the set of bounded idempotent selfadjoint operators on $\mathcal{H}$ or still, the set of orthogonal projections on $\mathcal{H}$.

Let $(X,\mathcal{S})$ be a measurable space (so $\mathcal{S}$ is a σ-algebra of subsets of X) -- then a <u>spectral measure</u> on $\mathcal{S}$ is a function $E:\mathcal{S} \to \text{Pro}_{\mathcal{H}}$ such that

$$E(\emptyset) = 0,\ E(X) = 1\ (\equiv I)$$

and

$$E\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} E(S_n)$$

in the strong operator topology whenever $\{S_n\}$ is a disjoint sequence of sets in $\mathcal{S}$.

2.1 <u>EXAMPLE</u> Take $X = \underline{R}^n$, $\mathcal{S} = \text{Bor}(\underline{R}^n)$ and $\mathcal{H} = L^2(\underline{R}^n,\mu)$, where μ is a σ-finite measure on $\text{Bor}(\underline{R}^n)$ -- then the prescription

$$\begin{cases} \mathcal{S} \to \text{Pro}_{\mathcal{H}} \\ S \to E(S) \end{cases},\ E(S)\psi = \chi_S\psi$$

is a spectral measure.

2.2 <u>LEMMA</u> Suppose that $E:\mathcal{S} \to \text{Pro}_{\mathcal{H}}$ is a spectral measure -- then

$$E(S) \le E(T) \text{ and } E(T - S) = E(T) - E(S)$$

if $S \subset T$.

2.3 <u>LEMMA</u> Suppose that $E:\mathcal{S} \to \text{Pro}_{\mathcal{H}}$ is a spectral measure — then

$$E(S \cup T) + E(S \cap T) = E(S) + E(T).$$

2.4 <u>LEMMA</u> Suppose that $E:\mathcal{S} \to \text{Pro}_{\mathcal{H}}$ is a spectral measure — then

$$E(S \cap T) = E(S)E(T).$$

2.5 <u>REMARK</u> Spectral measures are continuous from above and below:

$$S_1 \supset S_2 \supset \cdots \supset S: \bigcap_n S_n = S$$

=>

$$E(S) = \lim E(S_n) \quad \text{(strong operator topology)}$$

and

$$S_1 \subset S_2 \subset \cdots \subset S: \bigcup_n S_n = S$$

=>

$$E(S) = \lim E(S_n) \quad \text{(strong operator topology).}$$

2.6 <u>CRITERION</u> A function $E:\mathcal{S} \to \text{Pro}_{\mathcal{H}}$ such that $E(\emptyset) = 0$, $E(X) = 1$ is a spectral measure iff $\forall\, x,y \in \mathcal{H}$, the function

$$\mu_{x,y}(S) = \langle x, E(S)y \rangle$$

is a complex measure on $\mathcal{S}$.

Specialize to the case when $X = \underline{R}$, $S = \text{Bor}(\underline{R})$ and fix a spectral measure E. Let $I_\lambda =]-\infty,\lambda]$ and $E_\lambda = E(I_\lambda)$ — then $\forall\ x \in H$, $F_x(\lambda) = \langle x,E_\lambda x\rangle$ is an increasing right continuous function on $\underline{R}$. By definition (cf. 2.6),

$$\mu_{x,x}(]a,b]) = \langle x,E(]a,b])x\rangle$$

$$= \langle x,E(I_b - I_a)x\rangle$$

$$= \langle x,E(I_b)x\rangle - \langle x,E(I_a)x\rangle$$

$$= F_x(b) - F_x(a),$$

thus $\mu_{x,x}$ is the Stieltjes measure induced by F_x (and F_x is the cumulative distribution function of $\mu_{x,x}$).

[Note: In general, the function $\lambda \rightarrow \langle x,E_\lambda y\rangle$ is of bounded variation (as can be seen by polarization) and $\mu_{x,y}$ is the associated Stieltjes measure. Symbolically: $d\mu_{x,y}(\lambda) = d\langle x,E_\lambda y\rangle$.]

Suppose that $f:\underline{R} \rightarrow \underline{C}$ is a bounded Borel function — then it is clear that there exists a unique bounded linear operator $A_f:H \rightarrow H$ such that $\forall\ x,y \in H$,

$$\langle x,A_f y\rangle = \int_{\underline{R}} f(\lambda)d\langle x,E_\lambda y\rangle.$$

Here

$$||A_f|| = \text{ess sup}_E |f| \ (= \inf_{S:E(S) = 0} \sup_{\lambda \notin S} |f(\lambda)|).$$

Moreover $A_f = A_g$ iff $f = g$ E - a.e., i.e., iff $E(\{\lambda:f(\lambda) \neq g(\lambda)\}) = 0$.

We shall call A_f the <u>integral</u> of f w.r.t. E and write

$$A_f = \int_{\underline{R}} f \, dE_\lambda .$$

[Note: The result of applying $\int_{\underline{R}} f \, dE_\lambda$ to a vector x is usually denoted by $\int_{\underline{R}} f \, dE_\lambda x$ rather than $(\int_{\underline{R}} f \, dE_\lambda)x$.]

<u>Properties of the Integral</u> The arrow $f \to A_f$ is a linear map from the bounded Borel functions on $\underline{R}$ to the bounded linear operators on $\mathcal{H}$. In addition:

1. $(\int_{\underline{R}} f \, dE_\lambda)^* = \int_{\underline{R}} \bar{f} \, dE_\lambda .$

2. $(\int_{\underline{R}} f \, dE_\lambda)(\int_{\underline{R}} g \, dE_\lambda) = \int_{\underline{R}} fg \, dE_\lambda .$

3. $< \int_{\underline{R}} f \, dE_\lambda x, \int_{\underline{R}} g \, dE_\lambda y > = \int_{\underline{R}} \bar{f} g \, d< x, E_\lambda y > .$

2.7 <u>REMARK</u> The operator A_f is always normal. It is unitary if $\forall \lambda$, $f(\lambda) \in \underline{S}^1$ and it is selfadjoint if $\forall \lambda$, $f(\lambda) \in \underline{R}$.

2.8 <u>EXAMPLE</u> $\forall$ Borel set S,

$$\int_{\underline{R}} \chi_S \, dE_\lambda = E(S).$$

Consequently,

$$\mu_{x,A_f y}(S) = < x, E(S) A_f y >$$

$$= < E(S)x, A_f y >$$

$$= < \int_{\underline{R}} \chi_S \, dE_\lambda x, \int_{\underline{R}} f \, dE_\lambda y >$$

$$= \int_{\underline{R}} \chi_S f \, d\langle x, E_\lambda y \rangle = \int_S f \, d\langle x, E_\lambda y \rangle.$$

To eliminate the boundedness restriction, consider an arbitrary Borel function $f:\underline{R} \to \underline{C}$. Put

$$D_f = \{x \in H: \int_{\underline{R}} |f(\lambda)|^2 d\langle x, E_\lambda x \rangle < \infty\}.$$

Then D_f is a linear subspace of H:

$$\mu_{cx + y, cx + y}(S) \le 2|c|^2 \mu_{x,x}(S) + 2\mu_{y,y}(S) \quad (c \in \underline{C}).$$

Furthermore, D_f is dense. To see this, fix $x \in H$ and let $x_n = E(S_n)x$, where $S_n = \{\lambda: |f(\lambda)| \le n\}$. Since $S_n \subset S_{n+1}$ and $\bigcup_n S_n = \underline{R}$, it follows that $E(S_n)x \to E(\underline{R})x$ or still, $x_n \to x$. But $x_n \in D_f$:

$$\int_{\underline{R}} |f(\lambda)|^2 d\langle x_n, E_\lambda x_n \rangle$$

$$= \int_{\underline{R}} |f(\lambda)|^2 d\langle E(S_n)x, E(I_\lambda)E(S_n)x \rangle$$

$$= \int_{\underline{R}} |f(\lambda)|^2 d\langle x, E(S_n \cap I_\lambda)x \rangle$$

$$= \int_{\underline{R}} |f(\lambda)|^2 \chi_{S_n}(\lambda) d\langle x, E_\lambda x \rangle$$

$$\le n^2 ||x||^2.$$

So D_f is indeed dense.

To construct A_f, let $x \in D_f$ and choose a sequence $\{f_n\}$ of bounded Borel functions such that

$$\lim_{n \to \infty} \int_{\underline{R}} |f_n - f|^2 d< x,E_\lambda x > = 0.$$

Set $x_n = \int_{\underline{R}} f_n \, dE_\lambda x$ -- then

$$||A_{f_n} x - A_{f_m} x||^2$$

$$\leq 2 \int_{\underline{R}} |f_n - f|^2 d< x,E_\lambda x > + 2 \int_{\underline{R}} |f_m - f|^2 d< x,E_\lambda x >.$$

Therefore the sequence $\{A_{f_n} x\}$ is Cauchy, thus has a limit in $\mathcal{H}$ which, by a similar argument, is independent of the approximating sequence $\{f_n\}$. The prescription

$$A_f x = \lim_{n \to \infty} A_{f_n} x \quad (x \in D_f)$$

then defines a linear operator, the <u>integral</u> of f w.r.t. E, written

$$A_f = \int_{\underline{R}} f \, dE_\lambda.$$

Accordingly,

$$||A_f x||^2 = \lim_{n \to \infty} ||A_{f_n} x||^2$$

$$= \lim_{n \to \infty} \int_{\underline{R}} |f_n|^2 d< x,E_\lambda x >$$

$$= \int_{\underline{R}} |f|^2 d< x,E_\lambda x >.$$

[Note: To establish that A_f is really linear, choose the f_n subject to

$$\begin{bmatrix} f_n \to f \\ |f_n| \leq |f| \end{bmatrix} \quad E - \text{a.e.}$$

(e.g. $f_n = \chi_{S_n} f$) -- then f_n is independent of $x \in D_f$

and by dominated convergence,

$$\lim_{n \to \infty} \int_{\underline{R}} |f_n - f|^2 d\langle x, E_\lambda x \rangle = 0.]$$

2.9 LEMMA Let $x \in \mathcal{H}$, $y \in D_f$ — then f is integrable w.r.t. $\mu_{x,y}$ and

$$\mu_{x,A_f y}(S) = \int_S f(\lambda) d\mu_{x,y}(\lambda).$$

Therefore $\forall\ x \in \mathcal{H}$ & $\forall\ y \in D_f$,

$$\begin{aligned} \langle x, A_f y \rangle &= \langle x, E(\underline{R}) A_f y \rangle \\ &= \mu_{x,A_f y}(\underline{R}) \\ &= \int_{\underline{R}} f(\lambda) d\mu_{x,y}(\lambda) \\ &= \int_{\underline{R}} f(\lambda) d\langle x, E_\lambda y \rangle, \end{aligned}$$

which is the defining property of A_f when f is bounded.

2.10 EXAMPLE Take $\mathcal{H} = L^2(\underline{R},\mu)$, where μ is a σ-finite measure on $\mathcal{S} = \text{Bor}(\underline{R})$ and let $E(S)\psi = \chi_S \psi$ (cf. 2.1). Suppose that $f:\underline{R} \to \underline{C}$ is Borel and consider its associated multiplication operator Q_f, viz. $\psi \to f\psi$ with

$$\text{Dom}(Q_f) = \{\psi \in L^2(\underline{R},\mu) : \int_{\underline{R}} |f|^2 |\psi|^2 d\mu < \infty\}.$$

Then $\text{Dom}(A_f) = \text{Dom}(Q_f)$ and

$$A_f = \int_{\underline{R}} f\, dE_\lambda = Q_f.$$

Properties of the Integral The unbounded situation is complicated by domain issues. It is certainly true that $A_{cf} = cA_f$ $(c \in \underline{C})$. As for addition and multiplication, we have

$$\begin{cases} A_{f+g} = \overline{A_f + A_g} \\ A_{fg} = \overline{A_f A_g}. \end{cases}$$

And it is still the case that

$$(\int_{\underline{R}} f \, dE_\lambda)^* = \int_{\underline{R}} \bar{f} \, dE_\lambda,$$

hence $\int_{\underline{R}} f \, dE_\lambda$ is selfadjoint whenever f is real (and normal in general).

[Note: If f and g are real valued, then $A_{f + \sqrt{-1}\, g} = A_f + \sqrt{-1}\, A_g$.]

2.11 LEMMA Let $f:\underline{R} \to \underline{C}$ be Borel -- then $A_{f^k} = A_f^k$ $(k = 1,2,\ldots)$. In addition, given complex numbers $c_0, c_1, \ldots, c_n$, we have

$$A_{c_0 + c_1 f + \cdots + c_n f^n} = c_0 + c_1 A_f + \cdots + c_n A_f^n.$$

So, by way of a corollary, if f is real, then the powers A_f^k $(k = 1,2,\ldots)$ are selfadjoint.

2.12 SPECTRAL THEOREM If A is selfadjoint, then $\exists$ a unique spectral measure E such that $A = \int_{\underline{R}} \lambda \, dE_\lambda$.

This is the central result of the theory. In order to help place it in perspective, it will be convenient to review some standard terminology.

Let A be a densely defined linear operator, which we shall assume is closed — then the <u>spectrum</u> $\sigma(A)$ of A is that subset of $\underline{C}$ consisting of those λ such that $A - \lambda$ is not a bijection $\text{Dom}(A) \to H$.

[Note: It may very well be the case that $\sigma(A)$ is empty.]

Suppose that $\lambda \in \sigma(A)$ — then there are two possibilities:

1. $A - \lambda$ is not injective.

2. $A - \lambda$ is injective but not surjective.

The elements $\lambda \in \sigma(A)$ corresponding to the first case are the eigenvalues of A. They constitute the <u>point spectrum</u> $\sigma_p(A)$ of A. The elements $\lambda \in \sigma(A)$ corresponding to the second case fall into two classes: The <u>continuous spectrum</u> $\sigma_c(A)$ consists of those λ such that $\text{Ran}(A - \lambda)$ is dense in H and the <u>residual spectrum</u> $\sigma_r(A)$ consists of those λ such that $\overline{\text{Ran}(A - \lambda)} \neq H$. Thus there is a disjoint decomposition

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

2.13 <u>LEMMA</u> $\sigma(A)$ is a closed subset of $\underline{C}$.

[Note: The spectrum of a selfadjoint operator is a closed subset of $\underline{R}$ while the spectrum of a unitary operator is a closed subset of $\underline{T}$.]

2.14 <u>EXAMPLE</u> (Annihilation and Creation) Agreeing to use the notation of 1.19 and 1.20, define linear operators

$$\begin{cases} \exp(z\underline{a}) \\ \exp(z\underline{c}) \end{cases} \quad (z \in \underline{C}) \text{ on D by}$$

$$\begin{cases} \exp(z\underline{a}) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \underline{a}^k \\ \exp(z\underline{c}) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \underline{c}^k. \end{cases}$$

Since

$$\left.\begin{matrix} ||\underline{a}^k e_n|| \\ ||\underline{c}^k e_n|| \end{matrix}\right\} \leq [(n+k)!]^{1/2},$$

these definitions make sense. Recalling that

$$e_n = \frac{\underline{c}^n}{\sqrt{n!}} e_0 \quad (n \geq 1),$$

we have

$$\exp(z\underline{c})e_0 = \sum_{k=0}^{\infty} \frac{z^k}{k!} \underline{c}^k e_0 = \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} e_k.$$

Obviously, then,

$$\sum_{n=0}^{\infty} n| < e_n, \exp(z\underline{c})e_0 > |^2 < \infty.$$

Therefore $\exp(z\underline{c})e_0 \in \mathrm{Dom}(\bar{a})$. Since

$$\bar{a}(\exp(z\underline{c})e_0) = z(\exp(z\underline{c})e_0)$$

and since $z \in \underline{C}$ is arbitrary, the conclusion is that $\sigma_p(\bar{a}) = \underline{C}$. On the other hand, $\sigma_p(\bar{c}) = \emptyset$ while $\sigma_r(\bar{c}) = \underline{C}$.

[Note: In passing, observe that

$$||\exp(z\underline{c})e_0||^2 = \sum_{n=0}^{\infty} |\frac{z^n}{\sqrt{n!}}|^2 = e^{|z|^2}.]$$

Assume henceforth that A is normal -- then the residual spectrum is empty: $\sigma_r(A) = \emptyset$. Turning to the point spectrum, one can show that $\lambda \in \sigma_p(A)$ iff $\bar{\lambda} \in \sigma_p(A^*)$ with

$$\mathrm{Ker}(A - \lambda) = \mathrm{Ker}(A^* - \bar{\lambda}).$$

And the eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.

The spectrum of A is said to be pure point if there is an orthonormal basis $\{e_i : i \in I\}$ for $\mathcal{H}$ consisting of eigenvectors for $A: Ae_i = \lambda_i e_i$.

2.15 LEMMA If A is a normal operator whose spectrum is pure point, then $\sigma(A) = \overline{\sigma_p(A)}$ and

$$\mathrm{Dom}(A) = \{x: \sum_i |\lambda_i|^2 |< e_i, x >|^2 < \infty\}.$$

2.16 EXAMPLE Consider $\bar{N}$, the closure of the number operator N (cf. 1.19) -- then $\bar{N}$ is selfadjoint and $\bar{N}e_n = ne_n$ $(n \geq 0)$. Therefore the spectrum of $\bar{N}$ is pure point and

$$\mathrm{Dom}(\bar{N}) = \{x \in \mathcal{H}: \sum_0^{\infty} n^2 |< e_n, x >|^2 < \infty\}.$$

2.17 EXAMPLE Take $\mathcal{H}$ separable and let $\{e_n\}$ be an orthonormal basis. Define a linear operator A on the linear span of the e_n by $Ae_n = \frac{1}{n} e_n$ — then $\bar{A}$ is selfadjoint (cf. 1.13). But $\bar{A}e_n = \frac{1}{n} e_n$. Therefore the spectrum of $\bar{A}$ is pure point and $\sigma(A) = \overline{\{\frac{1}{n} : n \in \underline{N}\}} = \{0\} \cup \{\frac{1}{n} : n \in \underline{N}\}$, so $\sigma_c(A) = \{0\}$.

[Note: Let F be an infinite subset of $\underline{Q}$ -- then a simple variation on this theme gives rise to a selfadjoint operator whose spectrum is pure point and coincides with F.]

2.18 <u>CRITERION</u> Suppose that A is normal -- then $\lambda \in \sigma(A)$ iff $\exists$ a sequence of unit vectors $x_n \in \text{Dom}(A)$ such that $(A - \lambda)x_n \to 0$.

2.19 <u>EXAMPLE</u> Take $H = L^2(\underline{R})$ and let $A = Q$, the position operator -- then $\sigma(Q) = \underline{R}$. For Q is selfadjoint, so $\sigma(Q) \subset \underline{R}$. This said, fix $\lambda \in \underline{R}$ and put $f_n = \sqrt{n}\, \chi_{I_n}$, where $I_n = [\lambda, \lambda + \frac{1}{n}]$ -- then $||f_n|| = 1$ and $||(Q - \lambda)f_n|| = \frac{1}{\sqrt{3n}} \to 0$, thus 2.18 is applicable.

[Note: Obviously, $\sigma_p(Q) = \emptyset$, hence $\sigma(Q) = \sigma_c(Q)$.]

2.20 <u>LEMMA</u> If $A:\text{Dom}(A) \to H$ is selfadjoint and if $U:H \to H$ is unitary, then $\sigma(UAU^{-1}) = \sigma(A)$.

[Note: Recall that $UAU^{-1}:U\text{Dom}(A) \to H$ is selfadjoint (cf. 1.10).]

2.21 <u>EXAMPLE</u> Take $H = L^2(\underline{R})$ and let $A = P$, the momentum operator -- then

$\sigma(P) = \underline{R}$. In fact, $\sigma(P) = \sigma(U_F^{-1} Q U_F) = \sigma(Q) = \underline{R}$.

Let A be selfadjoint -- then in the notation of the spectral theorem (cf. 2.12), $\exists$ a unique spectral measure E such that $A = \int_{\underline{R}} \lambda \, dE_\lambda$.

[Note: Bear in mind that in this context, the domain of E is $\text{Bor}(\underline{R})$.]

2.22 REMARK The spectrum $\sigma(A)$ of A is a nonempty closed subset of $\underline{R}$. Moreover $E(\underline{R} - \sigma(A)) = 0$ and, in fact, E is supported by $\sigma(A)$.

[Note: A symmetric operator is selfadjoint iff its spectrum is real.]

2.23 LEMMA $\lambda \in \sigma(A)$ iff $E(]\lambda - \varepsilon, \lambda + \varepsilon[) \neq 0 \ \forall \ \varepsilon > 0$.

2.24 LEMMA $\lambda \in \sigma_p(A)$ iff $E(\{\lambda\}) \neq 0$.

[Note: The range of $E(\{\lambda\})$ is the corresponding eigenspace.]

2.25 REMARK Any isolated point of $\sigma(A)$ is an eigenvalue.

2.26 EXAMPLE Suppose that A is pure point -- then there is an orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\lambda \in \sigma_p(A)} \text{Ker}(A - \lambda)$$

and the spectral measure determined by A is given by the rule

$$E(S) = \sum_{\lambda \in \sigma_p(A)} \chi_S(\lambda) E(\{\lambda\}),$$

where the convergence is in the strong operator topology.

Since $\sigma(A)$ is closed, it contains its limit points: $\sigma(A) \supset \sigma(A)'$. The essential spectrum $\sigma_{ess}(A)$ of A is then by definition $\sigma(A)'$ together with the eigenvalues of infinite multiplicity.

2.27 LEMMA $\lambda \in \sigma_{ess}(A)$ iff the dimension of $E(]\lambda - \varepsilon, \lambda + \varepsilon[)$ is infinite $\forall\ \varepsilon > 0$.

[Note: This implies that $\sigma_{ess}(A)$ is a closed subset of $\underline{R}$.]

There is a decomposition

$$\sigma(A) = \sigma_p(A) \cup \sigma_{ess}(A),$$

hence

$$\sigma_c(A) = \sigma_{ess}(A) - \sigma_p(A).$$

The complement

$$\sigma_d(A) = \sigma(A) - \sigma_{ess}(A)$$

is called the discrete spectrum of A. It consists of all isolated eigenvalues of finite multiplicity. If the essential spectrum is empty, then $\sigma(A) = \sigma_d(A) = \sigma_p(A)$ and the spectrum of A is pure point. However, it may very well be the case that the spectrum of A is pure point, yet the discrete spectrum is empty.

Working still with the spectral measure attached to A, let $f:\underline{R} \to \underline{R}$ be Borel -- then $A_f = \int_{\underline{R}} f\, dE_\lambda$ is selfadjoint and $\forall\ x \in H\ \&\ \forall\ y \in D_f$, we have

$$< x, A_f y > = \int_{\underline{R}} f(\lambda) d < x, E_\lambda y >.$$

[Note: In this context, it is customary to write f(A) in place of A_f.]

2.28 EXAMPLE Given $x \in \mathcal{H}$, let $x_n = E(S_n)x$, where $S_n = \{\lambda : |\lambda| \leq n\}$ -- then x_n is analytic for A. In fact,

$$\begin{aligned} ||A^k x_n||^2 &= \int_{\underline{R}} |\lambda|^{2k} d < x_n, dE_\lambda x_n > \\ &= \int_{\underline{R}} |\lambda|^{2k} \chi_{S_n}(\lambda) d < x, E_\lambda x > \\ &\leq n^{2k} ||x||^2. \end{aligned}$$

Therefore the power series

$$\sum_{k=0}^{\infty} \frac{||A^k x_n||}{k!} t^k$$

is absolutely convergent for all t.

[Note: Since $x_n \to x$ and x is arbitrary, the set of analytic vectors for A is dense.]

2.29 LEMMA The spectral measure attached to f(A) is the assignment $S \to E(f^{-1}(S))$.

2.30 LEMMA Suppose that f is continuous -- then $\sigma(f(A)) = \overline{f(\sigma(A))}$.

We shall term A nonnegative if $< x, Ax > \geq 0 \ \forall \ x \in \text{Dom}(A)$. When this is so,

$\sigma(A) \subset \underline{R}_{\geq 0}$ and A admits a unique nonnegative n^{th} root $A^{1/n}$, viz.

$$A^{1/n} = \int_0^\infty \lambda^{1/n} \, dE_\lambda .$$

2.31 EXAMPLE Consider $\bar{N}$, the closure of the number operator N (cf. 1.19) -- then $\bar{N}$ is nonnegative and

$$\mathrm{Dom}(\bar{N}^{1/2}) = \{x \in H: \sum_{n=0}^{\infty} n| < e_n, x > |^2 < \infty\}.$$

I.e.: $\mathrm{Dom}(\bar{N}^{1/2}) = \bar{D}$, the common domain of $\bar{a}$ and $\bar{c}$ (cf. 1.20).

2.32 LEMMA Suppose that A is selfadjoint and nonnegative -- then

$$A^{1/2} = \overline{A^{1/2}|\mathrm{Dom}(A)},$$

i.e., Dom(A) is a domain of essential selfadjointness for $A^{1/2}$.

2.33 LEMMA If A is selfadjoint, then $|A|$ ($= \int_{\underline{R}} |\lambda| \, dE_\lambda$) is nonnegative and $\mathrm{Dom}(|A|) = \mathrm{Dom}(A)$ (thus $|A| = A$ if A is nonnegative). And: $|A| = (A^2)^{1/2}$.

If $f:\underline{R} \to \underline{C}$ is Borel, then A_f is normal with $A_f^* = A_{\bar{f}}$ or still, f(A) is normal with $f(A)^* = \bar{f}(A)$.

2.34 EXAMPLE Suppose that x is an analytic vector for A, hence $\exists \, R_x > 0$:

$$\sum_{k=0}^{\infty} \frac{||A^k x||}{k!} |t|^k < \infty$$

if $|t| < R_x$. We then claim that

$$x \in \mathrm{Dom}(e^{zA}) \text{ and } e^{zA}x = \sum_{k=0}^{\infty} \frac{z^k}{k!} A^k x$$

provided $|z| < R_x$. For $x \in \mathrm{Dom}(e^{zA})$ iff

$$\int_{\underline{R}} |e^{z\lambda}|^2 \, d\langle x, E_\lambda x \rangle < \infty.$$

And $|z| < R_x$ =>

$$\left[\int_{-n}^{n} |e^{z\lambda}|^2 \, d\langle x, E_\lambda x \rangle\right]^{1/2}$$

$$= \left|\left|\int_{-n}^{n} e^{z\lambda} \, dE_\lambda x\right|\right|$$

$$= \left|\left|\int_{-n}^{n} \sum_{k=0}^{\infty} \frac{(z\lambda)^k}{k!} \, dE_\lambda x\right|\right|$$

$$\leq \sum_{k=0}^{\infty} \frac{|z|^k}{k!} \left|\left|\int_{-n}^{n} \lambda^k \, dE_\lambda x\right|\right|$$

$$\leq \sum_{k=0}^{\infty} \frac{|z|^k}{k!} ||A^k x|| < \infty$$

=>

$$\int_{\underline{R}} |e^{z\lambda}|^2 \, d\langle x, E_\lambda x \rangle < \infty.$$

Now write

$$e^{zA}x = \int_{\underline{R}} e^{z\lambda} \, dE_\lambda x$$

$$= \sum_{k=0}^{K} \frac{z^k}{k!} \int_{\underline{R}} \lambda^k \, dE_\lambda x + \int_{\underline{R}} \sum_{k=K+1}^{\infty} \frac{(z\lambda)^k}{k!} \, dE_\lambda x$$

and observe that

$$\left|\left| \sum_{k=K+1}^{\infty} \frac{(z\lambda)^k}{k!} \, dE_\lambda x \right|\right|$$

$$\leq \sum_{k=K+1}^{\infty} \frac{|z|^k}{k!} \left|\left| \int_{\underline{R}} \lambda^k \, dE_\lambda x \right|\right|$$

$$= \sum_{k=K+1}^{\infty} \frac{|z|^k}{k!} \, ||A^k x|| \to 0 \text{ as } K \to \infty.$$

Therefore

$$e^{zA}x = \sum_{k=0}^{\infty} \frac{z^k}{k!} A^k f.$$

2.35 <u>REMARK</u> If A is selfadjoint, then its set of analytic vectors is

$$\bigcup_{t>0} \text{Dom}(e^{t|A|}).$$

§3. *ONE PARAMETER UNITARY GROUPS*

Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space. Denote by $\mathcal{U}(\mathcal{H})$ the set of all unitary operators on $\mathcal{H}$ -- then $\mathcal{U}(\mathcal{H})$ is a group under operator multiplication and is a topological group when equipped with the strong operator topology.

3.1 EXAMPLE The strong limit of a sequence of unitary operators need not be unitary. To see this, take $\mathcal{H} = \ell^2(\underline{N})$ and define $U_k:\mathcal{H} \rightarrow \mathcal{H}$ $(k > 1)$ by

$$U_k(\{x_n\}) = (x_k,x_1,x_2,\ldots,x_{k-1},x_{k+1},x_{k+2},\ldots).$$

Then the U_k are unitary and converge strongly to T, where

$$T(\{x_n\}) = (0,x_1,x_2,\ldots).$$

Suppose that U is unitary -- then $\sigma(U)$ is a closed subset of $\{z:|z| = 1\}$.

3.2 SPECTRAL THEOREM If U is unitary, then $\exists$ a spectral measure E such that $E(]-\infty,0[) = 0$, $E([2\pi,\infty[) = 0$, and

$$U = \int_{\underline{R}} e^{\sqrt{-1}\,\lambda}\, dE_\lambda.$$

[Note: As in 2.12, the domain of E is $\text{Bor}(\underline{R})$. Incidentally, these conditions determine E uniquely.]

3.3 EXAMPLE Let $U_F:L^2(\underline{R}) \rightarrow L^2(\underline{R})$ be the unitary operator provided by the Plancherel theorem -- then

$$P_0 = \frac{1}{4}(I + U_F + U_F^2 + U_F^3),$$

$$P_1 = \frac{1}{4}(I - \sqrt{-1}\,U_F - U_F^2 + \sqrt{-1}\,U_F^3),$$

$$P_2 = \frac{1}{4}(I - U_F + U_F^2 - U_F^3),$$

$$P_3 = \frac{1}{4}(I + \sqrt{-1}\,U_F - U_F^2 - \sqrt{-1}\,U_F^3)$$

are pairwise orthogonal nonzero projections whose sum is I. Since $U_F P_k = (\sqrt{-1})^k P_k$ $(k = 0,1,2,3)$, it follows that $\sigma(U_F) = \{1, \sqrt{-1}, -1, -\sqrt{-1}\}$ and the spectrum of U is pure point. The spectral measure determined by U_F is given by the rule

$$E(S) = \frac{1}{4} \sum_{j,k=0}^{3} \chi_S(\frac{\pi k}{2})(\sqrt{-1})^{jk}\, U_F^j.$$

[Note: Each of the eigenvalues ± 1, $\pm\sqrt{-1}$ is of infinite multiplicity.]

3.4 <u>LEMMA</u> Suppose that U is unitary. Put $A_U = \int_0^{2\pi} \lambda\, dE_\lambda$ -- then $U = e^{\sqrt{-1}\,A_U}$.

[Note: Here E is the spectral measure per 3.2.]

Let G be a topological group -- then a <u>unitary representation</u> U of G on $\mathcal{H}$ is a continuous homomorphism $U:G \to \mathcal{U}(\mathcal{H})$.

[Note: Spelled out, the continuity of U is the requirement that $\forall\, x \in \mathcal{H}$, the map $\sigma \to U(\sigma)x$ from G to $\mathcal{H}$ is continuous.]

Specialize to the case when $G = \underline{R}$ -- then a unitary representation U of $\underline{R}$

on $\mathcal{H}$ is called a <u>one parameter unitary group</u>, thus $U:\underline{R} \to \mathcal{U}(\mathcal{H})$ is a continuous homomorphism and we have $U(0) = I$, $U(-t) = U(t)^{-1} = U(t)^*$.

3.5 <u>REMARK</u> Suppose that $U:\underline{R} \to \mathcal{U}(\mathcal{H})$ is a homomorphism -- then to check strong continuity it suffices to work at $t = 0$ and for this weak continuity at $t = 0$ is enough. Proof:

$$||U(t)x - x||^2 = ||U(t)x||^2 - < U(t)x,x > - < x,U(t)x > + ||x||^2$$

$$\to 2||x||^2 - 2||x||^2 = 0.$$

[Note: When $\mathcal{H}$ is separable, one can get away with less, viz. if for all $x,y \in \mathcal{H}$, the function $t \to < U(t)x,y >$ is Borel, then the function $t \to U(t)$ is strongly continuous. Here the separability assumption is necessary: Without it, strong continuity may fail.]

3.6 <u>EXAMPLE</u> Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_s : s \in \underline{R}\}$ in a one-to-one correspondence with $\underline{R}$. Put $U(t)e_s = e_{t+s}$ -- then the assignment $t \to U(t)$ is a homomorphism from $\underline{R}$ to $\mathcal{U}(\mathcal{H})$ but it is not a unitary representation of $\underline{R}$ on $\mathcal{H}$.

Given a one parameter unitary group U, let D_U be the set of all $x \in \mathcal{H}$ for which

$$\lim_{t \to 0} \frac{U(t) - I}{t} x$$

exists. Define a linear operator A on D_U by

$$Ax = \lim_{t \to 0} \frac{U(t) - I}{\sqrt{-1}\, t} x.$$

Then A is called the generator of U. Its domain Dom(A) (= D_U) is invariant under U and $\forall$ x $\in$ Dom(A),

$$AU(t)x = U(t)Ax$$

$$= \lim_{h \to 0} \frac{U(t+h) - U(t)}{\sqrt{-1}\, h} x$$

$$= - \sqrt{-1} \frac{dU}{dt}(t)x.$$

3.7 LEMMA Suppose that A is a selfadjoint operator. Put

$$U(t) = e^{\sqrt{-1}\, tA} \quad (= \int_{\underline{R}} e^{\sqrt{-1}\, t\lambda}\, dE_\lambda).$$

Then U is a one parameter unitary group and its generator is A.

PROOF It is clear that the U(t) are unitary and $\forall$ x,y $\in$ H,

$$< x, U(t_1)U(t_2)y >$$

$$= < U(t_1)^* x, U(t_2)y >$$

$$= < \int_{\underline{R}} e^{- \sqrt{-1}\, t_1\lambda} dE_\lambda x, \int_{\underline{R}} e^{\sqrt{-1}\, t_2\lambda} dE_\lambda y >$$

$$= \int_{\underline{R}} e^{\sqrt{-1}\, t_1\lambda} e^{\sqrt{-1}\, t_2\lambda} d< x, E_\lambda y >$$

$$= \int_{\underline{R}} e^{\sqrt{-1}\,(t_1 + t_2)\lambda} d< x,E_\lambda y >$$

$$= < x,U(t_1 + t_2)y >$$

=>

$$U(t_1)U(t_2) = U(t_1 + t_2).$$

This shows that $U:\underline{R} \to \mathcal{U}(\mathcal{H})$ is a homomorphism. To check strong continuity at $t = 0$, write

$$||U(t)x - x||^2 = \int_{\underline{R}} |e^{\sqrt{-1}\,t\lambda} - 1|^2 d< x,E_\lambda x >.$$

Since $|e^{\sqrt{-1}\,t\lambda} - 1|^2 \leq 4$ (which is integrable), an application of dominated convergence gives $\lim_{t \to 0} ||U(t)x - x||^2 = 0$. Assume now that $x \in \text{Dom}(A)$ -- then

$$|| \frac{e^{\sqrt{-1}\,tA} - I}{\sqrt{-1}\,t} x - Ax||^2$$

$$= \int_{\underline{R}} | \frac{e^{\sqrt{-1}\,t\lambda} - 1}{\sqrt{-1}\,t} - \lambda|^2 d <x,E_\lambda x >.$$

But

$$| \frac{e^{\sqrt{-1}\,t\lambda} - 1 - \sqrt{-1}\,t\lambda}{\sqrt{-1}\,t} |$$

$$\leq \frac{|e^{\sqrt{-1}\,t\lambda} - 1| + |t\lambda|}{|t|} \leq \frac{|t\lambda| + |t\lambda|}{|t|} \leq 2|\lambda|$$

and

$$x \in \mathrm{Dom}(A) \Rightarrow \int_{\underline{R}} \lambda^2 d\langle x, E_\lambda x\rangle < \infty,$$

so another application of dominated convergence gives

$$\lim_{t \to 0} \frac{U(t) - I}{\sqrt{-1}\, t} x = Ax.$$

Therefore $\mathrm{Dom}(A) \subset D_U$. To reverse this, let $x \in D_U$ and put

$$y = \lim_{t \to 0} \frac{U(t) - I}{\sqrt{-1}\, t} x.$$

Then for all sufficiently small $t \neq 0$, we have

$$\int_{\underline{R}} \left| \frac{e^{\sqrt{-1}\, t\lambda} - 1}{\sqrt{-1}\, t} \right|^2 d\langle x, E_\lambda x\rangle < (1 + ||y||)^2$$

$\Rightarrow$

$$\left| \frac{e^{\sqrt{-1} t\lambda} - 1}{\sqrt{-1}\, t} \right|^2 \in L^1(\underline{R}, \mu_{x,y}).$$

On the other hand,

$$\lim_{t \to 0} \left| \frac{e^{\sqrt{-1}\, t\lambda} - 1}{\sqrt{-1}\, t} \right|^2 = |\lambda|^2.$$

Fatou's lemma then implies that $|\lambda|^2 \in L^1(\underline{R}, \mu_{x,y})$, thus $x \in \mathrm{Dom}(A)$. Consequently, $\mathrm{Dom}(A) = D_U$ and A is the generator of U.

3.8 <u>EXAMPLE</u> (The Free Propagator) Take $\mathcal{H} = L^2(\underline{R}^n)$, $A = \Delta$ — then $\forall\, f \in \mathcal{S}(\underline{R}^n)$,

$$(e^{\sqrt{-1}\, t\Delta} f)(x) = \frac{1}{(4\pi\sqrt{-1}\, t)^{n/2}} \int_{\underline{R}^n} e^{\sqrt{-1}\, |x - y|^2/4t} f(y)\,dy.$$

3.9 <u>THEOREM</u> (Stone) Let U be a one parameter unitary group -- then there is a unique selfadjoint operator A such that $U(t) = e^{\sqrt{-1}\, tA}$.

The uniqueness of A is immediate (cf. 3.7). As for the existence of A, one can either proceed directly (there are various approaches) or one can cite a far more general result which goes as follows.

Let G be a locally compact abelian group, Γ its dual. Suppose that U is a unitary representation of G on $\mathcal{H}$ -- then there exists a unique spectral measure $E:\text{Bor}(\Gamma) \to \text{Pro}_{\mathcal{H}}$ such that

$$U(\sigma) = \int_{\Gamma} \chi(\sigma)\, dE_{\chi} \quad (\sigma \in G).$$

When specialized to the case when $G = \underline{R}$ (hence $\Gamma = \underline{R}$), this says that

$$U(t) = \int_{\underline{R}} e^{\sqrt{-1}\, t\lambda}\, dE_{\lambda}$$

or still,

$$U(t) = e^{\sqrt{-1}\, tA},$$

where $A = \int_{\underline{R}} \lambda\, dE_{\lambda}$.

3.10 <u>EXAMPLE</u> Take $\mathcal{H} = L^2(\underline{R})$ and let $A = Q$, the position operator -- then

$$e^{\sqrt{-1}\, tQ}\psi(\lambda) = e^{\sqrt{-1}\, t\lambda}\psi(\lambda) \quad \text{(cf. 2.10)}.$$

3.11 <u>EXAMPLE</u> Take $\mathcal{H} = L^2(\underset{\sim}{R})$ and let $A = P$, the momentum operator — then $P = U_F^{-1} Q U_F$ (cf. 1.11), hence

$$e^{\sqrt{-1}\, tP}\psi(\lambda) = \psi(\lambda + t).$$

3.12 <u>LEMMA</u> Suppose that U is a one parameter unitary group with generator A. Let $D \subset \text{Dom}(A)$ be a dense linear subspace of $\mathcal{H}$ which is invariant under U -- then $A|D$ is essentially selfadjoint and $\overline{A|D} = A$.

<u>PROOF</u> The restriction $A|D:D \to \mathcal{H}$ is symmetric (A being selfadjoint). To prove that $A|D$ is essentially selfadjoint, it suffices to show that the range of $A|D \pm \sqrt{-1}$ is dense in $\mathcal{H}$ and for this, it suffices to show that

$$\text{Ker}((A|D)^* \pm \sqrt{-1}) = \{0\}.$$

Thus let $y \in \text{Dom}((A|D)^*)$ and assume that $(A|D)^*y = \sqrt{-1}\, y$ -- then $\forall\, x \in D$, we have

$$\begin{aligned} &\frac{d}{dt} < y, U(t)x > \\ &\quad = < y, \sqrt{-1}\,(A|D)U(t)x > \\ &\quad = \sqrt{-1} < (A|D)^*y, U(t)x > \\ &\quad = \sqrt{-1} < \sqrt{-1}\, y, U(t)x > \\ &\quad = < y, U(t)x >. \end{aligned}$$

Therefore the complex valued function $f(t) = < y,U(t)x >$ satisfies the differential equation $f' = f$, hence $f(t) = f(0)e^t$. But $|f(t)|$ is bounded, so $f(0) = < y,x > = 0$.

As this holds for all $x \in D$ and D is dense in $\mathcal{H}$, it follows that $y = 0$. I.e.: The kernel of $(A|D)^* - \sqrt{-1}$ is $\{0\}$. Analogous considerations show that the kernel of $(A|D)^* + \sqrt{-1}$ is likewise $\{0\}$. Conclusion: $A|D$ is essentially selfadjoint. And: $\overline{A|D} = A$ (cf. 1.14).

3.13 <u>EXAMPLE</u> Take $\mathcal{H} = L^2(\underline{R})$ and let

$$(U(t)\psi)(\lambda) = e^{t/2}\psi(e^t\lambda) \quad (\psi \in L^2(\underline{R})).$$

Then the assignment $t \rightarrow U(t)$ is a one parameter unitary group and its generator A is given on $C_c^\infty(\underline{R})$ by

$$Af = (QP - \frac{\sqrt{-1}}{2})f.$$

Since $C_c^\infty(\underline{R})$ is invariant under U, an application of 3.12 implies that

$$A = \overline{(QP - \frac{\sqrt{-1}}{2})|C_c^\infty(\underline{R})}$$

or still,

$$A = \overline{\frac{1}{2}(QP + PQ)|C_c^\infty(\underline{R})}.$$

3.14 <u>EXAMPLE</u> Take $\mathcal{H} = L^2(\underline{R})$ and let

$$(U(t)\psi)(\lambda) = e^{\sqrt{-1}\, t(2\lambda + t)/2}\psi(\lambda + t) \quad (\psi \in L^2(\underline{R})).$$

Then the assignment $t \rightarrow U(t)$ is a one parameter unitary group and its generator A is given on $\mathcal{S}(\underline{R})$ by

$$Af = (P + Q)f.$$

Since $S(\underline{R})$ is invariant under U, an application of 3.12 implies that

$$A = \overline{(P + Q)|S(\underline{R})}.$$

[Note: The domain of P + Q is Dom(P) $\cap$ Dom(Q) and there, P + Q is symmetric. But

$$P + Q \supset (P + Q)|S(\underline{R})$$

$$\Rightarrow$$

$$\overline{P + Q} = A \text{ (cf. 1.14)}.$$

Therefore P + Q is essentially selfadjoint. On the other hand, $P + Q \subset T^{-1}PT$, where T is the unitary multiplication operator

$$(T\psi)(\lambda) = \exp(\frac{\sqrt{-1}}{2}\lambda^2)\psi(\lambda) \quad (\psi \in L^2(\underline{R})).$$

Since $T^{-1}PT$ is selfadjoint (cf. 1.10), it follows that $A = T^{-1}PT$.]

Let G be a Lie group. Suppose that U is a unitary representation of G on $\mathcal{H}$. Fix an $X \in \underline{g}$ -- then the assignment $t \to U(\exp(tX))$ is a one parameter unitary group, thus there is a unique selfadjoint operator dU(X) such that

$$U(\exp(tX)) = e^{\sqrt{-1}\, t dU(X)}.$$

3.15 EXAMPLE Working in $\mathcal{H} = L^2(\underline{R}^3)$, put

$$\begin{cases} X = z\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial z} \\ Y = x\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial x} \\ Z = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}. \end{cases}$$

Let $\{Q_x, Q_y, Q_z\}$ be the position operators and let $\{P_x, P_y, P_z\}$ be the momentum operators (cf. 1.12) — then $\forall\ f \in C_c^\infty(\underline{R}^3)$,

$$\begin{cases} Q_y P_z f - Q_z P_y f = \sqrt{-1}\ Xf \\ Q_z P_x f - Q_x P_z f = \sqrt{-1}\ Yf \\ Q_x P_y f - Q_y P_x f = \sqrt{-1}\ Zf. \end{cases}$$

Consider the canonical unitary representation U of $\underline{SO}(3)$ on $L^2(\underline{R}^3)$ arising from the right action of $\underline{SO}(3)$ on $\underline{R}^3$ (viewed as row vectors) and note that $C_c^\infty(\underline{R}^3)$ is invariant under U. Let

$$E_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ E_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ E_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the usual basis vectors for $\underline{so}(3)$, thus

$$E_x = [E_y, E_z],\ E_y = [E_z, E_x],\ E_z = [E_x, E_y].$$

Then there are selfadjoint operators $dU(E_x)$, $dU(E_y)$, $dU(E_x)$ characterized by the relations

$$\begin{cases} U(\exp(tE_x)) = e^{\sqrt{-1}\ t dU(E_x)} \\ U(\exp(tE_y)) = e^{\sqrt{-1}\ t dU(E_y)} \\ U(\exp(tE_z)) = e^{\sqrt{-1}\ t dU(E_z)}. \end{cases}$$

Since for any $f \in C_c^\infty(\underline{R}^3)$,

$$\sqrt{-1}\, dU(E_x) f(\vec{r}) = \frac{d}{dt} f(\vec{r} \exp(tE_x))\Big|_{t=0} = Xf,$$

it follows from 3.12 that

$$(Q_y P_z - Q_z P_y) | C_c^\infty(\underline{R}^3)$$

is essentially selfadjoint with

$$- dU(E_x) = \overline{(Q_y P_z - Q_z P_y) | C_c^\infty(\underline{R}^3)}.$$

Ditto for the other two. Set

$$L_x = - dU(E_x), \ L_y = - dU(E_y), \ L_z = - dU(E_z).$$

Then L_x, L_y, L_z are called the <u>angular momentum operators</u>. On $C_c^\infty(\underline{R}^3)$, we have

$$\sqrt{-1}\, L_x = [L_y, L_x], \ \sqrt{-1}\, L_y = [L_z, L_x], \ \sqrt{-1}\, L_z = [L_x, L_y].$$

E.g.:

$$\begin{aligned} [L_x, L_y] &= [\sqrt{-1}\, X, \sqrt{-1}\, Y] \\ &= - [X,Y] = - Z \\ &= - \frac{\sqrt{-1}}{\sqrt{-1}} Z = \sqrt{-1}\, L_z. \end{aligned}$$

3.16 <u>THEOREM</u> (Trotter Product Formula) If A and B are selfadjoint and if A + B is essentially selfadjoint, then

$$\lim_{n \to \infty} (e^{\sqrt{-1}\, tA/n} e^{\sqrt{-1}\, tB/n})^n = e^{\sqrt{-1}\, t\, \overline{(A + B)}}$$

in the strong operator topology.

3.17 EXAMPLE Let $V \in L^2(\underline{R}^3) + L^\infty(\underline{R}^3)$ be real valued — then $-\Delta + V$ is selfadjoint on $\mathrm{Dom}(-\Delta)$ ($= \mathrm{Dom}(\Delta)$). To see this, we shall use 1.27, taking $A = -\Delta$ (cf. 1.15) and $B = V$ (meaning multiplication by V, a selfadjoint operator). Thus write $V = V_2 + V_\infty$ ($V_2 \in L^2(\underline{R}^3)$, $V_\infty \in L^\infty(\underline{R}^3)$) — then

$$||Vf|| \leq ||V_2||\,||f||_\infty + ||V_\infty||_\infty ||f||,$$

which shows that $\mathrm{Dom}(-\Delta) \subset \mathrm{Dom}(V)$ (every element of $\mathrm{Dom}(-\Delta)$ is necessarily a bounded continuous function vanishing at infinity). But $\forall\, a > 0$, $\exists\, b > 0$: $\forall\, f \in \mathrm{Dom}(-\Delta)$,

$$||f||_\infty \leq a||-\Delta f|| + b||f||.$$

Therefore

$$||Vf|| \leq a||V_2||\,||-\Delta f|| + (b + ||V_\infty||_\infty)||f||,$$

so $-\Delta + V$ is indeed selfadjoint. Now put $H_0 = -\Delta$ — then according to 3.8, $\forall\, f \in S(\underline{R}^3)$,

$$(e^{-\sqrt{-1}\,tH_0}f)(x) = \frac{1}{(4\pi\sqrt{-1}\,t)^{3/2}} \int_{\underline{R}^3} e^{\sqrt{-1}\,|x - y|^2/4t} f(y)\,dy.$$

On the other hand, $H_0 + V$ is selfadjoint, hence by the Trotter product formula,

$$e^{-\sqrt{-1}\,t(H_0 + V)} f = \lim_{n \to \infty} (e^{-\sqrt{-1}\,tH_0/n} e^{-\sqrt{-1}\,tV/n})^n f.$$

Inserting the explicit expressions for $e^{-\sqrt{-1}\,tH_0/n}$ and $e^{-\sqrt{-1}\,tV/n}$ then gives

$$(e^{-\sqrt{-1}\, t(H_0 + V)} f)(x_0)$$

$$= \lim_{n \to \infty} \left(\frac{4\pi \sqrt{-1}\, t}{n} \right)^{-3n/2} \int_{\underline{R}^3} \cdots \int_{\underline{R}^3} \exp(\sqrt{-1}\, S_n(x_0,\ldots,x_n,t)) f(x_n) dx_n \ldots dx_1,$$

where

$$S_n(x_0,x_1,\ldots,x_n,t) = \sum_{i=1}^{n} \frac{t}{n} \left[\frac{1}{4} \left(\frac{|x_i - x_{i-1}|^2}{t/n} \right) - V(x_i) \right].$$

A conjugate linear bijection $U:\mathcal{H} \to \mathcal{H}$ is said to be <u>antiunitary</u> if $< Ux,Uy > = < y,x >$ for all x,y in $\mathcal{H}$. A <u>conjugation</u> is an antiunitary operator $C:\mathcal{H} \to \mathcal{H}$ such that $C^2 = I$.

- Suppose that U is antiunitary -- then

$$< Ux,Uy > = < y,x >$$

$\Rightarrow$

$$< y,U^*Ux > = < y,x >$$

$$\Rightarrow U^*U = I \Rightarrow U^* = U^{-1}.$$

- Suppose that C is a conjugation -- then

$$\begin{aligned} < x,C^*y > &= < y,Cx > \\ &= < C^2x,Cy > \\ &= < x,Cy > \end{aligned}$$

$$\Rightarrow C^* = C.$$

§4. COMMUTATIVITY

Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space. Let T_1, T_2 be bounded linear operators on $\mathcal{H}$ -- then T_1, T_2 commute iff $[T_1, T_2] = 0$.

4.1 LEMMA Suppose that A_1, A_2 are bounded and selfadjoint. Let E_1, E_2 be their spectral measures -- then A_1, A_2 commute iff for all Borel sets S_1, S_2,

$$[E_1(S_1), E_2(S_2)] = 0.$$

This motivates the following definition: Two selfadjoint operators A_1, A_2 are said to commute if their spectral measures commute, i.e., if for all Borel sets S_1, S_2,

$$[E_1(S_1), E_2(S_2)] = 0.$$

4.2 EXAMPLE Suppose that A is selfadjoint and let E be its spectral measure. Fix Borel functions $f, g: \underline{R} \to \underline{R}$ -- then $f(A), g(A)$ are selfadjoint and, moreover, they commute. In fact, the spectral measure attached to $f(A)$ is the assignment $S \to E(f^{-1}(S))$ and the spectral measure attached to $g(A)$ is the assignment $S \to E(g^{-1}(S))$ (cf. 2.29). So, for all Borel sets S_1, S_2 (cf. 2.4),

$$E(f^{-1}(S_1))E(g^{-1}(S_2))$$

$$= E(f^{-1}(S_1) \cap g^{-1}(S_2))$$

$$= E(g^{-1}(S_2) \cap f^{-1}(S_1))$$

$$= E(g^{-1}(S_2))E(f^{-1}(S_1))$$

$\Rightarrow$

$$[E(f^{-1}(S_1)),E(g^{-1}(S_2))] = 0.$$

4.3 <u>LEMMA</u> Let A be a selfadjoint operator, E its spectral measure. Let T be a bounded linear operator -- then $[E(S),T] = 0$ for all Borel sets S iff $[E_\lambda,T] = 0$ for all real numbers λ.

4.4 <u>LEMMA</u> Suppose that A is selfadjoint and let E be its spectral measure -- then a bounded linear operator T commutes with the $U(t) = e^{\sqrt{-1}\,tA}$ iff for all Borel sets S, $[E(S),T] = 0$.

<u>PROOF</u> First, if $[E(S),T] = 0$ for all S, then $\forall\ x,y \in H$,

$$< x,U(t)Ty > = \int_{\underline{R}} e^{\sqrt{-1}\,t\lambda}\, d< x,E_\lambda Ty >$$

$$= \int_{\underline{R}} e^{\sqrt{-1}\,t\lambda}\, d< x,TE_\lambda y >$$

$$= \int_{\underline{R}} e^{\sqrt{-1}\,t\lambda}\, d< T^*x,E_\lambda y >$$

$$= < T^*x,U(t)y >$$

$$= < x,TU(t)y >$$

=>

$$U(t)T = TU(t) \ \forall \ t.$$

Turning to the converse, fix λ and choose a sequence $\{p_n\}$ of trigonmetric polynomials such that p_n converges pointwise to $\chi_{]-\infty,\lambda]}$ subject to $|p_n| \leq C$ $\forall$ n — then $p_n(A)x \rightarrow E_\lambda x$ for all $x \in \mathcal{H}$, hence

$$U(t)T = TU(t) \ \forall \ t$$

=>

$$p_n(A)T = Tp_n(A) \ \forall \ n$$

=>

$$\begin{aligned} TE_\lambda x &= T \lim p_n(A)x \\ &= \lim Tp_n(A)x \\ &= \lim p_n(A)Tx \\ &= E_\lambda Tx \end{aligned}$$

=>

$$TE_\lambda = E_\lambda T.$$

But λ is arbitrary, so T commutes with all the E(S) (cf. 4.3).

4.5 <u>CRITERION</u> Suppose that A_1, A_2 are selfadjoint — then A_1, A_2 commute iff $\forall \ t_1, t_2$,

$$e^{\sqrt{-1} \ t_1 A_1} e^{\sqrt{-1} \ t_2 A_2} = e^{\sqrt{-1} \ t_2 A_2} e^{\sqrt{-1} \ t_1 A_1}.$$

[In view of 4.4, this is clear.]

4.6 LEMMA If A_1,A_2 are selfadjoint and if A_1,A_2 commute, then $\exists$ a selfadjoint operator A and Borel functions $f_1,f_2:\underline{R} \to \underline{R}$ such that $A_1 = f_1(A)$, $A_2 = f_2(A)$.

If A_1,A_2 are selfadjoint, then $A_1 + A_2$ need not be selfadjoint. However, let us assume that A_1,A_2 commute and, in addition, are nonnegative -- then $A_1 + A_2$ is selfadjoint. To see this, write

$$A_1 = \int_{\underline{R}} f_1 \, dE_\lambda, \ A_2 = \int_{\underline{R}} f_2 \, dE_\lambda,$$

where E is the spectral measure of A and $f_1 \geq 0$, $f_2 \geq 0$. On general grounds, $A_1 + A_2 \subset (f_1 + f_2)(A)$ (indeed, $(f_1 + f_2)^2 \leq 2(f_1^2 + f_2^2)$). But here $f_1^2 + f_2^2 \leq (f_1 + f_2)^2$, hence $(f_1 + f_2)(A) \subset A_1 + A_2$. Therefore $A_1 + A_2 = (f_1 + f_2)(A)$ and, of course, $(f_1 + f_2)(A)$ is selfadjoint.

[Note: The commutativity of A_1,A_2 does not imply that $A_1 + A_2$ is selfadjoint (e.g., take $A_2 = - A_1$). Still, the commutativity of A_1,A_2 does imply that $A_1 + A_2$ is essentially selfadjoint (cf. 4.13).]

4.7 LEMMA If A_1,A_2 are selfadjoint and if A_1,A_2 commute, then

$$(A_1A_2 - A_2A_1)x = 0 \quad (x \in \text{Dom}(A_1A_2) \cap \text{Dom}(A_2A_1)).$$

PROOF Per 4.6, write $A_1 = f_1(A)$, $A_2 = f_2(A)$. Bearing in mind that

$$\begin{cases} f_1(A)f_2(A) \subset (f_1f_2)(A) \\ f_2(A)f_1(A) \subset (f_2f_1)(A), \end{cases}$$

we have

$$A_1A_2x = f_1(A)f_2(A)x$$

$$= (f_1f_2)(A)x = (f_2f_1)(A)x$$

$$= f_2(A)f_1(A)x = A_2A_1x.$$

[Note: It will be shown below that $\text{Dom}([A_1,A_2])$ is dense (cf. 4.12).]

Suppose given two selfadjoint operators A_1,A_2 and a dense linear subspace D of $\mathcal{H}$ such that

1. $D \subset \text{Dom}(A_1) \cap \text{Dom}(A_2)$;
2. $A_1D \subset D$, $A_2D \subset D$;
3. $A_1A_2x = A_2A_1x \ \forall \ x \in D$;
4. $\overline{A_1|D} = A_1$, $\overline{A_2|D} = A_2$.

Then it is FALSE in general that A_1,A_2 commute.

[Note: Conditions 1 and 2 imply that $D \subset \text{Dom}([A_1,A_2])$.]

4.8 EXAMPLE (Fuglede) Take $\mathcal{H} = L^2(\underline{R})$ and let D be the linear subspace of $\mathcal{H}$ generated by the functions

$$x^n \exp(-rx^2 + cx) \quad (n \in \underline{N},\ r > 0,\ c \in \underline{C}).$$

Put

$$A_1 = e^{\sqrt{2\pi}\, Q},\ A_2 = e^{-\sqrt{2\pi}\, P}.$$

Then A_1, A_2 are selfadjoint and

$$U_F A_1 U_F^{-1} = A_2.$$

Points 1 and 2 are straightforward to establish. As regards 3, note that $\forall\ f \in D$,

$$(A_1A_2 - A_2A_1)f\Big|_\lambda$$

$$= e^{\sqrt{2\pi}\ \lambda}\ f(\lambda + \sqrt{-1}\ \sqrt{2\pi}) - e^{\sqrt{2\pi}\ (\lambda + \sqrt{-1}\ \sqrt{2\pi})}\ f(\lambda + \sqrt{-1}\ \sqrt{2\pi})$$

$$= 0.$$

Point 4 asserts that D is a domain of essential selfadjointness for both A_1 and A_2. Since $A_2 = U_F A_1 U_F^{-1}$ and $U_F D = D$, it suffices to consider A_1, the claim being that $(A_1|D)^* \subset A_1$. So suppose that $(A_1|D)^*\psi = \phi$. Since $f,g \in D \Rightarrow fg \in D$, we have

$$< \psi, A_1(fg) > \ = \ < \psi, (A_1|D)(fg) >$$

$$= \ < (A_1|D)^*\psi, fg >$$

$$= \ < \phi, fg > \ = \ < \phi\bar{f}, g >.$$

But

$$< \psi, A_1(fg) > \ = \ < \psi, (A_1 f) g >$$

$$= \int_{\underline{R}} \overline{\psi(\lambda)}\ e^{\sqrt{2\pi}\ \lambda}\ f(\lambda) g(\lambda) d\lambda$$

$$= \int_{\underline{R}} \overline{\psi(\lambda)}\ \overline{\overline{e^{\sqrt{2\pi}\ \lambda}\ \overline{f(\lambda)}}}\ g(\lambda) d\lambda$$

$$= < \psi(A_1\bar{f}), g >.$$

Therefore

$$\phi\bar{f} = \psi(A_1\bar{f}) = (A_1\psi)\bar{f}$$

or still, $\phi = A_1\psi$, which implies that $(A_1|D)^* \subset A_1$. It remains to prove that A_1, A_2 do not commute. To get a contradiction, suppose they did. Write $A_1 = f_1(A)$, $A_2 = f_2(A)$ (cf. 4.6), where $f_1 > 0$, $f_2 > 0$ -- then the spectral measures of $f_1(A)$, $f_2(A)$ commute (cf. 4.2), thus the same holds for the spectral measures of $\log f_1(A)$, $\log f_2(A)$. In other words, $\sqrt{2\pi}\, Q$, $-\sqrt{2\pi}\, P$ must commute, which is nonsense: On $\mathcal{S}(\underline{R})$,

$$[Q,P] = \sqrt{-1} \Rightarrow [\sqrt{2\pi}\, Q, -\sqrt{2\pi}\, P] = -2\pi\sqrt{-1}.$$

Let A be a selfadjoint operator -- then a bounded linear operator T is said to <u>commute</u> with A if $T\mathrm{Dom}(A) \subset \mathrm{Dom}(A)$ and $TAx = ATx \ \forall\ x \in \mathrm{Dom}(A)$.

4.9 <u>LEMMA</u> Suppose that A is selfadjoint -- then a bounded linear operator T commutes with A iff $[E(S),T] = 0$ for all Borel sets S.

<u>PROOF</u> Put $U(t) = e^{\sqrt{-1}\, tA}$ -- then the condition $[E(S),T] = 0 \ \forall\ S$ implies that $TU(t) = U(t)T \ \forall\ t$ (cf. 4.4), thus $\forall\ x \in \mathcal{H}$,

$$\frac{U(t) - I}{\sqrt{-1}\, t} Tx$$

$$= \frac{U(t)Tx - Tx}{\sqrt{-1}\, t}$$

$$= \frac{TU(t)x - Tx}{\sqrt{-1}\, t}$$

$$= T\,\frac{U(t) - I}{\sqrt{-1}\, t}\,x,$$

and so $\forall\, x \in \text{Dom}(A)$,

$$\lim_{t \to 0} \frac{U(t) - I}{\sqrt{-1}\, t}\,Tx$$

$$= \lim_{t \to 0} T\,\frac{U(t) - I}{\sqrt{-1}\, t}\,x$$

$$= T \lim_{t \to 0} \frac{U(t) - I}{\sqrt{-1}\, t}\,x$$

$$= TAx.$$

Consequently, $Tx \in \text{Dom}(A)$ and $ATx = TAx$. As for the converse, it's a bit technical, hence will be postponed to the end of the §.

4.10 EXAMPLE If A is selfadjoint and if S is a bounded Borel set, then $E(S)\mathcal{H} \subset \text{Dom}(A)$. But for any Borel set S', $[E(S),E(S')] = 0$, thus $E(S)Ax = AE(S)x$ $\forall\, x \in \text{Dom}(A)$ (cf. 4.9).

4.11 REMARK Suppose that A_1, A_2 are selfadjoint and A_2 is bounded — then there is a potential inconsistency in that one now has two notions of "commute". Thanks to 4.9, though, they coincide. To check this, assume first that

$$[E_1(S_1), E_2(S_2)] = 0$$

for all Borel sets S_1, S_2 -- then $\forall\ S_1$ and $\forall\ x,y \in \mathcal{H}$,

$$\begin{aligned} \langle x, A_2 E_1(S_1) y \rangle &= \int_{\underline{R}} \lambda \, d\langle x, E_\lambda^2 E_1(S_1) y \rangle \\ &= \int_{\underline{R}} \lambda \, d\langle x, E_1(S_1) E_\lambda^2 y \rangle \\ &= \int_{\underline{R}} \lambda \, d\langle E_1(S_1) x, E_\lambda^2 y \rangle \\ &= \langle E_1(S_1) x, A_2 y \rangle \\ &= \langle x, E_1(S_1) A_2 y \rangle \end{aligned}$$

=>

$$[E_1(S_1), A_2] = 0.$$

Therefore $A_2 \text{Dom}(A_1) \subset \text{Dom}(A_1)$ and $A_2 A_1 x = A_1 A_2 x\ \forall\ x \in \text{Dom}(A_1)$. Conversely, this condition implies that $[E_1(S_1), A_2] = 0$ for all Borel sets S_1. To prove it, fix λ and choose a sequence $\{p_n\}$ of polynomials such that $E_\lambda^2 = \lim p_n(A_2)$ in the strong operator topology (possible, A_2 being bounded) -- then

$$E_1(S_1) A_2 = A_2 E_1(S_1)$$

=>

$$E_1(S_1) p_n(A_2) = p_n(A_2) E_1(S_1)$$

=>

$$E_1(S_1) E_\lambda^2 = E_\lambda^2 E_1(S_1).$$

But λ is arbitrary, so $E_1(S_1)$ commutes with all the $E_2(S_2)$ (cf. 4.3).

4.12 LEMMA If A_1, A_2 are selfadjoint and if A_1, A_2 commute, then

$$\mathrm{Dom}([A_1,A_2]) = \mathrm{Dom}(A_1A_2) \cap \mathrm{Dom}(A_2A_1)$$

is dense.

PROOF Let D be the subset of $\mathcal{H}$ consisting of those x for which $\exists$ bounded Borel sets S_1, S_2 such that $x = E_1(S_1)E_2(S_2)x$.

• D is dense in $\mathcal{H}$. In fact, given any $x \in \mathcal{H}$,

$$\begin{cases} E_1([-n,n])x \to x \\ E_2([-n,n])x \to x \end{cases} \quad (n \to \infty),$$

hence by the sequential continuity of multiplication in the strong operator topology,

$$E_1([-n,n])E_2([-n,n])x \to x.$$

But

$$E_1([-n,n])E_2([-n,n])E_1([-n,n])E_2([-n,n])x$$

$$= E_1([-n,n])E_1([-n,n])E_2([-n,n])E_2([-n,n])x$$

$$= E_1([-n,n])E_2([-n,n])x$$

=>

$$E_1([-n,n])E_2([-n,n])x \in D.$$

• $D \subset \text{Dom}(A_1A_2) \cap \text{Dom}(A_2A_1)$. For suppose that $x \in D$, say $x = E_1(S_1)E_2(S_2)x$ — then

$$\int_{\underline{R}} \lambda^2 \, d\langle x, E_\lambda^2 x\rangle$$

$$= \int_{\underline{R}} \lambda^2 \, d\langle x, E_\lambda^2 E_1(S_1)E_2(S_2)x\rangle$$

$$= \int_{\underline{R}} \lambda^2 \, d\langle x, E_1(S_1)E_\lambda^2 E_2(S_2)x\rangle$$

$$= \int_{\underline{R}} \lambda^2 \, d\langle E_1(S_1)x, E_2(S_2)E_\lambda^2 x\rangle$$

$$= \int_{\underline{R}} \lambda^2 \, d\langle E_1(S_1)x, E_2(S_2 \cap I_\lambda)x\rangle$$

$$= \int_{\underline{R}} \lambda^2 \, \chi_{S_2}(\lambda) d\langle E_1(S_1)x, E_\lambda x\rangle$$

$$< \infty$$

$\Rightarrow$

$$x \in \text{Dom}(A_2).$$

Consider now $A_2x = A_2E_1(S_1)E_2(S_2)x$. Obviously, $E_2(S_2)x \in \text{Dom}(A_2)$. On the other hand, A_2 commutes with $E_1(S_1)$ (cf. 4.9), so

$$A_2E_1(S_1)E_2(S_2)x = E_1(S_1)A_2E_2(S_2)x.$$

But

$$E_1(S_1)A_2E_2(S_2)x \in \text{Dom}(A_1).$$

Therefore

$$x \in D \Rightarrow x \in \mathrm{Dom}(A_1A_2).$$

And, analogously,

$$x \in D \Rightarrow x \in \mathrm{Dom}(A_2A_1).$$

[Note: Some assumption on A_1, A_2 is necessary (recall that $\exists$ a pair of selfadjoint operators with the property that the domain of their commutator is $\{0\}$ (cf. 1.25)).]

4.13 <u>REMARK</u> If A_1, A_2 are selfadjoint and if A_1, A_2 commute, then $A_1 + A_2$ is essentially selfadjoint.

[In the notation of 4.12, the elements of D are analytic vectors for $A_1 + A_2$, so 1.18 is applicable.]

* * * * * * * * * *

Given $z \in \underline{C} - \underline{R}$, put

$$R_A(z) = (A - z)^{-1}.$$

Then $R_A(z)$ is a bounded linear operator on $\mathcal{H}$ with range Dom(A).

4.14 <u>LEMMA</u> Suppose that T commutes with A -- then $\forall\ z \in \underline{C} - \underline{R}$, $[R_A(z), T] = 0$.

<u>PROOF</u> If $x \in \mathcal{H}$, then $R_A(z)x \in \mathrm{Dom}(A)$ and

$$(A - z)TR_A(z)x = T(A - z)R_A(z)x = Tx$$

=>

$$R_A(z)(A - z)TR_A(z)x = R_A(z)Tx$$

=>

$$TR_A(z)x = R_A(z)Tx.$$

From the definitions,

$$R_A(z) = \int_{\underline{R}} \frac{1}{\lambda - z} dE_\lambda.$$

So, $\forall\ x,y \in \mathcal{H}$,

$$< x,R_A(z)Ty > = \int_{\underline{R}} \frac{1}{\lambda - z} d< x,E_\lambda Ty >.$$

But if T commutes with A, then $\forall\ x,y \in \mathcal{H}$,

$$< x,R_A(z)Ty > = < x,TR_A(z)y >$$

and

$$< x,TR_A(z)y > = < T^*x,R_A(z)y >$$

$$= \int_{\underline{R}} \frac{1}{\lambda - z} d< T^*x,E_\lambda y >$$

$$= \int_{\underline{R}} \frac{1}{\lambda - z} d< x,TE_\lambda y >.$$

Accordingly, $\forall\ z \in \underline{C} - \underline{R}$,

$$\int_{\underline{R}} \frac{1}{\lambda - z} (d\langle x, E_\lambda T y\rangle - d\langle x, T E_\lambda y\rangle) = 0.$$

And from this, we want to conclude that $[E_\lambda, T] = 0 \ \forall \ \lambda$, hence that $[E(S), T] = 0$ $\forall \ S$ (cf. 4.3).

4.15 LEMMA Suppose that $\alpha:\underline{R} \to \underline{R}$ is right continuous, of bounded variation, and $\lim_{t \to -\infty} \alpha(\lambda) = 0$. Put

$$f(z) = \int_{\underline{R}} \frac{1}{\lambda - z} d\alpha(\lambda) \quad (\text{Im } z > 0).$$

Then $\forall \ \lambda$,

$$\alpha(\lambda) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{-\infty}^{\lambda + \delta} \text{Im } f(t + \sqrt{-1}\ \varepsilon) dt.$$

PROOF Write

$$\text{Im } f(t + \sqrt{-1}\ \varepsilon) = \int_{\underline{R}} \text{Im}(\lambda - t - \sqrt{-1}\ \varepsilon)^{-1} d\alpha(\lambda)$$

$$= \int_{\underline{R}} \frac{\varepsilon}{(\lambda - t)^2 + \varepsilon^2} d\alpha(\lambda).$$

Then by Fubini,

$$\int_{-\infty}^{r} \text{Im } f(t + \sqrt{-1}\ \varepsilon) dt$$

$$= \int_{\underline{R}} \int_{-\infty}^{r} \frac{\varepsilon}{(\lambda - t)^2 + \varepsilon^2} dt\, d\alpha(\lambda)$$

$$= \int_{\underline{R}} [\text{Arc Tan } \frac{r - \lambda}{\varepsilon} + \frac{\pi}{2}] d\alpha(\lambda).$$

Since

$$|\text{Arc Tan } \frac{r - \lambda}{\varepsilon} + \frac{\pi}{2}| \le \pi$$

and since

$$\text{Arc Tan}\,\frac{r-\lambda}{\varepsilon} + \frac{\pi}{2} \to \begin{cases} \pi & (r > \lambda) \\ \frac{\pi}{2} & (r = \lambda) \\ 0 & (r < \lambda) \end{cases}$$

as $\varepsilon \downarrow 0$, an application of dominated convergence leads to

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{r} \text{Im } f(t + \sqrt{-1}\,\varepsilon)dt$$

$$= \int_{]-\infty,r[} \pi \, d\alpha(\lambda) + \int_{\{r\}} \frac{\pi}{2} \, d\alpha(\lambda) + \int_{]r,\infty[} 0 \, d\alpha(\lambda)$$

$$= \pi\alpha(r^-) + \frac{\pi}{2}(\alpha(r) - \alpha(r^-))$$

$$= \frac{\pi}{2}(\alpha(r) + \alpha(r^-)).$$

To finish the proof, replace r by $\lambda + \delta$ $(\delta > 0)$ and then let $\delta \downarrow 0$.

The obvious corollary to this is that $f \equiv 0 \Rightarrow \alpha \equiv 0$.

4.16 <u>LEMMA</u> Suppose that $\alpha: \underline{R} \to \underline{C}$ is right continuous, of bounded variation, and $\lim_{\lambda \to -\infty} \alpha(\lambda) = 0$. Assume:

$$\forall\, z \in \underline{C} - \underline{R}, \int_{\underline{R}} \frac{1}{\lambda - z} \, d\alpha(\lambda) = 0.$$

Then $\alpha(\lambda) = 0$ for all $\lambda \in \underline{R}$.

PROOF If Im z > 0, then

$$\int_{\underline{R}} \frac{1}{\lambda - z}\, d\alpha(\lambda) = 0$$

and

$$\int_{\underline{R}} \frac{1}{\lambda - z}\, d\bar{\alpha}(\lambda) = \overline{\left[\int_{\underline{R}} \frac{1}{\lambda - \bar{z}}\, d\alpha(\lambda) \right]} = 0.$$

Therefore

$$\begin{cases} \int_{\underline{R}} \frac{1}{\lambda - z}\, d(\mathrm{Re}\ \alpha(\lambda)) = 0 \\ \\ \int_{\underline{R}} \frac{1}{\lambda - z}\, d(\mathrm{Im}\ \alpha(\lambda)) = 0 \end{cases} \qquad (\mathrm{Im}(z) > 0)$$

=>

$$\begin{cases} \mathrm{Re}\ \alpha \equiv 0 \\ \\ \mathrm{Im}\ \alpha \equiv 0 \end{cases} \quad \Rightarrow \alpha \equiv 0.$$

Returning now to the equation

$$\int_{\underline{R}} \frac{1}{\lambda - z}\, (d\langle x, E_\lambda T y\rangle - d\langle x, T E_\lambda y\rangle) = 0,$$

the difference

$$\alpha(\lambda) = \langle x, E_\lambda T y\rangle - \langle x, T E_\lambda y\rangle$$

has the properties required in 4.16, thus α is identically zero. So, $\forall\ \lambda$, $E_\lambda T = T E_\lambda$ or still, $\forall\ \lambda$, $[E_\lambda, T] = 0$.

§5. TENSOR PRODUCTS

Given complex Hilbert spaces $\mathcal{H}_1,\ldots,\mathcal{H}_n$ with respective inner products $<\ ,\ >_1,\ldots,<\ ,\ >_n$, denote by $\mathcal{H}_1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}_n$ their tensor product in the sense of Hilbert space theory, i.e., the completion of the underlying algebraic tensor product $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ per

$$< x,y > = \prod_{k=1}^{n} < x_k,y_k >_k ,$$

where

$$\begin{cases} x = x_1 \otimes \cdots \otimes x_n \\ y = y_1 \otimes \cdots \otimes y_n. \end{cases}$$

5.1 <u>LEMMA</u> If S_k is total in $\mathcal{H}_k$, then the set

$$\{x_1 \otimes \cdots \otimes x_n : x_k \in \mathcal{H}_k\}$$

is total in $\mathcal{H}_1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}_n$.

5.2 <u>LEMMA</u> If $\{e_{k,i} : i \in I_k\}$ is an orthonormal basis for $\mathcal{H}_k$, then

$$\{e_{1,i_1} \otimes \cdots \otimes e_{n,i_n} : i_1 \in I_1,\ldots,i_n \in I_n\}$$

is an orthonormal basis for $\mathcal{H}_1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}_n$.

5.3 EXAMPLE Let $\Omega_1 \subset \underline{R}^{n_1}, \Omega_2 \subset \underline{R}^{n_2}$ be Borel. Suppose that $\begin{cases} \mu_1 \\ \mu_2 \end{cases}$ is a σ-finite measure on $\begin{cases} \text{Bor}(\Omega_1) \\ \text{Bor}(\Omega_2) \end{cases}$ — then

$$L^2(\Omega_1,\mu_1) \hat{\otimes} L^2(\Omega_2,\mu_2)$$

is isometrically isomorphic to

$$L^2(\Omega_1 \times \Omega_2,\mu_1 \times \mu_2).$$

In particular: $L^2(\underline{R}^{n_1}) \hat{\otimes} L^2(\underline{R}^{n_2})$ can be identified with $L^2(\underline{R}^{n_1 + n_2})$.

5.4 EXAMPLE Take H separable, let $\Omega \subset \underline{R}^n$ be Borel, and suppose that μ is a σ-finite measure on Bor(Ω) -- then

$$L^2(\Omega,\mu) \hat{\otimes} H$$

is isometrically isomorphic to

$$L^2(\Omega,\mu;H).$$

Assume henceforth that $H_1,\ldots,H_n$ are infinite dimensional and let $A_1,\ldots,A_n$ be densely defined linear operators on $H_1,\ldots,H_n$. Denote by $\text{Dom}(A_1) \otimes \cdots \otimes \text{Dom}(A_n)$ the set of finite linear combinations of vectors of the form $x_1 \otimes \cdots \otimes x_n$, where $x_k \in \text{Dom}(A_k)$ -- then $\text{Dom}(A_1)\otimes \cdots \otimes \text{Dom}(A_n)$ is dense in $H_1 \hat{\otimes} \cdots \hat{\otimes} H_n$ (cf. 2.1). Define $A_1 \otimes \cdots \otimes A_n$ on $\text{Dom}(A_1) \otimes \cdots \otimes \text{Dom}(A_n)$ by

$$(A_1 \otimes \cdots \otimes A_n)(x_1 \otimes \cdots \otimes x_n)$$

$$= A_1x_1 \otimes \cdots \otimes A_nx_n$$

and extend by linearity.

[Note: This makes sense, i.e., the definition of $A_1 \otimes \cdots \otimes A_n$ is independent of the representation of a vector in $\mathrm{Dom}(A_1) \otimes \cdots \otimes \mathrm{Dom}(A_n)$.]

Note that

$$A_1^* \otimes \cdots \otimes A_n^* \subset (A_1 \otimes \cdots \otimes A_n)^*,$$

the inclusion being strict in general.

5.5 <u>LEMMA</u> If $A_1,\ldots,A_n$ admit closure, then so does $A_1 \otimes \cdots \otimes A_n$ and we have

$$\bar{A}_1 \otimes \cdots \otimes \bar{A}_n \subset \overline{A_1 \otimes \cdots \otimes A_n}.$$

5.6 <u>REMARK</u> If $A_1,\ldots,A_n$ are bounded (and everywhere defined), then $A_1 \otimes \cdots \otimes A_n$ is bounded (and densely defined). Therefore $A_1 \otimes \cdots \otimes A_n$ has a unique extension to a bounded linear operator on $H_1 \hat{\otimes} \cdots \hat{\otimes} H_n$, viz. $\overline{A_1 \otimes \cdots \otimes A_n}$. Here

$$||\overline{A_1 \otimes \cdots \otimes A_n}|| = ||A_1|| \cdots ||A_n||.$$

[Note: If each A_k is selfadjoint, unitary, or a projection, then $\overline{A_1 \otimes \cdots \otimes A_n}$ is selfadjoint, unitary, or a projection.]

5.7 EXAMPLE Represent $L^2(\underline{R}^n)$ as $L^2(\underline{R}) \hat{\otimes} \cdots \hat{\otimes} L^2(\underline{R})$ — then

$$\overline{U_F \otimes \cdots \otimes U_F}$$

is the unitary operator on $L^2(\underline{R}^n)$ provided by the Plancherel theorem.

5.8 LEMMA Let A_1, A_2 be selfadjoint — then $A_1 \otimes A_2$ is essentially selfadjoint.

PROOF From the definitions, it is clear that $A_1 \otimes A_2$ is symmetric. This said, to establish that $A_1 \otimes A_2$ is essentially selfadjoint, it will be enough to show that $\text{Dom}(A_1 \otimes A_2)$ contains a dense set of analytic vectors (cf. 1.18). Let $S_1 \subset \text{Dom}(A_1^2)$, $S_2 \subset \text{Dom}(A_2^2)$ be the set of analytic vectors for A_1^2, A_2^2 — then S_1 is dense in H_1 and S_2 is dense in H_2 (cf. 2.28) and we claim that the

$$x_1 \otimes x_2 \quad (x_1 \in S_1, x_2 \in S_2)$$

are analytic vectors for $A_1 \otimes A_2$, which suffices (cf. 5.1). Thus fix $t_0 > 0$:

$$\begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{||A_1^{2k} x_1||}{k!} |t|^k < \infty \\ \\ \displaystyle\sum_{k=0}^{\infty} \frac{||A_2^{2k} x_2||}{k!} |t|^k < \infty \end{cases} \qquad (|t| < t_0).$$

Then $\forall\ t: |t| < t_0$,

$$\sum_{k=0}^{\infty} \frac{||(A_1 \otimes A_2)^k (x_1 \otimes x_2)||}{k!} |t|^k$$

$$= \sum_{k=0}^{\infty} \frac{||A_1^k x_1|| \; ||A_2^k x_2||}{k!} |t|^k$$

$$\leq \left[\sum_{k=0}^{\infty} \frac{||A_1^k x_1||^2}{k!} |t|^k \quad \sum_{k=0}^{\infty} \frac{||A_2^k x_2||^2}{k!} |t|^k \right]^{1/2}$$

$$= \left[\sum_{k=0}^{\infty} \frac{< x_1, A_1^{2k} x_1 >}{k!} |t|^k \quad \sum_{k=0}^{\infty} \frac{< x_2, A_2^{2k} x_2 >}{k!} |t|^k \right]^{1/2}$$

$$\leq ||x_1||^{1/2} ||x_2||^{1/2} \left[\sum_{k=0}^{\infty} \frac{||A_1^{2k} x_1||}{k!} |t|^k \quad \sum_{k=0}^{\infty} \frac{||A_2^{2k} x_2||}{k!} |t|^k \right]^{1/2}$$

$$< \infty.$$

Therefore $x_1 \otimes x_2$ is an analytic vector for $A_1 \otimes A_2$.

5.9 <u>LEMMA</u> Let A_1, A_2 be essentially selfadjoint -- then $A_1 \otimes A_2$ is essentially selfadjoint.

<u>PROOF</u> By hypothesis, $\bar{A}_1, \bar{A}_2$ are selfadjoint, thus $\bar{A}_1 \otimes \bar{A}_2$ is essentially selfadjoint (cf. 5.8). On the other hand,

$$A_1 \otimes A_2 \subset \bar{A}_1 \otimes \bar{A}_2 \subset \overline{A_1 \otimes A_2} \quad \text{(cf. 5.5)}.$$

But

$$A_1 \otimes A_2 \text{ symmetric} \Rightarrow \overline{A_1 \otimes A_2} \text{ symmetric}.$$

Therefore (cf. 1.14)

$$\overline{A_1 \otimes A_2} = \overline{\overline{A_1 \otimes A_2}} = \overline{\bar{A}_1 \otimes \bar{A}_2},$$

which implies that $\overline{A_1 \otimes A_2}$ is selfadjoint.

5.10 <u>EXAMPLE</u> Take $\mathcal{H}_1 = L^2(\underline{R})$, $\mathcal{H}_2 = L^2(\underline{R})$ and let A_1 = multiplication by x_1, A_2 = multiplication by x_2 — then A_1, A_2 are selfadjoint (cf. 1.9) and $\overline{A_1 \otimes A_2}$ is multiplication by $x_1 x_2$ in $L^2(\underline{R}^2)$.

Let $A_1, \ldots, A_n$ be densely defined linear operators on $\mathcal{H}_1, \ldots, \mathcal{H}_n$. Let I_k be the identity map of $\mathcal{H}_k$ $(k = 1, \ldots, n)$ -- then the domain of

$$A_1 \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_n$$

is $\mathrm{Dom}(A_1) \otimes \cdots \otimes \mathrm{Dom}(A_n)$.

Note that

$$A_1^* \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_n^*$$

$$\subset (A_1 \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_n)^*,$$

the inclusion being strict in general.

5.11 <u>LEMMA</u> If $A_1, \ldots, A_n$ admit closure, then so does

$$A_1 \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_n$$

and we have

$$\bar{A}_1 \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes \bar{A}_n$$

$$\subset \overline{A_1 \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_n}.$$

5.12 <u>REMARK</u> If $A_1,\ldots,A_n$ are bounded (and everywhere defined), then

$$A_1 \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_n$$

is bounded (and densely defined). Therefore

$$A_1 \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_n$$

has a unique extension to a bounded linear operator on $\mathcal{H}_1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}_n$, viz.

$$\overline{A_1 \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_n}.$$

Here

$$||\overline{A_1 \otimes I_2 \otimes \cdots \otimes I_n + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_n}||$$

$$\leq ||A_1|| \cdots ||A_n||.$$

5.13 <u>LEMMA</u> Let A_1,A_2 be selfadjoint — then $A_1 \otimes I_2 + I_1 \otimes A_2$ is essentially selfadjoint.

<u>PROOF</u> Since $A_1 \otimes I_2 + I_1 \otimes A_2$ is symmetric, one may proceed as in 5.8 but this time with $S_1 \subset \mathrm{Dom}(A_1)$, $S_2 \subset \mathrm{Dom}(A_2)$ the set of analytic vectors for A_1,A_2. Choose $x_1 \in S_1$, $x_2 \in S_2$ and fix $t_0 > 0$:

$$\begin{cases} \sum\limits_{k=0}^{\infty} \dfrac{||A_1^k x_1||}{k!} |t|^k < \infty \\ \sum\limits_{k=0}^{\infty} \dfrac{||A_2^k x_2||}{k!} |t|^k < \infty \end{cases} \qquad (|t| < t_0).$$

Then $\forall\ t: |t| < t_0$,

$$\sum_{k=0}^{\infty} ||(A_1 \otimes I_2 + I_1 \otimes A_2)^k\, x_1 \otimes x_2|| \frac{|t|^k}{k!}$$

$$\leq \sum_{k=0}^{\infty} || \sum_{\ell=0}^{k} \binom{k}{\ell} A_1^{\ell} x_1 \otimes A_2^{k-\ell} x_2 || \frac{|t|^k}{k!}$$

$$\leq \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{k!}{\ell!(k-\ell)!} ||A_1^{\ell} x_1||\ ||A_2^{k-\ell} x_2|| \frac{|t|^k}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{||A_1^{\ell} x_1||}{\ell!} |t|^{\ell} \frac{||A_2^{k-\ell} x_2||}{(k-\ell)!} |t|^{k-\ell}$$

$$= \sum_{\ell=0}^{\infty} \left[\frac{||A_1^{\ell} x_1||}{\ell!} |t|^{\ell} \sum_{k=\ell}^{\infty} \frac{||A_2^{k-\ell} x_2||}{(k-\ell)!} |t|^{k-\ell} \right]$$

$$= \sum_{\ell=0}^{\infty} \frac{||A_1^{\ell} x_1||}{\ell!} |t|^{\ell} \sum_{k=0}^{\infty} \frac{||A_2^{k} x_2||}{k!} |t|^{k}$$

$$< \infty.$$

Therefore $x_1 \otimes x_2$ is an analytic vector for $A_1 \otimes I_2 + I_1 \otimes A_2$.

5.14 LEMMA Let A_1, A_2 be essentially selfadjoint -- then $A_1 \otimes I_2 + I_2 \otimes A_2$ is essentially selfadjoint.

5.15 EXAMPLE Take $\mathcal{H}_1 = L^2(\underline{R})$, $\mathcal{H}_2 = L^2(\underline{R})$ and let A_2 = multiplication by x_1, A_2 = multiplication by x_2 — then A_1, A_2 are selfadjoint (cf. 1.9) and

$\overline{A_1 \otimes I_2 + I_1 \otimes A_2}$ is multiplication by $x_1 + x_2$ in $L^2(\underline{R}^2)$.

Given selfadjoint operators A_1,A_2 on $\mathcal{H}_1,\mathcal{H}_2$, put

$$\begin{cases} \underline{A}_1 = \overline{A_1 \otimes I_2} \\ \underline{A}_2 = \overline{I_1 \otimes A_2}. \end{cases}$$

Then $\underline{A}_1,\underline{A}_2$ are selfadjoint (cf. 5.8).

Let E_1,E_2 be the spectral measures attached to A_1,A_2 — then the assignments

$$\begin{cases} S \to \overline{E_1(S) \otimes I_2} \\ S \to \overline{I_1 \otimes E_2(S)} \end{cases} \qquad (S \in \text{Bor}(\underline{R}))$$

define spectral measures

$$\underline{E}_1,\underline{E}_2:\text{Bor}(\underline{R}) \to \text{Pro}_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}.$$

5.16 LEMMA The spectral measure attached to $\underline{A}_1$ is $\underline{E}_1$ and the spectral measure attached to $\underline{A}_2$ is $\underline{E}_2$.

Since for all Borel sets S_1,S_2,

$$[\underline{E}_1(S_1),\underline{E}_2(S_2)] = 0,$$

it follows that $\underline{A}_1, \underline{A}_2$ commute.

5.17 <u>REMARK</u> We have

$$\left[\begin{array}{l} A_1 \otimes I_2 \subset \overline{A_1 \otimes I_2} = \underline{A}_1 \\ \\ I_1 \otimes A_2 \subset \overline{I_1 \otimes A_2} = \underline{A}_2 \end{array}\right.$$

$\Rightarrow$

$$A_1 \otimes I_2 + I_1 \otimes A_2 \subset \underline{A}_1 + \underline{A}_2.$$

Because $\underline{A}_1, \underline{A}_2$ commute, their sum $\underline{A}_1 + \underline{A}_2$ is essentially selfadjoint (cf. 4.13). On the other hand, $A_1 \otimes I_2 + I_1 \otimes A_2$ is also essentially selfadjoint. Therefore (cf. 1.14)

$$\overline{A_1 \otimes I_2 + I_1 \otimes A_2} = \overline{\underline{A}_1 + \underline{A}_2}.$$

5.18 <u>LEMMA</u> Let

$$\left[\begin{array}{l} U_1(t) = e^{\sqrt{-1}\, tA_1} \\ \\ U_2(t) = e^{\sqrt{-1}\, tA_2}. \end{array}\right.$$

Then the assignment $t \to \overline{U_1(t) \otimes U_2(t)}$ is a one parameter unitary group and its generator is $\overline{\underline{A}_1 + \underline{A}_2}$.

[Note: The generator of $t \to \overline{U_1(t) \otimes I_2}$ is $\underset{\sim}{A}_1$ and the generator of $t \to \overline{I_1 \otimes U_2(t)}$ is $\underset{\sim}{A}_2$.]

5.19 <u>LEMMA</u> We have

$$\begin{cases} \sigma(A_1) = \sigma(\underset{\sim}{A}_1) \\ \sigma(A_2) = \sigma(\underset{\sim}{A}_2). \end{cases}$$

Let

$$\begin{cases} A_\Pi = \overline{A_1 \otimes A_2} \\ A_\Sigma = \overline{\underset{\sim}{A}_1 + \underset{\sim}{A}_2} \end{cases}$$

and let

$$\begin{cases} M_\Pi = \{\lambda_1\lambda_2 : \lambda_1 \in \sigma(A_1), \lambda_2 \in \sigma(A_2)\} \\ M_\Sigma = \{\lambda_1 + \lambda_2 : \lambda_1 \in \sigma(A_1), \lambda_2 \in \sigma(A_2)\}. \end{cases}$$

5.20 <u>LEMMA</u> We have

$$\begin{cases} \sigma(A_\Pi) = \overline{M_\Pi} \\ \sigma(A_\Sigma) = \overline{M_\Sigma}. \end{cases}$$

[Note: In general, the sets M_Π and M_Σ are not closed (simple examples illustrating this can be constructed using 1.13).]

As a final comment, we emphasize that while the preceding results were only formulated when n = 2, they can of course be extended to the case of arbitrary finite n.

§6. FOCK SPACE

Let $\mathcal{H}$ be a complex Hilbert space. For $n \geq 1$, let $\mathcal{H}^{\hat{\otimes} n}$ denote the n-fold tensor product of $\mathcal{H}$ and for $n = 0$, let $\mathcal{H}^{\hat{\otimes} 0} = \underline{C}$ — then

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\hat{\otimes} n}$$

is called the <u>Fock space</u> over $\mathcal{H}$.

[Note: The direct sum is in the sense of Hilbert space theory.]

If the norm in $\mathcal{H}^{\hat{\otimes} n}$ is indexed by n, then the elements of $\mathcal{F}(\mathcal{H})$ are sequences $X = \{X_n : n \geq 0\}$ with $X_n \in \mathcal{H}^{\hat{\otimes} n}$ such that $\sum_{n=0}^{\infty} ||X_n||_n^2 < \infty$.

[Note: The inner product in $\mathcal{F}(\mathcal{H})$ is given by

$$< X,Y > = \sum_{n=0}^{\infty} < X_n, Y_n >_n,$$

where $\forall$ n, $< , >_n$ is the inner product in $\mathcal{H}^{\hat{\otimes} n}$.]

6.1 <u>EXAMPLE</u> Take $\mathcal{H} = L^2(\underline{R})$ — then an element $\Psi \in \mathcal{F}(\mathcal{H})$ is a sequence of functions

$$\Psi = \{\psi_0, \psi_1(x), \psi_2(x_1,x_2), \ldots\}$$

such that

$$|\Psi_0|^2 + \sum_{n=1}^{\infty} \int_{\underline{R}^n} |\psi_n(x_1,\ldots,x_n)|^2 \, dx_1 \ldots dx_n < \infty.$$

Let $\sigma \in S_n$ (the symmetric group on n letters) — then there is a unitary operator $U_n(\sigma): \hat{\mathcal{H}^{\otimes n}} \to \hat{\mathcal{H}^{\otimes n}}$ with

$$U_n(\sigma)(x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

This said, put

$$P_n = \frac{1}{n!} \sum_{\sigma \in S_n} U_n(\sigma).$$

6.2 <u>LEMMA</u> P_n is an orthogonal projection.

Denote the range of P_n by $BO_n(\mathcal{H})$ (in particular, $BO_1(\mathcal{H}) = \mathcal{H}$ and, conventionally, $BO_0(\mathcal{H}) = \underline{C}$) -- then

$$BO(\mathcal{H}) = \bigoplus_{n=0}^{\infty} BO_n(\mathcal{H})$$

is the <u>bosonic Fock space</u> over $\mathcal{H}$.

[Note: The element $\Omega = \{1,0,0,\ldots\}$ is, by definition, the <u>vacuum</u>.]

6.3 <u>EXAMPLE</u> Take $\mathcal{H} = L^2(\underline{R})$ -- then $BO_n(\mathcal{H})$ is the subspace of $\hat{\mathcal{H}^{\otimes n}}$ $(= L^2(\underline{R}^n))$ consisting of those functions which are invariant under permutations of the coordinates (cf. 6.1).

6.4 <u>LEMMA</u> If $\mathcal{H}$ is separable, then $BO_n(\mathcal{H})$ is separable.

<u>PROOF</u> Let $e_1, e_2, \ldots$ be an orthonormal basis for $\mathcal{H}$. Take $n > 0$ and consider any sequence $\kappa = \{k_j\}$ of nonnegative integers, almost all of whose terms are zero,

with $\sum_j k_j = n$. Let

$$e_n(\kappa) = \left[\frac{n!}{k_1!k_2!\cdots}\right]^{1/2} P_n(e_1^{k_1} \otimes e_2^{k_2} \otimes \cdots).$$

Then the collection $\{e_n(\kappa)\}$ is an orthonormal basis for $BO_n(H)$.

[Note: Here it is understood that if $k_j = 0$, then $e_j^{k_j}$ does not appear in $e_n(\kappa)$.]

In the bosonic theory, it is traditional to denote the elements of H by $f, g, \ldots$ rather than $x, y, \ldots$.

6.5 LEMMA The linear span of the $f^{\otimes n}$ $(f \in H)$ is dense in $BO_n(H)$.

PROOF Take $n > 0$ — then the linear span of the $P_n(f_1 \otimes \cdots \otimes f_n)$ is dense in $BO_n(H)$. But

$$P_n(f_1 \otimes \cdots \otimes f_n)$$

$$= \frac{1}{2^n n!} \sum_{\varepsilon} \varepsilon_1 \cdots \varepsilon_n (\varepsilon_1 f_1 + \cdots + \varepsilon_n f_n)^{\otimes n},$$

the sum being over all $\varepsilon_i = \pm 1$ $(i = 1, \ldots, n)$.

Given $f \in H$, put

$$\underline{\exp}(f) = \sum_{n=0}^{\infty} \frac{f^{\otimes n}}{\sqrt{n!}},$$

the exponential vector attached to f. Special case: $\underline{\exp}(0) = \Omega$.

6.6 LEMMA Let $f,g \in \mathcal{H}$ — then

$$< \underline{\exp}(f), \underline{\exp}(g) > = e^{<f,g>}.$$

6.7 LEMMA The map $\underline{\exp}:\mathcal{H} \to BO(\mathcal{H})$ is injective and continuous.

PROOF Injectivity is obvious. As for continuity, note that

$$||\underline{\exp}(f) - \underline{\exp}(g)||^2$$

$$= e^{<f,f>} - e^{<f,g>} - e^{<g,f>} + e^{<g,g>}.$$

So if $f \to g$, then $\underline{\exp}(f) \to \underline{\exp}(g)$.

6.8 LEMMA The set of exponential vectors is linearly independent.

PROOF Fix distinct elements $f_1,\ldots,f_n$ in $\mathcal{H}$ and consider a dependence relation

$$\sum_{i=1}^{n} c_i \, \underline{\exp}(f_i) = 0 \quad (c_i \neq 0 \ \forall \ i).$$

Choose $f \in \mathcal{H}$ such that the $\theta_i = < f,f_i >$ $(i = 1,\ldots,n)$ are distinct — then for any $z \in \underline{C}$,

$$0 = < \underline{\exp}(\bar{z}f), \sum_{i=1}^{n} c_i \, \underline{\exp}(f_i) >$$

$$= \sum_{i=1}^{n} c_i < \underline{\exp}(\bar{z}f), \underline{\exp}(f_i) >$$

$$= \sum_{i=1}^{n} c_i \, e^{<\bar{z}f, f_i>}$$

$$= \sum_{i=1}^{n} c_i \, e^{z\theta_i}.$$

Since the exponentials of distinct linear functions are linearly independent over $\underline{C}$, it follows that $c_i = 0 \ \forall \ i$.

6.9 LEMMA The set of exponential vectors is total in $BO(H)$.

PROOF Let S be the closed linear subspace of $BO(H)$ generated by the set of exponential vectors — then in view of 6.5, it suffices to show that $\forall \ f \in H$, $f^{\otimes n} \in S$. And for this, one can proceed by induction:

$$f^{\otimes(n + 1)}$$

$$= \sqrt{(n + 1)!} \lim_{t \to 0} t^{-(n + 1)} [\underline{\exp}(tf) - \bigoplus_{k=0}^{n} \frac{t^k f^{\otimes k}}{\sqrt{k!}}].$$

6.10 EXAMPLE Take $H = \underline{C}$. Bearing in mind that $\otimes^n \underline{C}$ can be identified with $\underline{C}$ itself, we have

$$BO(\underline{C}) = \underline{C} \oplus \underline{C} \oplus \cdots = \ell^2(\underline{Z}_{\geq 0}).$$

Here, $\forall \ z \in \underline{C}$,

$$\underline{\exp}(z) = \{1, z, \ldots, (n!)^{-1/2} z^n, \ldots\}.$$

Let $d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ — then there exists an isometric isomorphism

$$T:BO(\underline{C}) \to L^2(\underline{R},\gamma)$$

characterized by the relation

$$(T\ \underline{\exp}(z))(x) = e^{zx - \frac{1}{2}z^2}.$$

In fact, the functions e^{zx} $(z \in \underline{C})$ are total in $L^2(\underline{R},\gamma)$ and

$$\int_{\underline{R}} e^{\bar{z}_1 x - \frac{1}{2}\bar{z}_1^2} \cdot e^{z_2 x - \frac{1}{2} z_2^2} d\gamma(x)$$

$$= e^{\bar{z}_1 z_2} = e^{\langle z_1, z_2\rangle} = \langle \underline{\exp}(z_1), \underline{\exp}(z_2) \rangle.$$

[Note: Define polynomials $H_n(x)$ by the prescription

$$e^{zx - \frac{1}{2}z^2} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) \qquad (\text{so } H_n = n^{\text{th}} \text{ Hermite polynomial}).$$

Then

$$T\{0,\ldots,0,1,0,\ldots\} = \frac{H_n}{\sqrt{n!}},$$

where 1 appears in the n^{th} position. Therefore the sequence $\{\frac{H_n}{\sqrt{n!}} : n \geq 0\}$ is an orthonormal basis for $L^2(\underline{R},\gamma)$.]

6.11 <u>LEMMA</u> Suppose that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ — then there is an isometric isomorphism

$$T:BO(\mathcal{H}) \to BO(\mathcal{H}_1) \hat{\otimes} BO(\mathcal{H}_2)$$

such that

$$T\ \underline{\exp}(f_1 \oplus f_2) = \underline{\exp}(f_1) \otimes \underline{\exp}(f_2).$$

[Note: This result extends to the case of a finite decomposition, say $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$.]

6.12 EXAMPLE Take $\mathcal{H} = \underline{C}^n$ — then

$$\begin{aligned} BO(\underline{C}^n) &= BO(\underline{C} \oplus \cdots \oplus \underline{C}) \\ &= BO(\underline{C}) \ \hat{\otimes} \cdots \hat{\otimes}\ BO(\underline{C}) \\ &= L^2(\underline{R},\gamma)\ \hat{\otimes} \cdots \hat{\otimes}\ L(\underline{R},\gamma) \\ &= L^2(\underline{R}^n,\gamma^{\times n}), \end{aligned}$$

where

$$\begin{aligned} d\gamma^{\times n}\,x &= \frac{1}{(2\pi)^{n/2}}\, e^{-x^2/2}\, dx \\ &= \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}}\, e^{-x_k^2/2}\, dx_k. \end{aligned}$$

[Note: Explicitly, the arrow

$$T:BO(\underline{C}^n) \to L^2(\underline{R}^n,\gamma^{\times n})$$

characterized by the relation

$$(T\ \underline{\exp}(z))(x) = \exp\Big(\sum_{k=1}^{n} z_k x_k - \frac{1}{2}\sum_{k=1}^{n} z_k^2\Big)$$

is an isometric isomorphism.]

6.13 REMARK Put

$$\underline{H}_{k_1,\ldots,k_n}(x_1,\ldots,x_n)$$

$$= \frac{H_{k_1}(x_1)}{\sqrt{k_1!}} \cdots \frac{H_{k_n}(x_n)}{\sqrt{k_n!}}.$$

Then the $\underline{H}_{k_1,\ldots,k_n}$ are an orthonormal basis for $L^2(\underline{R}^n,\gamma^{\times n})$.

Let $A:H \rightarrow H$ be a bounded linear operator — then A can be canonically extended to a bounded linear operator $A^{\hat{\otimes} n}:H^{\hat{\otimes} n} \rightarrow H^{\hat{\otimes} n}$, viz.

$$A^{\hat{\otimes} n} = \overline{A \otimes \cdots \otimes A} \quad \text{(cf. 5.6)}.$$

Here

$$||A^{\hat{\otimes} n}|| = ||A||^n.$$

[Note: When n = 0, the agreement is that $A^{\hat{\otimes} 0}$ is the identity on $H^{\hat{\otimes} 0} = \underline{C}$.]

From the definitions, it is clear that $A^{\hat{\otimes} n}$ induces a linear transformation $BO_n(H) \rightarrow BO_n(H)$, call if $\Gamma_n(A)$, and still,

$$||\Gamma_n(A)|| = ||A||^n.$$

6.14 LEMMA Suppose that $||A|| \leq 1$ — then the $\Gamma_n(A)$ combine and define a bounded linear operator

$$\Gamma(A):BO(\mathcal{H}) \rightarrow BO(\mathcal{H}).$$

[Note: By construction,

$$||\Gamma(A)|| = \sup_{n \geq 0} ||\Gamma_n(A)|| = \sup_{n \geq 0} ||A||^n = 1.]$$

For example, $\Gamma(cI)$ ($|c| \leq 1$) is multiplication by c^n on $BO_n(\mathcal{H})$.

6.15 <u>REMARK</u> If U is unitary, then the same is true of $\Gamma(U)$.

Let A be a densely defined linear operator on $\mathcal{H}$. Put

$$D_n(A) = \text{Dom}(A) \otimes \cdots \otimes \text{Dom}(A).$$

Then $D_n(A)$ is the domain of

$$\Sigma_n(A) = A \otimes I \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes A.$$

[Note: When $n = 1$, $D_1(A) = \text{Dom}(A)$ and $\Sigma_1(A) = A$. To complete the picture, take $D_0(A) = \underline{C}$ and let $\Sigma_0(A) = 0$.]

Fix $n \geq 1$ — then $\forall\ \sigma \in S_n$,

$$U_n(\sigma)D_n(A) \subset D_n(A)$$

$\Rightarrow$

$$P_nD_n(A) \subset D_n(A).$$

And

$$P_n \Sigma_n(A) = \Sigma_n(A) P_n$$

on $D_n(A)$. Proof: Let

$$\left|\begin{array}{l} f_1, \ldots, f_n \in \mathrm{Dom}(A) \\ \\ g_1, \ldots, g_n \in \mathrm{Dom}(A). \end{array}\right.$$

Then

$$< g_1 \otimes \cdots \otimes g_n, U_n(\sigma) \Sigma_n(A) (f_1 \otimes \cdots \otimes f_n) >_n$$

$$= \sum_{k=1}^{n} < g_{\sigma^{-1}(1)} \otimes \cdots \otimes g_{\sigma^{-1}(n)}, f_1 \otimes \cdots \otimes Af_k \otimes \cdots \otimes f_n >_n$$

$$= \sum_{k=1}^{n} < g_{\sigma^{-1}(k)}, Af_k > \prod_{\ell \neq k} < g_{\sigma^{-1}(\ell)}, f_\ell >$$

$$= \sum_{k=1}^{n} < g_k, Af_{\sigma(k)} > \prod_{\ell \neq k} < g_\ell, f_{\sigma(\ell)} >$$

$$= < g_1 \otimes \cdots \otimes g_n, \Sigma_n(A) U_n(\sigma) (f_1 \otimes \cdots \otimes f_n) >_n$$

$\Rightarrow$

$$U_n(\sigma) \Sigma_n(A) (f_1 \otimes \cdots \otimes f_n) = \Sigma_n(A) U_n(\sigma) (f_1 \otimes \cdots \otimes f_n).$$

Therefore

$$P_n \Sigma_n(A) = \Sigma_n(A) P_n$$

on $D_n(A)$.

Let

$$D(A) = \bigcup_{N=0}^{\infty} D_A(N),$$

where

$$D_A(N) = \{X(N) \in F(H) : X(N) = \{X_0, \ldots, X_N, 0, \ldots\} : X_n \in D_n(A)\}.$$

Then D(A) is a dense linear subspace of $F(H)$. Define a linear operator $\Sigma(A)$ on D(A) slotwise, i.e.,

$$\Sigma(A)X(N) = \{\Sigma_n(A)X_n\}.$$

From the above, $PD(A) \subset D(A)$ and

$$P\Sigma(A) = \Sigma(A)P$$

on D(A).

[Note: P is the orthogonal projection onto $BO(H)$, so, e.g.,

$$\begin{aligned} P\Sigma(A)X(N) &= \{P_n\Sigma_n(A)X_n\} \\ &= \{\Sigma_n(A)P_nX_n\} \\ &= \Sigma(A)PX(N).] \end{aligned}$$

These considerations imply that the restriction

$$\Sigma(A)\,|PD(A)$$

is a densely defined linear operator on $BO(H)$.

6.16 LEMMA Suppose that A is selfadjoint — then $\Sigma(A)$ and $\Sigma(A)\,|PD(A)$ are essentially selfadjoint.

PROOF The operator $\Sigma(A)$ is symmetric. On the other hand, $\forall$ n, $\Sigma_n(A)$ is essentially selfadjoint (cf. 5.13), hence the range of $\Sigma_n(A) \pm \sqrt{-1}$ is dense in $\mathcal{H}^{\hat{\otimes} n}$. But from this it follows that the range of $\Sigma(A) \pm \sqrt{-1}$ is dense in $\mathcal{F}(\mathcal{H})$. Therefore $\Sigma(A)$ is essentially selfadjoint, thus $\Sigma(A)|PD(A)$ is too.

By way of notation, put

$$d\Gamma(A) = \overline{\Sigma(A)|PD(A)}.$$

6.17 EXAMPLE Let

$$NX = \{nX_n\},$$

where

$$\text{Dom}(N) = \{X \in \mathcal{F}(\mathcal{H}): \sum_{n=0}^{\infty} n^2||X_n||_n^2 < \infty\}.$$

Then N is selfadjoint and its spectrum is pure point: $\sigma(N) = \{0,1,\ldots\}$. Obviously, $\text{PDom}(N) \subset \text{Dom}(N)$ and

$$PN = NP$$

on Dom(N). Therefore $N|\text{PDom}(N)$ is selfadjoint. To interpret this, in the foregoing take $A = I$ — then $d\Gamma(I) = N|\text{PDom}(N)$.

[Note: $d\Gamma(I)$ is called the number operator (often denoted by N as well). It is selfadjoint and its spectrum is pure point: $\sigma(d\Gamma(I)) = \{0,1,\ldots\}$.]

Suppose that $t \to U(t)$ is a one parameter unitary group with generator A —

then $t \to \Gamma(U(t))$ is a one parameter unitary group with generator $d\Gamma(A)$:

$$\Gamma(U(t)) = e^{\sqrt{-1}\, t d\Gamma(A)}$$

or still,

$$\Gamma(e^{\sqrt{-1}\, tA}) = e^{\sqrt{-1}\, t d\Gamma(A)}.$$

6.18 LEMMA If A is selfadjoint and if $f \in \mathrm{Dom}(A)$, then $\underline{\exp}(f) \in \mathrm{Dom}(d\Gamma(A))$.

PROOF It suffices to show that the function

$$t \to e^{\sqrt{-1}\, t d\Gamma(A)} \underline{\exp}(f)$$

is differentiable at $t = 0$. But the function $t \to e^{\sqrt{-1}\, tA} f$ is differentiable at $t = 0$ and

$$\underline{\exp}(e^{\sqrt{-1}\, tA} f) = e^{\sqrt{-1}\, t d\Gamma(A)} \underline{\exp}(f).$$

6.19 REMARK On occasion it is necessary to work over $\underline{R}$ rather than $\underline{C}$. In this connection, note that if H is a real Hilbert space and if $H_{\underline{C}}$ is its complexification, then $BO(H_{\underline{C}})$ is isometrically isomorphic to $BO(H)_{\underline{C}}$ (the complexification of $BO(H)$).

§7. FIELD OPERATORS

Let H be a complex Hilbert space, which we shall assume is separable -- then $\forall$ n, $BO_n(H)$ is separable (cf. 6.4). Denote by $BO_F(H)$ the algebraic direct sum of the $BO_n(H)$.

Fix $f \neq 0$ in H -- then one can associate with f two unbounded linear operators

$$\begin{cases} \underline{a}(f):BO_F(H) \to BO(H) \\ \underline{c}(f):BO_F(H) \to BO(H) \end{cases}$$

termed annihilation and creation operators, respectively.

[Note: Matters are trivial if $f = 0$: Take $\underline{a}(f) = 0$, $\underline{c}(f) = 0$.]

It will be simplest to start with $\underline{c}(f)$ and proceed in stages. Thus put

$$\underline{c}_0(f)\Omega = f$$

and for $n > 0$, let

$$\underline{c}_n(f)P_n(f_1 \otimes \cdots \otimes f_n) = \sqrt{n+1}\, P_n(f \otimes f_1 \otimes \cdots \otimes f_n).$$

Write D_n for the linear span of the $P_n(f_1 \otimes \cdots \otimes f_n)$ -- then D_n is dense in $BO_n(H)$ and

$$\underline{c}_n(f):D_n \to BO_{n+1}(H).$$

7.1 LEMMA There exists a dense linear subspace $D_n(f) \subset D_n$ such that $\forall\ X_n \in D_n(f)$,

$$||\underline{c}_n(f)X_n|| \leq \sqrt{n+1}\ ||f||\ ||X_n||.$$

PROOF Set $e_1 = f/||f||$ and choose an orthonormal basis $e_2, e_3, \ldots$ for $\{\underline{C}e_1\}^{\perp}$. Construct from this data an orthonormal basis $\{e_n(\kappa)\}$ for $BO_n(H)$ (cf. 6.4). Let $D_n(f)$ be the linear span of the $e_n(\kappa)$ — then by direct computation, we find that $\forall\ X_n \in D_n(f)$,

$$||\underline{c}_n(f)X_n|| \leq \sqrt{n+1}\ ||f||\ ||X_n||.$$

Since $f^{\otimes n} \in D_n(f)$ and since

$$||\underline{c}_n(f)f^{\otimes n}|| = \sqrt{n+1}\ ||f||\ ||f^{\otimes n}||,$$

it follows that $\underline{c}_n(f)$ extends to a bounded linear operator $BO_n(H) \rightarrow BO_{n+1}(H)$ of norm $\sqrt{n+1}\ ||f||$, which we shall again denote by $\underline{c}_n(f)$. Define now a linear operator $\underline{c}(f):BO_F(H) \rightarrow BO(H)$ by demanding that

$$\underline{c}(f)|BO_n(H) = \underline{c}_n(f).$$

Then $\underline{c}(f)$ is densely defined but unbounded.

[Note: There is a small technicality which has been glossed over. While there is no question that $\underline{c}_n(f)|D_n(f)$ extends to a bounded linear operator $BO_n(H) \rightarrow BO_{n+1}(H)$ of norm $\sqrt{n+1}\ ||f||$, one can still ask: Why does the restriction of this extension to D_n agree with the original definition of $\underline{c}_n(f)$? That it does can be settled by a straightforward limiting argument.]

7.2 REMARK From its very definition, $\underline{c}(f)BO_F(H) \subset BO_F(H)$, hence the elements of $BO_F(H)$ are C^∞ vectors for $\underline{c}(f)$. In fact, the elements of $BO_F(H)$ are analytic vectors for $\underline{c}(f)$. To see this, let $X_n \in BO_n(H)$ — then

$$||\underline{c}(f)^k X_n|| \leq \left[\frac{(n+k)!}{n!}\right]^{1/2} ||f||^k ||X_n||.$$

Therefore

$$\sum_{k=0}^{\infty} \frac{||\underline{c}(f)^k X_n||}{k!} |t|^k$$

$$\leq ||X_n|| \sum_{k=0}^{\infty} \left[\frac{(n+k)!}{n!}\right]^{1/2} \frac{(||f||\ |t|)^k}{k!},$$

which is convergent for all t.

7.3 EXAMPLE Take $H = L^2(\underline{R})$ and let $\psi_n \in BO_n(H)$ $(n > 0)$ (cf. 6.3) -- then for any $\psi \neq 0$ in H,

$$(\underline{c}(\psi)\psi_n)\ (x_1,\ldots,x_{n+1})$$

$$= \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \psi(x_i)\psi_n(x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}).$$

Because

$$\underline{c}_n(f):BO_n(H) \rightarrow BO_{n+1}(H)$$

is bounded, it has a bounded adjoint

$$\underline{c}_n(f)^*:BO_{n+1}(H) \rightarrow BO_n(H).$$

7.4 LEMMA The domain of $\underline{c}(f)^*$ contains $BO_F(H)$.

PROOF Fix $Y \in BO_{n+1}(H)$ and put $Y^* = \underline{c}_n(f)^*Y$. Let $X \in BO_F(H)$ -- then

$$\begin{cases} < Y^*,X > = 0 \text{ unless } X_n \neq 0 \\ < Y,\underline{c}(f)X > = 0 \text{ unless } X_n \neq 0. \end{cases}$$

On the other hand, if $X_n \neq 0$, then

$$\begin{aligned} < Y^*,X > &= < Y^*,X_n > \\ &= < \underline{c}_n(f)^*Y,X_n > \\ &= < Y,\underline{c}_n(f)X_n > \\ &= < Y,\underline{c}(f)X_n > \\ &= < Y,\underline{c}(f)X >. \end{aligned}$$

Therefore

$$Y^* = \underline{c}(f)^*Y.$$

Consequently, $\underline{c}(f)^*$ is densely defined, thus $\underline{c}(f)$ admits closure (cf. 1.5).

7.5 LEMMA We have

$$\underline{c}_n(f)^*(P_{n+1}(g_1 \otimes \cdots \otimes g_{n+1}))$$

$$= \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} < f,g_i > P_n(g_1 \otimes \cdots \otimes g_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_{n+1}).$$

<u>PROOF</u> Let $f_1 = f$ — then

$$< P_{n+1}(g_1 \otimes \cdots \otimes g_{n+1}), \underline{c}_n(f)P_n(f_2 \otimes \cdots \otimes f_{n+1}) >$$

$$= < P_{n+1}(g_1 \otimes \cdots \otimes g_{n+1}), \sqrt{n+1}\, P_{n+1}(f_1 \otimes \cdots \otimes f_{n+1}) >$$

$$= \frac{\sqrt{n+1}}{(n+1)!} n! \sum_{i=1}^{n+1} < g_i, f >$$

$$\times < P_n(g_1 \otimes \cdots \otimes g_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_{n+1}), P_n(f_2 \otimes \cdots \otimes f_n) >$$

$$= \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \overline{< f, g_i >} < P_n(g_1 \otimes \cdots \otimes g_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_{n+1}), P_n(f_2 \otimes \cdots \otimes f_n) >$$

$$= < \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} < f, g_i > P_n(g_1 \otimes \cdots \otimes g_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_{n+1}), P_n(f_2 \otimes \cdots \otimes f_n) >.$$

But D_n is dense in $BO_n(H)$, from which the lemma.

Let

$$\underline{a}(f) = \underline{c}(f)^* | BO_F(H).$$

Then

$$\underline{a}(f)\Omega = 0$$

and for $n > 0$,

$$\underline{a}(f)P_n(f_1 \otimes \cdots \otimes f_n)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} < f, f_i > P_{n-1}(f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_n).$$

Note too that

$$||\underline{a}(f)|BO_{n+1}(H)||^2$$

$$= ||\underline{c}(f)|BO_n(H)||^2$$

$$= (n + 1)||f||^2.$$

7.6 REMARK The elements of $BO_F(H)$ are analytic vectors for $\underline{a}(f)$ (cf. 7.2).

7.7 EXAMPLE Take $H = L^2(\underline{R})$ and let $\psi_n \in BO_n(H)$ $(n > 0)$ (cf. 6.3) — then for any $\psi \neq 0$ in H,

$$(\underline{a}(\psi)\psi_n)(x_1,\ldots,x_{n-1})$$

$$= \sqrt{n} \int_{\underline{R}} \overline{\psi(x)}\psi_n(x,x_1,\ldots,x_{n-1})dx.$$

7.8 LEMMA Let $f,g \in H$ — then on $BO_F(H)$,

$$\begin{cases} [\underline{a}(f),\underline{a}(g)] = 0 \\ [\underline{c}(f),\underline{c}(g)] = 0 \end{cases}$$

and

$$[\underline{a}(f),\underline{c}(g)] = \langle f,g \rangle.$$

7.9 LEMMA Let $X \in BO_F(H)$ -- then

$$||\underline{c}(f)X||^2 = ||\underline{a}(f)X||^2 + ||f||^2||X||^2.$$

PROOF In fact,

$$\begin{aligned}
||\underline{c}(f)X||^2 &= < \underline{c}(f)X, \underline{c}(f)X > \\
&= < \underline{a}(f)\underline{c}(f)X,X > \\
&= < \underline{c}(f)\underline{a}(f)X,X > + < ||f||^2X,X > \quad (\text{cf. } 7.8) \\
&= ||\underline{a}(f)X||^2 + ||f||^2||X||^2.
\end{aligned}$$

Let

$$\begin{cases} \tilde{a}(f) = \underline{c}(f)^* \\ \tilde{c}(f) = \tilde{a}(f)^*. \end{cases}$$

Then

$$\begin{cases} \tilde{a}(f)|BO_F(H) = \underline{a}(f) \\ \tilde{c}(f)|BO_F(H) = \underline{c}(f). \end{cases}$$

7.10 LEMMA $\tilde{a}(f)$ is the adjoint of $\tilde{c}(f)$.

PROOF One has only to note that

$$\begin{aligned}
\tilde{c}(f)^* &= \tilde{a}(f)^{**} \\
&= (\underline{c}(f)^{**})^*
\end{aligned}$$

$$= (\overline{\underline{c}(f)})^* \quad (\text{cf. } 1.6)$$

$$= \underline{c}(f)^* \quad (\text{cf. } 1.6)$$

$$= \tilde{a}(f).$$

Therefore

$$\begin{cases} X \in \text{Dom}(\tilde{a}(f)) \\ Y \in \text{Dom}(\tilde{c}(f)) \end{cases}$$

$\Rightarrow$

$$< \tilde{a}(f)X,Y > = < X,\tilde{c}(f)Y >.$$

7.11 <u>LEMMA</u> We have

$$\begin{cases} \tilde{a}(f) = \overline{\underline{a}(f)} \\ \tilde{c}(f) = \overline{\underline{c}(f)}. \end{cases}$$

Let

$$D_f = \{X \in BO(H): \sum_n ||\underline{c}(f)X_n||^2 < \infty\}.$$

Then (cf. 7.9)

$$D_f = \{X \in BO(H): \sum_n ||\underline{a}(f)X_n||^2 < \infty\}.$$

7.12 <u>LEMMA</u> The operators $\tilde{a}(f)$ and $\tilde{c}(f)$ have the same domain, viz. D_f.

<u>PROOF</u> Suppose that $X \in \mathrm{Dom}(\tilde{a}(f))$ and let

$$\tilde{a}(f)X = \sum_n Y_n.$$

Then $\forall\ Z_n \in BO_n(H)$,

$$\begin{aligned} < Y_n, Z_n > &= < \tilde{a}(f)X, Z_n > \\ &= < X, \underline{c}(f)Z_n > \\ &= < X_{n+1}, \underline{c}(f)Z_n > \\ &= < \underline{a}(f)X_{n+1}, Z_n > \end{aligned}$$

=>

$$Y_n = \underline{a}(f)X_{n+1}.$$

But $\sum_n ||Y_n||^2 < \infty$. Therefore $X \in D_f$. Conversely, suppose that $X \in D_f$ -- then, as $N \to \infty$,

$$\sum_{n=0}^{N} X_n \to X = \sum_{n=0}^{\infty} X_n$$

and

$$\tilde{a}(f)\left(\sum_{n=0}^{N} X_n\right) \to Y = \sum_{n=0}^{\infty} \tilde{a}(f)X_n,$$

thus $X \in \mathrm{Dom}(\tilde{a}(f))$ ($\tilde{a}(f)$ being closed).

In other words,

$$D_f = \mathrm{Dom}(\tilde{a}(f))$$

and, analogously,

$$D_f = \mathrm{Dom}(\tilde{c}(f)).$$

7.13 REMARK The results formulated in 7.8 and 7.9 remain valid if $\underline{a}(f)$ and $\underline{c}(f)$ are replaced by $\tilde{a}(f)$ and $\tilde{c}(f)$ and $BO_F(H)$ is replaced by D_f.

Let

$$\tilde{D} = \mathrm{Dom}(\sqrt{d\Gamma(I)}),$$

where $d\Gamma(I)$ is the number operator (cf. 6.17) — then $X \in \tilde{D}$ iff

$$\sum_{n=0}^{\infty} n||X_n||^2 < \infty.$$

7.14 LEMMA $\forall$ f,

$$\tilde{D} \subset D_f.$$

PROOF Let $X \in \tilde{D}$ — then

$$\begin{aligned}\sum_n ||\underline{c}(f)X_n||^2 &= \sum_n ||\underline{c}_n(f)X_n||^2 \\ &\leq \sum_n (\sqrt{n+1}\ ||f||\ ||X_n||)^2 \\ &= ||f||^2 (\sum_n (n+1)||X_n||^2) < \infty.\end{aligned}$$

[Note: Accordingly,

$$\tilde{D} \subset \bigcap_f D_f.]$$

The set of exponential vectors is evidently contained in $\tilde{D}$.

7.15 LEMMA We have

$$\begin{cases} \tilde{a}(f)\underline{\exp}(g) = \langle f,g \rangle \underline{\exp}(g) \\ \tilde{c}(f)\underline{\exp}(g) = \frac{d}{dt} \underline{\exp}(g + tf)\Big|_{t=0} . \end{cases}$$

7.16 LEMMA Suppose that $U:H \to H$ is unitary -- then

$$\begin{cases} \Gamma(U)\tilde{a}(f)\Gamma(U)^{-1} = \tilde{a}(Uf) \\ \Gamma(U)\tilde{c}(f)\Gamma(U)^{-1} = \tilde{c}(Uf) \end{cases}$$

on $BO_F(H)$.

PROOF For

$$\Gamma(U)\underline{a}(f)\Gamma(U)^{-1}P_n(f_1 \otimes \cdots \otimes f_n)$$

$$= \Gamma(U)\underline{a}(f)P_n(U^{-1}f_1 \otimes \cdots \otimes U^{-1}f_n)$$

$$= \Gamma(U) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \langle f,U^{-1}f_i \rangle P_{n-1}(U^{-1}f_1 \otimes \cdots \otimes U^{-1}f_{i-1} \otimes U^{-1}f_{i+1} \otimes \cdots \otimes U^{-1}f_n)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} < Uf, f_i > \Gamma(U) P_{n-1}(U^{-1}f_1 \otimes \cdots \otimes U^{-1}f_{i-1} \otimes U^{-1}f_{i+1} \otimes \cdots \otimes U^{-1}f_n)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} < Uf, f_i > P_{n-1}(f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_n)$$

$$= \underline{a}(Uf) P_n(f_1 \otimes \cdots \otimes f_n),$$

which leads at once to the first relation. Taking adjoints then gives the second.

Let $f \in H$ -- then the field operators attached to f are the combinations

$$\begin{cases} Q(f) = \frac{1}{\sqrt{2}} (\tilde{c}(f) + \tilde{a}(f)) \\ P(f) = \frac{\sqrt{-1}}{\sqrt{2}} (\tilde{c}(f) - \tilde{a}(f)). \end{cases}$$

In what follows, it will be enough to deal with $Q(f)$ (since $P(f) = Q(\sqrt{-1}\, f)$).

[Note: The domain of $Q(f)$ is $\text{Dom}(\tilde{c}(f)) \cap \text{Dom}(\tilde{a}(f))$, i.e., is D_f (cf. 7.12).]

7.17 LEMMA $Q(f)$ is symmetric.

PROOF On general grounds,

$$Q(f)^* \supset \frac{1}{\sqrt{2}} (\tilde{c}(f)^* + \tilde{a}(f)^*).$$

But $\tilde{c}(f)^* = \tilde{a}(f)$, $\tilde{a}(f)^* = \tilde{c}(f)$, hence $Q(f)^* \supset Q(f)$.

7.18 LEMMA $Q(f)$ is essentially selfadjoint.

PROOF This is an application of 1.18: The elements of $BO_F(H)$ are analytic vectors for $Q(f)$. Indeed, the restriction of $Q(f)$ to $BO_n(H)$ is bounded and

$$||Q(f)x_n|| \le \frac{1}{\sqrt{2}} (||\underline{c}(f)x_n|| + ||\underline{a}(f)x_n||)$$

$$\le \frac{1}{\sqrt{2}} (\sqrt{n+1}\, ||f||\, ||x_n|| + \sqrt{n}\, ||f||\, ||x_n||)$$

$$\le \sqrt{2(n+1)}\, ||f||\, ||x_n||.$$

Proceeding from here by induction, we then get

$$||Q(f)^k x_n|| \le 2^{k/2} \left[\frac{(n+k)!}{n!} \right]^{1/2} ||f||^k ||x_n||.$$

Therefore $\forall$ t,

$$\sum_{k=0}^{\infty} \frac{||Q(f)^k x_n||}{k!} |t|^k < \infty.$$

7.19 REMARK It is clear that $Q(f)BO_F(H) \subset BO_F(H)$, thus $Q(f)|BO_F(H)$ is essentially selfadjoint (cf. 1.21).

Thanks to 7.18, the closures

$$\begin{bmatrix} \overline{Q(f)} \\ \overline{P(f)} \end{bmatrix}$$

are selfadjoint. And, of course,

$$D_f \subset \mathrm{Dom}(\overline{Q(f)}) \cap \mathrm{Dom}(\overline{P(f)}).$$

7.20 <u>LEMMA</u> We have

$$D_f = \mathrm{Dom}(\overline{Q(f)}) \cap \mathrm{Dom}(\overline{P(f)}).$$

<u>PROOF</u> Let $X \in \mathrm{Dom}(\overline{Q(f)}) \cap \mathrm{Dom}(\overline{P(f)})$ — then $\forall\ Y \in D_f$,

$$< X,\tilde{a}(f)Y > = < X, \frac{1}{\sqrt{2}} (Q(f) + \sqrt{-1}\ P(f))Y >$$

$$= \frac{1}{\sqrt{2}} < X,Q(f)Y > + \frac{\sqrt{-1}}{\sqrt{2}} < X,P(f)Y >$$

$$= \frac{1}{\sqrt{2}} < \overline{Q(f)}X,Y > + \frac{\sqrt{-1}}{\sqrt{2}} < \overline{P(f)}X,Y >$$

$$= < \frac{1}{\sqrt{2}} \overline{Q(f)}X - \frac{\sqrt{-1}}{\sqrt{2}} \overline{P(f)}X,Y > ,$$

so

$$X \in \mathrm{Dom}(\tilde{a}(f)^*) = \mathrm{Dom}(\tilde{c}(f)) = D_f.$$

7.21 <u>LEMMA</u> The set

$$\{Q(f_1) \cdots Q(f_n)\Omega\},$$

where the $f_i \in H$ and n are arbitrary, is total in $BO(H)$.

<u>PROOF</u> The linear span of the

$$Q(f_1) \cdots Q(f_n)\Omega$$

is the same as the linear span of the

$$\underline{c}(f_1) \cdots \underline{c}(f_n)\Omega.$$

But

$$\underline{c}(f_1) \cdots \underline{c}(f_n)\Omega = \sqrt{n!}\, P_n(f_1 \otimes \cdots \otimes f_n).$$

7.22 LEMMA On $BO_F(H)$,

$$[Q(f), Q(g)] = \sqrt{-1} \operatorname{Im} < f,g >.$$

PROOF In view of 7.8,

$$[Q(f),Q(g)]$$

$$= [\frac{1}{\sqrt{2}} (\underline{c}(f) + \underline{a}(f)), \frac{1}{\sqrt{2}} (\underline{c}(g) + \underline{a}(g))]$$

$$= \frac{1}{2} (< f,g > - < g,f >)$$

$$= \frac{1}{2} (< f,g > - \overline{< f,g >})$$

$$= \sqrt{-1} \operatorname{Im} < f,g >.$$

7.23 REMARK On $\operatorname{Dom}([\overline{Q(f)}, \overline{Q(g)}])$,

$$[\overline{Q(f)}, \overline{Q(g)}] = \sqrt{-1} \operatorname{Im} < f,g >.$$

To check this, fix $X \in \operatorname{Dom}([\overline{Q(f)}, \overline{Q(g)}])$ and let $Y \in BO_F(H)$ be arbitrary -- then

$$< [\overline{Q(f)}, \overline{Q(g)}]X,Y >$$

$$= < \overline{Q(f)}\ \overline{Q(g)}\ X - \overline{Q(g)}\ \overline{Q(f)}\ X,Y >$$

$$= < X, \overline{Q(g)}\ \overline{Q(f)}\ Y - \overline{Q(f)}\ \overline{Q(g)}\ Y >$$

$$= < X, Q(g)Q(f)\ Y - Q(f)Q(g)\ Y >$$

$$= < X, [Q(g), Q(f)]Y >$$

$$= < X, - \sqrt{-1}\ \mathrm{Im} < f,g > Y >$$

$$= < \sqrt{-1}\ \mathrm{Im} < f,g > X,Y >$$

=>

$$[\overline{Q(f)}, \overline{Q(g)}]X = \sqrt{-1}\ \mathrm{Im} < f,g >.$$

7.24 <u>EXAMPLE</u> Fix an orthonormal basis $\{e_n\}$ for H -- then

$$\begin{cases} [Q(e_i), Q(e_j)] = 0 \\ [P(e_i), P(e_j)] = 0 \end{cases}, \quad [Q(e_i), P(e_j)] = \sqrt{-1}\ \delta_{ij}$$

on $BO_F(H)$.

7.25 <u>LEMMA</u> Suppose that $U:H \rightarrow H$ is unitary -- then

$$\Gamma(U)\overline{Q(f)}\ \Gamma(U)^{-1} = \overline{Q(Uf)}$$

on $\mathrm{Dom}(\overline{Q(Uf)})$.

PROOF Owing to 7.16,

$$\Gamma(U)Q(f)\Gamma(U)^{-1} = Q(Uf)$$

on $BO_F(H)$. Furthermore

$$\Gamma(U)Q(f)\Gamma(U)^{-1}|BO_F(H)$$

and

$$Q(Uf)\,|BO_F(H)$$

are essentially selfadjoint (cf. 7.19), thus their respective closures are equal (cf. 1.14). But

$$\overline{\Gamma(U)Q(f)\Gamma(U)^{-1}|BO_F(H)}$$

$$= \overline{\Gamma(U)Q(f)\Gamma(U)^{-1}}$$

$$= \Gamma(U)\overline{Q(f)}\;\Gamma(U)^{-1}.$$

[Note: A priori, the domain of $\Gamma(U)\overline{Q(f)}\;\Gamma(U)^{-1}$ is $\Gamma(U)\mathrm{Dom}(\overline{Q(f)})$ which, therefore, is precisely $\mathrm{Dom}(\overline{Q(Uf)})$.]

7.26 EXAMPLE Let $U = \sqrt{-1}\,I$ -- then

$$\Gamma(U)\overline{Q(f)}\;\Gamma(U)^{-1} = \overline{P(f)},$$

so $\overline{Q(f)}$ and $\overline{P(f)}$ are unitarily equivalent.

If $r \in \underline{R}$, then

$$\overline{Q(rf)} = r\overline{Q(f)}.$$

The behavior of sums, however, is a little more complicated.

7.27 <u>LEMMA</u> $\forall$ f,g $\in H$,

$$\overline{Q(f + g)} = \overline{(\overline{Q(f)} + \overline{Q(g)})}.$$

<u>PROOF</u> Since $\overline{Q(f)}$ and $\overline{Q(g)}$ are selfadjoint (cf. 7.18) and since $\text{Dom}(\overline{Q(f)} + \overline{Q(g)})$ is dense, $\overline{Q(f)} + \overline{Q(g)}$ is necessarily symmetric:

$$(\overline{Q(f)} + \overline{Q(g)})^* \supset \overline{Q(f)} + \overline{Q(g)}.$$

But

$$\begin{aligned}
&(\overline{Q(f)} + \overline{Q(g)}) | BO_F(H) \\
&\quad = (Q(f) + Q(g)) | BO_F(H) \\
&\quad = Q(f + g) | BO_F(H),
\end{aligned}$$

the latter being essentially selfadjoint (cf. 7.19). Therefore (cf. 1.14)

$$\begin{aligned}
&\overline{(\overline{Q(f)} + \overline{Q(g)})} \\
&\quad = \overline{(\overline{Q(f)} + \overline{Q(g)}) | BO_F(H)} \\
&\quad = \overline{Q(f + g) | BO_F(H)} \\
&\quad = \overline{Q(f + g)}.
\end{aligned}$$

§8. COMPUTATIONS IN $BO(\underline{C})$

Take $H = \underline{C}$ and let $d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ -- then, as we know (cf. 6.10), there exists an isometric isomorphism

$$T: BO(\underline{C}) \to L^2(\underline{R},\gamma)$$

characterized by the relation

$$(T\,\underline{\exp}(z))(x) = e^{zx - \frac{1}{2}z^2}.$$

Noting that

$$\begin{cases} \tilde{a}(z) = \bar{z}\tilde{a}(1) \\ \tilde{c}(z) = z\tilde{c}(1), \end{cases}$$

put

$$\begin{cases} \tilde{a} = \tilde{a}(1) \\ \tilde{c} = \tilde{c}(1). \end{cases}$$

Then our initial problem will be to calculate the action of

$$\begin{cases} T\tilde{a}T^{-1} \\ T\tilde{c}T^{-1} \end{cases}$$

on $L^2(\underline{R},\gamma)$ (or, more precisely, on a certain dense subspace thereof).

<u>Calculation of $T\tilde{a}T^{-1}$</u> We have

$$T\tilde{a}T^{-1}\,[e^{zx - \frac{1}{2}z^2}]$$

$$= T\tilde{a}\ \underline{\exp}(z)$$

$$= T < 1,z > \underline{\exp}(z)$$

$$= ze^{zx - \frac{1}{2}z^2}$$

$$= \frac{d}{dx}\ [e^{zx - \frac{1}{2}z^2}]$$

=>

$$T\tilde{a}T^{-1} = \frac{d}{dx}\ .$$

<u>Calculation of $T\tilde{c}T^{-1}$</u> We have

$$T\tilde{c}T^{-1}\ [e^{zx - \frac{1}{2}z^2}]$$

$$= T\tilde{c}\ \underline{\exp}(z)$$

$$= T\ \frac{d}{dt}\ \underline{\exp}(z + t)\Big|_{t=0}$$

$$= \frac{d}{dt}\ [e^{(z + t)x - \frac{1}{2}(z + t)^2}]\Big|_{t=0}$$

$$= e^{zx - \frac{1}{2}z^2}\ \frac{d}{dt}\ [\exp(tx - tz - \frac{1}{2}t^2)]\Big|_{t=0}$$

$$= e^{zx - \frac{1}{2}z^2}(x - z)$$

$$= xe^{zx - \frac{1}{2}z^2} - \frac{d}{dx}[e^{zx - \frac{1}{2}z^2}]$$

=>

$$T\tilde{c}T^{-1} = x - \frac{d}{dx} .$$

8.1 <u>REMARK</u> Since

$$T\{0,\ldots,0,1,0,\ldots\} = \frac{H_n}{\sqrt{n!}} ,$$

where 1 appears in the n^{th} position, and since

$$x^n = n! \sum_{k=0}^{[n/2]} \frac{H_{n-2k}(x)}{2^k k!(n-2k)!} ,$$

it follows that the image of $BO_F(\underline{C})$ under T is simply the set of polynomials and there the preceding expressions for $\left[\begin{array}{l} T\tilde{a}T^{-1} \\ T\tilde{c}T^{-1} \end{array}\right.$ are equally valid.

8.2 <u>EXAMPLE</u> $\forall$ n,

$$(x - \frac{d}{dx})^n 1 = H_n(x) .$$

The above considerations can be transferred to $L^2(\underline{R})$ via the isometric isomorphism

$$T_G : L^2(\underline{R},\gamma) \to L^2(\underline{R})$$

which sends f to f·G, where

$$G(x) = \frac{1}{(2\pi)^{1/4}} \exp(-\frac{x^2}{4}).$$

<u>Calculation of $T_G(\frac{d}{dx})T_G^{-1}$</u> We have

$$T_G(\frac{d}{dx})T_G^{-1}(\psi)$$

$$= T_G(\frac{d}{dx})(\psi \cdot \frac{1}{G})$$

$$= T_G(\psi' \cdot \frac{1}{G} - \psi \cdot \frac{1}{G^2} \cdot G')$$

$$= T_G(\psi' \cdot \frac{1}{G} - \psi \cdot \frac{1}{G^2} \cdot G \cdot (-\frac{x}{2}))$$

$$= \psi' + (\frac{x}{2})\psi$$

=>

$$T_G(\frac{d}{dx})T_G^{-1} = \frac{d}{dx} + \frac{x}{2}.$$

<u>Calculation of $T_G(x)T_G^{-1}$</u> We have

$$T_G(x)T_G^{-1}(\psi)$$

$$= T_G(x)(\psi \cdot \frac{1}{G})$$

$$= T_G(x \cdot \psi \cdot \frac{1}{G})$$

$$= (x)\psi$$

=>

$$T_G(x)T_G^{-1} = x.$$

Therefore

$$\begin{cases} T_G T\tilde{a}T^{-1}T_G^{-1} = \frac{x}{2} + \frac{d}{dx} \\ \\ T_G T\tilde{c}T^{-1}T_G^{-1} = \frac{x}{2} - \frac{d}{dx} . \end{cases}$$

Given $r > 0$, define a unitary operator

$$U_r : L^2(\underline{R}) \to L^2(\underline{R})$$

by

$$U_r\psi(x) = \sqrt{r}\ \psi(rx).$$

Then

$$U_r^{-1} = U_{1/r}.$$

<u>Calculation of $U_r[\frac{x}{2} \pm \frac{d}{dx}]U_r^{-1}$</u> We have

$$U_r[\frac{x}{2} \pm \frac{d}{dx}]U_r^{-1}(\psi)$$

$$= U_r[\frac{x}{2} \pm \frac{d}{dx}]\ \frac{1}{\sqrt{r}}\ \psi(\frac{x}{r})$$

$$= U_r((\frac{x}{2})\ \frac{1}{\sqrt{r}}\ \psi(\frac{x}{r})\ \pm \frac{1}{\sqrt{r}} \cdot \frac{1}{r}\ \psi'(\frac{x}{r}))$$

$$= \frac{r}{2}\, x\psi(x) \pm \frac{1}{r}\, \psi'(x)$$

=>

$$U_r[\frac{x}{2} \pm \frac{d}{dx}]U_r^{-1} = \frac{r}{2}\, x \pm \frac{1}{r}\frac{d}{dx}\ .$$

Therefore

$$\begin{cases} U_r T_G T\tilde{a}T^{-1}T_G^{-1}U_r^{-1} = \frac{r}{2}\, x + \frac{1}{r}\frac{d}{dx} \\ U_r T_G T\tilde{c}T^{-1}T_G^{-1}U_r^{-1} = \frac{r}{2}\, x - \frac{1}{r}\frac{d}{dx}\ . \end{cases}$$

8.3 REMARK The image

$$U_r T_G TBO_F(\underline{C})$$

is the linear subspace L_r of $L^2(\underline{R})$ consisting of the functions

$$p(x)\exp(-\frac{1}{4}\, r^2x^2),$$

where p is a polynomial.

Let

$$\begin{cases} Q = x \\ P = -\sqrt{-1}\,\frac{d}{dx}\ . \end{cases}$$

Take $r = \sqrt{2}$ -- then

$$\begin{cases} U_{\sqrt{2}} T_G T\tilde{a}T^{-1}T_G^{-1}U_{\sqrt{2}}^{-1} = \frac{1}{\sqrt{2}}\,(Q + \sqrt{-1}\, P) \equiv A \\ U_{\sqrt{2}} T_G T\tilde{c}T^{-1}T_G^{-1}U_{\sqrt{2}}^{-1} = \frac{1}{\sqrt{2}}\,(Q - \sqrt{-1}\, P) \equiv C, \end{cases}$$

the traditional choice for the annihilation and creation operators in $L^2(\underline{R})$.

[Note: These formulas are valid on $L_{\sqrt{2}}$ (or $S(\underline{R})$).]

8.4 REMARK The sequence $\{\frac{H_n}{\sqrt{n!}} : n \geq 0\}$ is an orthonormal basis for $L^2(\underline{R},\gamma)$ (cf. 6.10). Put

$$h_n = \frac{1}{\sqrt{n!}} U_{\sqrt{2}} T_G H_n.$$

Then

$$h_n(x) = \frac{1}{\sqrt{n!}} \frac{1}{\pi^{1/4}} e^{-x^2/2} H_n(\sqrt{2}\, x)$$

is the n^{th} Hermite function and the sequence $\{h_n : n \geq 0\}$ is an orthonormal basis for $L^2(\underline{R})$. The h_n are eigenfunctions of U_F (cf. 3.3), viz.

$$U_F h_n = (-\sqrt{-1})^n h_n,$$

and satisfy the differential equation

$$(-\frac{d^2}{dx^2} + x^2) h_n = (2n+1) h_n.$$

Put $e_0 = \Omega$, $e_n = 1^{\otimes n}$ $(n \geq 1)$ -- then $\{e_n : n \geq 0\}$ is an orthonormal basis for $BO(\underline{C})$, so the machinery developed in 1.19 is applicable. Agreeing to use the notation thereof, the role of D is now played by $BO_F(\underline{C})$ and (cf. 1.20, 7.11)

$$\begin{cases} \tilde{a} = \bar{a} \\ \tilde{c} = \bar{c}\,. \end{cases}$$

From the definitions,

$$\begin{cases} Q(1) = \frac{1}{\sqrt{2}} (\tilde{c} + \tilde{a}) \\ P(1) = \frac{\sqrt{-1}}{\sqrt{2}} (\tilde{c} - \tilde{a}). \end{cases}$$

Consequently,

$$\begin{cases} Q(1)e_n = \frac{1}{\sqrt{2}} (\sqrt{n+1}\, e_{n+1} + \sqrt{n}\, e_{n-1}) \\ P(1)e_n = \frac{\sqrt{-1}}{\sqrt{2}} (\sqrt{n+1}\, e_{n+1} - \sqrt{n}\, e_{n-1}). \end{cases}$$

8.5 <u>LEMMA</u> On $L_{\sqrt{2}}$ (or $S(\underline{R})$),

$$\begin{cases} U_{\sqrt{2}} T_G T Q(1) T^{-1} T_G^{-1} U_{\sqrt{2}}^{-1} = Q \\ U_{\sqrt{2}} T_G T P(1) T^{-1} T_G^{-1} U_{\sqrt{2}}^{-1} = P. \end{cases}$$

Consider $\bar{N}$ (cf. 2.31) -- then $\text{Dom}(\bar{N}^{1/2}) = \bar{D}$, the common domain of $\tilde{a}$ ($= \tilde{c}^*$) and $\tilde{c}$($= \tilde{a}^*$) (cf. 7.10).

[Note: In this context, $\bar{N} = d\Gamma(I)$ (cf. 6.17) and, being nonnegative,

$$\bar{N}^{1/2} = \overline{\bar{N}^{1/2}|\text{Dom}(\bar{N})} \quad \text{(cf. 2.32).]}$$

8.6 <u>LEMMA</u> We have

$$\tilde{c}\tilde{a}|\mathrm{Dom}(\bar{N}) = \bar{N}$$

or still,

$$\tilde{c}\tilde{c}^*|\mathrm{Dom}(\bar{N}) = \bar{N}.$$

Therefore

$$T\bar{N}T^{-1} = -L,$$

where

$$L = \frac{d^2}{dx^2} - x\frac{d}{dx}.$$

[Note: Later on it will be seen that L is the generator of the Ornstein-Uhlenbeck semigroup.]

8.7 <u>LEMMA</u> On $L_{\sqrt{2}}$ (or $S(\underline{R})$),

$$U_{\sqrt{2}} T_G T(\bar{N} + \frac{1}{2}) T^{-1} T_G^{-1} U_{\sqrt{2}}^{-1}$$

$$= \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right),$$

the hamiltonian of the harmonic oscillator.

Let

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right).$$

Then

$$H = \frac{1}{2} (P^2 + Q^2)$$

and is selfadjoint.

[Note: H is essentially selfadjoint on $L_{\sqrt{2}}$ (or $S(\underline{R})$).]

8.8 EXAMPLE Consider the one parameter unitary group $t \rightarrow e^{-\sqrt{-1}\, Ht}$ and let $0 < t < \pi$ -- then $\forall$ $f \in S(\underline{R})$,

$$(e^{-\sqrt{-1}\, tH} f)(x)$$

$$= \frac{1}{\sqrt{2\pi \sqrt{-1} \sin t}} \int_{\underline{R}} \exp(\sqrt{-1} \frac{x^2 + y^2}{2} \frac{\cos t}{\sin t} - \sqrt{-1} \frac{xy}{\sin t}) f(y) dy.$$

8.9 REMARK The operator

$$-\frac{d^2}{dx^2} + x^2 + 1$$

figures in distribution theory. In fact, any tempered distribution on the line necessarily has the form

$$(- \frac{d^2}{dx^2} + x^2 + 1)^n f,$$

where n is a nonnegative integer and f is a bounded continuous function.

Given $t > 0$, write $BO_t(\underline{C})$ for $BO(\underline{C}_t)$, where $\underline{C}_t$ is $\underline{C}$ equipped with the inner product

$$< z,w >_t = \frac{<z,w>}{t} = \frac{\bar{z}w}{t}$$

The formation of the exponential vector is purely algebraic. Viewed in $BO_t(\underline{C})$, we have

$$< \underline{\exp}(z), \underline{\exp}(w) >_t$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{<z,w>^n}{t^n}$$

$$= \exp(\frac{<z,w>}{t})$$

$$= e^{<z,w>_t}.$$

I.e.:

$$< \underline{\exp}(z), \underline{\exp}(w) >_t$$

$$= < \underline{\exp}(\frac{z}{\sqrt{t}}), \underline{\exp}(\frac{w}{\sqrt{t}}) >.$$

Let $d\gamma_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$ -- then there exists an isometric isomorphism

$$T_t : BO_t(\underline{C}) \to L^2(\underline{R}, \gamma_t)$$

characterized by the relation

$$(T_t \underline{\exp}(z))(x) = \exp(\frac{zx}{t} - \frac{1}{2t} z^2).$$

[Note: In terms of the Hermite polynomials,

$$\exp\left(\frac{zx}{t} - \frac{1}{2t} z^2\right)$$

$$= \exp\left(\frac{z}{\sqrt{t}} \frac{x}{\sqrt{t}} - \frac{1}{2} \left(\frac{z}{\sqrt{t}}\right)^2\right)$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{1}{(\sqrt{t})^n} H_n\left(\frac{x}{\sqrt{t}}\right).]$$

Let $\iota_t : \underline{C} \to \underline{C}_t$ be the isometric isomorphism defined by the rule

$$\iota_t z = \sqrt{t}\, z.$$

Let $U_t : L^2(\underline{R}, \gamma_t) \to L^2(\underline{R}, \gamma)$ be the isometric isomorphism defined by the rule

$$U_t \psi(x) = \psi(\sqrt{t}\, x).$$

Then the following diagram

$$\begin{array}{ccc} BO_t(\underline{C}) & \xrightarrow{T_t} & L^2(\underline{R}, \gamma_t) \\ \Gamma(\iota_t) \uparrow & & \downarrow U_t \\ BO(\underline{C}) & \xrightarrow[T]{} & L^2(\underline{R}, \gamma) \end{array}$$

is commutative. In fact,

$$U_t T_t \Gamma(\iota_t) \underline{\exp}(z) \Big|_x$$

$$= U_t T_t \, \underline{\exp}(\sqrt{t}\, z) \Big|_x$$

$$= T_t \, \underline{\exp}(\sqrt{t}\, z) \Big|_{\sqrt{t}\, x}$$

$$= \exp\left(\frac{(\sqrt{t}\, z)(\sqrt{t}\, x)}{t} - \frac{1}{2t}(\sqrt{t}\, z)^2\right)$$

$$= \exp(zx - \frac{1}{2} z^2)$$

$$= T \, \underline{\exp}(z) \Big|_x .$$

8.10 <u>REMARK</u> Everything that has been said above is valid with no essential change when $\underline{C}$ is replaced by $\underline{C}^n$. Thus the point of departure is the fact that there exists an isometric isomorphism

$$T:BO(\underline{C}^n) \to L^2(\underline{R}^n, \gamma^{\times n})$$

characterized by the relation

$$(T\, \underline{\exp}(z))(x) = \exp(\sum_{k=1}^{n} z_k x_k - \frac{1}{2} \sum_{k=1}^{n} z_k^2) \quad \text{(cf. 6.12)}.$$

One then computes that on, e.g., $\mathcal{S}(\underline{R}^n)$

$$\begin{cases} T\tilde{a}(z)T^{-1} = \sum_{k=1}^{n} \bar{z}_k \frac{\partial}{\partial x_k} \\ T\tilde{c}(z)T^{-1} = \sum_{k=1}^{n} z_k (x_k - \frac{\partial}{\partial x_k}). \end{cases}$$

And so forth.

§9. *WEYL OPERATORS*

Let H be a separable complex Hilbert space — then $\forall\ f \in H$, the field operator $Q(f)$ is essentially selfadjoint (cf. 7.18). Therefore $\overline{Q(f)}$ is selfadjoint, thus it makes sense to form

$$W(f) = \exp(\sqrt{-1}\ \overline{Q(f)}),$$

the <u>Weyl operator</u> attached to f.

[Note: $W(f)$ is a unitary operator on $BO(H)$, $W(0)$ being, in particular, the identity.]

9.1 <u>LEMMA</u> $\forall\ f,g \in H$,

$$W(f)W(g) = \exp\left(- \frac{\sqrt{-1}}{2} \operatorname{Im} < f,g > \right)W(f + g).$$

<u>PROOF</u> Let $X \in BO_F(H)$ — then X is an analytic vector for $\overline{Q(g)}$ (cf. 7.18), hence (cf. 2.34)

$$e^{\sqrt{-1}\ \overline{Q(g)}}X = \sum_{\ell=0}^{\infty} \frac{(\sqrt{-1}\ \overline{Q(g)})^{\ell}}{\ell!} X.$$

The estimates established in 7.18 imply that

$$\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{||\overline{Q(f)}^{k}\ \overline{Q(g)}^{\ell}X||}{k!\ell!} |t|^{k}|t|^{\ell}$$

is convergent for all t. But $\forall$ k,

$$e^{\sqrt{-1}\ \overline{Q(g)}}X \in \operatorname{Dom}(\overline{Q(f)}^{k}).$$

Therefore $e^{\sqrt{-1}\,\overline{Q(g)}}X$ is an analytic vector for $\overline{Q(f)}$ and

$$e^{\sqrt{-1}\,\overline{Q(f)}}\, e^{\sqrt{-1}\,\overline{Q(g)}}X = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(\sqrt{-1}\,\overline{Q(f)})^{k}\,(\sqrt{-1}\,\overline{Q(g)})^{\ell}}{k!\,\ell!}X.$$

I.e.:

$$W(f)W(g)X = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(\sqrt{-1}\,Q(f))^{k}\,(\sqrt{-1}\,Q(g))^{\ell}}{k!\,\ell!}X.$$

Recall now that on $BO_F(H)$,

$$[Q(f),Q(g)] = \sqrt{-1}\,\mathrm{Im} < f,g > \quad (\text{cf. } 7.22).$$

With this in mind, we can then write

$$W(f+g)X = e^{\sqrt{-1}\,\overline{Q(f+g)}}X$$

$$= \sum_{n=0}^{\infty} \frac{(\sqrt{-1}\,\overline{Q(f+g)})^{n}}{n!}X$$

$$= \sum_{n=0}^{\infty} \frac{(\sqrt{-1}\,Q(f+g))^{n}}{n!}X$$

$$= \sum_{n=0}^{\infty} (\sqrt{-1})^{n} \frac{(Q(f)+Q(g))^{n}}{n!}X$$

$$= \sum_{n=0}^{\infty} (\sqrt{-1})^{n} \sum_{k+\ell+2m=n} \frac{Q(f)^{k}}{k!}\frac{Q(g)^{\ell}}{\ell!}\frac{1}{m!}\left(-\frac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g > \right)^{m}X$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g > \right)^{m} \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(\sqrt{-1}\,Q(f))^{k}\,(\sqrt{-1}\,Q(g))^{\ell}}{k!\,\ell!}X$$

$$= \exp(\frac{\sqrt{-1}}{2} \operatorname{Im} < f,g >) \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(\sqrt{-1}\, Q(f))^k \, (\sqrt{-1}\, Q(g))^{\ell}}{k!\ell!} X$$

$$= \exp(\frac{\sqrt{-1}}{2} \operatorname{Im} < f,g >) W(f)W(g)X.$$

Here are two corollaries:

- $$\begin{cases} W(f + g) = \exp(\frac{\sqrt{-1}}{2} \operatorname{Im} < f,g >)W(f)W(g) \\ W(g + f) = \exp(\frac{\sqrt{-1}}{2} \operatorname{Im} < g,f >)W(g)W(f) \end{cases}$$

=>

$$W(f)W(g) = \exp(- \sqrt{-1} \operatorname{Im} < f,g >)W(g)W(f).$$

- $W(f)W(- f) = W(0) = 1 \ (\equiv I)$

=>

$$W(f)^* = W(- f).$$

9.2 LEMMA The arrow

$$\begin{cases} H \rightarrow U(BO(H)) \\ f \rightarrow W(f) \end{cases}$$

is continuous.

PROOF The claim is that $\forall\ X \in BO(H)$, the arrow

$$\begin{cases} H \rightarrow BO(H) \\ f \rightarrow W(f)X \end{cases}$$

is continuous. And for this, it suffices to take $X \in BO_F(H)$, there being no loss of generality in assuming that $X \in BO_n(H)$, say $X = X_n$. But then

$$||(W(f) - 1)X_n|| = ||\sum_{k=1}^{\infty} \frac{(\sqrt{-1}\ \overline{Q(f)})^k}{k!} X_n||$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k!} ||Q(f)^k X_n||$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k!} 2^{k/2} \left[\frac{(n+k)!}{n!}\right]^{1/2} ||f||^k ||X_n||$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k!} 2^{k/2} \left[\frac{(n+k)!}{n!}\right]^{1/2} ||f||\ ||X_n||$$

provided $||f|| \leq 1$. Therefore

$$||(W(f) - 1)X_n|| \to 0$$

as $f \to 0$. To treat the general case, note that

$$||(W(f) - W(g))X_n||$$

$$= ||W(g)(W(-g)W(f) - 1)X_n||$$

$$\leq ||(W(-g)W(f) - 1)X_n||$$

$$= ||(\exp(-\frac{\sqrt{-1}}{2} \operatorname{Im} < -g,f >)W(-g+f) - 1)X_n||$$

$$= ||(\exp(-\frac{\sqrt{-1}}{2} \operatorname{Im} < f,g >)W(f-g) - 1)X_n||$$

$$= \|\exp(-\tfrac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g >)W(f - g)$$

$$- \exp(-\tfrac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g >)\exp(\tfrac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g >)X_n\|$$

$$= \|W(f - g) - \exp(\tfrac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g >)X_n\|$$

$$= \|(W(f - g) - 1)X_n$$

$$- (\exp(\tfrac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g >) - 1)X_n\|$$

$$\leq \|(W(f - g) - 1)X_n\|$$

$$+ \|(\exp(\tfrac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g >) - 1)X_n\|$$

$$\leq \|(W(f - g) - 1)X_n\|$$

$$+ |\exp(\tfrac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g >) - 1|\ \|X_n\|.$$

If $f \to g$, then $f - g \to 0$, hence by the above,

$$\|(W(f - g) - 1)X_n\| \to 0.$$

And, of course,

$$f \to g \Rightarrow \mathrm{Im} < f,g > \to \mathrm{Im} < g,g > = 0$$

$$\Rightarrow$$

$$|\exp(\tfrac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g >) - 1|\ \|X_n\| \to 0.$$

9.3 <u>REMARK</u> It is false that

$$f \to 0 \Rightarrow ||W(f) - 1|| \to 0.$$

Thus fix $f \neq 0$ and in the relation

$$W(g))^*W(f)W(g) = \exp(- \sqrt{-1} \operatorname{Im} < f,g >)W(f)$$

take $g = \sqrt{-1}\ \theta f\ ||f||^2$ to get

$$\sigma(W(f)) = e^{- \sqrt{-1}\ \theta}\sigma(W(f)).$$

Since θ is arbitrary, this implies that the spectrum of $W(f)$ is invariant under rotations, hence is the entire unit circle. But according to the spectral radius formula,

$$\lim_{n \to \infty} ||(W(f) - 1)^n||^{1/n}$$

is equal to the maximum distance from 1 to the points of $\sigma(W(f))$ which, in the case at hand, is 2. On the other hand, $W(f) - 1$ is normal, so

$$\begin{aligned} ||(W(f) - 1)^{2^n}||^2 &= ||(W(f)^* - 1)^{2^n}(W(f) - 1)^{2^n}|| \\ &= ||((W(f)^* - 1)(W(f) - 1))^{2^n}|| \\ &= ||((W(f)^* - 1)(W(f) - 1))^{2^{n-1}}||^2 \\ &= \cdots \\ &= ||(W(f)^* - 1)(W(f) - 1)||^{2^n} \\ &= ||W(f) - 1||^{2^{n+1}}. \end{aligned}$$

Therefore

$$2 = \lim_{n \to \infty} ||(W(f) - 1)^{2^n}||^{1/2^n}$$

$$= \lim_{n \to \infty} (||W(f) - 1||^{2^n})^{1/2^n}$$

$$= ||W(f) - 1||.$$

9.4 LEMMA We have

$$W(f)\underline{\exp}(g)$$

$$= \exp(- \frac{1}{4} ||f||^2 + \frac{\sqrt{-1}}{\sqrt{2}} < f,g >)\underline{\exp}(\frac{\sqrt{-1}}{\sqrt{2}} f + g).$$

PROOF Observe first that on the set of exponential vectors, the series defining

$$\begin{cases} e^{\tilde{a}(f)} \\ e^{\tilde{c}(f)} \end{cases}$$

are strongly convergent and

$$\begin{cases} e^{\tilde{a}(f)}\underline{\exp}(g) = e^{<f,g>}\underline{\exp}(g) \\ e^{\tilde{c}(f)}\underline{\exp}(g) = \underline{\exp}(f + g). \end{cases} \qquad \text{(cf. 7.15)}$$

Next, on purely formal grounds,

$$e^{A + B} = e^A e^B e^{- \frac{1}{2} [A,B]}$$

if the operators A and B satisfy

$$\begin{cases} [A,[A,B]] = 0 \\ [B,[A,B]] = 0. \end{cases}$$

This said, take

$$\begin{cases} A = \frac{\sqrt{-1}}{\sqrt{2}}\tilde{a}(f) \\ B = \frac{\sqrt{-1}}{\sqrt{2}}\tilde{c}(f). \end{cases}$$

Since

$$[\tilde{a}(f),\tilde{c}(f)] = < f,f >,$$

the identity is applicable on the exponential domain, where then W(f) admits the factorization

$$W(f) = \exp\left(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{a}(f) + \frac{\sqrt{-1}}{\sqrt{2}}\tilde{c}(f)\right)$$

$$= \exp\left(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{a}(f)\right)\exp\left(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{c}(f)\right)\exp\left(-\frac{1}{2}\left[\frac{\sqrt{-1}}{\sqrt{2}}\tilde{a}(f), \frac{\sqrt{-1}}{\sqrt{2}}\tilde{c}(f)\right]\right)$$

$$= \exp\left(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{a}(f)\right)\exp\left(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{c}(f)\right)\exp\left(\left(-\frac{1}{2}\right)\left(\frac{\sqrt{-1}}{\sqrt{2}}\right)^2[\tilde{a}(f),\tilde{c}(f)]\right)$$

$$= \exp\left(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{a}(f)\right)\exp\left(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{c}(f)\right)\exp\left(\frac{1}{4}||f||^2\right).$$

Therefore

$$W(f)\underline{\exp}(g)$$

$$= \exp(\frac{1}{4}||f||^2)\exp(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{a}(f))\exp(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{c}(f))\underline{\exp}(g)$$

$$= \exp(\frac{1}{4}||f||^2)\exp(\frac{\sqrt{-1}}{\sqrt{2}}\tilde{a}(f))\underline{\exp}(\frac{\sqrt{-1}}{\sqrt{2}}f + g)$$

$$= \exp(\frac{1}{4}||f||^2)\exp(\tilde{a}(-\frac{\sqrt{-1}}{\sqrt{2}}f))\underline{\exp}(\frac{\sqrt{-1}}{\sqrt{2}}f + g)$$

$$= \exp(\frac{1}{4}||f||^2)\exp(< -\frac{\sqrt{-1}}{\sqrt{2}}f, \frac{\sqrt{-1}}{\sqrt{2}}f + g >)\underline{\exp}(\frac{\sqrt{-1}}{\sqrt{2}}f + g)$$

$$= \exp(\frac{1}{4}||f||^2)\exp(-\frac{1}{2}||f||^2)\exp(\frac{\sqrt{-1}}{\sqrt{2}} < f,g >)\underline{\exp}(\frac{\sqrt{-1}}{\sqrt{2}}f + g)$$

$$= \exp(-\frac{1}{4}||f||^2 + \frac{\sqrt{-1}}{\sqrt{2}} < f,g >)\underline{\exp}(\frac{\sqrt{-1}}{\sqrt{2}}f + g).$$

9.5 EXAMPLE Take $g = 0$ -- then $\underline{\exp}(0) = \Omega$, hence

$$< \Omega, W(f)\Omega > = e^{-\frac{1}{4}||f||^2}.$$

[Note: Here is a direct approach. Thus, working through the definitions, one finds that

$$< \Omega, Q(f)^{2k+1}\Omega > = 0$$

and

$$< \Omega, Q(f)^{2k}\Omega > = \frac{(2k)!}{k!2^{2k}}||f||^{2k}.$$

Consequently,

$$<\Omega, W(f)\Omega> = \sum_{k=0}^{\infty} \frac{(\sqrt{-1})^k}{k!} <\Omega, Q(f)^k \Omega>$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{(2k)!}{k!2^{2k}} ||f||^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (-\frac{1}{4}||f||^2)^k$$

$$= e^{-\frac{1}{4}||f||^2} .]$$

Suppose that K is a complex Hilbert space. Let T be a set of bounded linear operators on K — then a vector $\zeta \in K$ is a cyclic vector for T if the set $\{T\zeta\}$, where T is in the algebra generated by T, is dense in K.

9.6 LEMMA Ω is a cyclic vector for the set $\{W(f) : f \in H\}$.

PROOF Indeed,

$$W(f)\Omega = W(f)\underline{\exp}(0)$$

$$= \exp(-\frac{1}{4}||f||^2)\underline{\exp}(\frac{\sqrt{-1}}{\sqrt{2}} f)$$

and the set of exponential vectors is total in $BO(H)$ (cf. 6.9).

9.7 LEMMA $\forall\ U \in U(H)$,

$$\Gamma(U)W(f)\Gamma(U)^{-1} = W(Uf).$$

PROOF Thanks to 7.25,

$$\Gamma(U)\overline{Q(f)}\Gamma(U)^{-1} = \overline{Q(Uf)},$$

so

$$\Gamma(U)W(f)\Gamma(U)^{-1}$$

$$= \Gamma(U)\exp(\sqrt{-1}\ \overline{Q(f)})\Gamma(U)^{-1}$$

$$= \exp(\sqrt{-1}\ \Gamma(U)\overline{Q(f)}\Gamma(U)^{-1})$$

$$= \exp(\sqrt{-1}\ \overline{Q(Uf)})$$

$$= W(Uf).$$

9.8 EXAMPLE Take

$$U = e^{\sqrt{-1}\ tI}.$$

Then

$$\Gamma(e^{\sqrt{-1}\ tI}) = e^{\sqrt{-1}\ td\Gamma(I)} = e^{\sqrt{-1}\ tN} \quad \text{(cf. 6.17)}$$

=>

$$e^{\sqrt{-1}\ tN}W(f)e^{-\sqrt{-1}\ tN} = W(e^{\sqrt{-1}\ tI}f) = W(e^{\sqrt{-1}\ t}f).$$

[Note: On $BO_F(H)$,

$$NW(f) = \frac{1}{\sqrt{-1}}\frac{d}{dt}\, e^{\sqrt{-1}\ tN}W(f)\Big|_{t=0}$$

$$= \frac{1}{\sqrt{-1}}\frac{d}{dt}\, W(e^{\sqrt{-1}\ t}f)e^{\sqrt{-1}\ tN}\Big|_{t=0}$$

$$= W(f)N + \frac{1}{\sqrt{-1}} \frac{d}{dt} \sum_{k=0}^{\infty} \frac{(\sqrt{-1}\, Q(e^{\sqrt{-1}\, t} f))^k}{k!} \Bigg|_{t=0}$$

$$= W(f)N + W(f)(P(f) + ||f||^2/2).]$$

Suppose that K is a complex Hilbert space. Let T be a set of bounded linear operators on K which is closed under the formation of adjoints (i.e., $T \in T \Rightarrow T^* \in T$) -- then T is said to be irreducible if it leaves no nontrivial closed linear subspace invariant.

9.9 SCHUR'S LEMMA T is irreducible iff the only bounded linear operators which commute with each $T \in T$ are the scalar multiples of the identity.

[Note: Suppose that T is irreducible and dim $K > 1$. Fix a nonzero $\zeta \in K$ -- then the set $\{T\zeta : T \in T\}$ is dense in K.]

9.10 SEGAL'S CRITERION Assume:

1. $\exists$ a nonnegative selfadjoint operator A on K such that

$$e^{\sqrt{-1}\, tA} T e^{-\sqrt{-1}\, tA} \subset T \ \forall\ t.$$

2. $\exists$ a nonzero vector $\zeta \in K$ (unique up to a multiplicative constant) which is annihilated by A.

Then T is irreducible provided ζ is cyclic for T.

[One can suppose from the outset that T is an algebra, hence that $T\zeta$ is dense in K. Let P denote the orthogonal projection of K onto a T-invariant subspace, so

$$T \in T \Rightarrow PT = TP$$

$$\Rightarrow \langle \zeta, PT\zeta \rangle = \langle \zeta, TP\zeta \rangle.$$

Since $e^{\sqrt{-1}\,tA}\zeta = \zeta$ (cf. 2.34) and

$$T \in \mathcal{T} \Rightarrow e^{\sqrt{-1}\,tA} T e^{-\sqrt{-1}\,tA} \in \mathcal{T},$$

for all $T \in \mathcal{T}$, we have

$$\langle \zeta, Pe^{\sqrt{-1}\,tA} T\zeta \rangle = \langle \zeta, Te^{-\sqrt{-1}\,tA} P\zeta \rangle \quad (t \in \underline{R}).$$

But, in view of the nonnegativity of A, the LHS of this equation can be extended to a bounded holomorphic function in the upper halfplane, while the RHS of this equation can be extended to a bounded holomorphic function in the lower halfplane. Therefore

$$\langle \zeta, Pe^{\sqrt{-1}\,tA} T\zeta \rangle$$

is independent of t. Because $\mathcal{T}\zeta$ is dense in $\mathcal{K}$, it follows that $\forall$ t,

$$e^{\sqrt{-1}\,tA} P\zeta = P\zeta.$$

This, however, implies that $P\zeta \in \text{Dom}(A)$ with

$$AP\zeta = 0.$$

Accordingly, $P\zeta = c\zeta$ for some $c \in \underline{C}$, thus $\forall\, x \in \mathcal{K}$,

$$\langle x, PT\zeta \rangle = \langle x, TP\zeta \rangle = \langle x, Tc\zeta \rangle = c \langle x, T\zeta \rangle$$

$$\Rightarrow$$

$$\langle Px, T\zeta \rangle = c \langle x, T\zeta \rangle$$

$$\Rightarrow$$

$$\langle Px, y \rangle = c \langle x, y \rangle \ \forall\, y \in \mathcal{K}$$

=>

$$Px = \bar{c}x$$

=>

$$P = 0 \text{ or } 1.]$$

9.11 LEMMA The set $\{W(f):f \in H\}$ is irreducible.

PROOF It is a matter of applying Segal's criterion, taking $T = \{W(f):f \in H\}$ (legitimate, since $W(f)^* = W(-f)$). To verify conditions 1 and 2, let $K = BO(H)$, $A = d\Gamma(I)$ (a.k.a. N), and $\zeta = \Omega$ -- then one has only to quote 9.6 and 9.8.

9.12 REMARK Fix an orthonormal basis $\{e_n\}$ for H -- then the set

$$\{W(te_n), W(\sqrt{-1}\, te_n):n = 1,2,\ldots, t \in \underline{R}\}$$

is irreducible.

[Note: Let E be the linear span of the e_n -- then the set

$$\{W(f)\Omega:f \in E\}$$

is dense in $BO(H)$ (cf. 9.9).]

9.13 LEMMA Let T be a bounded linear operator on $BO(H)$. Assume: T commutes with all the $\overline{Q(f)}$ $(f \in H)$ -- then T is a scalar multiple of the identity.

PROOF On the basis of 4.4 and 4.9, $\forall\, f \in H$,

$$T \exp(\sqrt{-1}\, t\, \overline{Q(f)}) = \exp(\sqrt{-1}\, t\, \overline{Q(f)})T \quad (t \in \underline{R}).$$

One can therefore apply 9.9 and 9.11.

9.14 LEMMA Suppose that $H = H_1 \oplus H_2$ -- then

$$TW(f_1 \oplus f_2) = W(f_1) \otimes W(f_2).$$

[Note:

$$T: BO(H) \to BO(H_1) \hat{\otimes} BO(H_2)$$

is the isometric isomorphism per 6.11.]

9.15 EXAMPLE Take $H = \underline{C}$ -- then, in the notation of §8,

$$TW(z)T^{-1} \quad (z \in \underline{C})$$

is a unitary operator on $L^2(\underline{R},\gamma)$. Explicated, let $z = a + \sqrt{-1}\, b$ and put

$$W_T(z) = TW(z)T^{-1}.$$

Then

$$W_T(z)\psi\Big|_x$$

$$= \exp(\sqrt{-1}\,(\frac{xa}{\sqrt{2}} + \frac{ab}{2}))\exp(-\frac{xb}{\sqrt{2}} - \frac{b^2}{2})\psi(x + \sqrt{2}\, b).$$

To confirm unitarity, write

$$< W_T(z)\psi, W_T(z)\psi' >$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \bar{\psi}(x + \sqrt{2}\, b)\psi'(x + \sqrt{2}\, b)\exp(-\sqrt{2}\, xb - b^2)e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \bar{\psi}(x + \sqrt{2}\, b)\psi'(x + \sqrt{2}\, b) e^{-(x + \sqrt{2}\, b)^2/2}\, dx$$

$$= \langle \psi, \psi' \rangle.$$

Here is another check on the work. From the definitions,

$$T\Omega = T\,\underline{\exp}(0) = 1.$$

But for any complex number $u + \sqrt{-1}\, v$,

$$\frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \exp((u + \sqrt{-1}\, v)x) e^{-x^2/2}\, dx$$

$$= \exp((u + \sqrt{-1}\, v)^2/2).$$

Therefore

$$\langle 1, W_T(z) 1 \rangle$$

$$= \exp\left(-\frac{b^2}{2} + \sqrt{-1}\,\frac{ab}{2}\right)\exp\left(\left(-\frac{b}{\sqrt{2}} + \sqrt{-1}\,\frac{a}{\sqrt{2}}\right)^2/2\right)$$

$$= \exp\left(-\frac{b^2}{2} + \sqrt{-1}\,\frac{ab}{2}\right)\exp\left(\frac{b^2}{4} - \sqrt{-1}\,\frac{ab}{2} - \frac{a^2}{4}\right)$$

$$= \exp\left(-\frac{1}{4}(a^2 + b^2)\right)$$

$$= \exp\left(-\frac{1}{4}\,|z|^2\right),$$

as predicted by 9.5. In practice, it is more convenient to deal with

$$\underline{W}_T(a,b) = TW\left(\sqrt{2}\, b, -\frac{a}{\sqrt{2}}\right)T^{-1}.$$

For later reference, note that

$$\underline{W}_T(a,b)\underline{W}_T(a',b')$$

$$= TW(\sqrt{2}\ b,\ -\frac{a}{\sqrt{2}})T^{-1}TW(\sqrt{2}\ b',\ -\frac{a'}{\sqrt{2}})T^{-1}$$

$$= TW(\sqrt{2}\ b,\ -\frac{a}{\sqrt{2}})W(\sqrt{2}\ b',\ -\frac{a'}{\sqrt{2}})T^{-1}$$

$$= \exp(\ -\frac{\sqrt{-1}}{2}\ \text{Im} < \sqrt{2}\ b - \sqrt{-1}\ \frac{a}{\sqrt{2}},\ \sqrt{2}\ b' - \sqrt{-1}\ \frac{a'}{\sqrt{2}} >)$$

$$\times\ TW(\sqrt{2}\ (b + b'),\ -\frac{(a + a')}{\sqrt{2}}\ T^{-1}$$

$$= \exp(\ -\frac{\sqrt{-1}}{2}\ \text{Im} < a + \sqrt{-1}\ b,\ a' + \sqrt{-1}\ b' >)\underline{W}_T(a + a',b + b').$$

Now let $\psi \in L^2(\underline{R},\gamma)$ -- then

$$\underline{W}_T(a,b)\psi\Big|_x$$

$$= \exp(\sqrt{-1}\ (xb - ab/2))[\exp(xa - a^2/2)]^{1/2}\psi(x - a).$$

Using the isometric isomorphism

$$T_G:L^2(R,\gamma) \rightarrow L^2(R) \quad (\text{cf. } §8),$$

these considerations can be transferred from $L^2(\underline{R},\gamma)$ to $L^2(\underline{R})$. So, $\forall\ \psi \in L^2(\underline{R})$,

$$T_G\underline{W}_T(a,b)T_G^{-1}\psi\Big|_x$$

$$= T_G \underline{W}_T(a,b)\ \left(\frac{\psi}{G}\right)\Big|_x$$

$$= \frac{1}{(2\pi)^{1/4}} \exp\left(-\frac{x^2}{4}\right) \exp(\sqrt{-1}\ (xb - ab/2))$$

$$\times\ [\exp(xa - a^2/2)]^{1/2} (2\pi)^{1/4} \exp\left(\frac{(x-a)^2}{4}\right) \psi(x-a)$$

$$= \exp(\sqrt{-1}\ (xb - ab/2))\psi(x-a).$$

§10. *WEYL SYSTEMS*

Let $E \neq 0$ be a real linear space equipped with a bilinear form σ -- then the pair (E,σ) is a symplectic vector space if σ is antisymmetric and nondegenerate (so either $\dim E = \infty$ or $\dim E = 2n$ $(n = 1,2,\ldots)$).

10.1 EXAMPLE Take for E a complex pre-Hilbert space, view E as a real linear space via restriction of scalars, and let

$$\sigma(f,g) = \text{Im} < f,g >.$$

A symplectic vector space (E,σ) is topological if E is a real topological vector space and σ is continuous.

10.2 EXAMPLE Let M and N be real topological vector spaces. Suppose that

$$B:M \times N \to \underline{R}$$

is a continuous nondegenerate bilinear form. Take $E = M \oplus N$ and let

$$\sigma((x,\lambda),(x',\lambda')) = B(x,\lambda') - B(x',\lambda).$$

Then the symplectic vector space (E,σ) is topological.

Let (E,σ) be a symplectic topological vector space. Suppose that K is a complex Hilbert space -- then a map

$$W:E \to U(K)$$

is said to satisfy the Weyl relations if $\forall$ $f,g \in E$:

$$W(f)W(g) = \exp(-\frac{\sqrt{-1}}{2}\sigma(f,g))W(f+g).$$

So, $\forall\ f \in E$ and $\forall\ t_1, t_2 \in \underline{R}$,

$$\begin{aligned} &W(t_1 f)W(t_2 f) \\ &\qquad = \exp(-\frac{\sqrt{-1}}{2} t_1 t_2 \sigma(f,f))W((t_1 + t_2)f) \\ &\qquad = W((t_1 + t_2)f). \end{aligned}$$

I.e.: The arrow

$$\begin{cases} \underline{R} \to \mathcal{U}(\mathcal{K}) \\ t \to W(tf) \end{cases}$$

is a homomorphism. One then says that the pair $(\mathcal{K},W)$ is a Weyl system over (E,σ) if, in addition, $\forall\ f \in E$, the arrow

$$\begin{cases} \underline{R} \to \mathcal{U}(\mathcal{K}) \\ t \to W(tf) \end{cases}$$

is continuous. Accordingly, when this is the case, $\{W(tf) : t \in \underline{R}\}$ is a one parameter unitary group, hence admits a generator $\Phi(f)$ (which, of course, is selfadjoint).

[Note: Unless stipulated to the contrary, a Weyl system over a complex pre-Hilbert space is a Weyl system over the underlying real topological vector space with $\sigma = \text{Im} < , >$.]

10.3 EXAMPLE (The Fock System) Take for E a separable complex Hilbert space $\mathcal{H}$ and let $K = \text{BO}(\mathcal{H})$ -- then the map

$$W:\mathcal{H} \to \mathcal{U}(\text{BO}(\mathcal{H}))$$

which sends $f \in \mathcal{H}$ to the Weyl operator

$$W(f) = \exp(\sqrt{-1}\ \overline{Q(f)})$$

is a Weyl system over $\mathcal{H}$ (cf. 9.1 and 9.2).

10.4 <u>EXAMPLE</u> (The Schrödinger System) The real topological vector space underlying $\underline{C}^n$ is $\underline{R}^{2n}$. Take $K = L^2(\underline{R}^n)$ and given $z = a + \sqrt{-1}\ b$ $(a,b \in \underline{R}^n)$, define a unitary operator $W(z)$ by

$$W(z)\psi\Big|_x$$

$$= \exp(\sqrt{-1}\ (< x,b > - < a,b > /2))\psi(x - a).$$

Then W is a Weyl system over $\underline{C}^n$ (cf. 9.15) which, moreover, is irreducible (cf. 9.11).

10.5 <u>CONSTRUCTION</u> Let M and N be real topological vector spaces. Suppose that

$$B:M \times N \to \underline{R}$$

is a continuous nondegenerate bilinear form. Let U and V be unitary representations of the additive groups of M and N respectively on a Hilbert space K such that

$$U(x)V(\lambda) = \exp(\sqrt{-1}\ B(x,\lambda))V(\lambda)U(x)$$

for all $x \in M, \lambda \in N$. Put

$$W(x \oplus \lambda) = \exp(\frac{\sqrt{-1}}{2} B(x,\lambda))U(-x)V(\lambda).$$

Then W defines a Weyl system over $E = M \oplus N$ (with σ per B as in 10.2). In fact,

$$W(x \oplus \lambda)W(x' \oplus \lambda')$$

$$= \exp(\frac{\sqrt{-1}}{2} B(x,\lambda))\exp(\frac{\sqrt{-1}}{2} B(x',\lambda'))$$

$$\cdot U(-x)V(\lambda)U(-x')V(\lambda')$$

$$= \exp(\frac{\sqrt{-1}}{2} (B(x,\lambda) + B(x',\lambda')))\exp(\sqrt{-1}\, B(x',\lambda))$$

$$\cdot U(-x-x')V(\lambda+\lambda').$$

On the other hand,

$$\exp(-\frac{\sqrt{-1}}{2} (B(x,\lambda') - B(x',\lambda)))W((x+x') \oplus (y+y'))$$

$$= \exp(-\frac{\sqrt{-1}}{2} (B(x,\lambda') - B(x',\lambda)))$$

$$\cdot \exp(\frac{\sqrt{-1}}{2} B(x+x',\lambda+\lambda'))U(-x-x')V(\lambda+\lambda').$$

And

$$-\frac{1}{2} B(x,\lambda') + \frac{1}{2} B(x',\lambda)$$

$$+ \frac{1}{2} (B(x,\lambda) + B(x,\lambda') + B(x',\lambda) + B(x',\lambda'))$$

$$= \frac{1}{2} (B(x,\lambda) + B(x',\lambda')) + B(x',\lambda).$$

10.6 EXAMPLE Take $M = \underline{R}^n$, $N = \underline{R}^n$, and let $B(x,\lambda) = < x,\lambda >$ be the usual inner product. Change the notation and replace x by a, λ by b. Take $K = L^2(\underline{R}^n)$ -- then the assignments

$$\begin{cases} a \to U(a) \\ b \to V(b), \end{cases}$$

where

$$\begin{cases} U(a)\psi(x) = \psi(x + a) \\ V(b)\psi(x) = e^{\sqrt{-1} < x,b >}\psi(x) \end{cases}$$

define unitary representations of $\underline{R}^n$ on $L^2(\underline{R}^n)$. Therefore the prescription

$$W(a,b) = \exp(\frac{\sqrt{-1}}{2} < a,b >)U(-a)V(b)$$

defines a Weyl system over $\underline{R}^{2n} = \underline{R}^n \oplus \underline{R}^n$.

[Note: With $z = a + \sqrt{-1}\, b$ and $W(z) = W(a,b)$, it follows that

$$W(z)\psi\Big|_x$$

$$= \exp(\frac{\sqrt{-1}}{2} < a,b >)\exp(\sqrt{-1} < x - a,b >)\psi(x - a)$$

$$= \exp(\sqrt{-1}\ (< x,b > - < a,b > /2))\psi(x - a).$$

The procedure thus recovers the Schrödinger system.]

10.7 LEMMA Let (K,W) be a Weyl system over (E,σ) — then the restriction of W to each finite dimensional subspace of E is continuous.

PROOF If $f_1,\ldots,f_n$ are elements of E, then

$$W(f_1)\cdots W(f_n) = \exp\left(-\frac{\sqrt{-1}}{2}\sum_{j<k}\sigma(f_j,f_k)\right)W(f_1+\cdots+f_n).$$

[Note: It is not necessarily true that $W:E \rightarrow \mathcal{U}(K)$ is continuous (cf. 10.14).]

10.8 LEMMA Let H be a separable complex Hilbert space,

$$W:H \rightarrow \mathcal{U}(BO(H))$$

the Fock system over H. Fix a real linear function $\Lambda:H \rightarrow \underline{R}$ and put

$$W_\Lambda(f) = e^{\sqrt{-1}\,\Lambda(f)}W(f).$$

Then W_Λ is a Weyl system over H. In addition, W_Λ is unitarily equivalent to W iff Λ is continuous.

[Suppose that W_Λ is unitarily equivalent to W -- then

$$f \rightarrow 0 \Rightarrow e^{\sqrt{-1}\,\Lambda(f)}X \rightarrow X \quad (X \in BO(H)).$$

If Λ were not continuous, then Ker Λ would be dense. Fix $X_0:\Lambda(X_0) = \pi$ and choose $X_n \in X_0 + \text{Ker}\,\Lambda:X_n \rightarrow 0$, thus

$$e^{\sqrt{-1}\,\Lambda(X_n)}X_0 \rightarrow e^{\sqrt{-1}\,\Lambda(X_0)}X_0 = -X_0,$$

a contradiction. To discuss the converse, write $\Lambda(f) = \text{Re} < f,x_\Lambda > (x_\Lambda \in H)$ and proceed as in 10.11.]

10.9 EXAMPLE In the context of 10.8, take H infinite dimensional, fix an orthonormal basis $\{e_n\}$ for H, and let H_0 be the linear span of the e_n (thus H_0 is a pre-Hilbert space). Suppose that

$$W':H \to U(BO(H))$$

is a Weyl system over H such that $W'|H_0 = W|H_0$ -- then $\exists$ a real linear function $\Lambda:H \to \underline{R}$ such that $W' = W_\Lambda$ with $\Lambda(H_0) = \{0\}$. First, in view of the Weyl relations,

$$(W'(f)W(f)^{-1})W(f_0) = W(f_0)(W'(f)W(f)^{-1}) \quad (f \in H, f_0 \in H_0).$$

But the set $\{W(f_0):f_0 \in H_0\}$ is irreducible (cf. 9.12), so $W'(f)W(f)^{-1}$ is a scalar multiple of the identity (cf. 9.9), hence $\exists$ a complex number $\chi(f)$ of modulus 1 such that

$$W'(f) = \chi(f)W(f) \quad (f \in H).$$

Since

$$\chi(f_1 + f_2) = \chi(f_1)\chi(f_2)$$

and since the arrow

$$\begin{cases} \underline{R} \to \underline{C} \\ t \to \chi(tf) \end{cases}$$

is continuous, there exists a unique real number $\chi(f)$:

$$\chi(tf) = e^{\sqrt{-1}\, t\Lambda(f)}.$$

As a function from H to $\underline{R}$, Λ is real linear. And: $W' = W_\Lambda$ with $\Lambda(H_0) = \{0\}$.

[Note: If $\Lambda \not\equiv 0$, then Λ is discontinuous.]

10.10 REMARK To construct a real linear function $\Lambda: H \to \underline{R}$ such that $\Lambda(H_0) = \{0\}$, enlarge $\{e_n\}$ to a Hamel basis $\{e_n\} \cup \{e_i\}$. Assign to each index i two real numbers a_i and b_i. Put $\Lambda(e_i) = a_i$, $\Lambda(\sqrt{-1}\, e_i) = b_i$. Finally, extend Λ to all of H by real linearity and the condition that $\Lambda(H_0) = \{0\}$.

10.11 EXAMPLE Fix a real linear function $\Lambda: H \to \underline{R}$ such that $\Lambda(H_0) = \{0\}$ -- then W_Λ is irreducible (cf. 9.12) but W_Λ is not unitarily equivalent to W if $\Lambda \not\equiv 0$ (cf. 10.8 or 10.12). Nevertheless, for any finite dimensional subspace $F \subset H$, the restriction $W_\Lambda|F$ is unitarily equivalent to the restriction $W|F$. In fact, let $x_{\Lambda,F}$ be the unique element of F such that

$$\Lambda(f) = \text{Re} < f, x_{\Lambda,F} > \quad (f \in F).$$

Then $\forall f \in F$,

$$W_\Lambda(f) = W(\sqrt{-1}\, x_{\Lambda,F})W(f)W(-\sqrt{-1}\, x_{\Lambda,F}).$$

Proof: We have

$$W(\sqrt{-1}\, x_{\Lambda,F})W(f)W(-\sqrt{-1}\, x_{\Lambda,F})$$

$$= W(\sqrt{-1}\, x_{\Lambda,F})\exp(-\frac{\sqrt{-1}}{2}\text{Im} < f, -\sqrt{-1}\, x_{\Lambda,F} >)W(f - \sqrt{-1}\, x_{\Lambda,F})$$

$$= \exp(\frac{\sqrt{-1}}{2}\text{Im} < f, \sqrt{-1}\, x_{\Lambda,F} >)\exp(-\frac{\sqrt{-1}}{2}\text{Im} < \sqrt{-1}\, x_{\Lambda,F}, f - \sqrt{-1}\, x_{\Lambda,F} >)W(f)$$

$$= \exp(\frac{\sqrt{-1}}{2}\text{Im}(< f, \sqrt{-1}\, x_{\Lambda,F} > - < \sqrt{-1}\, x_{\Lambda,F}, f >))W(f)$$

$$= \exp(\frac{\sqrt{-1}}{2}\text{Im}(< f, \sqrt{-1}\, x_{\Lambda,F} > - \overline{< f, \sqrt{-1}\, x_{\Lambda,F} >}))W(f)$$

$$= \exp(\frac{\sqrt{-1}}{2}\, 2 \text{ Re} < f, x_{\Lambda,F} >)W(f)$$

$$= \exp(\sqrt{-1} \text{ Re} < f, x_{\Lambda,F} >)W(f)$$

$$= e^{\sqrt{-1}\, \Lambda(f)} W(f)$$

$$= W_{\Lambda}(f).$$

10.12 REMARK Let $\Lambda_1, \Lambda_2 : H \to \underline{R}$ be real linear functions such that $\Lambda_1(H_0) = \{0\}$, $\Lambda_2(H_0) = \{0\}$ -- then W_{Λ_1} is unitarily equivalent to W_{Λ_2} iff $\Lambda_1 = \Lambda_2$.

[For suppose $\exists$ a unitary $U:BO(H) \to BO(H)$ such that

$$UW_{\Lambda_1}(f)U^{-1} = W_{\Lambda_2}(f) \quad (f \in H).$$

Then $\forall$ $f_0 \in H_0$,

$$UW_{\Lambda_1}(f_0)U^{-1} = W_{\Lambda_2}(f_0)$$

or still,

$$UW(f_0)U^{-1} = W(f_0).$$

Therefore U is a scalar multiple of the identity (cf. 10.9), hence $W_{\Lambda_1} = W_{\Lambda_2} \Rightarrow \Lambda_1 = \Lambda_2$.]

10.13 LEMMA Let H be a complex Hilbert space. Suppose that H_0 is a dense linear subspace of $\mathcal{H}$ and let

$$W_0:\mathcal{H}_0 \to \mathcal{U}(\mathcal{K})$$

be a Weyl system over $\mathcal{H}_0$. Assume: W_0 is continuous — then W_0 has a unique continuous extension to a Weyl system $W:\mathcal{H} \to \mathcal{U}(\mathcal{K})$.

10.14 <u>EXAMPLE</u> In the setting of 10.11, if $\Lambda \not\equiv 0$, then W_Λ, as a map from $\mathcal{H}$ to $\mathcal{U}(\mathrm{BO}(\mathcal{H}))$ is not continuous. For if it were, then the fact that $W_\Lambda|\mathcal{H}_0 = W|\mathcal{H}_0$ would, in view of 10.13, imply that $W_\Lambda = W$.

Let

$$W:E \to \mathcal{U}(\mathcal{K})$$

be a Weyl system over (E,σ) -- then a selfadjoint operator N on $\mathcal{K}$ is a <u>number operator</u> for W if $\forall\ t \in \underline{R}$:

$$e^{\sqrt{-1}\ tN}W(f)e^{-\sqrt{-1}\ tN} = W(e^{\sqrt{-1}\ t}f) \quad (f \in E).$$

10.15 <u>EXAMPLE</u> Let $\mathcal{H}$ be a separable complex Hilbert space. Consider the Fock system

$$W:\mathcal{H} \to \mathcal{U}(\mathrm{BO}(\mathcal{H})).$$

Then $d\Gamma(I)$ is a number operator in the sense of the preceding definition (cf. 9.8).

[Note: Put $N = d\Gamma(I)$ and fix an orthonormal basis $\{e_n\}$ for $\mathcal{H}$. Consider $\sum_{k=1}^{n} \tilde{c}(e_k)\tilde{a}(e_k)$ -- then $\tilde{c}(e_k)\tilde{a}(e_k) = \tilde{a}(e_k)^*\tilde{a}(e_k)$, thus is selfadjoint (cf. 1.30) and nonnegative. Moreover, $\tilde{c}(e_k)\tilde{a}(e_k)$ commutes with $\tilde{c}(e_\ell)\tilde{a}(e_\ell)$. Therefore $\sum_{k=1}^{n} \tilde{c}(e_k)\tilde{a}(e_k)$ is selfadjoint (see the discussion following 4.6). And $\forall$ t,

$$e^{\sqrt{-1}\, tN} = \lim_{n \to \infty} \exp(\sqrt{-1}\, t \sum_{k=1}^{n} \tilde{c}(c_k)\tilde{a}(e_k))$$

in the strong operator topology.]

10.16 EXAMPLE If $\Lambda \neq 0$, then W_Λ does not admit a number operator. To get a contradiction, assume the opposite, hence $\forall$ $f \in H$,

$$e^{\sqrt{-1}\, tN} W_\Lambda(f) e^{-\sqrt{-1}\, tN} = W_\Lambda(e^{\sqrt{-1}\, t} f),$$

so $\forall$ $f_0 \in H_0$,

$$e^{\sqrt{-1}\, tN} W(f_0) = W(e^{\sqrt{-1}\, t} f_0) e^{\sqrt{-1}\, tN}$$

or still,

$$e^{\sqrt{-1}\, tN} W(f_0) = e^{\sqrt{-1}\, t d\Gamma(I)} W(f_0) e^{-\sqrt{-1}\, t d\Gamma(I)} e^{\sqrt{-1}\, tN}$$

or still,

$$e^{-\sqrt{-1}\, t d\Gamma(I)} e^{\sqrt{-1}\, tN} W(f_0) = W(f_0) e^{-\sqrt{-1}\, t d\Gamma(I)} e^{\sqrt{-1}\, tN}.$$

Therefore (cf. 10.9)

$$e^{-\sqrt{-1}\, t d\Gamma(I)} e^{\sqrt{-1}\, tN} = c(t) I \quad (c(t) \in \underline{C}).$$

But then

$$e^{\sqrt{-1}\, tN} W_\Lambda(f) e^{-\sqrt{-1}\, tN}$$

$$= c(t)e^{\sqrt{-1}\, t d\Gamma(I)} W_\Lambda(f) c(t)^{-1} e^{-\sqrt{-1}\, t d\Gamma(I)}$$

$$= e^{\sqrt{-1}\, \Lambda(f)} W(e^{\sqrt{-1}\, t} f)$$

$$= \exp(\sqrt{-1}\, (\Lambda(f) - \Lambda(e^{\sqrt{-1}\, t} f)) W_\Lambda(e^{\sqrt{-1}\, t} f)$$

$$= \exp(\sqrt{-1}\, \Lambda((1 - e^{\sqrt{-1}\, t}) f)) W_\Lambda(e^{\sqrt{-1}\, t} f).$$

And this means that $\forall$ f and $\forall$ t,

$$\exp(\sqrt{-1}\, \Lambda((1 - e^{\sqrt{-1}\, t}) f)) = 1,$$

which is manifestly impossible.

10.17 THEOREM (Chaiken) Let $\mathcal{H}$ be a separable complex Hilbert space -- then a Weyl system W over $\mathcal{H}$ is unitarily equivalent to a direct sum of the Fock system over $\mathcal{H}$ iff W admits a number operator whose spectrum is a subset of the nonnegative integers.

10.18 LEMMA Let $\mathcal{H}$ be a separable complex Hilbert space. Suppose that W is an irreducible Weyl system over $\mathcal{H}$ which admits a number operator N whose spectrum is bounded below -- then W is unitarily equivalent to the Fock system over $\mathcal{H}$.

PROOF We have

$$e^{2\pi\sqrt{-1}\, N} W(f) e^{-2\pi\sqrt{-1}\, N} = W(e^{2\pi\sqrt{-1}} f) = W(f).$$

But, by assumption, the set $\{W(f): f \in \mathcal{H}\}$ is irreducible, thus

$$e^{2\pi\sqrt{-1}\, N} = e^{2\pi\sqrt{-1}\, a}I$$

for some real number a (cf. 9.9). Here $\rho \le a < \rho + 1$, where $\rho = \inf \sigma(N)$. So, if $\lambda \in \sigma(N)$, then $\lambda - a$ is a nonnegative integer, hence $N - aI$ is a selfadjoint operator with $\sigma(N - aI) \subset \underline{Z}_{\geq 0}$. Since $N - aI$ is obviously a number operator, an application of 10.17 leads to the desired conclusion.

[Note: Recall that the Fock system over H is irreducible (cf. 9.11).]

Suppose that F is a finite dimensional subspace of H and let P_F be the associated orthogonal projection — then $\forall\, f \in H$,

$$e^{\sqrt{-1}\, td\Gamma(P_F)} W(f) e^{-\sqrt{-1}\, td\Gamma(P_F)}$$

$$= \Gamma(e^{\sqrt{-1}\, tP_F}) W(f) \Gamma(e^{-\sqrt{-1}\, tP_F})$$

$$= W(e^{\sqrt{-1}\, tP_F} f) \quad \text{(cf. 9.7)}.$$

Therefore $d\Gamma(P_F)$ is a number operator for $W|F$.

10.19 <u>LEMMA</u> Fix an orthonormal basis $\{u_1,\ldots,u_n\}$ for F and let P_{u_i} be the orthogonal projection onto $\underline{C}u_i$ — then

$$d\Gamma(P_{u_i}) = \tilde{a}(u_i)^*\tilde{a}(u_i)$$

and

$$d\Gamma(P_F) = \sum_{i=1}^{n} \tilde{a}(u_i)^*\tilde{a}(u_i).$$

So, as a corollary, $d\Gamma(P_F)$ annihilates the vacuum.

10.20 REMARK If T_F is a selfadjoint operator on $BO(H)$ such that

$$e^{\sqrt{-1}\, tT_F} W(f) e^{-\sqrt{-1}\, tT_F} = W(e^{\sqrt{-1}\, tP_F} f)$$

for all $f \in H$ and all $t \in \underline{R}$, then by irreducibility

$$e^{-\sqrt{-1}\, td\Gamma(P_F)} e^{\sqrt{-1}\, tT_F} = e^{\sqrt{-1}\, at}I$$

for some real number a, hence

$$T_F = d\Gamma(P_F) + aI.$$

Consequently, $T_F = d\Gamma(P_F)$ provided $T_F\Omega = 0$.]

10.21 LEMMA $\forall\, X \in BO(H)$,

$$||\Gamma(P_F)X||^2$$

$$= \frac{1}{(2\pi)^n} \int_{\underline{R}^{2n}} \left| < W(\sum_{k=1}^{n} z_k u_k)\Omega, X > \right|^2 d^{2n}z.$$

§11. CANONICAL COMMUTATION RELATIONS

Let G be a locally compact abelian group, Γ its dual. Suppose that

$$\begin{cases} U:G \to \mathcal{U}(K) \\ V:\Gamma \to \mathcal{U}(K) \end{cases}$$

are unitary representations on a complex Hilbert space K — then U,V are said to satisfy the canonical commutation relations if

$$U(\sigma)V(\chi) = \chi(\sigma)V(\chi)U(\sigma)$$

for all $\sigma \in G$, $\chi \in \Gamma$.

11.1 EXAMPLE Define unitary representations U,V of G,Γ respectively on $L^2(G)$ by

$$\begin{cases} U(\sigma)\psi(x) = \psi(x + \sigma) \\ V(\chi)\psi(x) = \chi(x)\psi(x). \end{cases} \qquad (\psi \in L^2(G))$$

Then

$$U(\sigma)V(\chi) = \chi(\sigma)V(\chi)U(\sigma).$$

In addition, it can be shown that the set $\{U(\sigma),V(\chi):\sigma \in G,\chi \in \Gamma\}$ is irreducible.

[Note: The pair (U,V) is called the Schrödinger realization of the canonical commutation relations.]

11.2 THEOREM (Mackey) Suppose that

$$\begin{cases} U:G \to \mathcal{U}(K) \\ V:\Gamma \to \mathcal{U}(K) \end{cases}$$

are unitary representations on a complex Hilbert space K. Assume: (U,V) satisfies the canonical commutation relations -- then

- There is an orthogonal decomposition

$$K = \bigoplus_{i \in I} K_i$$

into closed subspaces K_i invariant w.r.t. the $U(\sigma)$ and the $V(\chi)$.

- There are unitary operators $T_i:K_i \to L^2(G)$ such that $\forall\ \psi \in L^2(G)$

$$\begin{cases} (T_i U(\sigma) T_i^{-1}\psi)(x) = \psi(x+\sigma) \\ (T_i V(\chi) T_i^{-1}\psi)(x) = \chi(x)\psi(x). \end{cases}$$

Let H_0 be a real pre-Hilbert space. Suppose that

$$\begin{cases} U:H_0 \to \mathcal{U}(K) \\ V:H_0 \to \mathcal{U}(K) \end{cases}$$

are unitary representations of the additive group of H_0 on a complex Hilbert space K -- then U,V are said to satisfy the <u>canonical commutation relations</u> if

$$U(f_0)V(g_0) = e^{\sqrt{-1}\,\langle f_0,g_0\rangle} V(g_0)U(f_0)$$

for all $f_0, g_0 \in H_0$.

[Note: H_0 is a topological group under addition.]

11.3 REMARK If $\bar{H}_0$ is the completion of H_0, then U,V can be uniquely extended to unitary representations

$$\begin{cases} \bar{U}:\bar{H}_0 \to U(K) \\ \bar{V}:\bar{H}_0 \to U(K) \end{cases}$$

which satisfy the canonical commutation relations whenever this is the case of U,V.

[Note: Apart from the obvious, there is one subtle difference between pre-Hilbert spaces and Hilbert spaces, namely every separable pre-Hilbert space has an orthonormal basis but a nonseparable pre-Hilbert space need not have an orthonormal basis.]

11.4 EXAMPLE Take $H_0 = \underline{R}^n$, $K = L^2(\underline{R}^n)$ and let

$$\begin{cases} U(a)\psi(x) = \psi(x + a) \\ V(b)\psi(x) = e^{\sqrt{-1}\,\langle x,b\rangle}\psi(x). \end{cases} \qquad (\psi \in L^2(\underline{R}^n))$$

Then

$$U(a)V(b) = e^{\sqrt{-1}\,\langle a,b\rangle}V(b)U(a).$$

Moreover, the set $\{U(a),V(b):a,b \in \underline{R}^n\}$ is irreducible (cf. 10.4).

[Note: The pair (U,V) is called the Schrödinger realization of the canonical commutation relations.]

11.5 <u>EXAMPLE</u> Let H be a separable complex Hilbert space,

$$W:H \rightarrow U(BO(H))$$

the Fock system over H. Fix an orthonormal basis $\{e_n\}$ for H and let H_0 be its real linear span -- then H_0 is a real pre-Hilbert space. Put

$$\begin{cases} U(f_0) = W(-f_0) & (f_0 \in H_0) \\ V(g_0) = W(\sqrt{-1}\, g_0) & (g_0 \in H_0). \end{cases}$$

Then in view of 9.1 and 9.2, the assignments

$$\begin{cases} f_0 \rightarrow U(f_0) \\ g_0 \rightarrow V(g_0) \end{cases}$$

are unitary representations of the additive group of H_0 on $BO(H)$ such that

$$U(f_0)V(g_0) = e^{\sqrt{-1}\,<f_0,g_0>} V(g_0)U(f_0).$$

Furthermore (cf. 9.12), the set $\{U(f_0),V(g_0):f_0,g_0 \in H_0\}$ is irreducible.

[Note: The pair (U,V) is called the <u>Fock realization</u> of the canonical commutation relations.]

11.6 <u>REMARK</u> In 10.5, take $M = H_0$, $N = \sqrt{-1}\, H_0$, $B = \text{Im} < , >$ -- then

$$\begin{aligned} B(f_0,\sqrt{-1}\, g_0) &= \text{Im} < f_0,\sqrt{-1}\, g_0 > \\ &= \text{Im} \sqrt{-1} < f_0,g_0 > \\ &= < f_0,g_0 >. \end{aligned}$$

And

$$\exp(\frac{\sqrt{-1}}{2} B(f_0,\sqrt{-1}\, g_0))U(- f_0)V(g_0)$$

$$= \exp(\frac{\sqrt{-1}}{2} \text{Im} < f_0,g_0 >)W(f_0)W(\sqrt{-1}\, g_0)$$

$$= \exp(\frac{\sqrt{-1}}{2} \text{Im} < f_0,\sqrt{-1}\, g_0 >)W(f_0)W(\sqrt{-1}\, g_0)$$

$$= W(f_0 + \sqrt{-1}\, g_0).$$

11.7 <u>THEOREM</u> (Stone-von Neumann) Suppose that

$$\begin{cases} U:\underline{R}^n \to \mathcal{U}(\mathcal{K}) \\ V:\underline{R}^n \to \mathcal{U}(\mathcal{K}) \end{cases}$$

are unitary representations of $\underline{R}^n$ on a complex Hilbert space $\mathcal{K}$. Assume: (U,V) satisfies the canonical commutation relations -- then

- There is an orthogonal decomposition

$$\mathcal{K} = \bigoplus_{i \in I} \mathcal{K}_i$$

into closed subspaces invariant w.r.t. the U(a) and the V(b) $(a,b \in \underline{R}^n)$.

- There are unitary operators $T_i:\mathcal{K}_i \to L^2(\underline{R}^n)$ such that $\forall\ \psi \in L^2(\underline{R}^n)$

$$\begin{cases} (T_i U(a) T_i^{-1}\psi)(x) = \psi(x + a) \\ (T_i V(b) T_i^{-1}\psi)(x) = e^{\sqrt{-1}\, <x,b>}\psi(x). \end{cases}$$

11.8 REMARK The Stone-von Neumann theorem is, of course, a special case of Mackey's theorem and was originally established by bare hand methods. Later on, after the development of appropriate machinery, the general case was obtained via an application of imprimitivity theory.

[Note: It is to be emphasized that no restrictions are placed on K, i.e., K may be nonseparable.]

Let H_0 be a real pre-Hilbert space. Suppose that

$$\begin{cases} U:H_0 \to U(K) \\ V:H_0 \to U(K) \end{cases}, \qquad \begin{cases} U':H_0 \to U(K') \\ V':H_0 \to U(K') \end{cases}$$

are unitary representations of the additive group of H_0 on complex Hilbert spaces K,K' respectively -- then (U,V) is unitarily equivalent to (U',V') if $\exists$ a unitary operator $T:K \to K'$ such that

$$\begin{cases} TUT^{-1} = U' \\ TVT^{-1} = V'. \end{cases}$$

11.9 REMARK If H_0 is a real pre-Hilbert space and if $\dim H_0 < \infty$, then H_0 is automatically complete and the Stone-von Neumann theorem implies that up to unitary equivalence, H_0 supports a unique irreducible realization of the canonical commutation relations, viz. the Schrödinger realization.

The situation when dim $\mathcal{H}_0 = \infty$ is far more complicated, as can be illustrated by example.

11.10 EXAMPLE Define

$$C:L^2(\underline{R}) \rightarrow L^2(\underline{R})$$

by

$$C\psi(x) = \overline{\psi(-x)} \quad (\psi \in L^2(\underline{R})).$$

Put

$$\mathcal{S}_C(\underline{R}) = \{f \in \mathcal{S}(\underline{R}) : Cf = f\}.$$

Then $\mathcal{S}_C(\underline{R})$ is a real pre-Hilbert space:

$$< f,g > = < Cf,Cg > = < g,f > = \overline{< f,g >}.$$

Given $m > 0$, let

$$\mu_m : \mathcal{S}_C(\underline{R}) \rightarrow \mathcal{S}_C(\underline{R})$$

be the multiplication operator $f \rightarrow \mu_m f$, where

$$(\mu_m f)(x) = \sqrt{m^2 + x^2}\, f(x).$$

Define unitary representations

$$\begin{cases} U_m : \mathcal{S}_C(\underline{R}) \rightarrow \mathcal{U}(BO(L^2(\underline{R}))) \\ V_m : \mathcal{S}_C(\underline{R}) \rightarrow \mathcal{U}(BO(L^2(\underline{R}))) \end{cases}$$

by

$$\begin{cases} U_m(f) = W(-\mu_m^{-1} f) \\ V_m(f) = W(\sqrt{-1}\, \mu_m f). \end{cases}$$

Then U_m, V_m satisfy the canonical commutation relations and the set $\{U_m(f), V_m(f) : f \in S_C(\underline{R})\}$ is irreducible (cf. 9.12) (one can always find an orthonormal basis for $L^2(\underline{R})$ which is contained in $S_C(\underline{R})$). Suppose now that $m \neq m'$ -- then (U_m, V_m) is not unitarily equivalent to $(U_{m'}, V_{m'})$. To see this, proceed by contradiction and assume that

$$T : BO(L^2(\underline{R})) \to BO(L^2(\underline{R}))$$

is a unitary operator such that

$$\begin{cases} TU_m T^{-1} = U_{m'} \\ TV_m T^{-1} = V_{m'}. \end{cases}$$

Given $b \in \underline{R}$, let

$$V(b)\psi(x) = e^{\sqrt{-1}\,\langle x,b\rangle}\psi(x) \quad (\psi \in L^2(\underline{R}))$$

and note that $S_C(\underline{R})$ is invariant under $V(b)$. Next, in view of 9.7, we have

$$\Gamma(V(b))W(\psi)\Gamma(V(b))^{-1} = W(V(b)\psi).$$

So, $\forall\ f \in S_C(\underline{R})$,

$$U_m(f)T^{-1}\Gamma(V(b))^{-1}T\Gamma(V(b))$$

$$= W(-\mu_m^{-1}f)T^{-1}\Gamma(V(b))^{-1}T\Gamma(V(b))$$

$$= T^{-1}(TW(-\mu_m^{-1}f)T^{-1})\Gamma(V(b))^{-1}T\Gamma(V(b))$$

$$= T^{-1}W(-\mu_{m'}^{-1}f)\Gamma(V(b))^{-1}T\Gamma(V(b))$$

$$= T^{-1}\Gamma(V(b))^{-1}W(V(b)(-\mu_{m'}^{-1}f))T\Gamma(V(b))$$

$$= T^{-1}\Gamma(V(b))^{-1}TW(V(b)(-\mu_m^{-1}f))\Gamma(V(b))$$

$$= T^{-1}\Gamma(V(b))^{-1}T\Gamma(V(b))W(-\mu_m^{-1}f)$$

$$= T^{-1}\Gamma(V(b))^{-1}T\Gamma(V(b))U_m(f).$$

And, analogously, $\forall\ f \in S_C(\underline{R})$,

$$V_m(f)T^{-1}\Gamma(V(b))^{-1}T\Gamma(V(b))$$

$$= T^{-1}\Gamma(V(b))^{-1}T\Gamma(V(b))V_m(f).$$

Therefore, by irreducibility (cf. 9.9),

$$T^{-1}\Gamma(V(b))^{-1}T\Gamma(V(b)) = \gamma_b I,$$

where $|\gamma_b| = 1$. But then

$$T^{-1}\Gamma(V(b))^{-1}T\Gamma(V(b))\Omega = \gamma_b\Omega$$

or still,

$$T^{-1}\Gamma(V(b))^{-1}T\Omega = \gamma_b\Omega$$

or still,

$$\Gamma(V(b))^{-1}T\Omega = \gamma_b T\Omega.$$

Let

$$\Psi = T\Omega = \{\psi_n : \psi_n \in BO_n(L^2(\underline{R}))\}.$$

Then

$$e^{-\sqrt{-1}\langle x_1 + \cdots x_n, b\rangle} \psi_n(x_1,\ldots,x_n)$$

$$= \gamma_b \psi_n(x_1,\ldots,x_n)$$

=>

$$\psi_n = 0 \quad (n \geq 1)$$

=>

$$T\Omega = \gamma_b \Omega.$$

Since this holds for every b and since $\gamma_0 = 1$, it follows that $T\Omega = \Omega$. On general grounds (cf. 9.5),

$$||\Phi(- \mu_m^{-1} f)\Omega||^2$$

$$= \frac{1}{2} ||- \mu_m^{-1} f||^2$$

$$= \frac{1}{2} \int_{\underline{R}} \frac{|f(x)|^2}{m^2 + x^2} dx.$$

On the other hand, we also have

$$||\Phi(- \mu_m^{-1} f)\Omega||^2$$

$$= ||T\Phi(- \mu_m^{-1} f)\Omega||^2$$

$$= ||\Phi(- \mu_{m'}^{-1} f)T\Omega||^2$$

$$= \|\Phi(-\mu_{m'}^{-1} f)\Omega\|^2$$

$$= \frac{1}{2} \int_{\underline{R}} \frac{|f(x)|^2}{(m')^2 + x^2}\, dx.$$

Thus $m = m'$, contrary to hypothesis.

[Note: The generator of the one parameter unitary group $t \to TU_m(tf)T^{-1}$ is

$$\overline{TQ(-\mu_m^{-1} f)T^{-1}},$$

while the generator of the one parameter unitary group $t \to U_{m'}(tf)$ is

$$\overline{Q(-\mu_{m'} f)}.$$

From the definitions,

$$\overline{TQ(-\mu_m^{-1} f)T^{-1}} = \overline{Q(-\mu_{m'} f)}$$

which implies that

$$\overline{TQ(-\mu_m^{-1} f)} \subset \overline{Q(-\mu_{m'} f)T},$$

a point used tacitly in the preceding computation.]

The term "unitary representation" carries with it a continuity requirement (cf. §3). Still, certain physical models lead one to consider homomorphisms

$$\begin{cases} U: H_0 \to \mathcal{U}(K) \\ V: H_0 \to \mathcal{U}(K) \end{cases}$$

with the property that

$$U(f_0)V(g_0) = e^{\sqrt{-1}\,<f_0,g_0>}V(g_0)U(f_0)$$

for all $f_0,g_0 \in H_0$ but where either U or V is discontinuous.

11.11 <u>EXAMPLE</u> Take $H_0 = \underline{R}$, $K = \ell^2(\underline{R})$ and for each $\lambda \in \underline{R}$, let χ_λ be the characteristic function of $\{\lambda\}$ -- then the set $\{\chi_\lambda : \lambda \in \underline{R}\}$ is an orthonormal basis for $\ell^2(\underline{R})$. Put

$$\begin{cases} U(a)\chi_\lambda = \chi_{\lambda-a} \\ V(b)\chi_\lambda = e^{\sqrt{-1}\,<\lambda,b>}\chi_\lambda. \end{cases} \qquad (a,b \in \underline{R})$$

Then U(a),V(b) admit unique extensions to unitary operators on $\ell^2(\underline{R})$ and we have

$$U(a)V(b)\chi_\lambda = e^{\sqrt{-1}\,<a,b>}V(b)U(a)\chi_\lambda.$$

Therefore U,V satisfy the canonical commutation relations.

- As a map from $\underline{R}$ to $U(\ell^2(\underline{R}))$, U is not continuous. Proof (cf. 3.5):

$$<\chi_\lambda, U(a)\chi_\lambda> = \begin{cases} 1 \text{ if } a = 0 \\ 0 \text{ if } a \neq 0. \end{cases}$$

- As a map from $\underline{R}$ to $U(\ell^2(R))$, V is continuous. Proof (cf. 3.5):

$$\lim_{b \to 0} < \chi_\lambda, V(b)\chi_\lambda > = \lim_{b \to 0} e^{\sqrt{-1} <\lambda,b>} = 1.$$

[Note: Let Q be the generator of the one parameter unitary group $b \to V(b)$, thus $V(b) = \exp(\sqrt{-1}\, bQ)$ and

$$Q\chi_\lambda = \lim_{b \to 0} \frac{V(b) - I}{\sqrt{-1}\, b} \chi_\lambda$$

$$= \lim_{b \to 0} \frac{e^{\sqrt{-1} <\lambda,b>} - 1}{\sqrt{-1}\, b} \chi_\lambda$$

$$= \frac{\sqrt{-1}\, \lambda}{\sqrt{-1}} \chi_\lambda = \lambda\chi_\lambda.$$

Thus, in this realization, the position operator exists (and its spectrum is pure point) but the momentum operator does not exist. There is also a variation on this theme which reverses these conclusions.]

§12. *SHALE'S THEOREM*

Let (E,σ) be a symplectic topological vector space -- then a symplectic automorphism of E is an $\underline{R}$-linear homeomorphism $T:E \to E$ such that

$$\sigma(Tf,Tg) = \sigma(f,g)$$

for all $f,g \in E$.

Specialize and assume that $\mathcal{H}$ is a separable complex Hilbert space. View $\mathcal{H}$ as a symplectic topological vector space with $\sigma = \mathrm{Im} < , >$ and denote by $SP(\mathcal{H})$ the set of all symplectic automorphisms of $\mathcal{H}$ -- then $SP(\mathcal{H})$ is a group under operator multiplication, the symplectic group of $\mathcal{H}$. Since

$$U \in U(\mathcal{H}) \Rightarrow \mathrm{Im} < Uf,Ug > = \mathrm{Im} < f,g >,$$

it follows that $U(\mathcal{H})$ is a subgroup of $SP(\mathcal{H})$.

Let $J:\mathcal{H} \to \mathcal{H}$ be multiplication by $\sqrt{-1}$. Suppose that $T:\mathcal{H} \to \mathcal{H}$ is $\underline{R}$-linear -- then there is a decomposition

$$T = T_1 + T_2,$$

where

$$\begin{cases} T_1 = \frac{1}{2}(T - JTJ) \\ T_2 = \frac{1}{2}(T + JTJ). \end{cases}$$

Here, $T_1J = JT_1$, thus T_1 is complex linear, and $T_2J = - JT_2$, thus T_2 is complex conjugate linear.

N.B. The adjoint T_1^* is given by $\langle f, T_1 g \rangle = \langle T_1^* f, g \rangle$ but the adjoint T_2^* is given by $\langle f, T_2 g \rangle = \langle g, T_2^* f \rangle$.

12.1 LEMMA Let $T \in SP(H)$ -- then

$$T^{-1} = T_1^* - T_2^*.$$

12.2 LEMMA Let $T \in SP(H)$ -- then

$$\begin{cases} T_1^* T_1 - T_2^* T_2 = I \\ T_1^* T_2 - T_2^* T_1 = 0 \end{cases}, \quad \begin{cases} T_1 T_1^* - T_2 T_2^* = I \\ T_2 T_1^* - T_1 T_2^* = 0. \end{cases}$$

Let $T \in SP(H)$ -- then

$$\begin{aligned} ||T_1 f||^2 &= || \; |T_1| f ||^2 \\ &= \langle f, T_1^* T_1 f \rangle \\ &= \langle f, (T_2^* T_2 + I) f \rangle \\ &= \langle f, T_2^* T_2 f \rangle + \langle f, f \rangle \\ &= \langle T_2 f, T_2 f \rangle + \langle f, f \rangle \\ &\geq ||f||^2. \end{aligned}$$

Therefore T_1 is invertible. And:

$$< f, |T_1|^2 f > \geq < f,f >$$

=>

$$|T_1|^2 \geq I \Rightarrow |T_1| = \sqrt{|T_1|^2} \geq \sqrt{I} = I.$$

12.3 <u>LEMMA</u> Let $T \in SP(H)$ -- then

$$\text{Ker}(|T_2|) = \{f: |T_1| f = f\}.$$

<u>PROOF</u> There are two points. First, $\left[\begin{array}{l} |T_1| \\ |T_2| \end{array}\right.$ are selfadjoint, hence

$$\left[\begin{array}{l} \text{Ker}(|T_1|) = \text{Ker}(|T_1|^2) \\ \text{Ker}(|T_2|) = \text{Ker}(|T_2|^2). \end{array}\right.$$

Second (cf. 12.2),

$$|T_1|^2 = |T_2|^2 + I.$$

Let $T_1 = U_1 |T_1|$ be the polar decomposition of T_1 -- then U_1 is unitary (and not merely a partial isometry).

12.4 <u>LEMMA</u> Let $T \in SP(H)$ -- then

$$U_1 \operatorname{Ker}(|T_2|) = \operatorname{Ker}(|T_2^*|).$$

<u>PROOF</u> We have (cf. 12.1)

$$T^{-1} = T_1^* - T_2^*$$

=>

$$\begin{cases} (T^{-1})_1 = T_1^* \\ (T^{-1})_2 = -T_2^*. \end{cases}$$

This said, replace T by T^{-1} in 12.3 to get:

$$\operatorname{Ker}(|T_2^*|) = \{f: |T_1^*|f = f\}.$$

Then

$$f \in \operatorname{Ker}(|T_2|)$$

=>

$$\begin{aligned} |T_1^*|U_1 f &= (U_1|T_1|U_1^{-1})U_1 f \\ &= U_1|T_1|f \\ &= U_1 f \quad (\text{cf. } 12.3) \end{aligned}$$

=>

$$U_1 f \in \operatorname{Ker}(|T_2^*|).$$

Conversely,

$$f \in \operatorname{Ker}(|T_2^*|)$$

=>

$$|T_1^*|f = f$$

=>

$$U_1|T_1|U_1^{-1}f = f$$

=>

$$|T_1|U_1^{-1}f = U_1^{-1}f$$

=>

$$U_1^{-1}f \in \mathrm{Ker}(|T_2|)$$

=>

$$f = U_1(U_1^{-1}f) \in U_1\mathrm{Ker}(|T_2|).$$

Let $T_2 = U_2|T_2|$ be the polar decomposition of T_2 -- then, as it stands, U_2 is a conjugate linear partial isometry which, for use below, is going to have to be modified.

Initially

$$U_2:\mathrm{Ran}(|T_2|) \to \mathrm{Ran}(T_2)$$

is defined by

$$U_2(|T_2|f) = T_2f.$$

Since

$$|| \ |T_2|f||^2 = ||T_2f||^2,$$

U_2 is isometric, thus extends to an isometry

$$U_2:\overline{\mathrm{Ran}(|T_2|)} \to \overline{\mathrm{Ran}(T_2)},$$

i.e., extends to an isometry

$$U_2 : \mathrm{Ker}(|T_2|)^{\perp} \to \mathrm{Ker}(|T_2^*|)^{\perp}.$$

The construction of the polar decomposition of T_2 is then completed by extending U_2 to all of $\mathcal{H}$ by taking it to be zero on $\mathrm{Ker}(|T_2|)$.

For our purposes, it is this last step that will not do. Instead, fix a conjugation $C_2:\mathrm{Ker}(|T_2|) \to \mathrm{Ker}(|T_2|)$ and then put

$$V_2 f = U_1 C_2 f \quad (f \in \mathrm{Ker}(|T_2|)).$$

Thanks to 12.4,

$$V_2 \mathrm{Ker}(|T_2|) = \mathrm{Ker}(|T_2^*|).$$

So, schematically,

$$\begin{array}{ccccc} \mathcal{H} = & \mathrm{Ker}(|T_2|)^{\perp} & \oplus & \mathrm{Ker}(|T_2|) \\ & U_2 \downarrow & & \downarrow V_2 \\ \mathcal{H} = & \mathrm{Ker}(|T_2^*|)^{\perp} & \oplus & \mathrm{Ker}(|T_2^*|). \end{array}$$

Now set $W_2 = U_2 \oplus V_2$ -- then W_2 is antiunitary and it is still the case that $T_2 = W_2|T_2|$ (bear in mind that $\mathrm{Ker}(|T_2|) = \mathrm{Ker}(T_2)$).

Let

$$C = W_2^{-1} U_1.$$

Then C is antiunitary.

12.5 <u>LEMMA</u> C commutes with $|T_1|$ and $|T_2|$.

PROOF We have

$$\begin{cases} |T_1^*|^2 = U_1|T_1|^2U_1^{-1} \\ |T_2^*|^2 = W_2|T_2|^2W_2^{-1}. \end{cases}$$

Therefore

$$U_1\exp(\sqrt{-1}\ t|T_1|^2)U_1^{-1}$$

$$= \exp(\sqrt{-1}\ t|T_1^*|^2)$$

$$= \exp(\sqrt{-1}\ t)\exp(\sqrt{-1}\ t|T_2^*|^2)$$

$$= \exp(\sqrt{-1}\ t)W_2\ \exp(-\sqrt{-1}\ t|T_2|^2)W_2^{-1}$$

$$= W_2\ \exp(-\sqrt{-1}\ t)\exp(-\sqrt{-1}\ t|T_2|^2)W_2^{-1}$$

$$= W_2\ \exp(-\sqrt{-1}\ t|T_1|^2)W_2^{-1}$$

=>

$$C\ \exp(\sqrt{-1}\ t|T_1|^2) = \exp(-\sqrt{-1}\ t|T_1|^2)C$$

=>

$$C|T_1|^2 = |T_1|^2C$$

=>

$$C|T_1| = |T_1|C \quad \text{(cf. 1.34)}.$$

And

$$|T_1|^2 = |T_2|^2 + I$$

=>

$$C|T_2|^2 = |T_2|^2C$$

=>

$$C|T_2| = |T_2|C \quad (\text{cf. } 1.34).$$

12.6 LEMMA The image

$$|T_2|\ |T_1|(\text{Ker}(|T_2|)^{\perp})$$

is a dense subspace of $\text{Ker}(|T_2|)^{\perp}$.

PROOF To begin with, $\text{Ker}(|T_2|)$ is invariant under $|T_1|$, thus $\text{Ker}(|T_2|)^{\perp}$ is too. Next,

$$|T_1|^2 = |T_2|^2 + I,$$

so $|T_1|$ and $|T_2|$ necessarily commute (cf. 1.36). Therefore $|T_1|\ |T_2| = |T_2|\ |T_1|$ is a bounded selfadjoint operator on H. But the restriction of $|T_2|\ |T_1|$ to $\text{Ker}(|T_2|)^{\perp}$ is injective, hence its range is dense.

12.7 LEMMA C is a conjugation.

PROOF C is antiunitary, so $C^* = C^{-1}$. If $f \in \text{Ker}(|T_2|)$, then

$$Cf = W_2^*U_1f$$

$$= V_2^*U_1f$$

$$= V_2^*U_1C_2C_2f$$

$$= V_2^* V_2 C_2 f = C_1 f.$$

On the other hand, if $f \in \text{Ker}(|T_2|)^{\perp}$, then $\forall\, g \in \text{Ker}(|T_2|)^{\perp}$,

$$< f, C^* |T_2|\ |T_1| g >$$

$$= < |T_2|\ |T_1| g, Cf >$$

$$= < |T_1|\ |T_2| g, Cf >$$

$$= < |T_2| g, |T_1| Cf >$$

$$= < |T_2| g, C |T_1| f > \quad \text{(cf. 12.5)}$$

$$= < |T_2| g, W_2^* U_1 |T_1| f >$$

$$= < U_1 |T_1| f, W_2 |T_2| g >$$

$$= < T_1 f, T_2 g >$$

$$= < f, T_1^* T_2 g >$$

$$= < f, T_2^* T_1 g > \quad \text{(cf. 12.2)}$$

$$= < T_1 g, T_2 f >$$

$$= < U_1 |T_1| g, W_2 |T_2| f >$$

$$= < |T_2| f, W_2^* U_1 |T_1| g >$$

$$= < |T_2|f, C|T_1|g >$$

$$= < f, |T_2|C|T_1|g >$$

$$= < f, C|T_2|\ |T_1|g > \quad \text{(cf. 12.5)}$$

=>

$$C^*|T_2|\ |T_1|g = C|T_2|\ |T_1|g$$

=>

$$C^*|\mathrm{Ker}(|T_2|)^{\perp} = C|\mathrm{Ker}(|T_2|)^{\perp} \quad \text{(cf. 12.6)}.$$

Consequently, $C^* = C$, from which the lemma.

12.8 <u>REMARK</u> By definition, $C = W_2^{-1}U_1$, thus

$$W_2C = U_1$$

=>

$$W_2 = U_1C^{-1} = U_1C \quad \text{(cf. 12.7)}.$$

Because cosh: $[0,\infty[\rightarrow [1,\infty[$ is bijective, $\exists$ a nonnegative selfadjoint operator S such that

$$\begin{cases} |T_1| = \cosh(S) \\ |T_2| = \sinh(S). \end{cases}$$

12.9 <u>LEMMA</u> Let $T \in SP(H)$ — then there exists a unitary operator U, a nonnegative selfadjoint operator S, and a conjugation C such that

$$T = U\cosh(S) + UC\sinh(S).$$

<u>PROOF</u> Write

$$\begin{aligned} T &= T_1 + T_2 \\ &= U_1|T_1| + W_2|T_2| \\ &= U_1\cosh(S) + U_1C\sinh(S) \quad \text{(cf. 12.8)} \\ &= U\cosh(S) + UC\sinh(S), \end{aligned}$$

where $U = U_1$.

[Note: T is unitary iff $T_2 = 0$ ($S = 0$ in 12.9).]

Denote by $SP_2(H)$ the subset of $SP(H)$ consisting of those T such that $T_2 \in \underline{L}_2(H)$.

12.10 <u>REMARK</u> $SP_2(H)$ is a group under multiplication. In fact,

$$\left[\begin{array}{l} T \in SP_2(H) \Rightarrow (T^{-1})_2 = -T_2^* \\ T',T'' \in SP_2(H) \Rightarrow (T'T'')_2 = T_1'T_2'' + T_2'T_1''. \end{array}\right.$$

[Note: $SP_2(H)$ is a topological group if one uses the operator norm topology on the complex linear part and the Hilbert-Schmidt topology on the complex conjugate linear part:

$$d_2(T',T'') = ||T_1' - T_1''|| + ||T_2' - T_2''||_2.]$$

12.11 LEMMA Let $T \in SP(H)$ — then $T \in SP_2(H)$ iff $|T_2| \in \underline{L}_2(H)$.

12.12 LEMMA Let $T \in SP(H)$ — then $T \in SP_2(H)$ iff $|T_1| - I \in \underline{L}_1(H)$.

PROOF We have

$$|T_2|^2 = |T_1|^2 - I$$

$$= (|T_1| - I)(|T_1| + I).$$

According to 12.11,

$$T \in SP_2(H) \iff |T_2| \in \underline{L}_2(H).$$

But the product of two Hilbert-Schmidt operators is trace class, hence

$$|T_2| \in \underline{L}_2(H) \Rightarrow |T_2|^2 \in \underline{L}_1(H)$$

$\Rightarrow$

$$|T_1| - I = |T_2|^2 \, (|T_1| + I)^{-1} \in \underline{L}_1(H).$$

Conversely,

$$|T_1| - I \in \underline{L}_1(H)$$

$\Rightarrow$

$$(|T_1| - I)(|T_1| + I) \in \underline{L}_1(H)$$

$\Rightarrow$

$$|T_2|^2 \in \underline{L}_1(H) \Rightarrow |T_2| \in \underline{L}_2(H).$$

If H is viewed as a real Hilbert space with inner product Re $< f,g >$, then the adjoint of an $\underline{R}$-linear operator A is denoted by A^+:

$$\text{Re} < f,Ag > = \text{Re} < A^+f,g >.$$

12.13 <u>LEMMA</u> Suppose that $T:H \to H$ is an $\underline{R}$-linear homeomorphism -- then $T \in SP(H)$ iff $T^+JT = J$.

12.14 <u>LEMMA</u> Let $T \in SP(H)$ -- then $T^+ \in SP(H)$ and $T^{-1} = JT^+J^{-1}$.

12.15 <u>LEMMA</u> Let $T \in SP(H)$ -- then $T \in SP_2(H)$ iff $T^+T - I$ is Hilbert-Schmidt.

12.16 <u>LEMMA</u> Let $T \in SP(H)$ -- then $T^+T - I$ is Hilbert-Schmidt iff $TJ - JT$ is Hilbert-Schmidt.

<u>PROOF</u> For

$T^+T - I$ Hilbert-Schmidt

=>

$(T^+T - I)J$ Hilbert-Schmidt

=>

$T^+TJ - T^+JT$ Hilbert-Schmidt (cf. 12.13)

=>

$T^+(TJ - JT)$ Hilbert-Schmidt

=>

$(T^+)^{-1}T^+(TJ- JT)$ Hilbert-Schmidt

=>

$TJ - JT$ Hilbert-Schmidt.

And conversely... .

12.17 <u>LEMMA</u> Let $T \in SP(H)$ -- then $TJ - JT$ is Hilbert-Schmidt iff $J - TJT^{-1}$ is Hilbert-Schmidt.

<u>PROOF</u> If $TJ - JT$ is Hilbert-Schmidt, then $T \in SP_2(H)$ (cf. 12.15 and 12.16), thus $T^{-1} \in SP_2(H)$ and so $T^{-1}J - JT^{-1}$ is Hilbert-Schmidt. Therefore $T(T^{-1}J - JT^{-1})$ is Hilbert-Schmidt or still, $J - TJT^{-1}$ is Hilbert-Schmidt. To establish the converse, just reverse the steps.

Let

$$W: H \to U(BO(H))$$

be the Fock system. Given $T \in SP(H)$, put

$$W_T(f) = W(Tf) \quad (f \in H).$$

Then W_T is a Weyl system over H which, moreover, is irreducible (cf. 9.11). But, contrary to what might be expected, W_T is not necessarily unitarily equivalent to W. One is thus led to say that T is <u>implementable</u> if $\exists\ \Gamma_T \in U(BO(H))$ such that

$$\Gamma_T W(f) \Gamma_T^{-1} = W_T(f) \ \forall\ f \in H.$$

12.18 <u>EXAMPLE</u> Let $U \in U(H)$ -- then U is implementable. In fact (cf. 9.7),

$$\Gamma(U) W(f) \Gamma(U)^{-1} = W(Uf) \ \forall\ f \in H.$$

The problem now is to characterize the $T \in SP(H)$ which are implementable.

12.19 <u>THEOREM</u> (Shale) Let $T \in SP(H)$ -- then T is implementable iff $T \in SP_2(H)$.

12.20 <u>REMARK</u> If $\dim H < \infty$, then Shale's theorem is a consequence of the Stone-von Neumann theorem (cf. 11.7).

We shall begin with the necessity, which requires some preparation.

By definition,

$$\begin{cases} Q(f) = \dfrac{1}{\sqrt{2}} (\tilde{c}(f) + \tilde{a}(f)) \\[2ex] P(f) = \dfrac{\sqrt{-1}}{\sqrt{2}} (\tilde{c}(f) - \tilde{a}(f)). \end{cases}$$

Furthermore, all operators in sight have the same domain, viz. D_f (cf. 7.12), thus

$$\begin{cases} \tilde{a}(f) = \dfrac{1}{\sqrt{2}} (Q(f) + \sqrt{-1}\, P(f)) \\[2ex] \tilde{c}(f) = \dfrac{1}{\sqrt{2}} (Q(f) - \sqrt{-1}\, P(f)). \end{cases}$$

12.21 <u>LEMMA</u> We have

$$\begin{cases} \tilde{a}(f) = \dfrac{1}{\sqrt{2}} (\overline{Q(f)} + \sqrt{-1}\, \overline{P(f)}) \\[2ex] \tilde{c}(f) = \dfrac{1}{\sqrt{2}} (\overline{Q(f)} - \sqrt{-1}\, \overline{P(f)}). \end{cases}$$

PROOF According to 7.20,

$$D_f = \mathrm{Dom}(\overline{Q(f)}) \cap \mathrm{Dom}(\overline{P(f)}),$$

which, of course, is the domain of

$$\begin{cases} \dfrac{1}{\sqrt{2}} (\overline{Q(f)} + \sqrt{-1}\,\overline{P(f)}) \\ \\ \dfrac{1}{\sqrt{2}} (\overline{Q(f)} - \sqrt{-1}\,\overline{P(f)}). \end{cases}$$

Let $X \in D_f$ -- then

$$\begin{aligned}\tilde{a}(f)X &= \frac{1}{\sqrt{2}} (Q(f) + \sqrt{-1}\,P(f))X \\ &= \frac{1}{\sqrt{2}} (Q(f)X + \sqrt{-1}\,P(f)X) \\ &= \frac{1}{\sqrt{2}} (\overline{Q(f)}X + \sqrt{-1}\,\overline{P(f)}X) \\ &= \frac{1}{\sqrt{2}} (\overline{Q(f)} + \sqrt{-1}\,\overline{P(f)})X.\end{aligned}$$

Ditto for $\tilde{c}(f)$.

Assume now that T is implementable, so $\exists\ \Gamma_T \in \mathcal{U}(\mathrm{BO}(\mathcal{H}))$ such that

$$\Gamma_T W(f) \Gamma_T^{-1} = W_T(f).$$

Then

$$\Gamma_T \overline{Q(f)} \Gamma_T^{-1} = \overline{Q(Tf)}.$$

12.22 <u>REMARK</u> In the relation

$$\Gamma_T \overline{Q(f)} \Gamma_T^{-1} = \overline{Q(Tf)},$$

replace f by $T^{-1}f$ to get

$$\Gamma_T \overline{Q(T^{-1}f)} = \overline{Q(f)} \Gamma_T.$$

Then

$$X \in BO_F(H) \Rightarrow X \in \mathrm{Dom}(\overline{Q(T^{-1}f)})$$

$$\Rightarrow \Gamma_T X \in \mathrm{Dom}(\overline{Q(f)}).$$

Since this holds $\forall$ f $\in$ H, it follows that

$$\Gamma_T BO_F(H) \subset \bigcap_f \mathrm{Dom}(\overline{Q(f)}) = \bigcap_f D_f.$$

In particular:

$$\Gamma_T \Omega \in \bigcap_f D_f.$$

And this implies that

$$(\tilde{a}(f) + \tilde{c}(g)) \Gamma_T \Omega = \tilde{a}(f) \Gamma_T \Omega + \tilde{c}(g) \Gamma_T \Omega$$

for all f,g in H.

[Note: $\Gamma_T \Omega = e^{\sqrt{-1}\,\theta}\Omega$ $(0 \le \theta < 2\pi)$ iff $T_2 = 0$, i.e., iff T is unitary.]

We have

$$\Gamma_T \tilde{a}(f) \Gamma_T^{-1}$$

$$= \Gamma_T \frac{1}{\sqrt{2}} (\overline{Q(f)} + \sqrt{-1}\, \overline{P(f)})\Gamma_T^{-1}$$

$$= \frac{1}{\sqrt{2}} (\Gamma_T \overline{Q(f)}\Gamma_T^{-1} + \sqrt{-1}\, \Gamma_T \overline{P(f)}\Gamma_T^{-1})$$

$$= \frac{1}{\sqrt{2}} (\Gamma_T \overline{Q(f)}\Gamma_T^{-1} + \sqrt{-1}\, \Gamma_T \overline{Q(\sqrt{-1}\, f)}\Gamma_T^{-1})$$

$$= \frac{1}{\sqrt{2}} (\overline{Q(Tf)} + \sqrt{-1}\, \overline{Q(T\sqrt{-1}\, f)})$$

=>

$$(\Gamma_T \tilde{a}(f)\Gamma_T^{-1})(\Gamma_T\Omega)$$

$$= \frac{1}{\sqrt{2}} (\overline{Q(Tf)}\Gamma_T\Omega + \sqrt{-1}\, \overline{Q(T\sqrt{-1}\, f)}\Gamma_T\Omega).$$

- $\Gamma_T\Omega \in \mathrm{Dom}(\overline{Q(Tf)}) \cap \mathrm{Dom}(\overline{Q(\sqrt{-1}\, Tf)})$ (cf. 12.22)

=>

$\Gamma_T\Omega \in D_{Tf}$ (cf. 7.20).

I.e.:

$\Gamma_T\Omega \in \mathrm{Dom}(Q(Tf))$

=>

$$\overline{Q(Tf)}\Gamma_T\Omega$$

$$= Q(Tf)\Gamma_T\Omega$$

$$= \frac{1}{\sqrt{2}} (\tilde{c}(Tf) + \tilde{a}(Tf))\Gamma_T\Omega.$$

- $\Gamma_T\Omega \in \text{Dom}(\overline{Q(T\sqrt{-1}\ f)}) \cap \text{Dom}(\overline{Q(\sqrt{-1}\ T\sqrt{-1}\ f)})$ (cf. 12.22)

=>

$\Gamma_T\Omega \in D_{T\sqrt{-1}\ f}$ (cf. 7.20).

I.e.:

$\Gamma_T\Omega \in \text{Dom}(Q(T\sqrt{-1}\ f))$

=>

$$\overline{Q(T\sqrt{-1}\ f)}\Gamma_T\Omega$$

$$= Q(T\sqrt{-1}\ f)\Gamma_T\Omega$$

$$= \frac{1}{\sqrt{2}}\ (\tilde{c}(T\sqrt{-1}\ f) + \tilde{a}(T\sqrt{-1}\ f))\Gamma_T\Omega.$$

Setting aside $\Gamma_T\Omega$ for the moment, note that

$$\frac{1}{\sqrt{2}}\ (\frac{1}{\sqrt{2}}\ (\tilde{c}(Tf) + \tilde{a}(Tf)) + \frac{\sqrt{-1}}{\sqrt{2}}\ (\tilde{c}(T\sqrt{-1}\ f) + \tilde{a}(T\sqrt{-1}\ f)))$$

$$= \frac{1}{2}\ (\tilde{c}(T_1f) + \tilde{c}(T_2f) + \tilde{a}(T_1f) + \tilde{a}(T_2f))$$

$$+ \frac{\sqrt{-1}}{2}\ (\sqrt{-1}\ \tilde{c}(T_1f) - \sqrt{-1}\ \tilde{c}(T_2f) - \sqrt{-1}\ \tilde{a}(T_1f) + \sqrt{-1}\ \tilde{a}(T_2f))$$

$$= \tilde{a}(T_1f) + \tilde{c}(T_2f).$$

Write

$$\Gamma_T\Omega = \{X_n\},$$

thus $X_0 = c_0\Omega$, where

$$c_0 = < \Omega, \Gamma_T\Omega >.$$

Then

$$0 = \tilde{a}(f)\Omega$$

$$= \Gamma_T\tilde{a}(f)\Omega$$

$$= (\Gamma_T\tilde{a}(f)\Gamma_T^{-1})\Gamma_T\Omega$$

$$= (\tilde{a}(T_1f) + \tilde{c}(T_2f))\Gamma_T\Omega$$

or still,

$$(\tilde{a}(f) + \tilde{c}(T_2(T_1)^{-1}f))\Gamma_T\Omega = 0$$

=>

$$\underline{a}(f)X_{n+1} + \underline{c}(T_2(T_1)^{-1}f)X_{n-1} = 0$$

=>

$$X_1 = 0 \Rightarrow X_{2k+1} = 0.$$

But $\Gamma_T\Omega \neq 0$, hence $c_0 \neq 0$.

12.23 <u>LEMMA</u> Let $f,g \in \mathcal{H}$ — then

$$\sqrt{2} < f \otimes g, X_2 > = - c_0 < g, T_2(T_1)^{-1}f >.$$

PROOF On the one hand,

$$< \tilde{c}(f)g,X_2 > = < \sqrt{2}\, P_2(f \otimes g),X_2 >$$

$$= \sqrt{2} < f \otimes g,P_2X_2 >$$

$$= \sqrt{2} < f \otimes g,X_2 >,$$

while on the other,

$$< \tilde{c}(f)g,X_2 > = < g,\tilde{a}(f)X_2 >$$

$$= - c_0 < g,T_2(T_1)^{-1}f >.$$

Now fix an orthonormal basis $\{e_n\}$ for H -- then

$$\infty > \frac{||X_2||^2}{|c_0|^2} = \sum_{n,m} | < e_n \otimes e_m,c_0^{-1}X_2 > |^2$$

$$= \frac{1}{2} \sum_{n,m} | < e_m,T_2(T_1)^{-1}e_n > |^2$$

$$= \frac{1}{2} ||T_2(T_1)^{-1}||_2^2$$

$$= \frac{1}{2} ||UC|T_2||T_1|^{-1}U^{-1}||_2^2 \quad \text{(cf. 12.9)}$$

$$= \frac{1}{2} ||C|T_2|\ |T_1|^{-1}||_2^2$$

$$= \frac{1}{2} ||\ |T_2|C|T_1|^{-1}||_2^2 \quad \text{(cf. 12.5)}.$$

Therefore

$$|T_2|C|T_1|^{-1}$$

is Hilbert-Schmidt or still,

$$|T_2| = |T_2|C|T_1|^{-1}(|T_1|C^{-1})$$

is Hilbert-Schmidt, so $T \in SP_2(H)$ (cf. 12.11).

It remains to deal with the sufficiency.

12.24 LEMMA Let $f,g \in H$ -- then

$$\tilde{a}(f)W(g)\Omega = W(g)(\tilde{a}(f) + \frac{\sqrt{-1}}{\sqrt{2}} < f,g >)\Omega$$

and

$$\tilde{c}(f)W(g)\Omega = W(g)(\tilde{c}(f) - \frac{\sqrt{-1}}{\sqrt{2}} < g,f >)\Omega.$$

Since $|T_2|$ is assumed to be Hilbert-Schmidt and since $|T_2|$ commutes with C (cf. 12.5), $\exists$ an orthonormal basis $\mathcal{O} = \{e\}$ for H consisting of eigenvectors of $|T_2|$ such that $Ce = e \ \forall \ e \in \mathcal{O}$.

Let F be a finite subset of $\mathcal{O}$ and let P_F be the orthogonal projection onto the linear span L_F of F. Fix a unit vector $u \in H$ and let P_u be the orthogonal projection onto $\underline{C}u$ -- then $\forall \ f \in L_F$,

$$||(\tilde{a}(T_1u) + \tilde{c}(T_2u))W(Uf)\Omega||^2$$

$$= ||\tilde{a}(T_1u)W(Uf)\Omega + \tilde{c}(T_2u)W(Uf)\Omega||^2$$

$$= ||W(Uf)(\tilde{a}(T_1u) + \frac{\sqrt{-1}}{\sqrt{2}} < T_1u,Uf >)\Omega$$

$$+ W(Uf)(\tilde{c}(T_2u) - \frac{\sqrt{-1}}{\sqrt{2}} < Uf,T_2u >)\Omega||^2$$

$$= ||(\tilde{a}(T_1u) + \frac{\sqrt{-1}}{\sqrt{2}} < T_1u,Uf >)\Omega$$

$$+ (\tilde{c}(T_2u) - \frac{\sqrt{-1}}{\sqrt{2}} < Uf,T_2u >)\Omega||^2$$

$$= ||(\frac{\sqrt{-1}}{\sqrt{2}} < T_1u,Uf > - \frac{\sqrt{-1}}{\sqrt{2}} < Uf,T_2u >)\Omega + T_2u||^2$$

$$= | \frac{\sqrt{-1}}{\sqrt{2}} < T_1u,Uf > - \frac{\sqrt{-1}}{\sqrt{2}} < Uf,T_2u > |^2 + ||T_2u||^2$$

$$= | \frac{\sqrt{-1}}{\sqrt{2}} < T_1u,Uf > - \frac{\sqrt{-1}}{\sqrt{2}} < Uf,T_2u > |^2 + || \, |T_2|u||^2$$

$$= \frac{1}{2} | < T_1u,Uf > - < Uf,T_2u > |^2 + || \, |T_2|u||^2.$$

But

$$| < T_1u,Uf > - < Uf,T_2u > |^2$$

$$= | < T_1u,Uf > - < f,U^{-1}T_2u > |^2$$

$$= | < U|T_1|u,Uf > - < f,C|T_2|u > |^2$$

$$= | < |T_1|u,f > - < CCf,C|T_2|u > |^2$$

$$= | < u,|T_1|f > - < |T_2|u,Cf > |^2$$

$$= | < u,|T_1|f > - < u,C|T_2|f > |^2$$

$$= | < u,|T_1|f - C|T_2|f > |^2.$$

Therefore

$$||(\tilde{a}(T_1u) + \tilde{c}(T_2u))W(Uf)\Omega||^2$$

$$\leq | < u,|T_1|f - C|T_2|f > |^2 + || \, |T_2|u||^2.$$

12.25 <u>LEMMA</u> L_F is invariant under $|T_1|$ and $C|T_2|$.

<u>PROOF</u> The definitions imply that L_F is invariant under C and $|T_2|$, hence L_F is invariant under $C|T_2|$. As for $|T_1|$, recall that $|T_1|^2 = |T_2|^2 + I$, so L_F is invariant under $|T_1|^2$, i.e., $P_F|T_1|^2 = |T_1|^2P_F$, thus $P_F|T_1| = |T_1|P_F$ (cf. 1.34), implying thereby that L_F is invariant under $|T_1|$.

Let $v_f = |T_1|f - C|T_2|f$ — then $v_f \in L_F$ and

$$| < u,|T_1|f - C|T_2|f > |^2 = | < u,v_f > |^2$$

$$= | < P_uu,P_Fv_f > |^2$$

$$= | < u,P_uP_Fv_f > |^2$$

$$\leq ||u||^2 ||P_u P_{\mathcal{F}} v_f||^2$$

$$= ||P_u P_{\mathcal{F}} v_f||^2$$

$$\leq ||P_u P_{\mathcal{F}}||^2 ||v_f||^2.$$

Write

$$||P_u P_{\mathcal{F}}||^2 \leq ||P_u P_{\mathcal{F}}||_2^2$$

$$= tr(|P_u P_{\mathcal{F}}|^2)$$

$$= tr((P_u P_{\mathcal{F}})^* P_u P_{\mathcal{F}})$$

$$= tr(P_{\mathcal{F}}^* P_u^* P_u P_{\mathcal{F}})$$

$$= tr(P_{\mathcal{F}}^* P_u^2 P_{\mathcal{F}})$$

$$= tr(P_{\mathcal{F}} P_u P_{\mathcal{F}})$$

$$= tr(P_{\mathcal{F}}^2 P_u)$$

$$= tr(P_{\mathcal{F}} P_u)$$

$$= tr(P_u P_{\mathcal{F}}).$$

Therefore

$$| < u, |T_1| f - C |T_2| f > |^2$$

$$\leq \mathrm{tr}(P_u c_f P_F),$$

where $c_f = ||v_f||^2$.

12.26 <u>LEMMA</u> Let $A \in \mathcal{B}(H)$ -- then AP_u is trace class (since P_u is trace class) and

$$\mathrm{tr}(P_u A) = < u, Au >.$$

Consequently,

$$|| \ |T_2|u||^2 = < |T_2|u, |T_2|u >$$

$$= < u, |T_2| |T_2|u >$$

$$= < u, |T_2|^2 u >$$

$$= < u, T_2^* T_2 u >$$

$$= \mathrm{tr}(P_u T_2^* T_2).$$

So, to recapitulate:

$$||(\tilde{a}(T_1 u) + \tilde{c}(T_2 u))W(Uf)\Omega||^2$$

$$\leq \mathrm{tr}(P_u c_f P_F) + \mathrm{tr}(P_u T_2^* T_2)$$

$$= \mathrm{tr}(P_u(c_f P_F + T_2^* T_2)).$$

To finish the proof of the sufficiency, we shall apply 10.18 and construct a number operator for W_T whose spectrum is bounded below by 0 (W_T is irreducible).

Let

$$\begin{cases} \tilde{a}_T(f) = \frac{1}{\sqrt{2}} (\overline{Q(Tf)} + \sqrt{-1}\ \overline{Q(T\sqrt{-1}\ f)}) \\ \tilde{c}_T(f) = \frac{1}{\sqrt{2}} (\overline{Q(Tf)} - \sqrt{-1}\ \overline{Q(T\sqrt{-1}\ f)}). \end{cases}$$

Suppose that F is a finite dimensional subspace of H and let P_F be the associated orthogonal projection. Fix an orthonormal basis $\{u_1,\ldots,u_n\}$ for F -- then the prescription

$$Q_{T,F}(f) = \sum_{i=1}^{n} ||\tilde{a}_T(u_i)f||^2 \quad (f \in \bigcap_{i=1}^{n} \mathrm{Dom}(\tilde{a}_T(u_i)))$$

is a densely defined nonnegative closed quadratic form on H which is independent of the choice of the u_i. Thus, on general grounds, $\exists$ a unique nonnegative self-adjoint operator $N_{T,F}$ such that

$$\mathrm{Dom}(Q_{T,F}) = \mathrm{Dom}(\sqrt{N_{T,F}})$$

and

$$Q_{T,F}(f) = < \sqrt{N_{T,F}}\ f, \sqrt{N_{T,F}}\ f >,$$

so, in particular,

$$Q_{T,F}(f) = < f, N_{T,F} f >$$

provided $f \in \mathrm{Dom}(N_{T,F})$.

12.27 <u>LEMMA</u> We have

$$N_{T,F} = \sum_{i=1}^{n} \tilde{a}_T(u_i)^*\tilde{a}_T(u_i).$$

The finite dimensional subspaces of H form a directed set when ordered by inclusion. This being the case, put

$$Q_T(f) = \sup_F Q_{T,F}(f),$$

where

$$\mathrm{Dom}(Q_T) = \bigcap_F \mathrm{Dom}(Q_{T,F})$$

subject to $Q_T(f) < \infty$. While Q_T is a nonnegative closed quadratic form on H, it is not a priori clear that $\mathrm{Dom}(Q_T)$ is dense (which, in the final analysis, is the crux of the matter).

12.28 LEMMA Given $f \in L_F$,

$$\sum_{i=1}^{n} ||\tilde{a}_T(u_i)W(Uf)\Omega||^2$$

$$\leq \mathrm{tr}(c(f)P_F + T_2^*T_2) < \infty.$$

PROOF In fact,

$$\sum_{i=1}^{n} ||\tilde{a}_T(u_i)W(Uf)\Omega||^2$$

$$\leq \sum_{i=1}^{n} \mathrm{tr}(P_{u_i}(c_f P_F + T_2^*T_2))$$

$$= \mathrm{tr}((\sum_{i=1}^{n} P_{u_i})(c_f P_F + T_2^*T_2))$$

$$= \operatorname{tr}(P_F(c_f P_F + T_2^* T_2))$$

$$\leq \operatorname{tr}(c_f P_F + T_2^* T_2) < \infty.$$

Every f in the linear span L_0 of 0 is, needless to say, in some L_F. Therefore

$$Q_T(W(Uf)\Omega) = \sup_F Q_{T,F}(W(Uf)\Omega) < \infty.$$

But $\{W(Uf)\Omega : f \in L_0\}$ is dense in BO(H) (cf. 9.12), so Q_T is densely defined.

Let N_T be the nonnegative selfadjoint operator corresponding to Q_T.

12.29 <u>LEMMA</u> In the strong operator topology,

$$\lim_F e^{\sqrt{-1}\, tN_{T,F}} = e^{\sqrt{-1}\, tN_T}$$

uniformly for t in finite intervals.

[Here is a sketch of the argument. First one proves that $N_{T,F} \rightarrow N_T$ in the strong resolvent sense (since the data is nonnegative, it suffices to show that $(N_{T,F} + I)^{-1} \rightarrow (N_T + I)^{-1}$ strongly). A wellknown theorem due to Trotter then implies that

$$\lim_F \|(e^{\sqrt{-1}\, tN_{T,F}} - e^{\sqrt{-1}\, tN_T})X\| = 0$$

for all $X \in$ BO(H), uniformly for t in finite intervals.]

12.30 <u>LEMMA</u> $\forall\, t \in \underline{R}$,

$$\lim_{F} e^{\sqrt{-1}\, tN_{T,F}} W_T(f) e^{-\sqrt{-1}\, tN_{T,F}}$$

$$= e^{\sqrt{-1}\, tN_T} W_T(f) e^{-\sqrt{-1}\, tN_T}.$$

PROOF Let $X \in BO(H)$ and fix $\varepsilon > 0$. Choose F_1 such that

$$F \supset F_1 \Rightarrow$$

$$\| e^{-\sqrt{-1}\, tN_{T,F}} X - e^{-\sqrt{-1}\, tN_T} X \| < \varepsilon/2.$$

Choose F_2 such that

$$F \supset F_2 \Rightarrow$$

$$\| e^{\sqrt{-1}\, tN_{T,F}} W_T(f) e^{-\sqrt{-1}\, tN_T} X$$

$$- e^{\sqrt{-1}\, tN_T} W_T(f) e^{-\sqrt{-1}\, tN_T} X \| < \varepsilon/2.$$

Then

$$F \supset F_1, F_2 \Rightarrow$$

$$\| e^{\sqrt{-1}\, tN_{T,F}} W_T(f) e^{-\sqrt{-1}\, tN_{T,F}} X$$

$$- e^{\sqrt{-1}\, tN_T} W_T(f) e^{-\sqrt{-1}\, tN_T} \|$$

$$= \| e^{\sqrt{-1}\, tN_{T,F}} W_T(f) e^{-\sqrt{-1}\, tN_{T,F}} X$$

$$- e^{\sqrt{-1}\, tN_{T,F}} W_T(f) e^{-\sqrt{-1}\, tN_T} X$$

$$+ e^{\sqrt{-1}\, tN_{T,F}} W_T(f) e^{-\sqrt{-1}\, tN_T} x$$

$$- e^{\sqrt{-1}\, tN_T} W_T(f) e^{-\sqrt{-1}\, tN_T} x||$$

$$\leq ||e^{-\sqrt{-1}\, tN_{T,F}} x - e^{-\sqrt{-1}\, tN_T} x||$$

$$+ ||e^{\sqrt{-1}\, tN_{T,F}} W_T(f) e^{-\sqrt{-1}\, tN_T} x$$

$$- e^{\sqrt{-1}\, tN_T} W_T(f) e^{-\sqrt{-1}\, tN_T} x||$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

12.31 <u>LEMMA</u> Let $f \in F$ -- then $\forall\, t \in \underline{R}$,

$$e^{\sqrt{-1}\, tN_{T,F}} W_T(f) e^{-\sqrt{-1}\, tN_{T,F}} = W_T(e^{\sqrt{-1}\, t} f).$$

Since the set of F containing a given f is cofinal in the set of all F, 12.30 and 12.31 imply that $\forall\, t \in \underline{R}$,

$$e^{\sqrt{-1}\, tN_T} W_T(f) e^{-\sqrt{-1}\, tN_T} = W_T(e^{\sqrt{-1}\, t} f).$$

This shows that N_T is a number operator for W_T. But its spectrum is bounded below by 0 (N_T being nonnegative). Therefore, thanks to 10.18, W_T is unitarily equivalent to W.

12.32 <u>REMARK</u> The proof of sufficiency is incomplete in several respects.

1. It depends on 10.18, which in turn depends on 10.17, whose proof was

omitted.

2. It depends on 12.29, whose proof was only sketched.

3. It depends on 12.31, whose proof was omitted.

There are other approaches that circumvent these difficulties (and avoid the use of number operators altogether) but I shall forgo the details.

12.33 EXAMPLE Take $\mathcal{H}$ infinite dimensional and fix a closed subset $\mathcal{H}_0 \subset \mathcal{H}$ such that:

1. $f,g \in \mathcal{H}_0 \Rightarrow \langle f,g \rangle \in \underline{R}$.

2. $f,g \in \mathcal{H}_0 \Rightarrow af + bg \in \mathcal{H}_0 \quad (a,b \in \underline{R})$.

3. $\mathcal{H} = \mathcal{H}_0 + \sqrt{-1}\, \mathcal{H}_0$.

Define $T_\rho : \mathcal{H} \to \mathcal{H}$ by

$$T_\rho(f + \sqrt{-1}\, g) = \rho f + \sqrt{-1}\, \rho^{-1} g \quad (f,g \in \mathcal{H}_0, \rho > 0).$$

Then T_ρ is symplectic and $T_\rho^+ = T_\rho$. Therefore

$$\begin{aligned}(T_\rho^+ T_\rho - I)(f + \sqrt{-1}\, g) \\ &= (T_\rho^2 - I)(f + \sqrt{-1}\, g) \\ &= (\rho^2 - 1)f + \sqrt{-1}\, (\rho^{-2} - 1)g,\end{aligned}$$

which is Hilbert-Schmidt iff $\rho = 1$, so T_ρ is implementable iff $\rho = 1$ (cf. 12.15).

Let $T \in SP_2(\mathcal{H})$ -- then $|T_1| - I$ is trace class (cf. 12.12), hence

$$|T_1| = (|T_1| - I) + I$$

has a determinant (which is necessarily nonzero).

12.34 <u>LEMMA</u> Let $T \in SP_2(H)$ -- then

$$| < \Omega, \Gamma_T \Omega > | = (\det(|T_1|))^{-1/2}.$$

§13. METAPLECTIC MATTERS

Let

$$W: \mathcal{H} \to \mathcal{U}(\mathrm{BO}(\mathcal{H}))$$

be the Fock system — then according to Shale's theorem (cf. 12.19), $\forall\ T \in SP_2(\mathcal{H})$,

$$W_T(f) = W(Tf) \quad (f \in \mathcal{H})$$

is implementable, i.e., $\exists\ \Gamma_T \in \mathcal{U}(\mathrm{BO}(\mathcal{H}))$ such that $\forall\ f \in \mathcal{H}$,

$$\Gamma_T W(f) \Gamma_T^{-1} = W_T(f).$$

Let $\underline{U}(1)$ denote the group of unitary scalar operators on $\mathrm{BO}(\mathcal{H})$ -- then, in view of the irreducibility of W (cf. 9.11), any two implementers Γ_T', Γ_T'' are congruent modulo $\underline{U}(1)$, thus we have an arrow

$$\begin{cases} SP_2(\mathcal{H}) \to \mathcal{U}(\mathrm{BO}(\mathcal{H}))/\underline{U}(1) \\ T \to [\Gamma_T], \end{cases}$$

where $[\Gamma_T]$ is the coset determined by Γ_T.

13.1 <u>LEMMA</u> The arrow

$$\begin{cases} SP_2(\mathcal{H}) \to \mathcal{U}(\mathrm{BO}(\mathcal{H}))/\underline{U}(1) \\ T \to [\Gamma_T] \end{cases}$$

is a homomorphism.

Suppose that dim $H < \infty$ -- then it is wellknown that one can attach to each $T \in SP(H)$ $(\equiv SP_2(H)!)$ a pair of unitary operators $\{\pm \Gamma_T\}$ which implement W_T and have the property that the arrow

$$\begin{cases} SP(H) \rightarrow U(BO(H))/\{\pm I\} \\ T \rightarrow \{\pm \Gamma_T\} \end{cases}$$

is a homomorphism.

13.2 <u>REMARK</u> This arrow is called the <u>metaplectic representation</u> of $SP(H)$ (it is a bona fide unitary representation of $MP(H)$, the double covering group of $SP(H)$).

The situation when H is infinite dimensional is different. Thus denote by $SP_+(H)$ the subset of $SP_2(H)$ consisting of those T such that $T_1 - I$ is trace class -- then $SP_+(H)$ is a normal subgroup of $SP_2(H)$.

13.3 <u>LEMMA</u> $SP_+(H)$ is a connected topological group if one uses the trace norm topology on the complex linear part and the Hilbert-Schmidt topology on the complex conjugate linear part:

$$d_+(T',T'') = ||T_1' - T_1''||_1 + ||T_2' - T_2''||_2.$$

13.4 <u>REMARK</u> Equip $SP_2(H)$ with its structure of a topological group per

12.10 -- then the inclusion $SP_+(H) \to SP_2(H)$ is a continuous homomorphism (the trace norm dominates operator norm). Now endow $U(H)$ with the operator norm topology -- then it can be shown that $SP_2(H)$ and $U(H)$ have the same homotopy type. But a classical theorem due to Kuiper says that $U(H)$ is contractible. Therefore in the infinite dimensional case, $SP_2(H)$ is simply connected which is in stark contrast to the situation in the finite dimensional case.

What was said when $\dim H < \infty$ goes through when $\dim H = \infty$ provided one works with $SP_+(H)$, i.e., one can attach to each $T \in SP_+(H)$ a pair of unitary operators $\{\pm \Gamma_T\}$ which implement W_T and have the property that the arrow

$$\begin{cases} SP_+(H) \to U(BO(H))/\{\pm I\} \\ T \to \{\pm \Gamma_T\} \end{cases}$$

is a homomorphism.

§14. *KERNELS*

Let X be a nonempty set -- then a map $K:X \times X \rightarrow \underline{C}$ is called a <u>kernel</u> if for all

$$\left[\begin{array}{l} x_1,\ldots,x_n \in X \\ c_1,\ldots,c_n \in \underline{C}, \end{array}\right.$$

we have

$$\sum_{i,j=1}^{n} \bar{c}_i c_j K(x_i,x_j) \geq 0.$$

14.1 <u>EXAMPLE</u> Take $X = H$, a complex Hilbert space -- then $K(x,y) = \langle x,y \rangle$ is a kernel on H.

14.2 <u>EXAMPLE</u> Let G be a group and let $U:G \rightarrow U(H)$ be a homomorphism. Given a unit vector $x \in H$, put $K_x(\sigma,\tau) = \langle x,U(\sigma^{-1}\tau)x \rangle$ $(\sigma,\tau \in G)$ -- then K_x is a kernel on G.

[Note: The function $\sigma \rightarrow \langle x,U(\sigma)x \rangle$ is positive definite.]

14.3 <u>EXAMPLE</u> Take $X = B(H)$ and suppose that $T \in \underline{L}_1(H)$ is nonnegative -- then $K_T(A,B) = tr(TA^*B)$ is a kernel on $B(H)$.

Let $A = [a_{ij}]$ be an n-by-n matrix $(a_{ij} \in \underline{C})$ -- then A is said to be <u>positive definite</u> if for every sequence $c_1,\ldots,c_n$ of n complex numbers,

$$\sum_{i,j=1}^{n} \bar{c}_i c_j a_{ij} \geq 0.$$

[Note: A positive definite n-by-n matrix determines a kernel on $\{1,\ldots,n\}$ (and vice-versa).]

14.4 <u>REMARK</u> If K is a kernel on X, then the matrix $[K(x_i,x_j)]$ is positive definite, hence in particular

$$K(x,y) = \overline{K(y,x)}.$$

14.5 <u>LEMMA</u> If $A = [a_{ij}]$ and $B = [b_{ij}]$ are positive definite, then so is $C = [a_{ij}b_{ij}]$ (the entrywise product of A and B).

<u>PROOF</u> Let

$$y_{ij} = c_i \bar{c}_j b_{ji}.$$

Then $Y = [y_{ij}]$ is positive definite:

$$\sum_{i,j=1}^{n} \bar{z}_i z_j y_{ij} = \sum_{i,j=1}^{n} \bar{z}_i z_j c_i \bar{c}_j b_{ji}$$

$$= \sum_{i,j=1}^{n} \overline{(\bar{z}_j c_j)} (\bar{z}_i c_i) b_{ji}$$

$$= \sum_{i,j=1}^{n} \overline{(\bar{z}_i c_i)} (\bar{z}_j c_j) b_{ij}$$

$$\geq 0.$$

Therefore $\text{tr}(AY) \geq 0$, i.e.,

$$\sum_{i,j=1}^{n} a_{ij}y_{ji} = \sum_{i,j=1}^{n} a_{ij}c_j\bar{c}_i b_{ij}$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j a_{ij} b_{ij}$$

$$\geq 0.$$

Denote by $K(X)$ the set whose elements are the kernels on X -- then 14.5 implies that $K(X)$ is closed under pointwise multiplication.

14.6 LEMMA If $A = [a_{ij}]$ is positive definite, then so is $[E(A)_{ij}]$, where

$$E(A)_{ij} = e^{a_{ij}}.$$

Corollary: $K \in K(X) \Rightarrow e^K \in K(X)$.

14.7 THEOREM (The Kolmogorov Construction) Let K be a kernel on X -- then $\exists$ a complex Hilbert space $\mathcal{H}_K$ (not necessarily separable) and a map $\Lambda: X \to \mathcal{H}_K$ such that

$$K(x,y) = \langle \Lambda(x), \Lambda(y) \rangle$$

and the set $\{\Lambda(x) : x \in X\}$ is total in $\mathcal{H}_K$.

PROOF Consider the vector space $\underline{C}^{(X)}$ of all complex valued functions $f: X \to \underline{C}$ such that $f(x) = 0$ except for at most a finite set of x. Put

$$< f,g > = \sum_{x,y} \overline{f(x)} g(y) K(x,y).$$

Then the pair $(\underline{C}^{(X)}, < \, , \, >)$ is a complex, potentially non Hausdorff, pre-Hilbert space. To get a genuine pre-Hilbert space, divide out by $N = \{f: < f,f > = 0\}$ and then take for H_K the completion of $\underline{C}^{(X)}/N$. As for Λ, simply note that

$$K(x,y) = < \delta_x, \delta_y >.$$

[Note: If H_K' is another Hilbert space and if $\Lambda':X \to H_K'$ is another map satisfying the preceding conditions, then there is an isometric isomorphism $T:H_K \to H_K'$ such that $T\Lambda(x) = \Lambda'(x) \ \forall \ x \in X$.]

14.8 <u>REMARK</u> If X is a topological space and if $K:X \times X \to \underline{C}$ is continuous, then $\Lambda:X \to H_K$ is continuous. In fact,

$$||\Lambda(x) - \Lambda(y)||^2$$

$$= < \Lambda(x) - \Lambda(y), \Lambda(x) - \Lambda(y) >$$

$$= K(x,x) + K(y,y) - 2\text{Re}\, K(x,y)$$

$$\to 0$$

if $x \to y$.

14.9 <u>EXAMPLE</u> Let H be a separable complex Hilbert space. Put

$$K(f,g) = e^{<f,g>} \quad (f,g \in H).$$

Then K is a kernel on H and $H_K = BO(H)$.

[Note: Here $\Lambda:H \rightarrow BO(H)$ is the map $f \rightarrow \underline{\exp}(f)$.]

14.10 EXAMPLE Let G be a group. Given a positive definite function $\chi:G \rightarrow \underline{C}$ with $\chi(e) = 1$, put $K_\chi(\sigma,\tau) = \chi(\sigma^{-1}\tau)$ $(\sigma,\tau \in G)$ — then K_χ is a kernel on G so, in view of 14.7, $\exists$ a complex Hilbert space H_χ, a homomorphism $U_\chi:G \rightarrow U(H_\chi)$, and a cyclic unit vector $x_\chi \in H_\chi$ such that $\forall\ \sigma \in G$,

$$\chi(\sigma) = < x_\chi, U_\chi(\sigma)x_\chi >.$$

Spelled out, x_χ is the image of δ_e and $U_\chi(\sigma)$ is the operator associated with $U(\sigma):\underline{C}^{(G)} \rightarrow \underline{C}^{(G)}$, where $(U(\sigma)f)(\tau) = f(\sigma^{-1}\tau)$:

$$< U(\sigma)f,U(\sigma)f >$$

$$= \sum_{x,y} \overline{(U(\sigma)f)(x)}\,(U(\sigma)f)(y)\chi(x^{-1}y)$$

$$= \sum_{x,y} \overline{f(\sigma^{-1}x)}\,f(\sigma^{-1}y)\chi(x^{-1}y)$$

$$= \sum_{x,y} \overline{f(x)}\,f(y)\chi(x^{-1}\sigma^{-1}\sigma y)$$

$$= \sum_{x,y} \overline{f(x)}\,f(y)\chi(x^{-1}y)$$

$$= < f,f >.$$

[Note: If G is a topological group and if χ is continuous, then $U_\chi: G \to \mathcal{U}(\mathcal{H}_\chi)$ is strongly continuous, i.e., is a unitary representation. Thus suppose that $\sigma \to e$ -- then

$$< \Lambda(\tau_1), U(\sigma)\Lambda(\tau_2) >$$

$$= \sum_{x,y} \overline{\delta_{\tau_1}(x)}\,\delta_{\tau_2}(\sigma^{-1}y)\,\chi(x^{-1}y)$$

$$= \chi(\tau_1^{-1}\sigma\tau_2)$$

$$\to \chi(\tau_1^{-1}\tau_2) = K(\tau_1,\tau_2)$$

$$= < \Lambda(\tau_1), \Lambda(\tau_2) >.$$

And this suffices ($U(\sigma)$ is unitary and $\Lambda(G)$ is total).]

14.11 <u>EXAMPLE</u> Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be complex Hilbert spaces with respective inner products $< , >_1, \ldots, < , >_n$. Put

$$K(x,y) = \prod_{k=1}^{n} < x_k, y_k >_k,$$

where

$$\begin{cases} x = (x_1, \ldots, x_n) \\ y = (y_1, \ldots, y_n). \end{cases}$$

Then K is a kernel on $H_1 \times \dots \times H_n$ and

$$H_K = H_1 \hat{\otimes} \cdots \hat{\otimes} H_n.$$

Suppose given a sequence of separable complex Hilbert spaces H_n and a sequence of unit vectors $u_n \in H_n$ $(n = 1,2,\dots)$. Let X be the set of sequences $x = \{x_n\}$:

$$x_n \in H_n \ \& \ x_n = u_n \quad (n >> 0).$$

Define $K: X \times X \rightarrow \underline{C}$ by

$$K(x,y) = \prod_{n=1}^{\infty} < x_n, y_n >_n .$$

Then K is a kernel on X. Now apply the Kolmogorov construction -- then the resulting Hilbert space H_K is called the <u>countable tensor product</u> of the H_n w.r.t. the <u>stabilizing sequence</u> u_n:

$$\bigotimes_{n=1}^{\infty} (H_n, u_n)$$

and we write

$$\Lambda(x) = x_1 \otimes x_2 \otimes \cdots \quad (x \in X).$$

If $E_n = \{e_{n0}, e_{n1}, \dots\}$ is an orthonormal basis for H_n such that $e_{n0} = u_n \ \forall \ n$, then the set $\{\Lambda(x) : x \in X \ \& \ x_n \in E_n \ \forall \ n\}$ is an orthonormal basis for $\bigotimes_{n=1}^{\infty} (H_n, u_n)$.

14.12 <u>REMARK</u> Abstractly, the countable tensor product of the H_n w.r.t.

the stabilizing sequence u_n is a system (H,u,T_Δ) consisting of a complex Hilbert space H, a unit vector $u \in H$, and for each finite subset $\Delta \subset \underline{N}$ an isometric map T_Δ from $\hat{\bigotimes}_{n\in\Delta} H_n$ into H with the following properties:

1. $\forall\ \Delta$,

$$T_\Delta(\bigotimes_{n\in\Delta} u_n) = u;$$

2. $\forall\ \Delta,\Delta':\Delta \subset \Delta' \Rightarrow$

$$T_{\Delta'}(\bigotimes_{n\in\Delta'} x_n) = T_\Delta(\bigotimes_{n\in\Delta} x_n)$$

if $x_n = u_n$ for $n \in \Delta' - \Delta$;

3. $\overline{\bigcup_\Delta \text{Ran}\ T_\Delta} = H.$

[Note: These properties characterize $\bigotimes_1^\infty (H_n,u_n)$ to within unitary equivalence.]

14.13 <u>EXAMPLE</u> Suppose that $H = \bigoplus_{n=1}^\infty H_n$ — then there is an isometric isomorphism

$$T:BO(H) \rightarrow \bigotimes_{n=1}^\infty (BO(H_n),\Omega_n).$$

E.g.:

$$\ell^2(\underline{N}) = \underline{C} \oplus \underline{C} \oplus \cdots$$

$\Rightarrow$

$$BO(\ell^2(\underline{N})) = BO(\underline{C}) \otimes BO(\underline{C}) \otimes \cdots,$$

where the countable tensor product is w.r.t. the stabilizing sequence of vacuum vectors.

14.14 <u>EXAMPLE</u> Let $\mathcal{H}_n = L^2(\Omega_n, \mathcal{A}_n, \mu_n)$, where $\forall$ n, μ_n is a probability measure on the σ-algebra $\mathcal{A}_n$. Consider the product probability space $(\Omega, \mathcal{A}, \mu)$ -- then $L^2(\Omega, \mathcal{A}, \mu)$ is the countable tensor product of the $L^2(\Omega_n, \mathcal{A}_n, \mu_n)$ w.r.t. the stabilizing sequence 1_n (= the constant function 1 on Ω_n).

§15. *C*-ALGEBRAS*

In this section, I shall give a more or less proofless summary of those definitions and facts from the theory that will be of use in the sequel.

Let A be a nonzero complex Banach algebra, $*:A \to A$ an involution — then the pair $(A,*)$ is said to be a C*-algebra if $\forall\ A \in A$,

$$||A^*A|| = ||A||^2.$$

It is then automatic that $||A^*|| = ||A||$.

[Note: A morphism of C*-algebras is a linear map $\phi:A \to B$ such that

$$\phi(A_1A_2) = \phi(A_1)\phi(A_2) \ \ \&\ \ \phi(A^*) = \phi(A)^*.$$

An isomorphism is a bijective morphism. Every morphism is automatically continuous: $||\phi(A)|| \leq ||A||\ \forall\ A \in A$. Furthermore, the kernel of ϕ is a closed ideal in A and the image of ϕ is a C*-subalgebra of B. Finally, ϕ injective => ϕ isometric: $||\phi(A)|| = ||A||\ \forall\ A \in A$.]

15.1 EXAMPLE Let X be a LCH space, $C_\infty(X)$ the algebra of complex valued continuous functions on X that vanish at infinity. Equip $C_\infty(X)$ with the sup norm and let the involution be complex conjugation — then the pair $(C_\infty(X),*)$ is a commutative C*-algebra.

[Note: If A is an arbitrary commutative C*-algebra, then $\exists$ a LCH space X and an isomorphism $A \to C_\infty(X)$. Such an X is unique up to homeomorphism and is compact when A is unital.]

15.2 EXAMPLE Let H be a complex Hilbert space, $\mathcal{B}(H)$ the algebra of bounded linear operators on H. Equip $\mathcal{B}(H)$ with the operator norm and let the involution $*$ be the adjunction -- then the pair $(\mathcal{B}(H),*)$ is a C*-algebra.

[Note: A norm closed *-subalgebra of $\mathcal{B}(H)$ is a C*-algebra. Conversely, every C*-algebra is isomorphic to a norm closed *-subalgebra of $\mathcal{B}(H)$ for some H.]

We shall assume henceforth that A is unital (i.e., has a unit I).

[Note: If A is a C*-algebra without a unit, then there exists a unital C*-algebra A_I, the unitization of A, and an injective morphism $A \to A_I$ such that $A_I/A = \underline{C}$.]

N.B. To reflect the assumption that our C*-algebras are unital, the term morphism will now carry the additional requirement that the units are respected.

Let A be a C*-algebra, H a complex Hilbert space -- then a representation π of A on H is a morphism $\pi:A \to \mathcal{B}(H)$ (thus π is automatically continuous: $||\pi(A)|| \leq ||A|| \ \forall \ A \in A$ (which sharpens to $||\pi(A)|| = ||A|| \ \forall \ A \in A$ if π is faithful)). In particular: $\pi(I) = I$.

15.3 LEMMA Every representation is a direct sum of cyclic representations.

[Note: A representation $\pi:A \to \mathcal{B}(H)$ is cyclic if $\exists \ x \in H:\pi(A)x = \{\pi(A)x:A \in A\}$ is dense in H.]

15.4 REMARK Every representation of a simple C*-algebra is faithful.

[Note: A C*-algebra is said to be simple if it has no nontrivial closed ideals. If A is simple, then A has no nontrivial ideals period and, in addition, is <u>central</u>, meaning that the center of A is $\{cI:c \in \underline{C}\}$.]

Let A be a C*-algebra.

- $A_{\underline{R}}$ is the collection of all selfadjoint elements in A, i.e.,

$$A_{\underline{R}} = \{A \in A : A^* = A\}.$$

- A^+ is the collection of all positive elements in A, i.e.,

$$A^+ = \{A^2 : A \in A_{\underline{R}}\}$$

or still,

$$A^+ = \{A^*A : A \in A\}.$$

A <u>state</u> on A is a linear functional $\omega : A \to \underline{C}$ such that

$$\begin{cases} \omega(A) \geq 0 & \forall\, A \in A^+ \\ \omega(I) = 1. \end{cases}$$

[Note: A state ω is necessarily hermitian: $\omega(A^*) = \overline{\omega(A)} \quad \forall\, A \in A$.]

Let $S(A)$ be the state space of A (meaning the set of states on A) -- then $S(A)$ is convex and its elements are continuous of norm 1, thus $S(A)$ is contained in the unit ball of the dual of A. It is easy to verify that $S(A)$ is closed in the weak* topology, so $S(A)$ is compact (Alaoglu).

15.5 <u>EXAMPLE</u> Suppose that π is a representation of A on H. Fix a unit vector $\Omega \in H$ -- then the linear functional

$$\omega(A) = < \Omega, \pi(A)\Omega >$$

is a state on A.

15.6 THEOREM (The GNS Construction) Let $\omega \in S(A)$ — then $\exists$ a cyclic representation π_ω of A on a Hilbert space H_ω with cyclic unit vector Ω_ω such that

$$\omega(A) = < \Omega_\omega, \pi_\omega(A)\Omega_\omega >.$$

PROOF In the Kolmogorov construction (cf. 14.7), take $X = A$, let $K(A,B) = \omega(A^*B)$, and put $H_\omega = H_K$. Denote by Ω_ω the image of δ_I and call $\pi_\omega(A)$ the operator associated with $\pi(A):\underline{C}^{(A)} \to \underline{C}^{(A)}$, where

$$\pi(A)f = \pi(A) \sum_{x \in A} c_x \delta_x$$

$$= \sum_{x \in A} c_x \delta_{Ax}.$$

Then

$$\omega(A) = \omega(I^*A)$$

$$= K(I,A)$$

$$= < \delta_I, \delta_A >$$

$$= < \delta_I, \pi(A)\delta_I >$$

$$= < \Omega_\omega, \pi_\omega(A)\Omega_\omega >.$$

[Note: If $(\pi_\omega', H_\omega', \Omega_\omega')$ is another triple of GNS data per ω, then there is an isometric isomorphism $T:H_\omega \to H_\omega'$ which intertwines π_ω and π_ω' and sends Ω_ω to Ω_ω'.]

15.7 REMARK Suppose that π is a cyclic representation of A. Take any cyclic unit vector Ω and perform the GNS construction on

$$\omega(A) = < \Omega, \pi(A)\Omega >.$$

Then π_ω is unitarily equivalent to π.

The <u>universal representation</u> π_{UN} of A is the direct sum of all its GNS representations $\pi_\omega(\omega \in S(A))$, thus

$$H_{UN} = \bigoplus_{\omega \in S(A)} H_\omega.$$

15.8 <u>LEMMA</u> $\forall\ A \in A_{\underline{R}}, \exists\ \omega \in S(A)$:

$$||A|| = |\omega(A)|.$$

15.9 <u>THEOREM</u> (Gelfand-Naimark) π_{UN} is faithful.

<u>PROOF</u> In fact,

$$\begin{aligned} &\pi_{UN}(A) = 0 \\ &\quad \Rightarrow \pi_\omega(A)\Omega_\omega = 0\ \forall\ \omega \\ &\quad \Rightarrow ||\pi_\omega(A)\Omega_\omega||^2 = 0\ \forall\ \omega \\ &\quad \Rightarrow \omega(A^*A) = 0\ \forall\ \omega \\ &\quad \Rightarrow A^*A = 0\ (\text{cf. } 15.8) \end{aligned}$$

$$\Rightarrow ||A^*A|| = ||A||^2 = 0$$

$$\Rightarrow A = 0.$$

15.10 REMARK Since π_{UN} is faithful, it is isometric:

$$||\pi_{UN}(A)|| = ||A|| \quad (A \in \mathcal{A}).$$

Suppose that $\alpha:\mathcal{A} \rightarrow \mathcal{A}$ is an automorphism of $\mathcal{A}$ -- then α induces a bijection $\alpha^*:S(\mathcal{A}) \rightarrow S(\mathcal{A})$, where $\alpha^*\omega = \omega \circ \alpha$.

15.11 LEMMA There exists an isometric isomorphism $T:\mathcal{H}_\omega \rightarrow \mathcal{H}_{\alpha^*\omega}$ such that

$$\pi_{\alpha^*\omega}(A) = T\pi_\omega(\alpha(A))T^{-1}$$

for all $A \in \mathcal{A}$.

Let $\omega \in S(\mathcal{A})$ -- then ω is pure iff it is an extreme point of $S(\mathcal{A})$.

[Note: A state that is not pure is called mixed. If $\mathcal{A}$ is commutative, then a state ω is pure iff it is multiplicative, i.e., iff

$$\omega(AB) = \omega(A)\omega(B)$$

for all $A,B \in \mathcal{A}$.]

15.12 THEOREM (Segal) The GNS representation π_ω associated with a state ω is irreducible iff ω is pure.

15.13 <u>REMARK</u> Assume that $\pi: A \to B(H)$ is irreducible, take any unit vector $\Omega \in H$, and let

$$\omega(A) = < \Omega, \pi(A)\Omega >.$$

Then Ω is cyclic (cf. 9.9), so π_ω is unitarily equivalent to π (cf. 15.7). In particular: π_ω is irreducible, thus ω is pure (cf. 15.12).

[Note: Therefore every irreducible representation of a C*-algebra comes from a pure state via the GNS construction.]

Denote by $P(A)$ the set of pure states on A.

15.14 <u>LEMMA</u> $\forall\, A \in A_{\underline{R}}, \exists\, \omega \in P(A): ||A|| = |\omega(A)|$.

The <u>atomic representation</u> π_{AT} of A is the direct sum of the GNS representations π_ω, where $\omega \in P(A)$. Because of 15.14, one can argue exactly as in 15.9 to conclude that π_{AT} is faithful.

Let $\hat{A}$ be the set of unitary equivalence classes of irreducible representations of A -- then the canonical arrow

$$\begin{cases} P(A) \to \hat{A} \\ \omega \to [\pi_\omega] \end{cases}$$

is surjective. It is bijective iff every irreducible representation of A is one dimensional, which is the case iff A is commutative.

Let $U \in A$ -- then U is said to be <u>unitary</u> if $U^*U = UU^* = I$.

[Note: Therefore U is invertible and $||U|| = 1$.]

15.15 <u>LEMMA</u> Let $\omega_1, \omega_2 \in P(A)$ -- then $\pi_{\omega_1}, \pi_{\omega_2}$ are unitarily equivalent (i.e., $[\pi_{\omega_1}] = [\pi_{\omega_2}]$) iff there is a unitary $U \in A$:

$$\omega_2(A) = \omega_1(UAU^{-1}) \quad (A \in A).$$

15.16 <u>REMARK</u> Suppose that $\pi: A \rightarrow B(H)$ is an irreducible representation. Let

$$\begin{cases} \Omega_1 \in H \\ \Omega_2 \in H \end{cases}$$

be unit vectors. Put

$$\begin{cases} \omega_1(A) = \langle \Omega_1, \pi(A)\Omega_1 \rangle \\ \omega_2(A) = \langle \Omega_2, \pi(A)\Omega_1 \rangle. \end{cases}$$

Then $\omega_1 = \omega_2$ iff $\exists\ c(|c| = 1): \Omega_2 = c\Omega_1$.

Let Rep A be the set of all representations of A -- then in Rep A there are three standard notions of "equivalence":

1. unitary equivalence;
2. geometric equivalence;
3. weak equivalence.

As we shall see, 1 => 2 => 3 and these implications are not reversible (except in certain special situations).

Let H be a complex Hilbert space -- then a <u>density operator</u> is a bounded linear operator W on H such that:

1. W is nonnegative (hence selfadjoint).

2. W is trace class with tr(W) = 1.

Let A be a C*-algebra, π a representation of A on H -- then the <u>folium</u> of π is the set $F(\pi)$ of states on A of the form

$$A \rightarrow \mathrm{tr}(\pi(A)W),$$

where W is a density operator on H.

[Note: The folium $F(\omega)$ of a state $\omega \in S(A)$ is, by definition, $F(\pi_\omega)$. Since the orthogonal projection onto $\underline{C}\Omega_\omega$ is a density operator, it follows that $\omega \in F(\omega)$.]

15.17 <u>LEMMA</u> Let π be a representation of A -- then

$$\mathrm{Ker}\ \pi = \bigcap_{\omega \in F(\pi)} \mathrm{Ker}\ \omega.$$

15.18 <u>THEOREM</u> (Fell) The folium of a faithful representation of A is weak* dense in the set of all states on A.

Let π_1, π_2 be representations of A -- then π_1, π_2 are said to be <u>geometrically equivalent</u> if $F(\pi_1) = F(\pi_2)$.

[Note: States ω_1, ω_2 are geometrically equivalent provided this is the case of $\pi_{\omega_1}, \pi_{\omega_2}$.]

15.19 <u>REMARK</u> If π_1, π_2 are geometrically equivalent, then Ker π_1 = Ker π_2 (cf. 15.17).

[Note: One says that π_1 is <u>weakly equivalent</u> to π_2 if Ker π_1 = Ker π_2. Accordingly,

"geometric equivalence" => "weak equivalence".]

15.20 <u>LEMMA</u> Representations π_1, π_2 are geometrically equivalent iff π_1 is unitarily equivalent to a subrepresentation of a multiple of π_2 and vice versa.

[Note: Therefore a given representation is geometrically equivalent to any of its multiples.]

In particular:

"unitary equivalence" => "geometric equivalence".

15.21 <u>LEMMA</u> Representations π_1, π_2 are geometrically equivalent iff $\exists$ a cardinal number n such that $n\pi_1$ is unitarily equivalent to $n\pi_2$.

15.22 <u>REMARK</u> If π_1 is irreducible and π_2 is geometrically equivalent to π_1, then π_2 is unitarily equivalent to a multiple of π_1. Thus if π_2 is also irreducible, then π_1 and π_2 are unitarily equivalent.

Let π_1, π_2 be representations of A — then π_1, π_2 are said to be <u>disjoint</u> if $F(\pi_1) \cap F(\pi_2) = \emptyset$.

[Note: States ω_1, ω_2 are disjoint provided this is the case of $\pi_{\omega_1}, \pi_{\omega_2}$.]

15.23 LEMMA Representations π_1, π_2 are disjoint iff π_1, π_2 have no geometrically equivalent subrepresentations or still, iff π_1, π_2 have no unitarily equivalent subrepresentations.

15.24 LEMMA Representations π_1, π_2 are geometrically equivalent iff π_1 has no subrepresentation disjoint from π_2 and vice versa.

A representation π of A is said to be primary if every subrepresentation of π is geometrically equivalent to π.

[Note: A state ω is primary if this is so of π_ω.]

If π is irreducible, then π is primary (as is $\pi \oplus \pi$ which, of course, is not irreducible).

15.25 LEMMA Two primary representations of A are either geometrically equivalent or disjoint.

15.26 LEMMA If π is primary and if $\omega \in F(\pi)$, then π is geometrically equivalent to π_ω.

Given a state $\omega \in S(A)$ and $A \in A$ such that $\omega(A^*A) > 0$, define $\omega_A \in S(A)$ by

$$\omega_A = \frac{\omega(A^* \cdot A)}{\omega(A^*A)} .$$

15.27 LEMMA Let $\omega \in S(A)$ — then $F(\omega)$ is the norm closed convex hull of the ω_A.

[Note: So, if $\omega_1,\omega_2 \in S(A)$, then $F(\omega_1) = F(\omega_2)$ iff $\omega_1 \in F(\omega_2)$ & $\omega_2 \in F(\omega_1)$.]

A folium in $S(A)$ is a norm closed convex subset F of $S(A)$ with the property that if $\omega \in F$, then $\omega_A \in F$ for all $A:\omega(A^*A) > 0$.

The terminology is consistent since the folium $F(\pi)$ of a representation π is a folium in $S(A)$.

15.28 REMARK If $\omega \in S(A)$, then $F(\omega)$ is the smallest folium containing ω (cf. 15.27).

15.29 LEMMA If F is a folium in $S(A)$, then $\exists$ a representation π of A, determined up to geometric equivalence, such that $F(\pi) = F$.

[One has only to take for π the direct sum of the GNS representations π_ω $(\omega \in F)$.]

[Note: The folia in $S(A)$ are thus in a one-to-one correspondence with the geometric equivalence classes in Rep A.]

15.30 EXAMPLE Let $\pi \in \text{Rep } A$ — then π is geometrically equivalent to the direct sum of the GNS representations π_ω $(\omega \in F(\pi))$.

Given representations π_1,π_2, write $\pi_1 \leq \pi_2$ if π_1 is geometrically equivalent to a subrepresentation of π_2 or still, if $F(\pi_1) \subset F(\pi_2)$.

15.31 <u>LEMMA</u> Every representation π of A is geometrically equivalent to a subrepresentation of the universal representation π_{UN}, hence $\pi \leq \pi_{UN}$ and

$$F(\pi) \subset F(\pi_{UN}) \equiv S(A).$$

§16. *SLAWNY'S THEOREM*

Let (E,σ) be a symplectic vector space -- then a <u>CCR realization</u> of (E,σ) is a unital C*-algebra $\mathcal{W}(E,\sigma)$ which is generated by nonzero elements $W(f)$ $(f \in E)$ subject to

$$W(f)^* = W(-f) \quad (f \in E)$$

and

$$W(f)W(g) = \exp\left(-\frac{\sqrt{-1}}{2}\sigma(f,g)\right)W(f+g) \quad (f,g \in E).$$

[Note: Obviously,

$$W(f)W(0) = W(f) = W(0)W(f),$$

so $W(0) = I$ is the unit of $\mathcal{W}(E,\sigma)$. Furthermore,

$$W(-f)W(f) = W(0) = W(f)W(-f).$$

Therefore $W(f)$ is unitary.]

16.1 <u>EXAMPLE</u> Let $\mathcal{H}$ be a separable complex Hilbert space. Consider the Fock system

$$W:\mathcal{H} \to \mathcal{U}(BO(\mathcal{H})).$$

Then the C*-subalgebra of $\mathcal{B}(BO(\mathcal{H}))$ generated by the $W(f)$ is a CCR realization of $(\mathcal{H}, \text{Im} < , >)$.

16.2 <u>THEOREM</u> (Slawny) The pair (E,σ) admits a CCR realization. Moreover, if $\mathcal{W}_1(E,\sigma)$ and $\mathcal{W}_2(E,\sigma)$ are two CCR realizations of (E,σ), then $\exists$ a unique isomorphism

$$\phi:\mathcal{W}_1(E,\sigma) \to \mathcal{W}_2(E,\sigma)$$

such that

$$\phi(W_1(f)) = W_2(f) \ \forall \ f \in E.$$

To establish the existence, consider

$$\ell^2(E) = \{\Lambda:E \to \underline{C}: \sum_{x\in E} |\Lambda(x)|^2 < \infty\}$$

and define $W(f) \in \mathcal{U}(\ell^2(E))$ by the rule

$$(W(f)\Lambda)(x) = \exp(-\frac{\sqrt{-1}}{2}\sigma(x,f))\Lambda(x+f) \quad (x,f \in E).$$

Then the norm closure of the set

$$\sum_{i=1}^{n} c_i W(f_i) \quad (c_i \in \underline{C}, f_i \in E)$$

in $\mathcal{B}(\ell^2(E))$ is a unital C*-algebra with the required properties.

To treat the uniqueness, it will be convenient to introduce some machinery.

[Note: In any event, it is clear that ϕ is unique if it exists.]

Let G be an abelian group (written additively) -- then a <u>multiplier</u> is a map

$$b:G \times G \to \underline{T}$$

such that

$$b(\sigma,0) = b(0,\sigma) = 1$$

and

$$b(\sigma_1,\sigma_2)b(\sigma_1+\sigma_2,\sigma_3) = b(\sigma_1,\sigma_2+\sigma_3)b(\sigma_2,\sigma_3).$$

Let $\mathcal{H}$ be a Hilbert space — then a projective representation of G on $\mathcal{H}$ with multiplier b is a map $U:G \rightarrow \mathcal{U}(\mathcal{H})$ such that $\forall\ \sigma,\tau \in G$,

$$U(\sigma)U(\tau) = b(\sigma,\tau)U(\sigma + \tau).$$

[Note:

$$U(\sigma)U(0) = b(\sigma,0)U(\sigma) = U(\sigma)$$

=>

$$U(\sigma)^{-1}U(\sigma)U(0) = U(\sigma)^{-1}U(\sigma)$$

=>

$$U(0) = I.]$$

16.3 EXAMPLE Let (E,σ) be a symplectic vector space — then

$$b(f,g) = \exp\left(- \frac{\sqrt{-1}}{2}\,\sigma(f,g)\right)$$

is a multiplier and, extending the terminology introduced in §10, a Weyl system over (E,σ) is a projective representation of E with multiplier b.

[Note: Suppose given a representation $\pi:\mathcal{W}(E,\sigma) \rightarrow \mathcal{B}(\mathcal{H})$ — then the arrow $f \rightarrow \pi(W(f))$ defines a Weyl system over (E,σ).]

Assume now that G is, in addition, locally compact — then the term "projective representation" presupposes that $b:G \times G \rightarrow \underline{T}$ is continuous and $U:G \rightarrow \mathcal{U}(\mathcal{H})$ is continuous (where, as usual, $\mathcal{U}(\mathcal{H})$ is equipped with the strong operator topology).

- Define

$$B:G \rightarrow \mathcal{U}(L^2(G))$$

by

$$(B(\sigma)f)(\tau) = b(\tau,\sigma)f(\tau + \sigma).$$

• Define

$$R:G \to \mathcal{U}(L^2(G))$$

by

$$(R(\sigma)f)(\tau) = f(\tau + \sigma).$$

16.4 LEMMA Let (b,U) be a projective representation of G on $\mathcal{H}$ -- then $(b,\overline{R \otimes U})$ is unitarily equivalent to $(b,\overline{B \otimes 1_{\mathcal{H}}})$.

PROOF By definition,

$$\begin{cases} \overline{R \otimes U} \\ \overline{B \otimes 1_{\mathcal{H}}} \end{cases}$$

operate on $L^2(G) \hat{\otimes} \mathcal{H}$ (cf. 5.6). This said, identify $L^2(G) \hat{\otimes} \mathcal{H}$ with $L^2(G;\mathcal{H})$ (permissible even though Haar measure on G is not necessarily σ-finite and $\mathcal{H}$ is not necessarily separable ...). Define

$$T:L^2(G;\mathcal{H}) \to L^2(G;\mathcal{H})$$

by

$$(Tf)(\sigma) = U(\sigma)f(\sigma).$$

Then T is unitary and intertwines $\overline{R \otimes U}$ and $\overline{B \otimes 1_{\mathcal{H}}}$:

$$T((\overline{R \otimes U})(\sigma)f)(\tau)$$

$$= U(\tau)((\overline{R \otimes U})(\sigma)f)(\tau)$$

$$= U(\tau)U(\sigma)f(\tau + \sigma)$$

$$= b(\tau,\sigma)U(\tau + \sigma)f(\tau + \sigma)$$

$$= b(\tau,\sigma)\,(Tf)\,(\tau + \sigma)$$

$$= ((B(\sigma) \otimes 1_{H})\,(Tf))\,(\tau).$$

Let Γ be the dual of G -- then the Fourier transform

$$\begin{bmatrix} L^2(G) \to L^2(\Gamma) \\ f \to \hat{f} \end{bmatrix}$$

implements a unitary equivalence between

$$R:G \to U(L^2(G))$$

and

$$\hat{}R:G \to U(L^2(\Gamma)),$$

where

$$(\hat{}R(\sigma)F)(\chi) = \chi(\sigma)F(\chi).$$

16.5 <u>LEMMA</u> Let (b,U) be a projective representation of G on H -- then the C*-algebra generated by $\overline{\hat{}R \otimes U}$ is isomorphic to the C*-algebra generated by B.

<u>PROOF</u> First, $\overline{R \otimes U}$ and $\overline{\hat{}R \otimes U}$ are unitarily equivalent, hence generate isomorphic C*-algebras. On the other hand, B and $\overline{B \otimes 1_{H}}$ also generate isomorphic C*-algebras, thus the result follows from 16.4.

Let $b:G \times G \rightarrow \underline{T}$ be a (continuous) multiplier -- then b determines a continuous homomorphism $\Phi_b:G \rightarrow \Gamma$, viz.

$$\Phi_b(\sigma)(\tau) = b(\sigma,\tau)b(\tau,\sigma)^{-1}.$$

[Note: Here is the verification that $\Phi_b(\sigma) \in \Gamma$:

$$\Phi_b(\sigma)(\tau_1 + \tau_2) = b(\sigma,\tau_1 + \tau_2)b(\tau_1 + \tau_2,\sigma)^{-1}$$

$$= (b(\sigma,\tau_1 + \tau_2)b(\tau_1,\tau_2))(b(\tau_1,\tau_2)b(\tau_1 + \tau_2,\sigma))^{-1}$$

$$= b(\sigma,\tau_1)b(\sigma + \tau_1,\tau_2)b(\tau_1,\tau_2 + \sigma)^{-1}b(\tau_2,\sigma)^{-1}$$

$$= b(\sigma,\tau_1)b(\tau_1 + \sigma,\tau_2)b(\tau_1,\sigma + \tau_2)^{-1}b(\tau_2,\sigma)^{-1}$$

$$= b(\sigma,\tau_1)b(\tau_1,\sigma)^{-1}(b(\tau_1,\sigma)b(\tau_1 + \sigma,\tau_2)$$

$$\times b(\tau_1,\sigma + \tau_2)^{-1})b(\tau_2,\sigma)^{-1}$$

$$= b(\sigma,\tau_1)b(\tau_1,\sigma)^{-1}b(\sigma,\tau_2)b(\tau_2,\sigma)^{-1}$$

$$= \Phi_b(\sigma)(\tau_1)\Phi_b(\sigma)(\tau_2).]$$

16.6 <u>LEMMA</u> Suppose that $\Phi_b:G \rightarrow \Gamma$ is injective -- then $\Phi_b(G)$ is dense in Γ.

<u>PROOF</u> In fact,

$$(\overline{\Phi_b(G)})^\wedge = G/\text{Ann } \Phi_b(G),$$

Ann standing for annihilator. But Ann $\Phi_b(G) = \{0\}$, Φ_b being injective. Therefore

$$(\overline{\Phi_b(G)})\hat{} = G$$

=>

$$(\overline{\Phi_b(G)})\hat{}\hat{} = \Gamma$$

=>

$$\overline{\Phi_b(G)} = \Gamma.$$

16.7 <u>LEMMA</u> Let (b,U) be a projective representation of G on H and suppose that $\Phi_b: G \to \Gamma$ is injective -- then the C*-algebra generated by U is isomorphic to the C*-algebra generated by $\overline{\hat{}R \otimes U}$:

$$(\overline{\hat{}R \otimes U})(\sigma) \longleftrightarrow U(\sigma).$$

<u>PROOF</u> If $f: G \to \underline{C}$ is a function with finite support, then

$$\left|\left| \sum_{\sigma \in G} f(\sigma)(\overline{\hat{}R \otimes U})(\sigma) \right|\right|$$

$$= \underset{\chi \in \Gamma}{\text{ess sup}} \left|\left| \sum_{\sigma \in G} f(\sigma)\chi(\sigma)U(\sigma) \right|\right|$$

$$= \underset{\tau \in G}{\text{ess sup}} \left|\left| \sum_{\sigma \in G} f(\sigma)\Phi_b(\tau)(\sigma)U(\sigma) \right|\right| \quad \text{(cf. 16.6)}$$

$$= \underset{\tau \in G}{\text{ess sup}} \left|\left| \sum_{\sigma \in G} f(\sigma)U(\tau)U(\sigma)U(\tau)^{-1} \right|\right|$$

$$= \underset{\tau \in G}{\text{ess sup}} \left|\left| U(\tau)\left(\sum_{\sigma \in G} f(\sigma)U(\sigma)\right)U(\tau)^{-1} \right|\right|$$

$$= \left|\left| \sum_{\sigma \in G} f(\sigma)U(\sigma) \right|\right|.$$

[Note: We have

$$U(\tau)U(\sigma)U(\tau)^{-1}$$

$$= b(\tau,\sigma)U(\sigma + \tau)U(\tau)^{-1}$$

$$= b(\tau,\sigma)U(\sigma + \tau)U(\sigma + \tau)^{-1}b(\sigma,\tau)^{-1}U(\sigma)$$

$$= b(\tau,\sigma)b(\sigma,\tau)^{-1}U(\sigma)$$

$$= \Phi_b(\tau)(\sigma).]$$

16.8 <u>LEMMA</u> Let

$$\begin{cases} (b,U_1) \\ (b,U_2) \end{cases}$$

be projective representations of G on

$$\begin{cases} \mathcal{H}_1 \\ \mathcal{H}_2 \end{cases}$$

and suppose that $\Phi_b:G \rightarrow \Gamma$ is injective -- then $\exists$ a unique isomorphism ϕ from the C*-algebra A_1 generated by U_1 to the C*-algebra A_2 generated by U_2 such that

$$\phi(U_1(\sigma)) = U_2(\sigma) \quad (\sigma \in G).$$

[Assemble the facts developed in 16.4, 16.5, and 16.7 (taking care to keep track of the various identifications).]

Specialize and take G = E (discrete topology), denoting the dual of E by $\hat{E}$. Let

$$b(f,g) = \exp\left(-\frac{\sqrt{-1}}{2}\,\sigma(f,g)\right) \quad (f,g \in E).$$

Then

$$\begin{aligned}\Phi_b(f)(g) &= b(f,g)b(g,f)^{-1} \\ &= \exp\left(-\frac{\sqrt{-1}}{2}\,\sigma(f,g)\right)\exp\left(\frac{\sqrt{-1}}{2}\,\sigma(g,f)\right) \\ &= \exp(-\sqrt{-1}\,\sigma(f,g)).\end{aligned}$$

16.9 <u>LEMMA</u> $\Phi_b : E \to \hat{E}$ is injective.

<u>PROOF</u> Suppose that $\Phi_b(f)(g) = 1\ \forall\, g$ — then the claim is that $f = 0$ and, for this, it need only be shown that $\sigma(f,g) = 0\ \forall\, g$ (σ being symplectic, hence nondegenerate). If $\sigma(f,g) \neq 0$, let $t = -\pi/\sigma(f,g)$ to get

$$1 = e^{-\sqrt{-1}\,\sigma(f,tg)} = e^{\sqrt{-1}\,\pi} = -1.$$

To finish the uniqueness, represent

$$\begin{cases} \mathcal{W}_1(E,\sigma) \text{ faithfully on } \mathcal{H}_1 \text{ by } \pi_1 \\ \mathcal{W}_2(E,\sigma) \text{ faithfully on } \mathcal{H}_2 \text{ by } \pi_2 \end{cases} \qquad \text{(cf. 15.9)}$$

and apply 16.8 to the arrows

$$\begin{cases} f \to \pi_1(W(f)) \\ f \to \pi_2(W(f)) \end{cases} \qquad \text{(cf. 16.3).}$$

16.10 LEMMA Let $f \in E$ $(f \neq 0)$ -- then

$$||W(f) - I|| = 2.$$

[Argue as in 9.3.]

16.11 LEMMA $\mathcal{W}(E,\sigma)$ is not separable.

PROOF Suppose that $W(n)$ $(n \in \underline{N})$ is a countable dense subset of $\mathcal{W}(E,\sigma)$. Fix $f \neq 0$ in $\mathcal{W}(E,\sigma)$ -- then $\forall\ t \in \underline{R}$, $\exists\ n_t$:

$$||W(tf) - W(n_t)|| < 1.$$

But

$$t_1 \neq t_2 \Rightarrow W(n_{t_1}) \neq W(n_{t_2}).$$

For otherwise, calling their common value W,

$$||W(t_1f) - W(t_2f)||$$

$$= ||W(t_1f) - W + W - W(t_2f)||$$

$$\leq ||W(t_1f) - W|| + ||W(t_2f) - W||$$

$$< 2.$$

Therefore

$$||W((t_1 - t_2)f) - I||$$

$$= ||W(t_1f - t_2f) - I||$$

$$= ||W(t_1f)W(- t_2f) - I||$$

$$= \|W(t_1f)W(t_2f)^{-1} - W(t_2f)W(t_2f)^{-1}\|$$

$$= \|(W(t_1f) - W(t_2f))W(t_2f)^{-1}\|$$

$$\leq \|W(t_1f) - W(t_2f\| < 2,$$

which contradicts 16.10. And $\underline{R}$ is not countable.

16.12 LEMMA $\mathcal{W}(E,\sigma)$ is simple.

PROOF Let $\pi:\mathcal{W}(E,\sigma) \to \mathcal{B}(\mathcal{H})$ be a representation of $\mathcal{W}(E,\sigma)$ — then $\pi(\mathcal{W}(E,\sigma))$ is a CCR realization of (E,σ), hence by 16.2, $\exists$ a unique isomorphism

$$\phi:\mathcal{W}(E,\sigma) \to \pi(\mathcal{W}(E,\sigma))$$

such that

$$\phi(W(f)) = \pi(W(f)) \ \forall \ f \in E.$$

But this implies that $\phi = \pi$, so the kernel of π is zero. Therefore, since π is arbitrary, $\mathcal{W}(E,\sigma)$ has no nontrivial closed ideals, thus is simple.

[Note:

$\mathcal{W}(E,\sigma)$ simple => $\mathcal{W}(E,\sigma)$ central (cf. 15.4).]

16.13 REMARK Let M be a subspace of E — then the C*-subalgebra of $\mathcal{W}(E,\sigma)$ generated by $\{W(f):f \in M\}$ is equal to $\mathcal{W}(E,\sigma)$ iff $M = E$.

Having derived the existence, essential uniqueness, and basic properties of $\mathcal{W}(E,\sigma)$, we shall now go back and take a look at certain structural issues of an algebraic nature.

Give E the discrete topology and let $\underline{C}^{(E)}$ be the vector space of all finitely supported complex valued functions $\zeta: E \to \underline{C}$. Define a product

$$\underline{C}^{(E)} \times \underline{C}^{(E)} \to \underline{C}^{(E)}$$

by

$$(\zeta_1\zeta_2)(f) = \sum_{x+y=f} b(x,y)\zeta_1(x)\zeta_2(y),$$

where

$$b(x,y) = \exp(-\frac{\sqrt{-1}}{2}\sigma(x,y))\ (x,y \in E).$$

Then in this way $\underline{C}^{(E)}$ acquires the structure of a complex associative algebra, denoted from here on by $W(E,\sigma)$.

It is clear that a basis for $W(E,\sigma)$ is the set $\{\delta_f : f \in E\}$. And:

1. δ_0 is the multiplicative identity of $W(E,\sigma)$.

2. δ_f is a unit with inverse δ_{-f}.

From the definitions,

$$\delta_f\delta_g = b(f,g)\delta_{f+g},$$

so $\forall\ \zeta$,

$$(\delta_f\zeta\delta_f^{-1})(g) = b(f,g)^2\zeta(g).$$

16.14 <u>LEMMA</u> The algebra $W(E,\sigma)$ is central, i.e., its center consists of the scalar multiples of δ_0.

<u>PROOF</u> Let ζ belong to the center of $W(E,\sigma)$. Take a nonzero $g \in E$ and

choose $f:b(f,g)^2 \neq 1$ -- then

$$b(f,g)^2\zeta(g) = (\delta_f\zeta\delta_f^{-1})(g)$$

$$= \zeta(g)$$

=>

$$\zeta(g) = 0$$

=>

$$\text{spt } \zeta \subset \{0\}.$$

16.15 <u>LEMMA</u> The algebra $W(E,\sigma)$ is simple, i.e., has no nontrivial ideals.

<u>PROOF</u> Let $I \subset W(E,\sigma)$ be a nonzero ideal -- then I is an additive subgroup of $W(E,\sigma)$ and is invariant under all inner automorphisms. Fix a nonzero $\zeta \in I$: The cardinality of spt ζ is minimal. We claim that $\#(\text{spt } \zeta) = 1$, thus ζ is a unit (so $I = W(E,\sigma)$). To see this, suppose that spt ζ contains distinct points x and y. Choose $f \in E$:

$$\begin{cases} b(f,x)^2 = 1 \\ b(f,y)^2 \neq 1. \end{cases}$$

Then

$$\zeta' \equiv \delta_f\zeta\delta_f^{-1} - \zeta \in I$$

and

$$\text{spt } \zeta' \subset \text{spt } \zeta.$$

But

- $\zeta'(x) = (b(f,x)^2 - 1)\zeta(x) = 0$

=>

$$\text{spt } \zeta' \neq \text{spt } \zeta$$

- $\zeta'(y) = (b(f,y)^2 - 1)\zeta(y) \neq 0$

=>

$$\zeta' \neq 0.$$

Therefore ζ' is a nonzero element of I with $\#(\text{spt } \zeta') < \#(\text{spt } \zeta)$, which is a contradiction.

16.16 <u>LEMMA</u> The algebra $W(E,\sigma)$ has no zero divisors and its units are the $c\delta_f$ $(c \in \underline{C}^\times, f \in E)$.

Let $\phi: W(E,\sigma) \to W(E,\sigma)$ be an algebra automorphism -- then ϕ sends units to units, hence ϕ gives rise to maps

$$\begin{cases} T: E \to E \\ \tau: E \to \underline{C}^\times \end{cases}$$

via the prescription

$$f \in E \Rightarrow \phi(\delta_f) = \tau(f)\delta_{Tf}.$$

And

$$\begin{cases} \phi(\delta_f)\phi(\delta_g) = b(Tf,Tg)\tau(f)\tau(g)\delta_{Tf + Tg} \\ \phi(\delta_f\delta_g) = b(f,g)\tau(f + g)\delta_{T(f + g)}, \end{cases}$$

=>

$$\begin{cases} T(f+g) = Tf + Tg \\ b(f,g)\tau(f+g) = b(Tf,Tg)\tau(f)\tau(g). \end{cases}$$

Therefore T is an automorphism of the additive group of E or still, T is an automorphism of E viewed as a rational vector space. More is true. Thus rewrite the relation

$$b(f,g)\tau(f+g) = b(Tf,Tg)\tau(f)\tau(g)$$

in the form

$$\frac{\tau(f+g)}{\tau(f)\tau(g)} = \frac{b(Tf,Tg)}{b(f,g)} .$$

Switching f,g leaves the LHS unchanged and inverts the RHS. Consequently,

$$\frac{b(Tf,Tg)}{b(f,g)} = \pm 1$$

$$\Rightarrow$$

$$\sigma(Tf,Tg) - \sigma(f,g) \in 2\pi\underline{Z}$$

$$\Rightarrow$$

$$\sigma(Tf,Tg) - \sigma(f,g) = 0,$$

T being $\underline{Q}$-linear. But then

$$\tau(f+g) = \tau(f)\tau(g).$$

16.17 <u>LEMMA</u> The algebra automorphisms of $W(E,\sigma)$ are the linear bijections $\phi:W(E,\sigma) \to W(E,\sigma)$ given by

$$\phi(\delta_f) = \tau(f)\delta_{Tf},$$

where

$$\tau : E \to \underline{C}^{\times}$$

is a homomorphism and

$$T : E \to E$$

is an additive automorphism of E which leaves σ invariant.

PROOF The preceding discussion shows that every algebra automorphism $\phi : W(E,\sigma) \to W(E,\sigma)$ determines a pair (τ, T) with the stated properties. Conversely, if ϕ is defined as above by (τ, T), then

$$\begin{aligned} \phi(\delta_f \delta_g) &= b(f,g)\tau(f + g)\delta_{T(f + g)} \\ &= b(Tf,Tg)\tau(f)\tau(g)\delta_{Tf + Tg} \\ &= \phi(\delta_f)\phi(\delta_g), \end{aligned}$$

thus ϕ is an algebra automorphism of $W(E,\sigma)$.

Given $\zeta \in W(E,\sigma)$, define ζ^* by

$$\zeta^*(f) = \overline{\zeta(- f)}.$$

Then the map $\zeta \to \zeta^*$ is conjugate linear and

$$\begin{cases} (\zeta^*)^* = \zeta \\ (\zeta_1\zeta_2)^* = \zeta_2^*\zeta_1^* . \end{cases}$$

Therefore $W(E,\sigma)$ is a unital $*$-algebra.

Because of this, we shall then agree that a representation π of $W(E,\sigma)$

on a complex Hilbert space $\mathcal{H}$ is a morphism $\pi:W(E,\sigma) \rightarrow \mathcal{B}(\mathcal{H})$ in the category of unital $*$-algebras, thus π is linear and

$$\pi(\zeta_1\zeta_2) = \pi(\zeta_1)\pi(\zeta_2) \ \& \ \pi(\zeta^*) = \pi(\zeta)^*$$

with $\pi(\delta_0) = I$.

[Note: π is necessarily faithful (cf. 16.15).]

16.18 REMARK If $\phi:W(E,\sigma) \rightarrow W(E,\sigma)$ is a $*$-automorphism (cf. 16.17), then $\tau \in \hat{E}$. Proof:

$$\tau(f)^{-1}\delta_{-Tf} = \tau(-f)\delta_{-Tf}$$

$$= \phi(\delta_{-f}) = \phi(\delta_f^*) = \phi(\delta_f)^*$$

$$= (\tau(f)\delta_{Tf})^*$$

$$= \overline{\tau(f)}\delta_{-Tf}$$

=>

$$\tau(f)^{-1} = \overline{\tau(f)}$$

=>

$$\tau(f) \in \underline{T}.]$$

Let $\pi:W(E,\sigma) \rightarrow \mathcal{B}(\mathcal{H})$ be a representation -- then the norm closure $\mathcal{W}_\pi(E,\sigma)$ of $\pi(W(E,\sigma))$ is a unital C*-algebra which is generated by the $\pi(\delta_f)$. Here

$$\pi(\delta_f)^* = \pi(\delta_f^*) = \pi(\delta_{-f})$$

and

$$\pi(\delta_f)\pi(\delta_g) = \pi(\delta_f\delta_g)$$

$$= \pi(b(f,g)\delta_{f+g})$$

$$= b(f,g)\pi(\delta_{f+g}).$$

Therefore $\mathcal{W}_\pi(E,\sigma)$ is a CCR realization of (E,σ).

Suppose that

$$\begin{cases} \pi_1:W(E,\sigma) \to \mathcal{B}(\mathcal{H}_1) \\ \pi_2:W(E,\sigma) \to \mathcal{B}(\mathcal{H}_2) \end{cases}$$

are representations of $W(E,\sigma)$ -- then by 16.2, $\exists$ a unique isomorphism

$$\phi:\mathcal{W}_{\pi_1}(E,\sigma) \to \mathcal{W}_{\pi_2}(E,\sigma)$$

such that

$$\phi(\pi_1(\delta_f)) = \pi_2(\delta_f) \ \forall\ f \in E.$$

So, $\forall\ \zeta \in W(E,\sigma)$,

$$||\pi_1(\zeta)|| = ||\phi(\pi_1(\zeta))|| = ||\pi_2(\zeta)||.$$

Accordingly, if $\pi:W(E,\sigma) \to \mathcal{B}(\mathcal{H})$ is a representation and if, by definition,

$$||\zeta||_\pi = ||\pi(\zeta)||,$$

then $||\zeta||_\pi$ is independent of the choice of π, call it $||\zeta||$, and the completion $\mathcal{W}(E,\sigma)$ of $W(E,\sigma)$ in this norm is a CCR realization of (E,σ).

16.19 REMARK As regards terminology, some authorities refer to $W(E,\sigma)$ as the Weyl algebra per (E,σ) while others reserve this term for $\mathcal{W}(E,\sigma)$, the latter convention being the one that we shall follow.

16.20 EXAMPLE Let $\mathcal{H}$ be a separable complex Hilbert space — then the Fock representation

$$\pi_F : W(\mathcal{H}, \text{Im} < , >) \to \mathcal{B}(\text{BO}(\mathcal{H}))$$

is characterized by the requirement that

$$\pi_F(\delta_f) = W(f),$$

where

$$W(f) = \exp(\sqrt{-1}\ \overline{Q(f)}) \quad (f \in \mathcal{H}).$$

It extends uniquely to a representation of $\mathcal{W}(\mathcal{H}, \text{Im} < , >)$ on $\text{BO}(\mathcal{H})$ (denoted still by π_F). The prescription

$$\omega_F(W) = < \Omega, \pi_F(W)\Omega > \quad (W \in \mathcal{W}(\mathcal{H}, \text{Im} < , >))$$

defines the vacuum state on $\mathcal{W}(\mathcal{H}, \text{Im} < , >)$. Since Ω is cyclic (cf. 9.6), it follows that π_F is the GNS representation associated with ω_F (cf. 15.6), so ω_F is pure (π_F being irreducible (cf. 9.11)).

[Note: $\forall\ f \in \mathcal{H}$,

$$\begin{aligned} \omega_F(\delta_f) &= < \Omega, \pi_F(\delta_f)\Omega > \\ &= < \Omega, W(f)\Omega > \\ &= e^{-\frac{1}{4}||f||^2} \quad \text{(cf. 9.5).]} \end{aligned}$$

An $\underline{R}$-linear bijection $T:E \to E$ is said to be symplectic if

$$\sigma(Tf,Tg) = \sigma(f,g) \ \forall \ f,g \in E.$$

16.21 LEMMA Given a symplectic map $T:E \to E$, $\exists$ a unique automorphism α_T of $\mathcal{W}(E,\sigma)$ such that

$$\alpha_T(W(f)) = W(Tf) \quad (f \in E).$$

PROOF The $W(Tf)$ satisfy the same general conditions as the $W(f)$ and both generate $\mathcal{W}(E,\sigma)$. Now apply Slawny's theorem.

The α_T are called Bogolubov automorphisms. They form a subgroup of Aut $\mathcal{W}(E,\sigma)$ and the arrow $T \to \alpha_T$ is a representation of the symplectic group of (E,σ) on $\mathcal{W}(E,\sigma)$.

16.22 EXAMPLE $\exists$ a unique automorphism Π of $\mathcal{W}(E,\sigma)$ such that

$$\Pi(W(f)) = W(-f) \quad (f \in E).$$

16.23 REMARK To define α_T, it suffices that T be an additive automorphism of E which leaves σ invariant.

16.24 EXAMPLE Let $\mathcal{H}$ be a separable complex Hilbert space. Fix $T \in SP(\mathcal{H})$ and put $\pi_{F,T} = \pi_F \circ \alpha_T$.

- TFAE:

1. $T \in SP_2(\mathcal{H})$;

2. $F(\pi_F) = F(\pi_{F,T})$;

3. π_F and $\pi_{F,T}$ are geometrically equivalent.

[Note: π_F and $\pi_{F,T}$ are irreducible, hence geometric equivalence and unitary equivalence are one and the same (cf. 15.22).]

• TFAE:

1. $T \notin SP_2(H)$;

2. $F(\pi_F) \cap F(\pi_{F,T}) = \emptyset$;

3. π_F and $\pi_{F,T}$ are disjoint.

16.25 LEMMA Suppose that $T:E \rightarrow E$ is symplectic -- then

$$\alpha_T : W(E,\sigma) \rightarrow W(E,\sigma)$$

is an inner automorphism iff $T = I$.

16.26 REMARK Let $\pi:W(E,\sigma) \rightarrow B(H)$ be a representation — then π is faithful (cf. 16.12), hence $\pi \circ \alpha_T \circ \pi^{-1}$ is an automorphism of $\pi(W(E,\sigma))$, which, in view of 16.25, is not inner $(T \neq I)$. Therefore $\pi(W(E,\sigma)) \neq B(H)$.

[Note: Every automorphism of $B(H)$ is inner. In fact,

$$\begin{cases} \text{Aut } B(H) \longleftrightarrow U(H)/\underline{U}(1) \\ \alpha \longleftrightarrow U, \end{cases}$$

where $\alpha(A) = UAU^{-1}$.]

Let

$$\left[\begin{array}{l}(E_1,\sigma_1)\\ \\ (E_2,\sigma_2)\end{array}\right.$$

be symplectic vector spaces. Suppose that $T:E_1 \to E_2$ is an $\underline{R}$-linear map such that

$$\sigma_2(Tf_1,Tg_1) = \sigma_1(f_1,g_1) \ \forall \ f_1,g_1 \in E_1.$$

[Note: T is necessarily one-to-one.]

16.27 <u>LEMMA</u> $\exists$ an injective morphism

$$\mathcal{W}(E_1,\sigma_1) \to \mathcal{W}(E_2,\sigma_2).$$

<u>PROOF</u> We have

$$W_2(Tf_1)W_2(Tg_1)$$

$$= \exp(-\frac{\sqrt{-1}}{2}\sigma_2(Tf_1,Tg_1))W_2(Tf_1 + Tg_1)$$

$$= \exp(-\frac{\sqrt{-1}}{2}\sigma_1(f_1,g_1))W_2(Tf_1 + Tg_1).$$

Therefore the C*-subalgebra of $\mathcal{W}(E_2,\sigma_2)$ generated by the $W_2(Tf_1)$ is a CCR realization of (E_1,σ_1).

§17. *THE PRE-SYMPLECTIC THEORY*

Let $E \neq 0$ be a real linear space equipped with a bilinear form σ -- then the pair (E,σ) is a pre-symplectic vector space if σ is antisymmetric.

N.B. Put

$$E_0 = \{f \in E:\sigma(f,g) = 0 \ \forall \ g \in E\}.$$

Then the pair (E,σ) is a symplectic vector space iff $E_0 = \{0\}$.

The construction of the unital *-algebra $W(E,\sigma)$ in the preceding § did not use the assumption that σ was symplectic and goes through verbatim when σ is merely pre-symplectic. On the other hand, the structure of $W(E,\sigma)$ in the pre-symplectic case is not the same as in the symplectic case. E.g.: If $E_0 \neq \{0\}$, then it is no longer true that the center of $W(E,\sigma)$ consists of scalar multiples of δ_0 alone (i.e., 16.14 fails). Indeed,

$$f \in E_0 \Rightarrow b(f,g) = 1 \ \forall \ g \in E.$$

Therefore δ_f is central.

[Note: We admit the possibility that σ is identically zero, thus $W(E,0)$ is commutative.]

Given a function $\chi:E \to \underline{C}$ with $\chi(0) = 1$, put

$$K_\chi(f,g) = \exp(\frac{\sqrt{-1}}{2}\sigma(f,g))\chi(g - f) \quad (f,g \in E).$$

Then χ is said to be σ positive definite if K_χ is a kernel on E, i.e., if for all

$$\left[\begin{array}{l} f_1,\ldots,f_n \in E \\ \\ c_1,\ldots,c_n \in \underline{C}, \end{array}\right.$$

we have

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \exp(\frac{\sqrt{-1}}{2}\sigma(f_i,f_j))\chi(f_j - f_i) \geq 0.$$

Write $PD(E,\sigma)$ for the set of σ positive definite functions on E and, as before, let

$$b(f,g) = \exp(-\frac{\sqrt{-1}}{2}\sigma(f,g)) \quad (f,g \in E).$$

17.1 LEMMA Suppose that (b,U) is a projective representation of E on $\mathcal{H}$. Fix a unit vector $x \in \mathcal{H}$ and put

$$\chi_x(f) = \langle x, U(f)x \rangle \quad (f \in E).$$

Then χ_x is σ positive definite, thus $\chi_x \in PD(E,\sigma)$.

PROOF In fact,

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \exp(\frac{\sqrt{-1}}{2}\sigma(f_i,f_j)) \langle x, U(f_j - f_i)x \rangle$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j \exp(\frac{\sqrt{-1}}{2}\sigma(f_i,f_j)) \langle x, U(-f_i + f_j)x \rangle$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j \overline{b(f_i,f_j)} \langle x, b(-f_i,f_j)^{-1}U(-f_i)U(f_j)x \rangle$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j \overline{b(f_i,f_j)}\ \overline{b(f_i,f_j)}^{-1} \langle U(f_i)x, U(f_j)x \rangle$$

$$= \langle \sum_{i=1}^{n} c_i U(f_i)x, \sum_{j=1}^{n} c_j U(f_j)x \rangle$$

$$\geq 0.$$

17.2 <u>EXAMPLE</u> Let H be a separable complex Hilbert space — then the Fock system $f \rightarrow W(f)$ defines a projective representation of H on $BO(H)$ with multiplier

$$\exp\left(- \frac{\sqrt{-1}}{2} \operatorname{Im} < f,g > \right).$$

Since

$$e^{-\frac{1}{4} ||f||^2} = < \Omega, W(f)\Omega > \quad \text{(cf. 9.5)},$$

it follows from 17.1 that

$$\exp\left(- \frac{1}{4} ||\cdot||^2\right) \in PD(H, \operatorname{Im} < \ , \ >).$$

17.3 <u>EXAMPLE</u> Let H be a separable complex Hilbert space. Fix $\lambda > 1$ -- then the function

$$f \rightarrow e^{-\frac{\lambda}{4} ||f||^2}$$

is in $PD(H, \operatorname{Im} < \ , \ >)$. To see this, pass to

$$BO(H) \hat{\otimes} BO(H).$$

Let

$$\begin{cases} \alpha = \left(\frac{\lambda + 1}{2}\right)^{1/2} \\ \\ \beta = \left(\frac{\lambda - 1}{2}\right)^{1/2} \end{cases} \quad \left(\Rightarrow \begin{cases} \alpha^2 + \beta^2 = \lambda \\ \alpha^2 - \beta^2 = 1 \end{cases} \right),$$

and let $C: H \rightarrow H$ be a conjugation. Put

$$W_\lambda(f) = \overline{W(\alpha f) \otimes W(\beta C f)}.$$

Then there are two claims:

1. W_λ defines a projective representation of H on $BO(H) \hat{\otimes} BO(H)$ with multiplier

$$\exp\left(- \frac{\sqrt{-1}}{2} \operatorname{Im} < f,g > \right).$$

2. $\forall\, f \in H,$

$$e^{-\frac{\lambda}{4} ||f||^2} = < \Omega \otimes \Omega, W_\lambda(f)(\Omega \otimes \Omega) >.$$

<u>Ad 1</u>: On $BO(H) \otimes BO(H)$ (cf. 5.6),

$$W_\lambda(f) \otimes W_\lambda(g)$$

$$= (W(\alpha f)W(\alpha g)) \otimes (W(\beta Cf) \otimes W(\beta Cg))$$

$$= \exp\left(- \frac{\sqrt{-1}}{2} \operatorname{Im} < \alpha f, \alpha g > \right) \exp\left(- \frac{\sqrt{-1}}{2} \operatorname{Im} < \beta Cf, \beta Cg > \right)$$

$$\times\, W(\alpha(f + g)) \otimes W(\beta C(f + g)).$$

And

$$- \frac{\sqrt{-1}}{2} \operatorname{Im} < \alpha f, \alpha g > - \frac{\sqrt{-1}}{2} \operatorname{Im} < \beta Cf, \beta Cg >$$

$$= - \frac{\sqrt{-1}}{2} (\alpha^2 \operatorname{Im} < f,g > + \beta^2 \operatorname{Im} < Cf, Cg >)$$

$$= - \frac{\sqrt{-1}}{2} (\alpha^2 \operatorname{Im} < f,g > + \beta^2 \operatorname{Im} < g,f >)$$

$$= - \frac{\sqrt{-1}}{2} (\alpha^2 \operatorname{Im} < f,g > + \beta^2 \operatorname{Im} \overline{< f,g >})$$

$$= - \frac{\sqrt{-1}}{2} (\alpha^2 \operatorname{Im} < f,g > - \beta^2 \operatorname{Im} < f,g >)$$

$$= -\frac{\sqrt{-1}}{2}\,((\alpha^2 - \beta^2)\,\mathrm{Im} < f,g >)$$

$$= -\frac{\sqrt{-1}}{2}\,\mathrm{Im} < f,g >.$$

Ad 2: We have

$$< \Omega \otimes \Omega, W_\lambda(f)(\Omega \otimes \Omega) >$$

$$= < \Omega \otimes \Omega, W(\alpha f)\Omega \otimes W(\beta Cf)\Omega >$$

$$= < \Omega, W(\alpha f)\Omega > < \Omega, W(\beta Cf)\Omega >$$

$$= \exp(-\frac{1}{4}\,||\alpha f||^2 - \frac{1}{4}\,||\beta Cf||^2)$$

$$= \exp(-\frac{1}{4}\,(\alpha^2 + \beta^2)\,||f||^2)$$

$$= e^{-\frac{\lambda}{4}\,||f||^2}.$$

[Note: Let μ be a probability measure on $[1,\infty[$ -- then the function

$$f \to \int_1^\infty e^{-\frac{\lambda}{4}\,||f||^2}\,d\mu(\lambda)$$

is in $PD(H,\mathrm{Im} < , >)$.]

17.4 LEMMA If $\chi:E \to \underline{C}$ is σ positive definite, then $\exists$ a complex Hilbert space H_χ, a projective representation (b,U_χ) of E on H_χ, and a cyclic unit vector $x_\chi \in H_\chi$ such that $\forall\ f \in E$,

$$\chi(f) = < x_\chi, U_\chi(f) x_\chi >.$$

[This is an obvious variant on the considerations detailed in 14.10.]

A <u>state</u> on $W(E,\sigma)$ is a linear functional $\omega: W(E,\sigma) \rightarrow \underline{C}$ such that

$$\forall\ \zeta,\ \omega(\zeta^*\zeta) \geq 0$$

subject to $\omega(\delta_0) = 1$.

Let $S(W(E,\sigma))$ stand for the set of states on $W(E,\sigma)$ -- then there is a canonical one-to-one correspondence between $PD(E,\sigma)$ and $S(W(E,\sigma))$, namely the extension to $W(E,\sigma)$ by linearity of a σ positive definite function χ gives rise to a state ω_χ while the restriction to E of a state ω defines a σ positive function χ_ω:

$$\begin{cases} \chi_{\omega_\chi} = \chi \\ \omega_{\chi_\omega} = \omega. \end{cases}$$

[Note: The arrow $f \rightarrow \delta_f$ injects E into $W(E,\sigma)$.]

17.5 <u>EXAMPLE</u> Define $\chi_{tr}: E \rightarrow \underline{C}$ by

$$\chi_{tr}(f) = \begin{cases} 1 & (f = 0) \\ 0 & (f \neq 0). \end{cases}$$

Then

$$\chi_{tr} \in PD(E,\sigma).$$

Denote the associated state by ω_{tr}, thus

$$\omega_{tr}(\zeta) = \zeta(0).$$

And

$$\omega_{tr}(\zeta^*\zeta) = \sum_f |\zeta(f)|^2.$$

[Note: ω_{tr} is a tracial state in the sense that

$$\omega_{tr}(\zeta_1\zeta_2) = \omega_{tr}(\zeta_2\zeta_1) \quad (\zeta_1,\zeta_2 \in W(E,\sigma)).]$$

Let $\text{Rep}_b\, E$ be the set of all projective representations of E with multiplier b and let Rep $W(E,\sigma)$ be the set of all representations of $W(E,\sigma)$ -- then

$$\text{Rep}_b\, E \longleftrightarrow \text{Rep}\, W(E,\sigma).$$

Thus let (b,U) be a projective representation of E on $\mathcal{H}$ -- then the prescription

$$\pi_U(\zeta) = \pi_U(\sum_{i=1}^n c_i\delta_{f_i})$$

$$= \sum_{i=1}^n c_i U(f_i)$$

defines a representation of $W(E,\sigma)$ on $\mathcal{H}$:

$$\pi_U(\delta_f\delta_g) = \pi_U(b(f,g)\delta_{f+g})$$

$$= b(f,g)\pi_U(\delta_{f+g})$$

$$= b(f,g)U(f+g)$$

$$= b(f,g)b(f,g)^{-1}U(f)U(g)$$

$$= U(f)U(g)$$

$$= \pi_U(\delta_f)\pi_U(\delta_g)$$

=>

$$\pi_U(\zeta_1\zeta_2) = \pi_U(\zeta_1)\pi_U(\zeta_2).$$

It is also clear that

$$\pi_U(\zeta^*) = \pi_U(\zeta)^*.$$

And trivially, $\pi_U(\delta_0) = U(0) = I$. Conversely, if π is a representation of $W(E,\sigma)$ on $\mathcal{H}$, then the prescription

$$U_\pi(f) = \pi(\delta_f)$$

defines a projective representation (b,U_π) of E on $\mathcal{H}$.

[Note: The formalism entails

$$\begin{cases} U_{\pi_U} = U \\ \pi_{U_\pi} = \pi. \end{cases}$$

17.6 <u>REMARK</u> A <u>Weyl system</u> over (E,σ) is a projective representation of E with multiplier b (cf. 16.3).

17.7 <u>LEMMA</u> Let $\omega \in S(W(E,\sigma))$ -- then $\exists$ a cyclic representation π_ω of $W(E,\sigma)$ on a Hilbert space H_ω with cyclic unit vector Ω_ω such that

$$\omega(\zeta) = < \Omega_\omega, \pi_\omega(\zeta)\Omega_\omega >.$$

[Note: The triple $(\pi_\omega, H_\omega, \Omega_\omega)$ is unique up to unitary equivalence.]

17.8 <u>EXAMPLE</u> Define a projective representation (b,B) of E on $\ell^2(E)$ by the rule

$$(B(f)\Lambda)(x) = b(x,f)\Lambda(x + f) \quad (x,f \in E).$$

Then

$$(\pi_B(\zeta)\Lambda)(x) = \sum_{i=1}^{n} c_i(B(f_i)\Lambda)(x)$$

$$= \sum_{i=1}^{n} c_i b(x,f_i)\Lambda(x + f_i).$$

Therefore

$$< \delta_0, \pi_B(\zeta)\delta_0 >$$

$$= \sum_{x \in E} \delta_0(x)(\pi_B(\zeta)\delta_0)(x)$$

$$= (\pi_B(\zeta)\delta_0)(0)$$

$$= \sum_{i=1}^{n} c_i b(0,f_i)\delta_0(f_i)$$

$$= \sum_{i=1}^{n} c_i \chi_{tr}(f_i)$$

$$= \omega_{tr}(\zeta).$$

The setup in 17.7 is thus realized by taking $\omega = \omega_{tr}$, $(\pi_\omega, H_\omega) = (\pi_B, \ell^2(E))$, and $\Omega_\omega = \delta_0$.

[Note: Change the notation and write π_{tr} in place of π_B. Let (b,U) be a projective representation of E -- then (cf. 16.4)

$$||\pi_U(\zeta)|| \leq ||\pi_{tr}(\zeta)|| \quad (\zeta \in W(E,\sigma)).]$$

A norm $||\cdot||$ on $W(E,\sigma)$ is said to be <u>algebraic</u> if $||\zeta_1\zeta_2|| \leq ||\zeta_1|| \, ||\zeta_2||$ for all $\zeta_1, \zeta_2 \in W(E,\sigma)$ and $||\delta_0|| = 1$. An algebraic norm $||\cdot||$ is called a <u>C*-norm</u> if $\forall \ \zeta \in W(E,\sigma)$,

$$||\zeta^*\zeta|| = ||\zeta||^2.$$

Put

1. $||\zeta||_1 = \sup\limits_{\omega} \omega(\zeta^*\zeta)^{1/2}$.

2. $||\zeta||_2 = \sup\limits_{(b,U)} ||\pi_U(\zeta)||$.

Here the first sup is taken over $S(W(E,\sigma))$ and the second sup is taken over Rep_b E.

17.9 <u>LEMMA</u> We have

$$||\cdot||_1 = ||\cdot||_2.$$

<u>PROOF</u> Let (b,U) be a projective representation of E on H -- then $\forall$ unit vector $x \in H$,

$$\chi_x(f) = \langle x, U(f)x \rangle \quad (f \in E)$$

is σ positive definite (cf. 17.1), thus

$$||\pi_U(\zeta)|| = \sup \{||\pi_U(\zeta)x|| : ||x|| = 1\}$$

$$= \sup \{< x, \pi_U(\zeta^*\zeta)x >^{1/2} : ||x|| = 1\}$$

$$= \sup \{\omega_{\chi_x}(\zeta^*\zeta)^{1/2} : ||x|| = 1\}$$

$$\leq ||\zeta||_1$$

=>

$$||\zeta||_2 \leq ||\zeta||_1.$$

On the other hand, a given state ω determines a σ positive function χ_ω and, in the notation of 17.4,

$$\chi_\omega(f) = < x_{\chi_\omega}, U_{\chi_\omega}(f)x_{\chi_\omega} >.$$

So

$$\omega(\zeta^*\zeta)^{1/2} = ||\pi_{U_{\chi_\omega}}(\zeta)x_{\chi_\omega}||$$

$$\leq ||\pi_{U_{\chi_\omega}}||$$

$$\leq ||\zeta||_2$$

=>

$$||\zeta||_1 \leq ||\zeta||_2.$$

Put

$$||\cdot|| = \left[\begin{array}{l} ||\cdot||_1 \\ \\ ||\cdot||_2. \end{array}\right.$$

Then $||\cdot||$ is a seminorm on $W(E,\sigma)$. But

$$||\zeta||^2 \geq \omega_{tr}(\zeta^*\zeta) = \sum_f |\zeta(f)|^2 \quad \text{(cf. 17.5).}$$

Therefore $||\cdot||$ is actually a norm on $W(E,\sigma)$, which is evidently algebraic. To see that it is a C*-norm, note first that

$$\begin{aligned} ||\zeta^*\zeta|| &\geq ||\pi_U(\zeta^*\zeta)|| \\ &= ||\pi_U(\zeta^*)\pi_U(\zeta)|| \\ &= ||\pi_U(\zeta)^*\pi_U(\zeta)|| \\ &= ||\pi_U(\zeta)||^2 \end{aligned}$$

=>

$$||\zeta^*\zeta||^{1/2} \geq ||\pi_U(\zeta)||$$

=>

$$||\zeta^*\zeta||^{1/2} \geq \sup_{(b,U)} ||\pi_U(\zeta)|| = ||\zeta||.$$

In the other direction,

$$||\zeta^*|| = \sup_{(b,U)} ||\pi_U(\zeta^*)||$$

$$= \sup_{(b,U)} ||\pi_U(\zeta)^*||$$

$$= \sup_{(b,U)} ||\pi_U(\zeta)|| = ||\zeta||$$

=>

$$||\zeta^*\zeta|| \leq ||\zeta^*|| \ ||\zeta|| = ||\zeta||^2$$

=>

$$||\zeta^*\zeta||^{1/2} \leq ||\zeta||.$$

17.10 LEMMA Let $\pi:W(E,\sigma) \rightarrow \mathcal{B}(\mathcal{H})$ be a representation -- then $\forall\ \zeta$,

$$||\pi(\zeta)|| \leq ||\zeta||.$$

PROOF For $\pi = \pi_U$, where (b,U) is a projective representation of E on $\mathcal{H}$.

17.11 REMARK If σ is symplectic, then as we have seen in §16, $\forall\ \zeta$,

$$||\pi(\zeta)|| = ||\zeta||.$$

17.12 LEMMA Let $||\cdot||'$ be a C*-norm on $W(E,\sigma)$ with the property that for every representation π,

$$||\pi(\zeta)|| \leq ||\zeta||' \quad (\zeta \in W(E,\sigma)).$$

Then $||\cdot||' = ||\cdot||$.

PROOF Let π' be a faithful representation of the $||\cdot||'$ completion of $W(E,\sigma)$ (cf. 15.9) -- then

$$||\pi'(\zeta)|| = ||\zeta||' \quad \text{(cf. 15.10)}.$$

But

$$||\pi'(\zeta)|| \leq ||\zeta||$$

=>

$$||\zeta||' \leq ||\zeta||.$$

To go the other way, let π be a faithful representation of the $||\cdot||$ completion of $W(E,\sigma)$ (cf. 15.9) -- then

$$||\pi(\zeta)|| = ||\zeta|| \quad \text{(cf. 15.10)}.$$

But

$$||\pi(\zeta)|| \leq ||\zeta||'$$

=>

$$||\zeta|| \leq ||\zeta||'.$$

The <u>Weyl algebra</u> per (E,σ) is the $||\cdot||$ completion $\mathcal{W}(E,\sigma)$ of $W(E,\sigma)$.

17.13 <u>REMARK</u> By construction, every representation of $W(E,\sigma)$ extends continuously to a representation of $\mathcal{W}(E,\sigma)$. Therefore every representation of $\mathcal{W}(E,\sigma)$ determines and is determined by an element of $\text{Rep}_b\ E$, i.e.,

$$\text{Rep}\ \mathcal{W}(E,\sigma) \longleftrightarrow \text{Rep}_b\ E.$$

17.14 <u>EXAMPLE</u> Let $\mathcal{H}$ be a separable complex Hilbert space. Fix $\lambda > 1$ and define W_λ as in 17.3 — then the <u>double Fock representation</u> (of parameter λ)

$$\pi_{F,\lambda}:\mathcal{W}(\mathcal{H},\mathrm{Im}\langle\ ,\ \rangle) \to \mathcal{B}(\mathrm{BO}(\mathcal{H}) \hat{\otimes} \mathrm{BO}(\mathcal{H}))$$

is characterized by the requirement that

$$\pi_{F,\lambda}(\delta_f) = W_\lambda(f).$$

In contrast to the Fock representation π_F (cf. 16.20), $\pi_{F,\lambda}$ is reducible. Indeed, $\forall\ f,g \in \mathcal{H}$,

$$(W(\alpha f) \otimes W(\beta Cf))(W(\beta g) \otimes W(\alpha Cg))$$

$$= (W(\beta g) \otimes W(\alpha Cg)) \otimes (W(\alpha f) \otimes W(\beta Cf)).$$

On the other hand, $\pi_{F,\lambda}$ is primary (cf. 20.14) but if $\mathcal{H}$ is infinite dimensional, then $\pi_{F,\lambda}$ is not geometrically equivalent to π_F and π_{F,λ_1} is not geometrically equivalent to π_{F,λ_2} $(\lambda_1 \neq \lambda_2)$ (cf. 21.9).]

17.15 LEMMA π_{tr} is a faithful representation of $\mathcal{W}(E,\sigma)$.

PROOF For any representation π of $\mathcal{W}(E,\sigma)$, we have (cf. 17.8)

$$||\pi(W)|| \leq ||\pi_{tr}(W)|| \quad (W \in \mathcal{W}(E,\sigma)).$$

And this implies that π_{tr} is faithful (since one can always choose π faithful (cf. 15.9)).

17.16 REMARK By construction, every state on $W(E,\sigma)$ extends continuously to a state on $\mathcal{W}(E,\sigma)$. Therefore every state on $\mathcal{W}(E,\sigma)$ determines and is determined by a σ positive definite function on E, i.e.,

$$\mathcal{S}(\mathcal{W}(E,\sigma)) \longleftrightarrow \mathcal{PD}(E,\sigma).$$

[Note: Give $S(\mathcal{W}(E,\sigma))$ the weak* topology and equip $\mathcal{PD}(E,\sigma)$ with the topology of pointwise convergence -- then the arrow

$$\begin{cases} S(\mathcal{W}(E,\sigma)) \rightarrow \mathcal{PD}(E,\sigma) \\ \omega \rightarrow \chi_\omega \end{cases}$$

is an affine homeomorphism, its inverse being the arrow

$$\begin{cases} \mathcal{PD}(E,\sigma) \rightarrow S(\mathcal{W}(E,\sigma)) \\ \chi \rightarrow \omega_\chi \end{cases}.]$$

17.17 <u>EXAMPLE</u> Let H be a separable complex Hilbert space. Fix $\lambda > 1$ and let ω_λ be the state on $\mathcal{W}(H, \text{Im} < , >)$ determined by the Im < , > positive definite function

$$f \rightarrow e^{-\frac{\lambda}{4}||f||^2} \quad (f \in H) \quad (\text{cf. } 17.3).$$

Since $\Omega \otimes \Omega$ is cyclic, $\pi_{F,\lambda}$ is the GNS representation associated with ω_λ (cf. 15.6).

17.18 <u>LEMMA</u> Let $f \in E$ $(f \neq 0)$ -- then

$$||\delta_f - \delta_0|| = 2.$$

[Note: More generally, $\forall$ $u,v \in \underline{C}$ and $\forall$ $f \neq g$ in E,

$$||u\delta_f + v\delta_g|| = |u| + |v|.]$$

17.19 <u>LEMMA</u> $\mathcal{W}(E,\sigma)$ is not separable.

17.20 LEMMA $\mathcal{W}(E,\sigma)$ is simple iff σ is symplectic.

These three lemmas are the analogs in the pre-symplectic situation of 16.10, 16.11, and 16.12, respectively.

17.21 LEMMA Let E' be a subspace of E and let σ' be the restriction of σ to E — then $W(E',\sigma')$ is a unital $*$-subalgebra of $W(E,\sigma)$. Moreover,

$$||\cdot||' = ||\cdot|| \Big| W(E',\sigma'),$$

so $\mathcal{W}(E',\sigma')$ is a unital C*-subalgebra of $\mathcal{W}(E,\sigma)$. Finally,

$$E' \neq E \Rightarrow \mathcal{W}(E',\sigma') \neq \mathcal{W}(E,\sigma).$$

[To see the last point, let $W' \in \mathcal{W}(E',\sigma')$, $f \in E - E'$ — then

$$\begin{aligned}
&||W' - \delta_f||^2 \\
&\geq \omega_{tr}((W' - \delta_f)^*(W' - \delta_f)) \\
&= \omega_{tr}((W')^*W') + \omega_{tr}(\delta_f^*\delta_f) \\
&\qquad - \omega_{tr}(W'\delta_f) - \omega_{tr}(\delta_f^*W') \\
&= \omega_{tr}((W')^*W') + 1 \\
&\geq 1.]
\end{aligned}$$

[Note: Compare this result with that mentioned in 16.13.]

17.22 LEMMA Let $\phi:W(E,\sigma) \rightarrow W(E,\sigma)$ be a $*$-automorphism — then ϕ is an isometry and extends continuously to an automorphism of $\mathcal{W}(E,\sigma)$ (denoted still by ϕ).

PROOF Fix a faithful representation π of $\mathcal{W}(E,\sigma)$ (cf. 15.9) -- then $\forall\ \zeta \in W(E,\sigma)$,

$$||\pi(\phi(\zeta))|| = ||\phi(\zeta)||.$$

But (cf. 17.10)

$$||(\pi \circ \phi)(\zeta)|| \leq ||\zeta||.$$

Therefore

$$||\phi(\zeta)|| \leq ||\zeta||.$$

And likewise

$$||\phi^{-1}(\zeta)|| \leq ||\zeta||.$$

So, $\forall\ \zeta \in W(E,\sigma)$,

$$\begin{aligned} ||\phi(\zeta)|| &\leq ||\zeta|| \\ &= ||\phi^{-1}(\phi(\zeta))|| \\ &\leq ||\phi(\zeta)|| \end{aligned}$$

=>

$$||\phi(\zeta)|| = ||\zeta||.$$

17.23 EXAMPLE Let $\tau:E \to \underline{T}$ be a character -- then the $*$-automorphism γ_τ

of $W(E,\sigma)$ satisfying the condition

$$\gamma_\tau(\delta_f) = \tau(f)\delta_f \quad (f \in E)$$

extends by continuity to an automorphism of $\mathcal{W}(E,\sigma)$.

17.24 <u>EXAMPLE</u> Let T be an additive automorphism of E which leaves σ invariant -- then the $*$-automorphism α_T of $W(E,\sigma)$ satisfying the condition

$$\alpha_T(\delta_f) = \delta_{Tf} \quad (f \in E)$$

extends by continuity to an automorphism of $\mathcal{W}(E,\sigma)$ (cf. 16.21 and 16.23).

Specialize now and take $\sigma = 0$ -- then $\mathcal{W}(E,0)$ is a commutative C*-algebra. In the weak* topology, $\mathcal{P}(\mathcal{W}(E,0))$ is a compact Hausdorff space and, via the Gelfand transform $W \to \hat{W}$, $\mathcal{W}(E,0)$ is isomorphic to $C(\mathcal{P}(\mathcal{W}(E,0)))$.

On general grounds (cf. 17.16),

$$\mathcal{PD}(E,0) \longleftrightarrow \mathcal{S}(\mathcal{W}(E,0))$$

and under this identification,

$$\hat{E} \longleftrightarrow \mathcal{P}(\mathcal{W}(E,0)).$$

[Note: $\hat{E}$ is a compact Hausdorff space and its topology is that of pointwise convergence, hence is the relative topology inherited from $\mathcal{PD}(E,0)$.]

Therefore $\mathcal{W}(E,0)$ is isomorphic to $C(\hat{E})$:

$$\hat{W}(\tau) = \omega_\tau(W) \quad (\tau \in \hat{E}, W \in \mathcal{W}(E,0)).$$

The state space $\mathcal{S}(C(\hat{E}))$ can be identified with the set $M_p(\hat{E})$ of Radon probability measures on $\hat{E}$ (the pure states corresponding to the $\delta_\tau (\tau \in \hat{E})$).

Consequently,

$$\mathcal{PD}(E,0) \longleftrightarrow \mathcal{S}(\mathcal{W}(E,0))$$

$$\longleftrightarrow \mathcal{S}(C(\hat{E})) \longleftrightarrow M_p(\hat{E}).$$

So in this way each $\chi \in \mathcal{PD}(E,0)$ determines an element μ_χ of $M_p(\hat{E})$ and vice versa. Explicated:

$$\omega_\chi(W) = \int_{\hat{E}} \hat{W}(\tau)\,d\mu_\chi(\tau).$$

17.25 <u>LEMMA</u> Let $\begin{cases} \sigma_1 \\ \sigma_2 \end{cases}$ be pre-symplectic structures on E. Let

$$\begin{cases} \chi_1 \in \mathcal{PD}(E,\sigma_1) \\ \chi_2 \in \mathcal{PD}(E,\sigma_2). \end{cases}$$

Then

$$\chi_1\chi_2 \in \mathcal{PD}(E,\sigma_1 + \sigma_2).$$

[This is because K(E) is closed under pointwise multiplication (cf. 14.5 and subsequent discussion).]

Accordingly, $\mathcal{PD}(E,\sigma)$ is closed under pointwise multiplication with the elements of $\mathcal{PD}(E,0)$.

§18. STATES ON THE WEYL ALGEBRA

Suppose that (E,σ) is a pre-symplectic vector space, $\mathcal{W}(E,\sigma)$ its Weyl algebra -- then the <u>characteristic function</u> of ω is the unique σ positive definite function $\chi_\omega \in \mathcal{PD}(E,\sigma)$ such that $\omega_{\chi_\omega} = \omega$ (cf. 17.16).

18.1 <u>LEMMA</u> $\forall\ f,g \in E$, we have

$$\frac{1}{2}\,|\chi_\omega(f) - \chi_\omega(g)|^2$$

$$\leq |\exp(\frac{\sqrt{-1}}{2}\,\sigma(f,g)) - 1| + |1 - \chi_\omega(f-g)|.$$

<u>PROOF</u> Let $(\pi_\omega,\mathcal{H}_\omega,\Omega_\omega)$ be GNS data per ω, so $\forall\ W \in \mathcal{W}(E,\sigma)$,

$$\omega(W) = \langle \Omega_\omega, \pi_\omega(W)\Omega_\omega \rangle.$$

Then

$$|\chi_\omega(f) - \chi_\omega(g)|^2$$

$$= |\omega(\delta_f) - \omega(\delta_g)|^2$$

$$= |\langle \Omega_\omega, \pi_\omega(\delta_f - \delta_g)\Omega_\omega \rangle|^2$$

$$\leq ||\pi_\omega(\delta_f - \delta_g)\Omega_\omega||^2$$

$$= \langle \Omega_\omega, \pi_\omega(\delta_f - \delta_g)^*\pi_\omega(\delta_f - \delta_g)\Omega_\omega \rangle$$

$$= \langle \Omega_\omega, \pi_\omega(\delta_f^* - \delta_g^*)\pi_\omega(\delta_f - \delta_g)\Omega_\omega \rangle$$

$$= \langle \Omega_\omega, \pi_\omega(\delta_{-f} - \delta_{-g})\pi_\omega(\delta_f - \delta_g)\Omega_\omega \rangle$$

$$= \langle \Omega_\omega, \pi_\omega(\delta_{-f}\delta_f + \delta_g\delta_{-g})\Omega_\omega \rangle$$

$$- \langle \Omega_\omega, \pi_\omega(\delta_{-f}\delta_g)\Omega_\omega \rangle$$

$$- \langle \Omega_\omega, \pi_\omega(\delta_{-g}\delta_f)\Omega_\omega \rangle$$

$$= 2 - \omega(\delta_{-f}\delta_g) - \omega(\delta_{-g}\delta_f)$$

$$= 2 - \omega((\delta_{-g}\delta_f)^*) - \omega(\delta_{-g}\delta_f)$$

$$= 2 - \omega(\delta_{-g}\delta_f) - \overline{\omega(\delta_{-g}\delta_f)}$$

$$= 2 - 2\ \mathrm{Re}(\omega(\delta_{-g}\delta_f))$$

$$= 2 - 2\ \mathrm{Re}(\exp(-\frac{\sqrt{-1}}{2}\sigma(-g,f))\omega(\delta_{-g+f}))$$

$$= 2 - 2\ \mathrm{Re}(\exp(-\frac{\sqrt{-1}}{2}\sigma(f,g))\omega(\delta_{f-g}))$$

$$\leq 2\left|1 - \exp(-\frac{\sqrt{-1}}{2}\sigma(f,g))\omega(\delta_{f-g})\right|$$

$$= 2\left|\exp(\frac{\sqrt{-1}}{2}\sigma(f,g)) - \omega(\delta_{f-g})\right|$$

$$= 2\left|\exp\left(\frac{\sqrt{-1}}{2}\,\sigma(f,g)\right) - 1 + 1 - \omega(\delta_{f-g})\right|$$

$$\leq 2\left|\exp\left(\frac{\sqrt{-1}}{2}\,\sigma(f,g)\right) - 1\right| + 2\left|1 - \chi_\omega(f - g)\right|.$$

Denote by $\mathcal{T}(E,\sigma)$ the set of all topologies τ on E such that:

1. $\forall\ f \in E$, the map

$$\begin{cases} E \to E \\ g \to f + g \end{cases}$$

is τ-continuous;

2. $\forall\ f \in E$, the map

$$\begin{cases} E \to \underline{R} \\ g \to \sigma(f,g) \end{cases}$$

is τ-continuous.

[Note: The discrete topology meets these requirements, hence $\mathcal{T}(E,\sigma)$ is not empty.]

18.2 EXAMPLE The finite topology on E is the final topology determined by the inclusions $F \to E$, where F is a finite dimensional linear subspace of E endowed with its natural euclidean topology. In other words, the finite topology on E is the largest topology for which each inclusion $F \to E$ is continuous. It is characterized by the property that if X is a topological space and if $f:E \to X$ is a function, then f is continuous iff $\forall\ F \subset E$, the restriction $f|F$ is continuous. Obviously, then, the finite topology on E is an element of $\mathcal{T}(E,\sigma)$.

[Note: The finite topology is, in general, not a vector topology (scalar multiplication $\underline{R} \times E \to E$ is continuous; vector addition $E \times E \to E$ is separately continuous but is jointly continuous iff dim E is $\leq$ aleph-naught).]

18.3 LEMMA If $\tau \in \mathcal{T}(E,\sigma)$ and if χ_ω is τ-continuous at the origin, then χ_ω is τ-continuous on all of E.

[This is an immediate consequence of 18.1.]

Given $\tau \in \mathcal{T}(E,\sigma)$, let

$$F_\tau = \{\omega \in S(W(E,\sigma)) : \chi_\omega \text{ is } \tau\text{-continuous}\}.$$

18.4 LEMMA F_τ is a folium in $S(W(E,\sigma))$.

[Note: If τ is the discrete topology, then $F_\tau = S(W(E,\sigma))$. And

$$\tau_1 \leq \tau_2 \Rightarrow F_{\tau_1} \subset F_{\tau_2}.]$$

A state $\omega \in S(W(E,\sigma))$ is said to be nonsingular provided χ_ω is continuous in the finite topology.

18.5 LEMMA If $\forall\, f \in E$, the function $t \to \chi_\omega(tf)$ $(t \in \underline{R})$ is continuous, then ω is nonsingular.

PROOF Working with the GNS representation π_ω attached to ω, $\forall\, f,g \in E$ and $\forall\, t \in \underline{R}$,

$$||(\pi_\omega(\delta_{tf}) - I)\pi_\omega(\delta_g)\Omega_\omega||^2$$

$$= 2 - e^{-\sqrt{-1}\, t\sigma(f,g)}\omega(\delta_{tf}) - e^{\sqrt{-1}\, t\sigma(f,g)}\omega(\delta_{-tf})$$

$$= 2 - e^{-\sqrt{-1}\, t\sigma(f,g)}\chi_\omega(tf) - e^{\sqrt{-1}\, t\sigma(f,g)}\chi_\omega(- tf).$$

Since Ω_ω is cyclic, it follows that $\forall\ f \in E$, $\pi_\omega(\delta_{tf})$ is strongly continuous in t, or still, $\forall\ f \in E$, $U_{\pi_\omega}(tf)$ is strongly continuous in t, which implies that U_{π_ω} is strongly continuous on finite dimensional subspaces of E (cf. 10.7). But

$$\omega(\delta_f) = \langle\ \Omega_\omega, \pi_\omega(\delta_f)\Omega_\omega\ \rangle,$$

i.e.,

$$\chi_\omega(f) = \langle\ \Omega_\omega, U_{\pi_\omega}(f)\Omega_\omega\ \rangle.$$

Therefore χ_ω is continuous in the finite topology.

[Note: The converse is, of course, trivial. Observe too that it suffices to check the continuity of $t \to \chi_\omega(tf)$ $(t \in \underline{R})$ at $t = 0$ (cf. 18.1).]

18.6 <u>EXAMPLE</u> Let H be a separable complex Hilbert space -- then the vacuum state ω_F is nonsingular:

$$\chi_F(f) = \omega_F(\delta_f) = e^{-\frac{1}{4}||f||^2} \quad \text{(cf. 16.20).}$$

18.7 <u>LEMMA</u> The set F_{ns} of all nonsingular states on $W(E,\sigma)$ is a folium in $S(W(E,\sigma))$.

If $\omega \in S(W(E,\sigma))$ is nonsingular, then $\forall\ f \in E$, the map

$$t \to \pi_\omega(\delta_{tf})$$

is a one parameter unitary group (see the proof of 18.5), hence admits a generator $\Phi_\omega(f)$:

$$\pi_\omega(\delta_{tf}) = \exp(\sqrt{-1}\, t\Phi_\omega(f)).$$

Unfortunately, however, it need not be true that $\Omega_\omega \in \text{Dom}(\Phi_\omega(f))$ but this difficulty can be dealt with by imposing an additional condition on ω: Call ω C^∞ if $\forall f \in E$, the function $t \to \chi_\omega(tf)$ is C^∞.

18.8 <u>LEMMA</u> If ω is C^∞, then $\forall f \in E$,

$$\pi_\omega(\delta_f)\Omega_\omega$$

is in the domain of

$$\Phi_\omega(f_1)\ldots\Phi_\omega(f_n)$$

for all $f_1,\ldots,f_n \in E$.

[Note: In particular, Ω_ω is in the domain of all the $\Phi_\omega(f)$.]

18.9 <u>REMARK</u> If $\forall f \in E$, the function $t \to \chi_\omega(tf)$ $(t \in \underline{R})$ is analytic, then Ω_ω is an analytic vector for $\Phi_\omega(f)$. To begin with, in view of 18.8,

$$\Omega_\omega \in \bigcap_{k=1}^{\infty} \text{Dom}(\Phi_\omega(f))^k.$$

I.e.: Ω_ω is a C^∞ vector for $\Phi_\omega(f)$. This said, there is an absolutely convergent expansion

$$\chi_\omega(tf) = \sum_{k=0}^{\infty} \frac{(\sqrt{-1})^k t^k}{k!} < \Omega_\omega, \Phi_\omega(f)^k \Omega_\omega > \quad (|t| < R_f),$$

so $\exists$ C > 0:

$$|t| < R_f$$

$$\Rightarrow$$

$$\left| \frac{t^k}{k!} < \Omega_\omega, \Phi_\omega(f)^k \Omega_\omega > \right| \leq C$$

$$\Rightarrow$$

$$\frac{|t|^{2k}}{(2k)!} ||\Phi_\omega(f)^k \Omega_\omega||^2 \leq C$$

$$\Rightarrow$$

$$\frac{|t|^k}{\sqrt{(2k)!}} ||\Phi_\omega(f)^k \Omega_\omega|| \leq \sqrt{C}.$$

We then claim that

$$|t| < R_f/2$$

$$\Rightarrow$$

$$\frac{|t|^k}{k!} ||\Phi_\omega(f)^k \Omega_\omega|| \leq \sqrt{C}.$$

Indeed,

$$\frac{|t|^k}{k!} ||\Phi_\omega(f)^k \Omega_\omega||$$

$$= \frac{|2t|^k}{\sqrt{(2k)!}} \left(\frac{(2k)!}{(k!)^2} \cdot \frac{1}{2^k} \right)^{1/2} ||\Phi_\omega(f)^k \Omega_\omega||$$

$$= \frac{|2t|^k}{\sqrt{(2k)!}} \left(\binom{2k}{k} / 2^k \right)^{1/2} ||\Phi_\omega(f)^k \Omega_\omega||$$

$$\leq \frac{|2t|^k}{\sqrt{(2k)!}} \left((1 + 1)^{2k}/2^k\right)^{1/2} ||\Phi_\omega(f)^k\Omega_\omega||$$

$$= \frac{|2t|^k}{\sqrt{(2k)!}} ||\Phi_\omega(f)^k\Omega_\omega||$$

$$\leq \sqrt{C}.$$

Therefore

$$|t| < R_f/4$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{|t|^k}{k!} ||\Phi_\omega(f)^k\Omega_\omega||$$

$$= \sum_{k=0}^{\infty} \frac{(R_f/4)^k}{k!} ||\Phi_\omega(f)^k\Omega_\omega|| \left(\frac{|t|}{R_f/4}\right)^k$$

$$\leq \sqrt{C} \sum_{k=0}^{\infty} \left(\frac{|t|}{R_f/4}\right)^k$$

$$< \infty.$$

The complement of F_{ns} in $S(W(E,\sigma))$ constitutes the set of <u>singular states</u>.

18.10 <u>EXAMPLE</u> Let ω_{tr} be the tracial state defined in 17.5 -- then ω_{tr} is singular. In fact, $\forall$ nonzero f in E,

$$\chi_{tr}(tf) = \begin{cases} 1 & (t = 0) \\ 0 & (t \neq 0). \end{cases}$$

Given a state $\omega \in S(\mathcal{W}(E,\sigma))$, put

$$L_\omega = \{f \in E : \chi_\omega(f) \in \underline{T}\}.$$

18.11 LEMMA If $f \in L_\omega$, then

$$\pi_\omega(\delta_f)\Omega_\omega = \chi_\omega(f)\Omega_\omega.$$

PROOF From the definitions,

$$\omega((\delta_f - \chi_\omega(f)\delta_0)^*(\delta_f - \chi_\omega(f)\delta_0))$$

$$= < \Omega_\omega, \pi_\omega((\delta_f - \chi_\omega(f)\delta_0)^*(\delta_f - \chi_\omega(f)\delta_0))\Omega_\omega >$$

$$= < \pi_\omega(\delta_f - \chi_\omega(f)\delta_0)\Omega_\omega, \pi_\omega(\delta_f - \chi_\omega(f)\delta_0)\Omega_\omega >$$

$$= ||\pi_\omega(\delta_f)\Omega_\omega - \chi_\omega(f)\Omega_\omega||^2.$$

But

$$\omega((\delta_f - \chi_\omega(f)\delta_0)^*(\delta_f - \chi_\omega(f)\delta_0))$$

$$= \omega((\delta_{-f} - \overline{\chi_\omega(f)}\delta_0)(\delta_f - \chi_\omega(f)\delta_0))$$

$$= \omega(\delta_0 - \overline{\chi_\omega(f)}\delta_f - \chi_\omega(f)\delta_{-f} + \overline{\chi_\omega(f)}\chi_\omega(f)\delta_0)$$

$$= 2 - \overline{\chi_\omega(f)}\omega(\delta_f) - \chi_\omega(f)\omega(\delta_{-f})$$

$$= 2 - \overline{\chi_\omega(f)}\chi_\omega(f) - \chi_\omega(f)\chi_\omega(- f)$$

$$= 2 - \overline{\chi_\omega(f)}\chi_\omega(f) - \chi_\omega(f)\overline{\chi_\omega(f)}$$

$$= 0$$

=>

$$\pi_\omega(\delta_f)\Omega_\omega = \chi_\omega(f)\Omega_\omega.$$

[Note: Suppose that $\pi_\omega(\delta_f)$ has Ω_ω as an eigenvector, say

$$\pi_\omega(\delta_f)\Omega_\omega = \lambda\Omega_\omega \quad (\text{so } |\lambda| = 1).$$

Then $\lambda = \chi_\omega(f)$, hence $f \in L_\omega$. For

$$\chi_\omega(f) = \omega(\delta_f) = \langle \Omega_\omega, \pi_\omega(\delta_f)\Omega_\omega \rangle$$

$$= \lambda \langle \Omega_\omega, \Omega_\omega \rangle$$

$$= \lambda.]$$

18.12 <u>LEMMA</u> L_ω is an additive subgroup of E on which

$$\sigma(L_\omega \times L_\omega) \subset 2\pi\underline{Z}.$$

Moreover, $\forall$ $f,g \in L_\omega$,

$$\chi_\omega(f)\chi_\omega(g) = (-1)^{\sigma(f,g)/2\pi}\chi_\omega(f + g).$$

<u>PROOF</u> We have

$$\pi_\omega(\delta_{f+g})\Omega_\omega = \pi_\omega(e^{\frac{\sqrt{-1}}{2}\sigma(f,g)}\delta_f\delta_g)\Omega_\omega$$

$$= e^{\frac{\sqrt{-1}}{2}\sigma(f,g)}\pi_\omega(\delta_f)\pi_\omega(\delta_g)\Omega_\omega$$

$$= e^{\frac{\sqrt{-1}}{2}\sigma(f,g)} \chi_\omega(f)\chi_\omega(g)\Omega_\omega.$$

Therefore $f + g \in L_\omega$ and

$$\chi_\omega(f + g) = e^{\frac{\sqrt{-1}}{2}\sigma(f,g)} \chi_\omega(f)\chi_\omega(g).$$

Reversing the roles of f and g then gives

$$e^{\frac{\sqrt{-1}}{2}\sigma(g,f)} = e^{\frac{\sqrt{-1}}{2}\sigma(f,g)}$$

or still,

$$e^{-\frac{\sqrt{-1}}{2}\sigma(f,g)} = e^{\frac{\sqrt{-1}}{2}\sigma(f,g)}$$

or still,

$$1 = e^{\sqrt{-1}\,\sigma(f,g)},$$

which implies that

$$\sigma(f,g) \in 2\pi\underline{\mathbb{Z}}.$$

Finally,

$$\sigma(f,g) = 2\pi n \quad (n \in \underline{\mathbb{Z}})$$

=>

$$\begin{aligned} e^{-\frac{\sqrt{-1}}{2}\sigma(f,g)} &= e^{-\sqrt{-1}\,\pi n} \\ &= (e^{-\sqrt{-1}\,\pi})^n \\ &= (e^{\sqrt{-1}\,\pi})^n \\ &= (-1)^n \end{aligned}$$

$$= (-1)^{2\pi n/2\pi}$$

$$= (-1)^{\sigma(f,g)/2\pi}.$$

18.13 <u>LEMMA</u> Take σ symplectic and suppose that ω is nonsingular — then $L_\omega = \{0\}$.

<u>PROOF</u> To get a contradiction, assume $\exists\ f \in L_\omega : f \neq 0$ — then

$$\pi_\omega(\delta_f)\Omega_\omega = \chi_\omega(f)\Omega_\omega \quad \text{(cf. 18.11)},$$

so $\forall\ g \in E$,

$$\begin{aligned}
\chi_\omega(tg) &= \overline{\chi_\omega(f)}\chi_\omega(f)\omega(\delta_{tg}) \\
&= \omega(\overline{\chi_\omega(f)}\delta_{tg}\chi_\omega(f)) \\
&= < \Omega_\omega, \pi_\omega(\overline{\chi_\omega(f)}\delta_{tg}\chi_\omega(f))\Omega_\omega > \\
&= < \Omega_\omega, \overline{\chi_\omega(f)}\pi_\omega(\delta_{tg})\chi_\omega(f)\Omega_\omega > \\
&= \overline{\chi_\omega(f)} < \Omega_\omega, \pi_\omega(\delta_{tg})\pi_\omega(\delta_f)\Omega_\omega > \\
&= < \chi_\omega(f)\Omega_\omega, \pi_\omega(\delta_{tg})\pi_\omega(\delta_f)\Omega_\omega > \\
&= < \pi_\omega(\delta_f)\Omega_\omega, \pi_\omega(\delta_{tg})\pi_\omega(\delta_f)\Omega_\omega > \\
&= < \Omega_\omega, \pi_\omega(\delta_{-f})\pi_\omega(\delta_{tg})\pi_\omega(\delta_f)\Omega_\omega >
\end{aligned}$$

$$= < \Omega_\omega, \pi_\omega(\delta_{-f}\delta_{tg}\delta_f)\Omega_\omega >$$

$$= b(-f,tg)^2 < \Omega_\omega, \pi_\omega(\delta_{tg})\Omega_\omega >$$

$$= e^{\sqrt{-1}\,\sigma(f,g)t}\chi_\omega(tg)$$

=>

$$(e^{\sqrt{-1}\,\sigma(f,g)t} - 1)\chi_\omega(tg) = 0.$$

Choose $g:\sigma(f,g) = 2\pi$ and let $-1 < t < 1$ -- then

$$e^{\sqrt{-1}\,2\pi t} - 1$$

is nonzero if t is nonzero, hence the restriction of $\chi_\omega(tg)$ to $]-1,1[$ is discontinuous:

$$\chi_\omega(tg) = \begin{cases} 1 & (t = 0) \\ 0 & (t \neq 0). \end{cases}$$

18.14 REMARK Take σ symplectic -- then a state $\omega \in S(W(E,\sigma))$ is said to be polarized if L_ω is maximal, i.e., if

$$\{f \in E:\sigma(f,L_\omega) \subset 2\pi\underline{Z}\} = L_\omega.$$

Every polarized state is necessarily singular (cf. 18.13). In addition, if ω is such a state, then ω is pure but its GNS Hilbert space H_ω is nonseparable.

[Note: Let ω_1, ω_2 be polarized states on $W(E,\sigma)$ -- then it can be shown that $\pi_{\omega_1}, \pi_{\omega_2}$ are unitarily equivalent iff

$$\begin{cases} L_{\omega_1} \cap L_{\omega_2} \text{ has finite index in } L_{\omega_1} \\ L_{\omega_1} \cap L_{\omega_2} \text{ has finite index in } L_{\omega_2} \end{cases}$$

and $\exists\ f \in E$ such that on $L_{\omega_1} \cap L_{\omega_2}$, $\chi_{\omega_1} = \chi_{\omega_2} e^{\sqrt{-1}\,\sigma(f,\cdot)}$.]

We shall conclude this section with an example which nicely illustrates the potential complexities that are hidden in the theory.

Thus take

$$\begin{cases} E = L^2(\underline{R}^3) \\ \sigma = \text{Im} < , > \end{cases}$$

and work with the associated Fock system (cf. 10.3):

$$W: L^2(\underline{R}^3) \to \mathcal{U}(BO(L^2(\underline{R}^3))).$$

Let

$$V_n = \frac{1}{8}\,[-\,n^{1/3}, n^{1/3}]^3,$$

a region of volume n, and set $f_n = \chi_{V_n}/\sqrt{n}$ — then $||f_n|| = 1$. Put

$$X_n = \frac{1}{\sqrt{n!}}\,\tilde{c}(f_n)^n\Omega,$$

an element of $BO(L^2(\underline{R}^3))$ of norm 1, and define

$$\chi_n: L^2(\underline{R}^3) \to \underline{C}$$

by

$$\chi_n(f) = < X_n, W(f) X_n >$$

or still,

$$\chi_n(f) = \frac{1}{n!} < \Omega, \tilde{a}(f_n)^n W(f) \tilde{c}(f_n)^n \Omega >.$$

[Note:

$$\chi_n \in \mathcal{PD}(L^2(\underline{R}^3), \mathrm{Im} < , >) \quad (\text{cf. } 17.1).]$$

18.15 RAPPEL The Laguerre polynomials L_n are given by

$$L_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} (-x)^{n-k}.$$

18.16 LEMMA $\forall\ f \in L^2(\underline{R}^3)$,

$$\chi_n(f) = \chi_F(f) L_n(\frac{1}{2} \mid < f_n, f > \mid^2),$$

where

$$\chi_F(f) = e^{-\frac{1}{4} ||f||^2} \quad (\text{cf. } 18.6).$$

PROOF First

$$\tilde{a}(f_n)^n W(f) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\sqrt{-1} < f_n, f >}{\sqrt{2}}\right)^{n-k} W(f) \tilde{a}(f_n)^k \quad (\text{cf. } 12.24),$$

thus

$$\chi_n(f) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\sqrt{-1} < f_n, f >}{\sqrt{2}}\right)^{n-k} < \Omega, W(f) \tilde{a}(f_n)^k \tilde{c}(f_n)^n \Omega >.$$

Next

$$\tilde{a}(f_n)^k \tilde{c}(f_n)^n \Omega = \frac{n!}{(n-k)!} \tilde{c}(f_n)^{n-k}\Omega,$$

so

$$\chi_n(f) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} \left(\frac{\sqrt{-1} < f_n, f >}{\sqrt{2}}\right)^{n-k} < \Omega, W(f)\tilde{c}(f_n)^{n-k}\Omega >.$$

Lastly

$$W(f)\tilde{c}(f_n)^{n-k} = \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} \left(\frac{-\sqrt{-1}\ \overline{< f_n, -f >}}{\sqrt{2}}\right)^{n-k-\ell} \tilde{c}(f_n)^{\ell} W(f) \quad \text{(cf. 12.24)},$$

and from the RHS, only the $\ell = 0$ term can contribute, hence

$$\chi_n(f) = \chi_F(f) \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} \left(-\frac{1}{2} \,|< f_n, f >|^2\right)^{n-k}$$

or still,

$$\chi_n(f) = \chi_F(f) L_n\left(\frac{1}{2} \,|< f_n, f >|^2\right).$$

18.17 <u>RAPPEL</u> We have

$$\lim_{n \to \infty} L_n(x/n) = J_0(2\sqrt{x}) \quad (x \geq 0),$$

where J_0 is the Bessel function:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-\sqrt{-1}(\alpha \cos\theta + \beta \sin\theta))d\theta$$

$$= J_0(\sqrt{\alpha^2 + \beta^2}).$$

18.18 <u>LEMMA</u> $\forall f \in C_c(\underline{R}^3)$,

$$\lim_{n \to \infty} \chi_n(f)$$

exists and equals

$$\chi_F(f) J_0((2\pi)^{3/2} \sqrt{2}\, |\hat{f}(0)|).$$

<u>PROOF</u> If $f \in C_c(\underline{R}^3)$ and if spt f is a proper subset of V_n, then

$$< f_n, f > = \frac{1}{\sqrt{n}} \int_{V_n} f$$

$$= \frac{1}{\sqrt{n}} \int_{\underline{R}^3} f$$

$$= \left(\frac{(2\pi)^3}{n}\right)^{1/2} \hat{f}(0).$$

So, for such an f,

$$\chi_n(f) = \chi_F(f) L_n\left(\frac{1}{2} \,|< f_n, f >|^2\right)$$

$$= \chi_F(f) L_n\left(\frac{(2\pi)^3 \,|\hat{f}(0)|^2}{2} \frac{1}{n}\right)$$

=>

$$\lim_{n \to \infty} \chi_n(f) = \chi_F(f) J_0((2\pi)^{3/2} \sqrt{2}\, |\hat{f}(0)|).$$

Obviously,

$$\chi_n | C_c(\underline{R}^3) \in \mathcal{PD}(C_c(\underline{R}^3), \mathrm{Im} < , >).$$

Now put

$$\chi_{GS}(f) = \lim_{n \to \infty} \chi_n(f) \quad (f \in C_c(\underline{R}^3)).$$

Then it is clear that

$$\chi_{GS} \in \mathcal{PD}(C_c(\underline{R}^3), \operatorname{Im} < , >).$$

Motivated by these considerations, extend χ_{GS} to $L^1(\underline{R}^3) \cap L^2(\underline{R}^3)$ by simply writing

$$\chi_{GS}(f) = \chi_F(f) J_0((2\pi)^{3/2} \sqrt{2}|\hat{f}(0)|).$$

While this makes sense, it is not immediately apparent that

$$\chi_{GS} \in \mathcal{PD}(L^1(\underline{R}^3) \cap L^2(\underline{R}^3), \operatorname{Im} < , >).$$

To resolve the issue, introduce

$$\mathcal{H}_{GS} = BO(L^2(\underline{R}^3)) \hat{\otimes} L^2(\underline{T}),$$

$\underline{T}$ being parameterized by $\theta \in [-\pi,\pi]$. Put

$$\Omega_{GS} = \Omega \otimes 1.$$

Define a Weyl system (cf. 16.3)

$$U_{GS}: L^1(\underline{R}^3) \cap L^2(\underline{R}^3) \to \mathcal{B}(\mathcal{H}_{GS})$$

over $(L^1(\underline{R}^3) \cap L^2(\underline{R}^3), \operatorname{Im} < , >)$ by

$$U_{GS}(f) = \overline{W(f) \otimes M_f}.$$

Here M_f is multiplication by

$$\exp(-\sqrt{-1}\ (2\pi)^{3/2} \sqrt{2}\ (\operatorname{Re} \hat{f}(0) \cos \theta + \operatorname{Im} \hat{f}(0) \sin \theta)).$$

[Note: As was detailed in §17, the Weyl system U_{GS} gives rise to a representation

$$\pi_{GS}:W(L^1(\underline{R}^3) \cap L^2(\underline{R}^3),\ \text{Im} < , >) \rightarrow \mathcal{B}(\mathcal{H}_{GS})$$

which extends continuously to a representation

$$\pi_{GS}:\mathcal{W}(L^1(\underline{R}^3) \cap L^2(\underline{R}^3),\ \text{Im} < , >) \rightarrow \mathcal{B}(\mathcal{H}_{GS}).]$$

18.19 <u>LEMMA</u> $\forall\ f \in L^1(\underline{R}^1) \cap L^2(\underline{R}^3)$,

$$\chi_{GS}(f) = < \Omega_{GS}, U_{GS}(f)\Omega_{GS} >.$$

<u>PROOF</u> The RHS equals

$$\chi_F(f)\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-\sqrt{-1}\ (2\pi)^{3/2}\ \sqrt{2}\ (\text{Re}\ \hat{f}(0)\cos\theta + \text{Im}\ \hat{f}(0)\sin\theta))d\theta$$

$$= \chi_F(f)J_0(((2\pi)^3 2((\text{Re}\ \hat{f}(0))^2 + (\text{Im}\ \hat{f}(0))^2)))^{1/2})$$

$$= \chi_F(f)J_0((2\pi)^{3/2}\ \sqrt{2}|\hat{f}(0)|)$$

$$= \chi_{GS}(f).$$

Therefore

$$\chi_{GS} \in \mathcal{PD}(L^1(\underline{R}^3) \cap L^2(\underline{R}^3),\ \text{Im} < , >) \quad \text{(cf. 17.1)}$$

and the associated state

$$\omega_{GS} \in \mathcal{S}(\mathcal{W}(L^1(\underline{R}^3) \cap L^2(\underline{R}^3),\ \text{Im} < , >))$$

is called the <u>ground state</u> of the infinite Bose gas.

[Note: ω_{GS} is nonsingular.]

18.20 <u>REMARK</u> π_{GS} is the GNS representation per ω_{GS}.

[It is a question of showing that Ω_{GS} is cyclic. For this purpose, let

$$f_{t,z}(x) = tz\sqrt{\pi/2}\,\frac{e^{-t|x|}}{|x|^2+1}\quad (x \in \underline{R}^3, t \in \underline{R}_{>0}, z \in \underline{C}).$$

Then

$$\begin{cases} \lim_{t \to 0} ||f_{t,z}|| = 0 \\ \lim_{t \to 0} \hat{f}_{t,z}(0) = z. \end{cases}$$

But

- Ω is cyclic for $\{W(f): f \in L^1(\underline{R}^3) \cap L^2(\underline{R}^3)\}$.

- 1 is cyclic for $\{e^{\sqrt{-1}\,(a \cos\theta + b \sin\theta)} : a,b \in \underline{R}\}$.]

§19. *COMPLEX STRUCTURES*

Let V be a vector space over $\underline{R}$ -- then a complex structure J on V is an $\underline{R}$-linear map $J:V \rightarrow V$ such that $J^2 = - I$.

[Note: If J is a complex structure, then so is - J.]

Suppose given a complex structure J on V -- then V can be turned into a vector space $V^{\sim}$ over $\underline{C}$ by stipulating that

$$(a + \sqrt{-1}\, b)v = av + bJv.$$

Of course, V and $V^{\sim}$ agree set theoretically.

19.1 REMARK Let W be a vector space over $\underline{C}$ -- then restriction of scalars gives rise to a vector space $W_{\underline{R}}$ over $\underline{R}$. On the other hand, multiplication by $\sqrt{-1}$ is a complex structure on $W_{\underline{R}}$ and it is clear that $W = W_{\underline{R}}^{\sim}$.

Let V be a vector space over $\underline{R}$ -- then the product $V \times V$ is a vector space over $\underline{R}$ and the map

$$J:V \times V \rightarrow V \times V$$

defined by

$$J(v,v') = (- v',v)$$

is a complex structure on $V \times V$. The complex vector space $(V \times V)^{\sim}$ is called the complexification of V and is denoted by $V_{\underline{C}}$. Since $(v,v') = (v,0) + J(v',0)$, one writes $v + \sqrt{-1}\, v'$ in place of (v,v'), thus

$$(a + \sqrt{-1}\, b)(v + \sqrt{-1}\, v') = av - bv' + \sqrt{-1}\, (av' + bv).$$

19.2 <u>EXAMPLE</u> Let X be a Hilbert space over $\underline{R}$. Suppose that J is a complex structure on X which is isometric:

$$||Jx|| = ||x|| \ \forall\, x \in X.$$

Noting that $J^* = -J$, put

$$< x,y >_J = < x,y > - \sqrt{-1} < x,Jy >.$$

Then $< \, , \, >_J$ is an inner product on $X^\sim$, so $X^\sim$ is actually a Hilbert space over $\underline{C}$.

[Note: Here is a special case. Form the direct sum $X \oplus X$, the inner product being

$$< (x,y),(x',y') > = < x,x' > + < y,y' >.$$

Then the complex structure $J(x,y) = (-y,x)$ is isometric. Now apply the preceding construction — then the upshot is that $X_{\underline{C}}$ is a complex Hilbert space with inner product

$$< x + \sqrt{-1}\, y, x' + \sqrt{-1}\, y' >_J$$

$$= < (x,y),(x',y') > - \sqrt{-1} < (x,y), J(x',y') >$$

$$= < x,x' > + < y,y' > + \sqrt{-1}\, (< x,y' > - < y,x' >).]$$

Let (E,σ) be a symplectic vector space — then a <u>Kähler structure</u> on (E,σ) is a complex structure $J:E \to E$ such that

$$\sigma(Jf,Jg) = \sigma(f,g) \quad (f,g \in E)$$

and

$$\sigma(f,Jf) > 0 \quad (f \in E, f \neq 0).$$

In the presence of a Kähler structure, $E^{\sim}$ is a complex pre-Hilbert space, the inner product being

$$< f,g >_J = \sigma(f,Jg) + \sqrt{-1}\,\sigma(f,g).$$

19.3 LEMMA Suppose that J is a Kähler structure on (E,σ) and $T:E \to E$ is symplectic -- then TJT^{-1} is also a Kähler structure on (E,σ).

19.4 REMARK In general, (E,σ) does not admit a Kähler structure. For example, let V be an infinite dimensional vector space over $\underline{R}$ and let $V^{\#}$ be its algebraic dual. Put $E = V \oplus V^{\#}$ and define $\sigma:E \times E \to \underline{R}$ by

$$\sigma((v,\lambda),(v',\lambda')) = \lambda'(v) - \lambda(v').$$

Then (E,σ) is a symplectic vector space but (E,σ) does not admit a Kähler structure.

19.5 EXAMPLE Suppose that E is a real Hilbert space and $\sigma:E \times E \to \underline{R}$ is continuous -- then the pair (E,σ) admits a Kähler structure.

19.6 LEMMA Suppose that J is a Kähler structure on (E,σ) and $T:E \to E$ is symplectic. Assume: $TJ = JT$ -- then $\forall\ f,g \in E$,

$$< Tf,Tg >_J = < f,g >_J.$$

[Note: The condition $TJ = JT$ amounts to saying that T is $\underline{C}$-linear. Write H_J for the completion of $E^{\sim}$ per $< , >_J$ -- then T extends uniquely to a unitary operator $U_T:\mathcal{H}_J \to \mathcal{H}_J$.]

19.7 EXAMPLE Take $T = J'$, where J' is another Kähler structure on (E,σ) -- then $J'J = JJ' \Rightarrow U_{J'} = \pm \sqrt{-1}\, I \Rightarrow J' = \pm J$. But

$$J' = - J \Rightarrow - \sigma(f,Jf) = \sigma(f,J'f) > 0 \quad (f \in E, f \neq 0).$$

Therefore $J' = J$.

[Note: $- J$ is not a Kähler structure per σ but $- J$ is a Kähler structure per $- \sigma$.]

19.8 REMARK The converse to 19.6 is also valid. Proof: $\forall\, f,g \in E$,

$$\begin{aligned} < TJf,Tg >_J &= < Jf,g >_J \\ &= < \sqrt{-1}\, f,g >_J \\ &= - \sqrt{-1} < f,g >_J \\ &= - \sqrt{-1} < Tf,Tg >_J \\ &= < \sqrt{-1}\, Tf,Tg >_J \\ &= < JTf,Tg >_J \end{aligned}$$

$\Rightarrow$

$$TJ = JT.$$

Let $t \to T_t$ be a one parameter group of symplectic maps -- then a Kähler structure J on E is said to be a unitarization of $\{T_t\}$ if $\forall$ t, $JT_t = T_tJ$, and $t \to T_t$ extends to a one parameter unitary group $U:\underline{R} \to U(H_J)$ such that U(t) $(\equiv U_{T_t}) = e^{\sqrt{-1}\, tH}$, where the generator H is positive and $0 \notin \sigma_p(H)$.

[Note: Let $x,y \in H_J$ -- then

$$< x,U(t)y > = \int_{\underline{R}} e^{\sqrt{-1}\, t\lambda}\, d< x,E_\lambda y >$$

$$= \int_{\underline{R}_{\geq 0}} e^{\sqrt{-1}\, t\lambda}\, d< x,E_\lambda y >$$

=>

$$< x,U(\sqrt{-1}\, t)y > = \int_{\underline{R}_{\geq 0}} e^{-t\lambda}\, d< x,E_\lambda y >$$

=>

$$\lim_{t \to \infty} < x,U(\sqrt{-1}\, t)y > = < x,Py >,$$

P the orthogonal projection onto the kernel of H. Since $0 \notin \sigma_p(H)$, the conclusion is that

$$\lim_{t \to \infty} < x,U(\sqrt{-1}\, t)y > = 0.]$$

19.9 THEOREM (Weinless) Let $t \to T_t$ be a one parameter group of symplectic maps. Suppose that J_1,J_2 are Kähler structures on E which are unitarizations of $\{T_t\}$ -- then $J_1 = J_2$.

PROOF Let $A = J_2J_1^{-1}$, thus Dom(A) = E, so A is a densely defined $\underline{R}$-linear operator from H_{J_1} to H_{J_2}. Call A^+ the adjoint of A when H_{J_1}, H_{J_2} are regarded as real Hilbert spaces:

$$\mathrm{Re} < A^+x,f >_{J_1} = \mathrm{Re} < x,Af >_{J_2}.$$

Then

$$- J_1A^{-1}J_2 \subset A^+,$$

hence A^+ is densely defined. Indeed,

$$\mathrm{Re} < - J_1A^{-1}J_2g,f >_{J_1}$$

$$= \sigma(- J_1A^{-1}J_2g,J_1f)$$

$$= - \sigma(A^{-1}J_2g,f)$$

$$= - \sigma(J_1J_2^{-1}J_2g,f)$$

$$= - \sigma(J_1g,f)$$

$$= - \sigma(- g,J_1f)$$

$$= \sigma(g,J_1f)$$

$$= \sigma(g,- (- J_1)f)$$

$$= \sigma(g,- J_1^{-1}f)$$

$$= \sigma(g, J_2 J_2 J_1^{-1} f)$$

$$= \sigma(g, J_2 A f)$$

$$= \mathrm{Re} < g, Af >_{J_2}.$$

Given $f \in E, x \in \mathrm{Dom}(A^+)$, put

$$\Phi_{x,f}(t) = < A^+ x, U_1(t) f >_{J_1} - < x, U_2(t) Af >_{J_2},$$

where

$$\begin{cases} U_1(t) = e^{\sqrt{-1}\, tH_1} \\ U_2(t) = e^{\sqrt{-1}\, tH_2}. \end{cases}$$

Then

$$\mathrm{Re} < A^+ x, U_1(t) f >_{J_1}$$

$$= \mathrm{Re} < A^+ x, T_t f >_{J_1}$$

$$= \mathrm{Re} < x, AT_t f >_{J_2}$$

$$= \mathrm{Re} < x, T_t Af >_{J_2}$$

$$= \mathrm{Re} < x, U_2(t) Af >_{J_2}$$

$\Rightarrow$

$$\mathrm{Re}\, \Phi_{x,f}(t) = 0.$$

Due to the assumptions on H_1 and H_2, $\Phi_{x,f}$ extends to a bounded holomorphic function in the upper half plane, so the Schwarz reflection principle implies that $\Phi_{x,f}$ extends to a bounded holomorphic function in the plane which, thanks to Liouville is a constant $C_{x,f}$. But, in view of the asymptotics that are present, $C_{x,f} = 0$. Now take $t = 0$ to get

$$< A^+x,f >_{J_1} = < x,Af >_{J_2} \quad (x \in \mathrm{Dom}(A^+), f \in E).$$

Then

$$< \sqrt{-1}\, x,Af >_{J_2} = - \sqrt{-1} < x,Af >_{J_2}$$

$$= - \sqrt{-1} < A^+x,f >_{J_1}$$

$$= < \sqrt{-1}\, A^+x,f >_{J_1}$$

$\Rightarrow$

$$\sqrt{-1}\, x \in \mathrm{Dom}(A^+) \ \&\ A^+(\sqrt{-1}\, x) = \sqrt{-1}\, A^+x.$$

Therefore A^+ is $\underline{C}$-linear. From this it follows that A^{++} is $\underline{C}$-linear:

$$\begin{bmatrix} \forall\ x \in \mathrm{Dom}(A^+) \\ \forall\ y \in \mathrm{Dom}(A^{++}), \end{bmatrix}$$

$$\mathrm{Re} < \sqrt{-1}\, y,A^+x >_{J_1}$$

$$= \mathrm{Re} - \sqrt{-1} < y,A^+x >_{J_1}$$

$$= \mathrm{Re} < y, -\sqrt{-1}\, A^{+}x >_{J_1}$$

$$= \mathrm{Re} < y, A^{+}(-\sqrt{-1}\, x) >_{J_1}$$

$$= \mathrm{Re} < A^{++}y, -\sqrt{-1}\, x >_{J_2}$$

$$= \mathrm{Re} - \sqrt{-1} < A^{++}y, x >_{J_2}$$

$$= \mathrm{Re} < \sqrt{-1}\, A^{++}y, x >_{J_2}$$

=>

$$\sqrt{-1}\, y \in \mathrm{Dom}(A^{++}) \;\&\; A^{++}(\sqrt{-1}\, y) = \sqrt{-1}\, A^{++}y.$$

Since A^{+} is densely defined, A admits closure (relative to the underlying real Hilbert space structures), and $\bar{A} = A^{++}$. Consequently, $\bar{A}$ is $\underline{C}$-linear, thus $\forall$ f $\in$ E,

$$\bar{A}J_1 f = J_2\bar{A}f$$

=>

$$AJ_1 f = J_2 Af$$

=>

$$J_2 J_1^{-1} J_1 f = J_2 J_2 J_1^{-1} f$$

=>

$$J_2 f = - J_1^{-1} f = J_1 f$$

=>

$$J_1 = J_2.$$

Suppose that J is a Kähler structure on (E,σ). Put

$$\mu(f,g) = \sigma(f,Jg).$$

Then the pair $(H_J, \mathrm{Re} < , >_J)$ is the completion of E per μ.

[Note: $\forall\ f,g \in E$,

$$\begin{aligned}\mu(Jf,Jg) &= \sigma(Jf,JJg)\\ &= \sigma(f,Jg)\\ &= \mu(f,g).]\end{aligned}$$

19.10 <u>LEMMA</u> We have

$$|\sigma(f,g)|^2 \leq \mu(f,f)\mu(g,g) \quad (f,g \in E).$$

<u>PROOF</u> In fact,

$$\begin{aligned}|\sigma(f,g)|^2 &= |\sigma(Jf,Jg)|^2\\ &= |\mu(Jf,g)|^2\\ &\leq |\mu(Jf,Jf)|^2|\mu(g,g)|^2\\ &= |\mu(f,f)|^2|\mu(g,g)|^2.\end{aligned}$$

Therefore σ admits a continuous extension σ_J to H_J as a bilinear form:

$$\sigma_J : H_J \times H_J \to \underline{R}.$$

19.11 LEMMA $\sigma_J = \text{Im} < , >_J$, hence is symplectic.

While it is not necessarily true that the Hilbert space H_J is separable, this does not impede the formation of $BO(H_J)$ and has little impact on the overall theory. In particular: It makes sense to consider the Fock representation

$$\pi_{F,J}: W(H_J, \sigma_J) \to \mathcal{B}(BO(H_J)) \quad \text{(cf. 16.20)}.$$

19.12 REMARK Put

$$\mu_J = \text{Re} < , >_J.$$

Then the characteristic function $\chi_{F,J}$ of $\omega_{F,J}$ is given by

$$\chi_{F,J}(x) = \omega_{F,J}(\delta_x) = \exp(-\frac{1}{4}\mu_J(x,x)) \quad \text{(cf. 18.6)}.$$

Suppose now that J_1, J_2 are Kähler structures on (E,σ). To simplify, abbreviate

$$\begin{bmatrix} \pi_{F,J_1}, \pi_{F,J_2} \\ \omega_{F,J_1}, \omega_{F,J_2} \\ \chi_{F,J_1}, \chi_{F,J_2} \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \pi_1, \pi_2 \\ \omega_1, \omega_2 \\ \chi_1, \chi_2 \end{bmatrix}$$

and let

$$\begin{bmatrix} \mu_1(f,g) = \sigma(f, J_1 g) \\ \mu_2(f,g) = \sigma(f, J_2 g). \end{bmatrix}$$

19.13 <u>LEMMA</u> If π_1 and π_2 are unitarily equivalent, then μ_1 and μ_2 are equivalent, i.e., $\exists\ C > 0,\ D > 0: \forall\ f \in E$,

$$C\mu_1(f,f) \leq \mu_2(f,f) \leq D\mu_1(f,f).$$

<u>PROOF</u> Assume there is a unitary $U:BO(H_{J_1}) \to BO(H_{J_2})$ such that $U\pi_1 U^{-1} = \pi_2$, yet $\nexists\ C > 0$:

$$\mu_1(f,f) \leq \mu_2(f,f)/C$$

for all $f \in E$. Choose a sequence $\{f_n\} \subset E$:

$$\begin{bmatrix} \mu_1(f_n,f_n) = 1\ \forall\ n \\ \mu_2(f_n,f_n) \to 0\ (n \to \infty) \end{bmatrix} \qquad \text{(see below).}$$

Then

$$W_2(f_n) - I_2 \to 0$$

in the strong operator topology (cf. 9.2). On the other hand,

$$< \Omega_1, W_1(f_n)\Omega_1 > = \exp(-\frac{1}{4}\mu_1(f,f)) = e^{-\frac{1}{4}}$$

=>

$$< \Omega_1, (W_1(f_n) - I_1)\Omega_1 > = e^{-\frac{1}{4}} - 1.$$

But

$$< \Omega_1, (W_1(f_n) - I_1)\Omega_1 >$$

$$= < U\Omega_1, U((W_1(f_n) - I_1)\Omega_1) >$$

$$= \langle U\Omega_1, UW_1(f_n)\Omega_1 - U\Omega_1 \rangle$$

$$= \langle U\Omega_1, W_2(f_n)U\Omega_1 - U\Omega_1 \rangle$$

$$= \langle U\Omega_1, (W_2(f_n) - I_2)U\Omega_1 \rangle$$

$$\rightarrow 0 \quad (n \rightarrow \infty),$$

a contradiction.

[Note: $\forall\, C > 0,\ \exists\, f_C \in E$:

$$\mu_1(f_C, f_C) > \mu_2(f_C, f_C)/C$$

$\Rightarrow$

$$\mu_1(f_C/||f_C||_1, f_C/||f_C||_1)$$

$$> \mu_2(f_C/||f_C||_1, f_C/||f_C||_1)/C$$

$\Rightarrow$

$$C > \mu_2(f_C/||f_C||_1, f_C/||f_C||_1).$$

Take $C = 1/n$ and let

$$f_n = f_{1/n}/||f_{1/n}||_1.$$

Then

$$\begin{cases} \mu_1(f_n, f_n) = 1 & \forall\, n \\ \mu_2(f_n, f_n) \rightarrow 0 & (n \rightarrow \infty).] \end{cases}$$

Assume henceforth that μ_1, μ_2 are equivalent — then there is no loss of generality in supposing that $H_{J_1} = H_{J_2}$ (as sets), label it H_μ. To maintain notational simplicity, denote the canonical extensions of J_1, J_2 to H_μ by J_1, J_2 (rather than U_{J_1}, U_{J_2}). As above (cf. 19.12), write

$$\begin{cases} \mu_{J_1} = \text{Re} < , >_{J_1} \\ \mu_{J_2} = \text{Re} < , >_{J_2}. \end{cases}$$

Then $\forall\, x,y \in H_\mu$:

$$\begin{cases} \mu_{J_1}(x,y) = \sigma_{J_1}(x,J_1y) \\ \mu_{J_2}(x,y) = \sigma_{J_2}(x,J_2y). \end{cases}$$

[Note: H_μ carries two real Hilbert space structures, namely those corresponding to μ_{J_1} and μ_{J_2} (here, of course, $\sigma_{J_1} = \sigma_{J_2}$).]

19.14 LEMMA Per μ_{J_1}, the operator $-(J_1J_2)$ is positive and selfadjoint.

PROOF $\forall\, x \in H_\mu (x \neq 0)$:

$$\begin{aligned} &\mu_{J_1}(x, -(J_1J_2)x) \\ &= \sigma_{J_1}(x, J_1(-(J_1J_2)x)) \\ &= \sigma_{J_1}(x, J_2x) \end{aligned}$$

$$= \sigma_{J_2}(x, J_2 x)$$

$$= \mu_{J_2}(x,x) > 0.$$

$\forall\, x,y \in H_\mu$:

$$\mu_{J_1}((-(J_1 J_2))^+ x, y)$$

$$= \mu_{J_1}(x, -(J_1 J_2) y)$$

$$= \sigma_{J_1}(x, J_1(-(J_1 J_2) y))$$

$$= \sigma_{J_1}(x, J_2 y)$$

$$= \sigma_{J_2}(x, J_2 y)$$

$$= \sigma_{J_2}(J_2 x, - y)$$

$$= \sigma_{J_2}(- J_2 x, y)$$

$$= \sigma_{J_1}(- J_2 x, y)$$

$$= \sigma_{J_1}(- (J_1 J_2) x, J_1 y)$$

$$= \mu_{J_1}(- (J_1 J_2) x, y)$$

$\Rightarrow$

$$(- (J_1 J_2))^+ = - (J_1 J_2).$$

19.15 <u>LEMMA</u> Suppose that $T:\mathcal{H}_{J_1} \rightarrow \mathcal{H}_{J_1}$ is an $\underline{R}$-linear homeomorphism which is selfadjoint per μ_{J_1} -- then $T \in \mathcal{SP}(\mathcal{H}_{J_1})$ iff $J_1TJ_1^{-1} = T^{-1}$ (cf. 12.13 and 12.14).

<u>PROOF</u>

Necessity: $\forall\ x,y \in \mathcal{H}_{J_1}$,

$$\mu_{J_1}(J_1TJ_1^{-1}x,y)$$

$$= \sigma_{J_1}(J_1TJ_1^{-1}x,J_1y)$$

$$= \sigma_{J_1}(TJ_1^{-1}x,y)$$

$$= \sigma_{J_1}(J_1^{-1}x,T^{-1}y)$$

$$= \sigma_{J_1}(x,J_1T^{-1}y)$$

$$= \mu_{J_1}(x,T^{-1}y)$$

$$= \mu_{J_1}((T^{-1})^{+}x,y)$$

$$= \mu_{J_1}(T^{-1}x,y)$$

=>

$$J_1TJ_1^{-1} = T^{-1}.$$

Sufficiency: $\forall\ x,y \in H_{J_1}$,

$$\sigma_{J_1}(Tx,Ty)$$

$$= \sigma_{J_1}(J_1Tx,J_1Ty)$$

$$= \mu_{J_1}(J_1Tx,Ty)$$

$$= \mu_{J_1}(T^+J_1Tx,y)$$

$$= \mu_{J_1}(TJ_1Tx,y)$$

$$= \mu_{J_1}(J_1T^{-1}J_1^{-1}J_1Tx,y)$$

$$= \mu_{J_1}(J_1x,y)$$

$$= \sigma_{J_1}(J_1x,J_1y)$$

$$= \sigma_{J_1}(x,y)$$

$\Rightarrow$

$$T \in SP(H_{J_1}).$$

As an application,

$$-(J_1J_2) \in SP(H_{J_1}).$$

Proof:

$$J_1(-(J_1J_2))J_1^{-1}$$

$$= J_1(-J_1)J_2J_1^{-1}$$

$$= J_2J_1^{-1}$$

$$= (-J_2)(-J_1^{-1})$$

$$= J_2^{-1}J_1$$

$$= (-(J_1J_2))^{-1}.$$

Let $T = (-(J_1J_2))^{1/2}$ — then per μ_{J_1}, T is selfadjoint.

19.16 LEMMA $\forall\, x,y \in H_\mu$:

$$\mu_{J_1}(Tx,Ty) = \mu_{J_2}(x,y).$$

PROOF In fact,

$$\mu_{J_1}(Tx,Ty) = \mu_{J_1}(T^2x,y)$$

$$= \mu_{J_1}(-(J_1J_2)x,y)$$

$$= \sigma_{J_1}(-(J_1J_2)x,J_1y)$$

$$= \sigma_{J_1}(- J_2x,y)$$

$$= \sigma_{J_2}(- J_2x,y)$$

$$= \sigma_{J_2}(x,J_2y)$$

$$= \mu_{J_2}(x,y).$$

By its very definition, $T:\mathcal{H}_{J_1} \to \mathcal{H}_{J_1}$ is an $\underline{R}$-linear homeomorphism and we claim that $T \in \mathcal{SP}(\mathcal{H}_{J_1})$ which, however, is a not so obvious point.

19.17 <u>LEMMA</u> Suppose that J is a Kähler structure on (E,σ). Let $S:\mathcal{H}_J \to \mathcal{H}_J$ be symplectic. Assume: S is positive and selfadjoint per μ_J — then $\exists$ a real Hilbert subspace $\mathcal{H}_0 \subset \mathcal{H}_J$ and a positive selfadjoint operator $A:\mathcal{H}_0 \to \mathcal{H}_0$ with a bounded inverse such that

$$\mathcal{H}_J = \mathcal{H}_0 \oplus J\mathcal{H}_0$$

and

$$S(x + Jy) = Ax + JA^{-1}y \quad (x,y \in \mathcal{H}_0).$$

<u>PROOF</u> Taking into account that the spectral theorem holds over the reals, let

$$\begin{cases} \mathcal{S}_- = \text{range of the spectral projection } E(]0,1[) \\ \mathcal{S}_0 = \text{range of the spectral projection } E(\{1\}) \\ \mathcal{S}_+ = \text{range of the spectral projection } E(]1,\infty[). \end{cases}$$

Since $S \in SP(H_J)$, one can use 19.15 (with J_1 replaced by J) to see that J maps S_+ onto S_-, S_- onto S_+, and leaves S_0 invariant (hence S_0 is a complex linear subspace of H_J). Fix a real Hilbert subspace $S_0' \subset S_0$ such that $S_0 = S_0' \oplus JS_0'$ and set $H_0 = S_+ \oplus S_0'$ -- then

$$H_J = H_0 \oplus JH_0$$

and

$$\begin{cases} SH_0 \subset H_0 \\ SJH_0 \subset JH_0. \end{cases}$$

Keeping in mind that $SJ = JS^{-1}$, these facts then lead to the existence of A with the stated properties.

19.18 <u>LEMMA</u> Suppose that J is a Kähler structure on (E,σ). Let $S:H_J \rightarrow H_J$ be symplectic. Assume: S is positive and selfadjoint per μ_J -- then

$$S^{1/2} \in SP(H_J).$$

<u>PROOF</u> In the notation of 19.17,

$$S = A \oplus JA^{-1}.$$

Therefore

$$S^{1/2} = A^{1/2} \oplus JA^{-1/2}$$

=>

$$JS^{1/2}J^{-1}(x + Jy)$$

$$= JS^{1/2}(y - Jx)$$

$$= J(A^{1/2}y - JA^{-1/2}x)$$

$$= A^{-1/2}x + JA^{1/2}y$$

$$= S^{-1/2}(x + Jy)$$

$\Rightarrow$

$$S^{-1/2} \in SP(H_J) \quad \text{(cf. 19.15)}.$$

Coming back to $T = (-(J_1J_2))^{1/2}$, in the above take $J = J_1$ and $S = -(J_1J_2)$ to conclude that $T \in SP(H_{J_1})$, from which the automorphism

$$\alpha_T: W(H_{J_1}, \sigma_{J_1}) \rightarrow W(H_{J_1}, \sigma_{J_1}) \quad \text{(cf. 16.21)}.$$

Proceeding,

$$\omega_{F,J_2}(x) = \exp(-\frac{1}{4}\mu_{J_2}(x,x))$$

$$= \exp(-\frac{1}{4}\mu_{J_1}(Tx,Tx)) \quad \text{(cf. 19.16)}.$$

I.e.:

$$\omega_{F,J_2} = \omega_{F,J_1} \circ T.$$

Therefore π_2 is unitarily equivalent to $\pi_1 \circ \alpha_T$. On the other hand, π_1 is unitarily equivalent to $\pi_1 \circ \alpha_T$ iff $T \in SP_2(H_{J_1})$ (cf. 16.24). And $T \in SP_2(H_{J_1})$

iff

$$T^+T - I = T^2 - I$$

$$= - (J_1J_2) - I$$

is Hilbert-Schmidt on H_μ per μ_{J_1} (cf. 12.15).

19.19 LEMMA - $(J_1J_2) - I$ is Hilbert-Schmidt iff $J_2 - J_1$ is Hilbert-Schmidt.

PROOF Write

$$J_2 - J_1 = J_1(- (J_1J_2) - I).$$

Then

$$- (J_1J_2) - I \text{ Hilbert-Schmidt}$$

$$\Rightarrow$$

$$J_2 - J_1 \text{ Hilbert-Schmidt.}$$

As for the converse, it suffices to note that J_1 is orthogonal:

$$\mu_{J_1}(J_1x, J_1y) = \mu_{J_1}(x,y) \quad (x,y \in H_\mu).$$

If $J_2 - J_1$ is Hilbert-Schmidt, then

$$(J_2 - J_1)(J_2 - J_1) = - (J_1J_2) - (J_2J_1) - 2I$$

is trace class.

[Note: Obviously,

$$(- (J_1J_2))^{-1} = - (J_2J_1).]$$

19.20 <u>LEMMA</u> If $A:H_\mu \to H_\mu$ is positive and selfadjoint with a bounded inverse, then

$$A + A^{-1} - 2I$$

is trace class iff

$$A - I$$

is Hilbert-Schmidt.

<u>PROOF</u>

$$A + A^{-1} - 2I \quad \text{trace class}$$

$\Rightarrow$

$$A(A + A^{-1} - 2I) \quad \text{trace class}$$

$\Rightarrow$

$$A^2 - 2A + I = (A - I)^2 \quad \text{trace class.}$$

But $A - I$ is selfadjoint, hence $A - I$ is Hilbert-Schmidt. Conversely,

$$A - I \qquad \text{Hilbert-Schmidt}$$

$\Rightarrow$

$$(A - I)^2 \qquad \text{trace class}$$

$\Rightarrow$

$$A^{-1}(A^2 - 2A + I) \qquad \text{trace class}$$

$\Rightarrow$

$$A + A^{-1} - 2I \qquad \text{trace class.}$$

We thus have the following chain of equivalences:

$$- (J_1 J_2) - I \quad \text{Hilbert-Schmidt}$$

<=>

$$J_2 - J_1 \quad \text{Hilbert-Schmidt}$$

<=>

$$- (J_1 J_2) - (J_2 J_1) - 2I \quad \text{trace class.}$$

19.21 THEOREM (Van Daele-Verbeure) Suppose that J_1, J_2 are Kähler structures on (E, σ). Assume: μ_1, μ_2 are equivalent -- then π_1, π_2 are unitarily equivalent iff $J_2 - J_1$ is Hilbert-Schmidt or still, iff $- (J_1 J_2) - (J_2 J_1) - 2I$ is trace class.

[This is simply a summary of the foregoing considerations.]

§20. *QUASIFREE STATES*

Let (E,σ) be a symplectic vector space and suppose that $\mu:E \times E \to \underline{R}$ is an inner product. Define $K_\mu:E \times E \to \underline{C}$ by

$$K_\mu(f,g) = \mu(f,g) + \sqrt{-1}\,\sigma(f,g).$$

20.1 LEMMA K_μ is a kernel on E iff $\forall$ $f,g \in E$,

$$|\sigma(f,g)|^2 \leq \mu(f,f)\mu(g,g).$$

PROOF Take E finite dimensional and consider the operator $A_\mu:E \to E$ defined by the relation

$$\sigma(f,g) = \mu(f,A_\mu g) \quad (f,g \in E).$$

In a suitable basis, the matrix of A_μ has block diagonal form

$$[A_\mu] = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{bmatrix} \quad (2n = \dim E),$$

where

$$A_i = \begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix} \quad (i = 1,\ldots,n).$$

Then K_μ is a kernel on E iff $\forall$ i,

$$\begin{bmatrix} 1 & \sqrt{-1}\, a_i \\ -\sqrt{-1}\, a_i & 1 \end{bmatrix}$$

is a positive definite 2-by-2 matrix, which is also the condition that $\forall$ f,g $\in$ E,

$$|\sigma(f,g)|^2 \leq \mu(f,f)\mu(g,g).$$

Write IP(E,σ) for the set of real valued inner products μ on E which dominate σ in the sense that

$$|\sigma(f,g)|^2 \leq \mu(f,f)\mu(g,g) \quad (f,g \in E).$$

20.2 <u>EXAMPLE</u> Suppose that J is a Kähler structure on (E,σ). Put

$$\mu(f,g) = \sigma(f,Jg).$$

Then $\mu \in$ IP(E,σ) (cf. 19.10).

20.3 <u>EXAMPLE</u> Let H be a complex Hilbert space. Suppose that A is a bounded selfadjoint operator on H such that A $\geq$ I, i.e., $\forall$ f $\in$ H,

$$< f,Af > \geq < f,f >.$$

Put

$$\mu_A(f,g) = \text{Re} < f,Ag > \quad (f,g \in H).$$

Then $\mu_A \in$ IP(H,Im < , >).

20.4 <u>REMARK</u> It can happen that IP(E,σ) is empty. Thus let V be an infinite dimensional vector space over $\underline{R}$ and let $V^{\#}$ be the algebraic dual of V. Put $E = V \oplus V^{\#}$ and define $\sigma: E \times E \to \underline{R}$ by

$$\sigma((v,\lambda),(v',\lambda')) = \lambda'(v) - \lambda(v').$$

Then (E,σ) is a symplectic vector space but there is no norm on E w.r.t. which σ is continuous. In fact, continuity of σ implies continuity of the map

$$V \times V^{\#} \to \underline{R}$$

that sends

$$(v,\lambda) \text{ to } \sigma(v \oplus 0, 0 \oplus \lambda) = \lambda(v).$$

Therefore every element of the algebraic dual $V^{\#}$ is a continuous linear functional on the normed linear space V. But this is possible only if V is finite dimensional.

[Note: It follows that (E,σ) does not admit a Kähler structure (cf. 19.4).]

20.5 LEMMA Let $\mu \in \mathrm{IP}(E,\sigma)$ -- then the function

$$\begin{cases} \chi_\mu : E \to \underline{R} \\ f \to \exp(-\frac{1}{4}\mu(f,f)) \end{cases}$$

is in $\mathcal{PD}(E,\sigma)$.

PROOF We have

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \exp(\frac{\sqrt{-1}}{2}\sigma(f_i,f_j))\chi_\mu(f_j - f_i)$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j \exp(\frac{\sqrt{-1}}{2}\sigma(f_i,f_j))$$

$$\times \exp(-\tfrac{1}{4}(\mu(f_j,f_j) - \mu(f_j,f_i) - \mu(f_i,f_j) + \mu(f_i,f_i)))$$

$$= \sum_{i,j=1}^{n} (\bar{c}_i \exp(-\tfrac{1}{4}\mu(f_i,f_i)))(c_j \exp(-\tfrac{1}{4}\mu(f_j,f_j)))$$

$$\times \exp(\tfrac{1}{2}\mu(f_i,f_j) + \tfrac{\sqrt{-1}}{2}\sigma(f_i,f_j))$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j \exp(\tfrac{1}{2}\mu(f_i,f_j) + \tfrac{\sqrt{-1}}{2}\sigma(f_i,f_j)).$$

But, according to 20.1, K_μ is a kernel on E, hence so is $e^{\frac{1}{2}K_\mu}$ (cf. 14.6), which implies that

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \exp(\tfrac{1}{2}\mu(f_i,f_j) + \tfrac{\sqrt{-1}}{2}\sigma(f_i,f_j)) \geq 0.$$

Recall now that

$$\mathcal{PD}(E,\sigma) \longleftrightarrow \mathcal{S}(\mathcal{W}(E,\sigma)),$$

thus

$$\chi_\mu \rightarrow \omega_{\chi_\mu} \equiv \omega_\mu.$$

This said, a state $\omega \in \mathcal{S}(\mathcal{W}(E,\sigma))$ is said to be quasifree if $\exists\ \mu \in IP(E,\sigma):\omega = \omega_\mu$.

20.6 REMARK Given $\mu \in IP(E,\sigma)$ and a symplectic $T:E \rightarrow E$, put $\mu_T = \mu \circ T$ — then $\mu_T \in IP(E,\sigma)$ and $\omega_\mu \circ \alpha_T = \omega_{\mu_T}$.

[Observe that

$$|\sigma(f,g)|^2 = |\sigma(Tf,Tg)|^2$$

$$\leq \mu(Tf,Tf)\mu(Tg,Tg)$$

$$= \mu_T(f,f)\mu_T(g,g).]$$

20.7 <u>LEMMA</u> A quasifree state is nonsingular.

[This is obvious (cf. 18.5).]

In fact, a quasifree state is necessarily C^∞, so 18.8 is applicable.

20.8 <u>LEMMA</u> Suppose that ω is quasifree, say $\omega = \omega_\mu$ $(\mu \in \mathbb{P}(E,\sigma))$ -- then

<u>n odd</u>:

$$< \Omega_\omega, \Phi_\omega(f_1) \cdots \Phi_\omega(f_n)\Omega_\omega > = 0;$$

<u>n even</u>:

$$< \Omega_\omega, \Phi_\omega(f_1) \cdots \Phi_\omega(f_n)\Omega_\omega >$$

$$= \Sigma \prod_{k=1}^{n/2} (\frac{1}{2}\mu(f_{i_k},f_{j_k}) + \frac{\sqrt{-1}}{2}\sigma(f_{i_k},f_{j_k})),$$

where the sum is over all partitions $\{P_1,\ldots,P_{n/2}\}$ of $\{1,\ldots,n\}$ such that $P_k = \{i_k,j_k\}$ with $i_k < j_k$ $(k = 1,\ldots,n/2)$.

[We have

$$< \Omega_\omega, \Phi_\omega(f_1) \cdots \Phi_\omega(f_n)\Omega_\omega >$$

$$= (-\sqrt{-1})^n \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \omega(\delta_{t_1 f_1} \cdots \delta_{t_n f_n}),$$

the derivative being taken at $t_1 = 0,\ldots,t_n = 0$. But

$$\omega(\delta_{t_1 f_1} \cdots \delta_{t_n f_n})$$

$$= \exp(-\frac{1}{4}\sum_{k=1}^{n} t_k^2 \mu(f_k,f_k))$$

$$\times \exp(\sum_{\ell>k} t_\ell t_k (-\frac{1}{2}\mu(f_\ell,f_k) - \frac{\sqrt{-1}}{2}\sigma(f_\ell,f_k)).$$

Inspection of the coefficient of $t_1 \cdots t_n$ in the power series expansion of the second factor then leads to the desired conclusion.]

20.9 <u>REMARK</u> If n is even, then

$$< \Omega_\omega, \Phi_\omega(f_1) \cdots \Phi_\omega(f_n)\Omega_\omega >$$

$$= \Sigma \prod_{k=1}^{n/2} < \Omega_\omega, \Phi_\omega(f_{i_k})\Phi_\omega(f_{j_k})\Omega_\omega >.$$

Therefore the 2-point functions

$$< \Omega_\omega, \Phi_\omega(f)\Phi_\omega(g)\Omega_\omega >$$

completely determine the n-point functions

$$< \Omega_\omega, \Phi_\omega(f_1) \cdots \Phi_\omega(f_n)\Omega_\omega >.$$

Given $\mu \in \mathrm{IP}(E,\mu)$, let H_μ be the completion of E per μ and denote by σ_μ the

μ-continuous extension of σ to H_μ -- then σ_μ is antisymmetric and there exists a unique bounded linear operator $A_\mu : H_\mu \to H_\mu$ such that

$$\sigma_\mu(x,y) = \mu(x,A_\mu y) \quad (x,y \in H_\mu).$$

20.10 LEMMA We have

$$A_\mu^+ = - A_\mu, \ ||A_\mu|| \le 1.$$

[Note: In general, $A_\mu E \not\subset E$.]

20.11 EXAMPLE Suppose that J is a Kähler structure on (E,σ) -- then in this context, $\sigma_\mu = \sigma_J$ and $\forall\ x,y \in H_\mu$,

$$\begin{aligned}\sigma_J(x,y) &= \sigma_J(x,J(- Jy)) \\ &= \mu_J(x,- Jy).\end{aligned}$$

Therefore $A_\mu = - J$.

[Note: View H_J as a real linear space via restriction of scalars -- then $H_\mu = H_J$ and $\mu_J = \text{Re} < , >_J$.]

20.12 LEMMA σ_μ is nondegenerate iff A_μ is injective.

[Note: Suppose that σ_μ is nondegenerate -- then the range of A_μ is dense $(\mu(x,A_\mu y) = 0\ \forall\ y \Rightarrow \sigma_\mu(x,y) = 0\ \forall\ y \Rightarrow x = 0)$, hence A_μ^{-1} is densely defined (but possibly unbounded).]

Therefore the pair (H_μ, σ_μ) is a symplectic vector space iff A_μ is injective.

20.13 REMARK Let $\mu \in \mathbb{IP}(E,\sigma)$ -- then it can be shown that ω_μ is primary iff σ_μ is symplectic.

20.14 EXAMPLE Let H be a separable complex Hilbert space. Fix $\lambda > 1$ and let $\mu(f,g) = \text{Re} < f,g >$ -- then $\lambda\mu \in \mathbb{IP}(H, \text{Im} < \ , \ >)$. In addition,

$$\begin{aligned}\sigma(f,g) &= \text{Im} < f,g > \\ &= \text{Re} < f, -\sqrt{-1}\, g > \\ &= \lambda\mu(f, -\frac{\sqrt{-1}}{\lambda} g)\end{aligned}$$

$$\Rightarrow$$

$$A_{\lambda\mu} = -\frac{\sqrt{-1}}{\lambda} I,$$

thus $A_{\lambda\mu}$ is injective and so $\omega_{\lambda\mu} \equiv \omega_\lambda$ is primary (cf. 20.13). Since $\pi_{F,\lambda}$ is the GNS representation associated with ω_λ (cf. 17.17), it follows that $\pi_{F,\lambda}$ is primary (cf. 17.14).

Bearing in mind that H_μ is a Hilbert space over $\underline{R}$ (not $\underline{C}$), assume that σ_μ is symplectic and let

$$A_\mu = J_\mu |A_\mu|$$

be the polar decomposition of A_μ (thus in this situation, J_μ is orthogonal).

Since $A_\mu^+ = - A_\mu$, A_μ is normal, hence J_μ and $|A_\mu|$ commute. And:

$$A_\mu^+ = |A_\mu|J_\mu^+ = - A_\mu = - J_\mu|A_\mu|$$

$$\Rightarrow$$

$$J_\mu|A_\mu|J_\mu^+ = - J_\mu^2|A_\mu|.$$

But $J_\mu|A_\mu|J_\mu^+$ is nonnegative, so the uniqueness of the polar decomposition gives

$$J_\mu^2 = - I.$$

20.15 <u>REMARK</u> (H_μ,σ_μ) is a symplectic vector space and $\pm J_\mu$ are complex structures on H_μ. If $|A_\mu| = I$, then

$$\begin{aligned}\sigma_\mu(- J_\mu x,- J_\mu y) &= \sigma_\mu(J_\mu x,J_\mu y)\\ &= \mu(J_\mu x,J_\mu J_\mu y)\\ &= \mu(x,J_\mu y)\\ &= \sigma_\mu(x,y) \quad (x,y \in H_\mu)\end{aligned}$$

and

$$\begin{aligned}\sigma_\mu(x,- J_\mu x) &= \mu(x,J_\mu(- J_\mu x))\\ &= \mu(x,x) > 0 \quad (x \in H_\mu,x \neq 0).\end{aligned}$$

Therefore $- J_\mu$ is a Kähler structure on (H_μ,σ_μ).

[Note: In general, $\pm J_\mu E \not\subset E$, thus $\pm J_\mu$ do not necessarily induce complex structures on E.]

Maintaining the assumption that σ_μ is symplectic, place on H_μ the structure of a complex Hilbert space via $- J_\mu$ (cf. 19.2):

$$< x,y >_{- J_\mu} = \mu(x,y) + \sqrt{-1}\, \mu(x,J_\mu y).$$

20.16 LEMMA A_μ is complex linear, i.e.,

$$A_\mu(- J_\mu) = (- J_\mu)A_\mu.$$

PROOF For

$$A_\mu = J_\mu|A_\mu|$$

=>

$$J_\mu^{-1}A_\mu = |A_\mu| \Rightarrow (- J_\mu)A_\mu = |A_\mu|.$$

On the other hand,

$$A_\mu = J_\mu|A_\mu|$$

=>

$$\begin{aligned} A_\mu(- J_\mu) &= J_\mu|A_\mu|(- J_\mu) \\ &= - J_\mu^2|A_\mu| \\ &= |A_\mu|. \end{aligned}$$

20.17 LEMMA The complex adjoint A_μ^* equals the real adjoint A_μ^+.

PROOF $\forall\ x,y \in H_\mu$,

$$< A_\mu^+ x,y >_{-J_\mu}$$

$$= \mu(A_\mu^+ x,y) + \sqrt{-1}\ \mu(A_\mu^+ x,J_\mu y)$$

$$= \mu(x,A_\mu y) + \sqrt{-1}\ \mu(x,A_\mu J_\mu y)$$

$$= \mu(x,A_\mu y) + \sqrt{-1}\ \mu(x,J_\mu A_\mu y)$$

$$= < x,A_\mu y >_{-J_\mu}.$$

Consequently, the symbol $|A_\mu|$ is unambiguous.

20.18 LEMMA $|A_\mu| \leq I$ and J_μ commutes with $(I \pm |A_\mu|)^{1/2}$.

PROOF The first point is clear (cf. 20.10). As for the second, J_μ commutes with $|A_\mu|$, hence J_μ commutes with $I \pm |A_\mu|$. But then J_μ commutes with $(I \pm |A_\mu|)^{1/2}$ (cf. 1.34).

20.19 THEOREM (Kay-Wald) There exists a complex Hilbert space K_μ and a real linear map $k_\mu : E \to K_\mu$ such that

(1) k_μ is one-to-one and $k_\mu E + \sqrt{-1}\ k_\mu E$ is dense in K_μ;

(2) $\forall\, f,g \in E,$

$$< k_\mu f, k_\mu g > = \mu(f,g) + \sqrt{-1}\,\sigma(f,g).$$

PROOF Fix an antiunitary operator $U: H_\mu \to H_\mu$ and define $k_\mu : E \to H_\mu \oplus H_\mu$ by

$$k_\mu f = \frac{1}{\sqrt{2}} (I + |A_\mu|)^{1/2} f \oplus \frac{1}{\sqrt{2}} U(I - |A_\mu|)^{1/2} f.$$

Then $\forall\, f,g \in E$, we have

$$< k_\mu f, k_\mu g >$$

$$= \frac{1}{2} < (I + |A_\mu|)^{1/2} f,\ (I + |A_\mu|)^{1/2} g >_{-J_\mu}$$

$$+ \frac{1}{2} < (I - |A_\mu|)^{1/2} g,\ (I - |A_\mu|)^{1/2} f >_{-J_\mu}$$

$$= \frac{1}{2} \mu((I + |A_\mu|)^{1/2} f,\ (I + |A_\mu|)^{1/2} g)$$

$$+ \frac{\sqrt{-1}}{2} \mu((I + |A_\mu|)^{1/2} f,\ J_\mu (I + |A_\mu|)^{1/2} g)$$

$$+ \frac{1}{2} \mu((I - |A_\mu|)^{1/2} g,\ (I - |A_\mu|)^{1/2} f)$$

$$+ \frac{\sqrt{-1}}{2} \mu((I - |A_\mu|)^{1/2} g,\ J_\mu (I - |A_\mu|)^{1/2} f)$$

$$= \frac{1}{2} \mu(f, (I + |A_\mu|) g) + \frac{\sqrt{-1}}{2} \mu(f, J_\mu (I + |A_\mu|) g)$$

$$+ \frac{1}{2}\,\mu(g,(I - |A_\mu|)f) + \frac{\sqrt{-1}}{2}\,\mu(g,J_\mu(I - |A_\mu|)f)$$

$$= \frac{1}{2}\,(\mu(f,g) + \mu(f,|A_\mu|g) + \mu(g,f) - \mu(g,|A_\mu|f))$$

$$+ \frac{\sqrt{-1}}{2}\,(\mu(f,J_\mu g) + \mu(f,J_\mu|A_\mu|g)$$

$$+ \mu(g,J_\mu f) - \mu(g,J_\mu|A_\mu|f))$$

$$= \mu(f,g)$$

$$+ \frac{\sqrt{-1}}{2}\,(\mu(f,A_\mu g) - \mu(g,A_\mu f)$$

$$+ \mu(f,J_\mu g) + \mu(g,J_\mu f)).$$

And:

- $$- \mu(g,A_\mu f) = - \mu(A_\mu^+ g,f)$$

$$= - \mu(- A_\mu g,f)$$

$$= \mu(A_\mu g,f)$$

$$= \mu(f,A_\mu g).$$

- $$\mu(g,J_\mu f) = \mu(J_\mu g,J_\mu J_\mu f)$$

$$= \mu(J_\mu g,- f)$$

$$= - \mu(f,J_\mu g).$$

Therefore

$$< k_\mu f, k_\mu g > = \mu(f,g) + \sqrt{-1}\,\mu(f, A_\mu g)$$

or still,

$$< k_\mu f, k_\mu g > = \mu(f,g) + \sqrt{-1}\,\sigma_\mu(f,g)$$

or still,

$$< k_\mu f, k_\mu g > = \mu(f,g) + \sqrt{-1}\,\sigma(f,g).$$

k_μ thus constructed is certainly one-to-one ($k_\mu f = 0 \Rightarrow \mu(f,f) = 0 \Rightarrow f = 0$), so to complete the proof, one has only to take

$$\mathcal{K}_\mu = \overline{\text{Ran } k_\mu + \sqrt{-1}\text{ Ran } k_\mu}.$$

20.20 LEMMA Let $\mathcal{K}_1, \mathcal{K}_2$ be complex Hilbert spaces. Let $D_1 \subset \mathcal{K}_1$, $D_2 \subset \mathcal{K}_2$ be real linear subspaces such that

$$\begin{cases} D_1 + \sqrt{-1}\,D_1 \text{ is dense in } \mathcal{K}_1 \\ D_2 + \sqrt{-1}\,D_2 \text{ is dense in } \mathcal{K}_2. \end{cases}$$

Let $T: D_1 \rightarrow D_2$ be a bijective real linear isometry: $\forall\ x,y \in D_1$,

$$< Tx, Ty >_{\mathcal{K}_2} = < x, y >_{\mathcal{K}_1}.$$

Then T can be extended to an isometric isomorphism $\mathcal{K}_1 \rightarrow \mathcal{K}_2$.

[Note: This extension is complex linear and unique.]

Suppose that

$$\left[\begin{array}{l} k_1 : E \to \mathcal{K}_1 \\ \\ k_2 : E \to \mathcal{K}_2 \end{array}\right.$$

are data per 20.19. Define $T : k_1 E \to k_2 E$ by the diagram

$$\begin{array}{ccc} E & \xrightarrow{k_1} & k_1 E \\ {\scriptstyle k_2}\downarrow & \swarrow & T. \\ k_2 E & & \end{array}$$

Then $\forall\ f,g \in E$,

$$< Tk_1 f, Tk_2 g >_{\mathcal{K}_2}$$

$$= < k_2 f, k_2 g >_{\mathcal{K}_2}$$

$$= \mu(f,g) + \sqrt{-1}\ \sigma(f,g)$$

$$= < k_1 f, k_2 g >_{\mathcal{K}_1}.$$

Consequently, in view of 20.20, $\exists$ a unique isometric isomorphism $\mathcal{K}_1 \to \mathcal{K}_2$ extending T.

In other words: The pair $(k_\mu, \mathcal{K}_\mu)$ is unique up to unitary equivalence.

20.21 <u>REMARK</u> The Kay-Wald theorem is valid for any $\mu \in IP(E,\sigma)$, i.e., it is not necessary to assume that σ_μ is symplectic but the preliminaries to the proof have to be modified. To this end, suppose that $\text{Ker}(A_\mu) \neq \{0\}$.

1. If $\dim \text{Ker}(A_\mu)$ is finite and odd, let $\mathcal{H}'_\mu = \mathcal{H}_\mu \oplus \underline{R}$ and $A'_\mu = A_\mu \oplus 0$.

2. If $\dim \text{Ker}(A_\mu)$ is finite and even or infinite, let $\mathcal{H}'_\mu = \mathcal{H}_\mu$ and $A'_\mu = A_\mu$.

Then $\dim \text{Ker}(A'_\mu)$ is either even or infinite and

$$\mathcal{H}'_\mu = \overline{\text{Ran}(A'_\mu)} \oplus \text{Ker}(A'_\mu).$$

Let $A'_\mu = U'_\mu |A'_\mu|$ be the polar decomposition of A'_μ thought of as a map from $\overline{\text{Ran}(A'_\mu)}$ to itself, thus

$$U'_\mu : \overline{\text{Ran}(A'_\mu)} \to \overline{\text{Ran}(A'_\mu)}$$

is orthogonal and $(U'_\mu)^2 = - I$. Put $J'_\mu = U'_\mu \oplus J$, where

$$J : \text{Ker}(A'_\mu) \to \text{Ker}(A'_\mu)$$

is orthogonal and $J^2 = - I$ — then $(J'_\mu)^2 = - I$. The rest of the analysis now goes through without change.

Fix $\mu \in \text{IP}(E,\sigma)$ and define $k_\mu : E \to K_\mu$ as above (taking into account 20.21). Let

$$\pi_F : \mathcal{W}(K_\mu, \text{Im} < \, , \, >) \to \mathcal{B}(\text{BO}(K_\mu))$$

be the Fock representation. Given $f \in E$, put

$$\pi_{F,\mu}(\delta_f) = \pi_F(\delta_{k_\mu f}) = W(k_\mu f).$$

Then $\forall f,g \in E$,

$$\pi_{F,\mu}(\delta_f)\pi_{F,\mu}(\delta_g)$$

$$= W(k_\mu f)W(k_\mu g)$$

$$= \exp(-\frac{\sqrt{-1}}{2} \text{Im} < k_\mu f, k_\mu g >)W(k_\mu f + k_\mu g)$$

$$= \exp(-\frac{\sqrt{-1}}{2} \sigma(f,g))\pi_{F,\mu}(\delta_{f+g}).$$

So $\pi_{F,\mu}$ gives rise to a representation of $\mathcal{W}(E,\sigma)$ on $BO(\mathcal{K}_\mu)$. But the fact that $k_\mu E + \sqrt{-1}\, k_\mu E$ is dense in $\mathcal{K}_\mu$ implies that Ω is cyclic. And $\forall\, f \in E$,

$$< \Omega, W(k_\mu f)\Omega >$$

$$= \exp(-\frac{1}{4} ||k_\mu f||^2)$$

$$= \exp(-\frac{1}{4} < k_\mu f, k_\mu f >)$$

$$= \exp(-\frac{1}{4} \mu(f,f))$$

$$= \chi_\mu(f)$$

$$= \omega_\mu(\delta_f).$$

Therefore $\pi_{F,\mu} = \pi_{\omega_\mu}$, the GNS representation associated with ω_μ.

[Note: Let $\omega = \omega_\mu$ -- then (cf. 20.8)

$$< \Omega_\omega, \Phi_\omega(f)\Phi_\omega(g)\Omega_\omega > = \frac{1}{2} (\mu(f,g) + \sqrt{-1}\, \sigma(f,g)).$$

Now take, as is permissible, $\Omega_\omega = \Omega$ and

$$\begin{cases} \Phi_\omega(f) = \overline{Q(k_\mu f)} \\ \Phi_\omega(g) = \overline{Q(k_\mu g)}. \end{cases}$$

Direct computation then gives

$$< \Omega, \overline{Q(k_\mu f)}\,\overline{Q(k_\mu g)}\,\Omega > = \frac{1}{2} < k_\mu f, k_\mu g >,$$

thereby providing a check on the work.]

20.22 <u>LEMMA</u> $\pi_{F,\mu}$ is irreducible iff $k_\mu E$ is dense in K_μ.

Let $\mu \in IP(E,\sigma)$ -- then μ is said to be <u>pure</u> if $\forall\, f \in E$,

$$\mu(f,f) = \sup_{g \in E - \{0\}} \frac{|\sigma(f,g)|^2}{\mu(g,g)}.$$

20.23 <u>EXAMPLE</u> Consider $(H, \text{Im} < , >)$, where H is a complex Hilbert space. Let $\mu(f,g) = \text{Re} < f,g >$ -- then μ is pure. In fact, $\forall\, f \neq 0$,

$$\begin{aligned}\sigma(f, \sqrt{-1}\, f) &= \text{Im} < f, \sqrt{-1}\, f > \\ &= \text{Re} < f, (-\sqrt{-1})\sqrt{-1}\, f > \\ &= \mu(f, (-\sqrt{-1})\sqrt{-1}\, f) \\ &= \mu(f,f)\end{aligned}$$

=>

$$\frac{|\sigma(f, \sqrt{-1}\, f)|^2}{\mu(\sqrt{-1}\, f, \sqrt{-1}\, f)} = \frac{\mu(f,f)^2}{\mu(f,f)} = \mu(f,f).$$

20.24 <u>LEMMA</u> μ is pure iff $k_\mu E$ is dense in K_μ.

[Use the relation

$$1 - \frac{\sigma(f,g)}{\mu(f,f)^{1/2}\mu(g,g)^{1/2}}$$

$$= \frac{1}{2} \left|\left|\sqrt{-1}\, \frac{k_\mu f}{||k_\mu f||} - \frac{k_\mu g}{||k_\mu g||}\right|\right|^2 .]$$

Therefore μ is pure iff ω_μ is pure, which justifies the terminology.

Given $\mu \in \mathrm{IP}(E,\sigma)$, let

$$A_\mu = U_\mu |A_\mu|$$

be the polar decomposition of A_μ ($U_\mu = J_\mu$ if σ_μ is symplectic).

20.25 <u>REMARK</u> Let $\mu \in \mathrm{IP}(E,\sigma)$ -- then μ is pure iff $|A_\mu| = I$.

[In fact,

$$|A_\mu| = I \Rightarrow (A_\mu^+ A_\mu)^{1/2} = I \Rightarrow A_\mu^+ A_\mu = I.$$

Thus A_μ is injective, so σ_μ is symplectic (cf. 20.12). That the condition is sufficient can then be seen by taking $g = J_\mu f$:

$$\frac{|\sigma(f,J_\mu f)|^2}{\mu(J_\mu f,J_\mu f)} = \frac{|\mu(f,J_\mu^2 f)|^2}{\mu(J_\mu f,J_\mu f)}$$

$$= \frac{\mu(f,-f)^2}{\mu(f,f)}$$

$$= \mu(f,f).$$

Conversely, if $|A_\mu| \neq I$, then $\sigma(|A_\mu|) \subset [0,1]$ but $\sigma(|A_\mu|) \neq \{1\}$. This being the case, fix $r_0 \in \sigma(|A_\mu|):r_0 < 1$ and choose $r:r_0 < r < 1$. Fix a nonzero $x \in E([0,r])(\mathcal{H})$ and choose a sequence $\{f_n \neq 0\} \subset E:f_n \to x$ in $\mathcal{H}_\mu$-- then $\forall\, g \neq 0$ in E,

$$\frac{|\sigma(f_n,g)|^2}{\mu(g,g)} = \frac{|\mu(f_n,A_\mu g)|^2}{\mu(g,g)}$$

$$= \frac{|\mu(A_\mu^* f_n,g)|^2}{\mu(g,g)}$$

$$= \frac{|\mu(-\,A_\mu f_n,g)|^2}{\mu(g,g)}$$

$$= \frac{|\mu(A_\mu f_n,g)|^2}{\mu(g,g)}$$

$$= \frac{|\mu(U_\mu|A_\mu|f_n,g)|^2}{\mu(g,g)}$$

$$\leq \mu(U_\mu|A_\mu|f_n,U_\mu|A_\mu|f_n)$$

$$\leq \mu(|A_\mu|f_n,|A_\mu|f_n).$$

Choose N:

$$n \geq N \Rightarrow$$

$$\mu(|A_\mu|f_n,|A_\mu|f_n)$$

$$< \mu(|A_\mu|x,|A_\mu|x) + r^2\mu(x,x).$$

Then

$n \geq N \Rightarrow$

$$\sup_{g \in E - \{0\}} \frac{|\sigma(f_n,g)|^2}{\mu(g,g)} < \mu(|A_\mu|x, |A_\mu|x) + r^2\mu(x,x)$$

$$< r^2\mu(x,x) + r^2\mu(x,x)$$

$$= 2r^2\mu(x,x).$$

Fix $\delta > 0$:

$$1 + \delta < \frac{1}{2r^2}.$$

Choose $N_\delta > N$:

$n \geq N_\delta \Rightarrow$

$$\frac{\mu(x,x)}{\mu(f_n,f_n)} < 1 + \delta.$$

Then

$n \geq N_\delta \Rightarrow$

$$\sup_{g \in E - \{0\}} \frac{|\sigma(f_n,g)|^2}{\mu(g,g)} < 2r^2\mu(x,x)$$

$$= 2r^2 \frac{\mu(x,x)}{\mu(f_n,f_n)} \mu(f_n,f_n)$$

$$< 2r^2(1 + \delta)\mu(f_n,f_n)$$

$$< \mu(f_n,f_n).$$

And this implies that μ is not pure.

Given $\mu \in \mathrm{IP}(E,\sigma)$, put

$$\mu_p(f,g) = \mu(f,|A_\mu|g) \quad (f,g \in E).$$

20.26 <u>LEMMA</u> $\mu_p \in \mathrm{IP}(E,\sigma)$.

<u>PROOF</u> We have

$$\begin{aligned}
|\sigma(f,g)|^2 &= |\mu(f,A_\mu g)|^2 \\
&= |\mu(f,U_\mu|A_\mu|g)|^2 \\
&= |\mu(f,-U_\mu^+|A_\mu|g)|^2 \\
&= |\mu(U_\mu f,|A_\mu|g)|^2 \\
&= |\mu(|A_\mu|^{1/2}U_\mu f,|A_\mu|^{1/2}g)|^2 \\
&\leq \mu(U_\mu|A_\mu|^{1/2}f,U_\mu|A_\mu|^{1/2}f) \\
&\qquad \times \mu(|A_\mu|^{1/2}g,|A_\mu|^{1/2}g) \\
&\leq \mu(|A_\mu|^{1/2}f,|A_\mu|^{1/2}f)\mu(|A_\mu|^{1/2}g,|A_\mu|^{1/2}g) \\
&= \mu(f,|A_\mu|f)\mu(g,|A_\mu|g) \\
&= \mu_p(f,f)\mu_p(g,g).
\end{aligned}$$

[Note: Since σ is symplectic,

$$\mu_p(f,f) = 0 \Rightarrow \sigma(f,g) = 0 \ \forall \ g$$

$$\Rightarrow f = 0.]$$

20.27 <u>LEMMA</u> μ_p is pure.

<u>PROOF</u> Fix $f \neq 0$ in E and write $f = f_{||} + f_{\perp}$, where

$$\begin{cases} f_{||} \in \text{Ker}(A_\mu) \\ f_{\perp} \in \text{Ker}(A_\mu)^{\perp} \ (\equiv \overline{\text{Ran}(A_\mu)}). \end{cases}$$

Let

$$\begin{cases} f^+ = f_{||} + \frac{1}{2}(I - U_\mu)f_{\perp} \\ f^- = \frac{1}{2}(I + U_\mu)f_{\perp}, \end{cases}$$

so that $f = f^+ + f^-$ -- then

$$\begin{cases} A_\mu f^+ = |A_\mu| f^- \\ A_\mu f^- = -|A_\mu| f^+ \end{cases}$$

and $\mu(f^+,f^-) = 0$. In addition,

$$\begin{cases} \mu(f^+,|A_\mu|f^-) = 0 \\ \mu(f^-,|A_\mu|f^+) = 0. \end{cases}$$

Choose a sequence $\{g_n \neq 0\} \subset E : g_n \to f^+ - f^-$ — then

$$\frac{|\sigma(f,g_n)|^2}{\mu_p(g_n,g_n)}$$

$$\to \frac{|\mu(f^+ + f^-, A_\mu(f^+ - f^-))|^2}{\mu(f^+ - f^-, |A_\mu|(f^+ - f^-))}$$

$$= \frac{|\mu(f^-, |A_\mu|f^-) - \mu(f^+, - |A_\mu|f^-))|^2}{\mu(f^+, |A_\mu|f^+) + \mu(f^-, |A_\mu|f^-)}$$

$$= \mu(f^+, |A_\mu|f^+) + \mu(f^-, |A_\mu|f^-)$$

$$= \mu(f, |A_\mu|f)$$

$$= \mu_p(f,f).$$

[Note: μ_p is called the purification of μ.]

Suppose that $\mu \in \text{IP}(E,\sigma)$ is pure — then $|A_\mu| = I$ (cf. 20.25) and on H_μ,

$$< x,y >_{-J_\mu} = \mu(x,y) + \sqrt{-1}\,\mu(x,J_\mu y)$$

$$= \mu(x,y) + \sqrt{-1}\,\sigma_\mu(x,y).$$

Furthermore, the construction in 20.19 simplifies considerably. Indeed, one can take $K_\mu = H_\mu$, $k_\mu : E \to H_\mu$ being the inclusion.

20.28 <u>REMARK</u> If μ_1, μ_2 are pure and if $\pi_{F,\mu_1}, \pi_{F,\mu_2}$ are unitarily equivalent, then μ_1, μ_2 are necessarily equivalent (cf. 19.13). Proceeding from here, one can extend 19.21 to the present setting. Precisely put: Suppose that μ_1, μ_2 are pure and equivalent -- then $\pi_{F,\mu_1}, \pi_{F,\mu_2}$ are unitarily equivalent iff $J_{\mu_2} - J_{\mu_1}$ is Hilbert-Schmidt or still, iff $-(J_{\mu_1} J_{\mu_2}) - (J_{\mu_2} J_{\mu_1}) - 2I$ is trace class.

§21. *QUESTIONS OF EQUIVALENCE*

Let (E,σ) be a symplectic vector space. Suppose that $\mu \in \text{IP}(E,\sigma)$ — then the complexification $H_{\mu_{\underline{C}}}$ $(= H_\mu + \sqrt{-1}\, H_\mu)$ is a complex Hilbert space with inner product $\mu_{\underline{C}}$ (cf. 19.2):

$$\mu_{\underline{C}}(x + \sqrt{-1}\, y, x' + \sqrt{-1}\, y')$$

$$= \mu(x,x') + \mu(y,y') + \sqrt{-1}\, (\mu(x,y') - \mu(y,x')).$$

<u>N.B.</u> There is a canonical arrow of extension

$$\begin{cases} B(H_\mu) \to B(H_{\mu_{\underline{C}}}) \\ \\ A \to A_{\underline{C}}, \end{cases}$$

viz. take $A \in B(H_\mu)$ and extend by complex linearity:

$$A_{\underline{C}}(x + \sqrt{-1}\, y) = Ax + \sqrt{-1}\, Ay.$$

Obviously, $(rA)_{\underline{C}} = rA_{\underline{C}}$ $(r \in \underline{R})$ and

$$(A + B)_{\underline{C}} = A_{\underline{C}} + B_{\underline{C}}, \quad (AB)_{\underline{C}} = A_{\underline{C}} B_{\underline{C}}.$$

In addition,

$$(A_{\underline{C}})^* = (A^+)_{\underline{C}}$$

$$\Rightarrow$$

$$(A_{\underline{C}})^* A_{\underline{C}} = (A^+)_{\underline{C}} A_{\underline{C}}$$

$$= (A^{+}A)_{\underline{C}}$$

$$\Rightarrow$$

$$|A_{\underline{C}}| = |A|_{\underline{C}}.$$

Now extend σ_μ to $\mathcal{H}_{\mu_{\underline{C}}}$ by taking it conjugate linear in the first variable, linear in the second variable. Calling this extension $\sigma_{\mu_{\underline{C}}}$, we have

$$\sigma_{\mu_{\underline{C}}}(x + \sqrt{-1}\, y, x' + \sqrt{-1}\, y') = \mu_{\underline{C}}(x + \sqrt{-1}\, y, (A_\mu)_{\underline{C}}(x' + \sqrt{-1}\, y')).$$

21.1 <u>REMARK</u> Assume that μ is pure ($\Rightarrow A_\mu = J_\mu$ (cf. 20.25)) and write

$$\mathcal{H}_{\mu_{\underline{C}}} = \mathcal{H}_\mu^{+} \oplus \mathcal{H}_\mu^{-},$$

where

$$\mathcal{H}_\mu^{\pm} = \{z \in \mathcal{H}_{\mu_{\underline{C}}} : (J_\mu)_{\underline{C}}\, z = \pm \sqrt{-1}\, z\}.$$

Let $P^{\pm}$ be the associated orthogonal projections. Define a real linear map $k_\mu : E \to \mathcal{H}_\mu^{-}$ by setting

$$k_\mu = \sqrt{2}\, P^{-}|E.$$

Then $\forall\, f,g \in E$,

$$< k_\mu f, k_\mu g > = \mu_{\underline{C}}(k_\mu f, k_\mu g)$$

$$= 2\mu_{\underline{C}}(P^{-} f, P^{-} g)$$

$$= 2\mu_{\underline{C}}(P^- f, -\frac{1}{\sqrt{-1}}(J_\mu)_{\underline{C}} P^- g)$$

$$= 2\sqrt{-1}\,\mu_{\underline{C}}(P^- f, (J_\mu)_{\underline{C}} P^- g)$$

$$= 2\sqrt{-1}\,\sigma_{\mu_{\underline{C}}}(P^- f, P^- g)$$

$$= 2\sqrt{-1}\,\sigma_{\mu_{\underline{C}}}(\frac{1}{2}(f + \sqrt{-1}\,(J_\mu)_{\underline{C}} f), \frac{1}{2}(g + \sqrt{-1}\,(J_\mu)_{\underline{C}} g))$$

$$= \frac{\sqrt{-1}}{2}\sigma_{\mu_{\underline{C}}}(f + \sqrt{-1}\,(J_\mu)_{\underline{C}} f, g + \sqrt{-1}\,(J_\mu)_{\underline{C}} g)$$

$$= \frac{\sqrt{-1}}{2}(\sigma_\mu(f,g) + \sigma_\mu(J_\mu f, J_\mu g)$$

$$+ \sqrt{-1}\,(\sigma_\mu(f, J_\mu g) - \sigma_\mu(J_\mu f, g)))$$

$$= \frac{\sqrt{-1}}{2}(\sigma(f,g) + \sigma(f,g) + \sqrt{-1}\,(-\mu(f,g) - \mu(f,g)))$$

$$= \mu(f,g) + \sqrt{-1}\,\sigma(f,g).$$

Since k_μ is one-to-one and $k_\mu E$ is dense in H_μ^-, this setup is another model for 20.19.

[Note: Working instead with $\sqrt{2}\,P^+|E$ leads to

$$\mu(f,g) - \sqrt{-1}\,\sigma(f,g).]$$

21.2 <u>LEMMA</u> $\exists$ a bounded linear operator S_μ on $H_{\mu_{\underline{C}}}$ such that $\forall\ z,z' \in H_{\mu_{\underline{C}}}$,

$$\mu_{\underline{C}}(z,z') + \sqrt{-1}\ \sigma_{\mu_{\underline{C}}}(z,z') = 2\mu_{\underline{C}}(z,S_\mu z').$$

Moreoever, S_μ is nonnegative and selfadjoint.

Explicated: $\forall\ z,z' \in H_{\mu_{\underline{C}}}$,

$$\mu_{\underline{C}}(z,z') + \sqrt{-1}\ \sigma_{\mu_{\underline{C}}}(z,z')$$

$$= \mu_{\underline{C}}(z,z') + \sqrt{-1}\ \mu_{\underline{C}}(z,(A_\mu)_{\underline{C}}z')$$

$$= \mu_{\underline{C}}(z,2S_\mu z')$$

=>

$$2S_\mu = I + \sqrt{-1}\ (A_\mu)_{\underline{C}}.$$

[Note: Write $z = x + \sqrt{-1}\ y$ and let $x + \sqrt{-1}\ y \longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$ -- then

$$\begin{bmatrix} I & -A_\mu \\ A_\mu & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x - A_\mu y \\ A_\mu x + y \end{bmatrix} \longleftrightarrow x - A_\mu y + \sqrt{-1}\ (A_\mu x + y)$$

$$= (I + \sqrt{-1}\ (A_\mu)_{\underline{C}})(x + \sqrt{-1}\ y)$$

$$= 2S_\mu(x + \sqrt{-1}\, y).$$

Therefore

$$2S_\mu \longleftrightarrow \begin{bmatrix} I & -A_\mu \\ A_\mu & I \end{bmatrix}.]$$

21.3 <u>LEMMA</u> Let $\mu \in \mathrm{IP}(E,\sigma)$ -- then μ is pure iff S_μ is an orthogonal projection.

<u>PROOF</u> If μ is pure, then $A_\mu = J_\mu$ (cf. 20.25), hence

$$S_\mu^2 = (\tfrac{1}{2}(I + \sqrt{-1}\,(J_\mu)_{\underline{C}}))^2$$

$$= \tfrac{1}{4}(I + 2\sqrt{-1}\,(J_\mu)_{\underline{C}} - (J_\mu)_{\underline{C}}^2)$$

$$= \tfrac{1}{4}(2I + 2\sqrt{-1}\,(J_\mu)_{\underline{C}})$$

$$= \tfrac{1}{2}(I + \sqrt{-1}\,(J_\mu)_{\underline{C}})$$

$$= S_\mu.$$

Conversely,

$$S_\mu^2 = S_\mu$$

$\Rightarrow$

$$\tfrac{1}{4}(I + 2\sqrt{-1}\,(A_\mu)_{\underline{C}} - (A_\mu)_{\underline{C}}^2) = \tfrac{1}{2}(I + \sqrt{-1}\,(A_\mu)_{\underline{C}})$$

$$\Rightarrow$$

$$I = -(A_\mu)_{\underline{C}}^2$$

$$= -(A_\mu)_{\underline{C}}(A_\mu)_{\underline{C}}$$

$$= ((A_\mu)_{\underline{C}})^*(A_\mu)_{\underline{C}}$$

$$= |(A_\mu)_{\underline{C}}|^2$$

$$\Rightarrow$$

$$|(A_\mu)_{\underline{C}}| = I$$

$$\Rightarrow$$

$$|A_\mu|_{\underline{C}} = I$$

$$\Rightarrow$$

$$|A_\mu| = I.$$

I.e.: μ is pure (cf. 20.25).

[Note: S_μ equals P^-, the orthogonal projection onto the eigenspace $\mathcal{H}_\mu^-$ (cf. 21.1).]

Let $\mu \in IP(E,\sigma)$ -- then $\forall\ f \in E$,

$$\omega_\mu(\delta_f) = \exp(-\frac{1}{4}\mu(f,f))$$

or still,

$$\exp(-\frac{1}{4}\mu(f,f)) = < \Omega, W(k_\mu f)\Omega >.$$

21.4 <u>LEMMA</u> Let $\mu_1,\mu_2 \in \mathbb{P}(E,\sigma)$. Suppose that the GNS representations π_1,π_2 per $\omega_{\mu_1},\omega_{\mu_2}$ are geometrically equivalent -- then μ_1,μ_2 are equivalent.

<u>PROOF</u> Realize π_1,π_2 as $\pi_{F,\mu_1},\pi_{F,\mu_2}$ -- then

$$F(\pi_{F,\mu_1}) = F(\pi_{F,\mu_2}),$$

which, on general grounds, is equivalent to the existence of an isomorphism

$$\phi:\pi_{F,\mu_1}(\mathcal{W}(E,\sigma))'' \to \pi_{F,\mu_2}(\mathcal{W}(E,\sigma))''$$

such that $\forall\ W \in \mathcal{W}(E,\sigma)$,

$$\phi(\pi_{F,\mu_1}(W)) = \pi_{F,\mu_2}(W).$$

Here the double prime denotes the bicommutant. Now write

$$\begin{aligned}\exp(-\frac{1}{4}\mu_2(f,f)) &= < \Omega_2, W(k_{\mu_2}f)\Omega_2 > \\ &= < \Omega_2, \pi_{F,\mu_2}(\delta_f)\Omega_2 > \\ &= < \Omega_2, \phi(\pi_{F,\mu_1}(\delta_f))\Omega_2 >.\end{aligned}$$

The last expression is continuous in the topology defined by μ_1, thus μ_2 is

μ_1-continuous. Analogously, μ_1 is μ_2-continuous. Therefore μ_1,μ_2 are equivalent.

Let $\mu_1,\mu_2 \in \mathrm{IP}(E,\sigma)$. Assume: μ_1,μ_2 are equivalent -- then there is no loss of generality in supposing that $H_{\mu_1} = H_{\mu_2}$ (as sets), label it H_μ, thus

$$\sigma_\mu = \begin{cases} \sigma_{\mu_1} \\ \\ \sigma_{\mu_2}. \end{cases}$$

21.5 <u>LEMMA</u> $\exists$ a bounded linear operator T_{μ_2} on $H_{\mu_{\underline{C}}}$ such that $\forall\ z,z' \in H_{\mu_{\underline{C}}}$,

$$\mu_{2,\underline{C}}(z,z') + \sqrt{-1}\ \sigma_{\mu_{\underline{C}}}(z,z') = 2\mu_{1,\underline{C}}(z,T_{\mu_2}z').$$

Moreover, T_{μ_2} is nonnegative and selfadjoint.

21.6 <u>EXAMPLE</u> Take σ_μ symplectic and write

$$\sigma_\mu(x,y) = \begin{cases} \mu_1(x,A_{\mu_1}y) \\ \\ \mu_2(x,A_{\mu_2}y). \end{cases} \qquad (x,y \in H_\mu)$$

Then $A_{\mu_1}^{-1},A_{\mu_2}^{-1}$ are densely defined and the product $A_{\mu_1}A_{\mu_2}^{-1}$ extends to a bounded linear operator on H_μ. In fact, $\forall\ x \in H_\mu$ & $\forall\ y \in \mathrm{Dom}(A_{\mu_2}^{-1})$,

$$\mu_1(x, A_{\mu_1} A_{\mu_2}^{-1} y) = \sigma_\mu(x, A_{\mu_2}^{-1} y)$$

$$= \mu_2(x,y).$$

So, $\forall\ z, z' \in H_{\mu_{\underline{C}}}$,

$$2\mu_{1,\underline{C}}(z, T_{\mu_2} z')$$

$$= \mu_{2,\underline{C}}(z,z') + \sqrt{-1}\ \sigma_{\mu_{\underline{C}}}(z,z')$$

$$= \mu_{2,\underline{C}}(z,z') + \sqrt{-1}\ \mu_{2,\underline{C}}(z, (A_{\mu_2})_{\underline{C}} z')$$

$$= \mu_{1,\underline{C}}(z, (A_{\mu_1} A_{\mu_2}^{-1})_{\underline{C}} z') + \sqrt{-1}\ \mu_{1,\underline{C}}(z, (A_{\mu_1})_{\underline{C}} z')$$

=>

$$2T_{\mu_2} = (A_{\mu_1} A_{\mu_2}^{-1})_{\underline{C}} + \sqrt{-1}\ (A_{\mu_1})_{\underline{C}}.$$

[Note: Write $z = x + \sqrt{-1}\ y$ and let $x + \sqrt{-1}\ y \longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$ -- then

$$\begin{bmatrix} A_{\mu_1} A_{\mu_2}^{-1} & -A_{\mu_1} \\ A_{\mu_1} & A_{\mu_1} A_{\mu_2}^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} A_{\mu_1}A_{\mu_2}^{-1}x - A_{\mu_1}y \\ \\ A_{\mu_1}x + A_{\mu_1}A_{\mu_2}^{-1}y \end{bmatrix}$$

$$\longleftrightarrow A_{\mu_1}A_{\mu_2}^{-1}x - A_{\mu_1}y + \sqrt{-1}\,(A_{\mu_1}x + A_{\mu_1}A_{\mu_2}^{-1}y)$$

$$= ((A_{\mu_1}A_{\mu_2}^{-1})_{\underline{C}} + \sqrt{-1}\,(A_{\mu_1})_{\underline{C}})(x + \sqrt{-1}\,y)$$

$$= 2T_{\mu_2}(x + \sqrt{-1}\,y).$$

Therefore

$$2T_{\mu_2} \longleftrightarrow \begin{bmatrix} A_{\mu_1}A_{\mu_2}^{-1} & -A_{\mu_1} \\ \\ A_{\mu_1} & A_{\mu_1}A_{\mu_2}^{-1} \end{bmatrix}.]$$

Keeping to the supposition that μ_1,μ_2 are equivalent, put

$$\begin{cases} S_1 = S_{\mu_1} \\ \\ T_2 = T_{\mu_2}. \end{cases}$$

21.7 <u>THEOREM</u> (Araki-Yamagami) Let $\mu_1,\mu_2 \in \text{IP}(E,\sigma)$. Assume: μ_1,μ_2 are equivalent — then π_1,π_2 are geometrically equivalent iff $\sqrt{S_1} - \sqrt{T_2}$ is

Hilbert-Schmidt.

[Note: Recall that π_1, π_2 are the GNS representations per $\omega_{\mu_1}, \omega_{\mu_2}$.]

The proof of this result is lengthy and involved, so I'm going to omit it. However, even upon specializing to the case when μ_1, μ_2 are pure, it is by no means obvious that one recovers the criterion set down in 20.28. This and other issues will be considered below.

21.8 LEMMA Let $\mathcal{H}$ be a Hilbert space. Suppose that $A, B \in \mathcal{B}(\mathcal{H})$ are nonnegative and selfadjoint -- then

$$\sqrt{A} - \sqrt{B} \in \underline{L}_2(\mathcal{H}) \Rightarrow 2(A-B) \in \underline{L}_2(\mathcal{H}).$$

PROOF Note that

$$(\sqrt{A} + \sqrt{B})(\sqrt{A} - \sqrt{B}) + (\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B})$$

$$= A - \sqrt{A}\sqrt{B} + \sqrt{B}\sqrt{A} - B$$

$$+ A + \sqrt{A}\sqrt{B} - \sqrt{B}\sqrt{A} - B$$

$$= 2(A-B).$$

21.9 EXAMPLE Let $\mathcal{H}$ be a separable complex Hilbert space. Take $\mathcal{H}$ infinite dimensional and consider the setup in 20.14 -- then we claim that π_{F,λ_1} is not geometrically equivalent to π_{F,λ_2} if $\lambda_1 \neq \lambda_2$. For if the opposite held, then 21.7 would imply that $\sqrt{S_1} - \sqrt{T_2}$ is Hilbert-Schmidt, hence by 21.8, that $2(S_1 - T_2)$

is Hilbert-Schmidt, hence by 21.8, that $2(S_1 - T_2)$ is Hilbert-Schmidt. But here

$$2S_1 = \begin{bmatrix} I & \frac{\sqrt{-1}}{\lambda_1} I \\ -\frac{\sqrt{-1}}{\lambda_1} I & I \end{bmatrix}$$

while

$$2T_2 = \begin{bmatrix} \frac{\lambda_2}{\lambda_1} I & \frac{\sqrt{-1}}{\lambda_1} I \\ -\frac{\sqrt{-1}}{\lambda_1} & \frac{\lambda_2}{\lambda_1} I \end{bmatrix}.$$

Therefore

$$2(S_1 - T_2) = \begin{bmatrix} (1 - \frac{\lambda_2}{\lambda_1}) I & 0 \\ 0 & (1 - \frac{\lambda_2}{\lambda_1}) I \end{bmatrix},$$

which is certainly not Hilbert-Schmidt if $\lambda_1 \neq \lambda_2$.

[Note: The same reasoning shows that $\pi_{F,\lambda}$ $(\lambda > 1)$ is not geometrically equivalent to π_F.]

21.10 <u>LEMMA</u> Let $\mathcal{H}$ be a Hilbert space. Suppose that $A,B \in \mathcal{B}(\mathcal{H})$ are nonnegative and selfadjoint -- then

$$A - B \in \underline{L}_1(H) \Rightarrow \sqrt{A} - \sqrt{B} \in \underline{L}_2(H).$$

<u>PROOF</u> Let

$$\begin{cases} S = \sqrt{A} - \sqrt{B} \\ T = \sqrt{A} + \sqrt{B}. \end{cases}$$

Then S is compact and selfadjoint, hence its spectrum is pure point. Fix an orthonormal basis $\{e_i\}$ for $H: Se_i = \lambda_i e_i$. Observing that $T \geq \pm S$ and $\frac{1}{2}(ST + TS) = A - B$, we have

$$\begin{aligned} ||A - B||_1 &= tr(|A - B|) \\ &= \sum_i \frac{1}{2} < e_i, |ST + TS| e_i > \\ &\geq \sum_i |\frac{1}{2} < e_i, (ST + TS) e_i >| \\ &= \sum_i |\lambda_i < e_i, Te_i >| \\ &\geq \sum_i \lambda_i^2 \\ &= \sum_i < e_i, S^2 e_i > \\ &= ||\sqrt{A} - \sqrt{B}||_2^2. \end{aligned}$$

21.11 <u>EXAMPLE</u> Take σ_μ symplectic (cf. 21.6) and put

$$\begin{cases} A_1 = A_{\mu_1} \\ A_2 = A_{\mu_2}. \end{cases}$$

Then

$$2(S_1 - T_2) = \begin{bmatrix} I - A_1A_2^{-1} & 0 \\ 0 & I - A_1A_2^{-1} \end{bmatrix}.$$

Consequently (cf. 21.10), $\sqrt{S_1} - \sqrt{T_2}$ is Hilbert-Schmidt provided $I - A_1A_2^{-1}$ is trace class, thus under this condition, π_1,π_2 are geometrically equivalent (cf. 21.7).

Assume now that μ_1,μ_2 are pure and equivalent -- then π_1,π_2 are unitarily equivalent iff $J_2 - J_1$ is Hilbert-Schmidt (cf. 20.28). On the other hand, according to 21.7, π_1,π_2 are unitarily equivalent iff $\sqrt{S_1} - \sqrt{T_2}$ is Hilbert-Schmidt. The problem then is: Why are these conditions the same?

[Note: Since π_1,π_2 are irreducible, "unitary equivalence" coincides with "geometric equivalence".]

If $\sqrt{S_1} - \sqrt{T_2}$ is Hilbert-Schmidt, then $2(S_1 - T_2)$ is Hilbert-Schmidt (cf. 21.8). But

$$2(S_1 - T_2) = \begin{bmatrix} I - J_1J_2^{-1} & 0 \\ 0 & I - J_1J_2^{-1} \end{bmatrix}.$$

Therefore

$$I - J_1 J_2^{-1} = I + J_1 J_2$$

is Hilbert-Schmidt, so the same is true of

$$J_2 - J_1 = J_1(-(J_1 J_2) - I).$$

Thus the criterion of Araki-Yamagami is sufficient. It remains to see why it is necessary. In other words, the claim is that

$$J_2 - J_1 \text{ Hilbert-Schmidt} \Rightarrow \sqrt{S_1} - \sqrt{T_2} \text{ Hilbert-Schmidt.}$$

And for this, a series of lemmas will be required.

Using the notation of 21.1, write

$$\left[\begin{array}{l} H_{\mu_{\underline{C}}} = H_{\mu_1}^{+} \oplus H_{\mu_1}^{-} \\ \\ H_{\mu_{\underline{C}}} = H_{\mu_2}^{+} \oplus H_{\mu_2}^{-} \end{array}\right.$$

with attendant orthogonal projections

$$\left[\begin{array}{l} P_1^{+}, P_1^{-} \\ \\ P_2^{+}, P_2^{-}. \end{array}\right.$$

To simplify, put

$$\left[\begin{array}{l} H_1^{+} = H_{\mu_1}^{+},\ H_1^{-} = H_{\mu_1}^{-} \\ \\ H_2^{+} = H_{\mu_2}^{+},\ H_2^{-} = H_{\mu_2}^{-}. \end{array}\right.$$

21.12 <u>LEMMA</u> We have

$$P_2^- P_1^+ = \frac{1}{4} (I + (J_2 J_1)_{\underline{C}} + \sqrt{-1} (J_2 - J_1)_{\underline{C}}).$$

<u>PROOF</u> In fact,

$$\begin{cases} P_1^+ = \frac{1}{2} (I - \sqrt{-1} (J_1)_{\underline{C}}) \\ P_2^- = \frac{1}{2} (I + \sqrt{-1} (J_2)_{\underline{C}}), \end{cases}$$

from which the result.

The assumption is that $J_2 - J_1$ is Hilbert-Schmidt. But

$$J_2 - J_1 = J_1(- (J_1 J_2) - I).$$

Therefore $- (J_1 J_2) - I$ is Hilbert-Schmidt. Since complexification does not alter the Hilbert-Schmidt status of an operator, it follows that $P_2^- P_1^+$ is Hilbert-Schmidt.

21.13 <u>LEMMA</u> $P_2^- P_1^+$ Hilbert-Schmidt => $P_2^- | H_1^+$ Hilbert-Schmidt.

Define

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} : \begin{matrix} H_1^+ \\ \oplus \\ H_1^- \end{matrix} \longrightarrow \begin{matrix} H_2^+ \\ \oplus \\ H_2^- \end{matrix}$$

by

$$A = P_2^+|\mathcal{H}_1^+,\ B = P_2^-|\mathcal{H}_1^+,\ C = P_2^+|\mathcal{H}_1^-,\ D = P_2^-|\mathcal{H}_1^- .$$

21.14 LEMMA We have

$$\begin{cases} A^*A - B^*B = I \text{ in } \mathcal{B}(\mathcal{H}_1^+,\mathcal{H}_1^+) \\ D^*D - C^*C = I \text{ in } \mathcal{B}(\mathcal{H}_1^-,\mathcal{H}_1^-) \\ B^*D - A^*C = 0 \text{ in } \mathcal{B}(\mathcal{H}_1^-,\mathcal{H}_1^+). \end{cases}$$

PROOF Let $z,z' \in \mathcal{H}_1^+$ -- then

$$\mu_{1,\underline{C}}(z,z') = \mu_{1,\underline{C}}(z, -\sqrt{-1}\,(J_1)_{\underline{C}}z')$$

$$= -\sqrt{-1}\,\mu_{1,\underline{C}}(z,(J_1)_{\underline{C}}z')$$

$$= -\sqrt{-1}\,\sigma_{\mu_{\underline{C}}}(z,z')$$

$$= -\sqrt{-1}\,\sigma_{\mu_{\underline{C}}}(P_2^+z,P_2^+z') - \sqrt{-1}\,\sigma_{\mu_{\underline{C}}}(P_2^-z,P_2^-z')$$

$$= -\sqrt{-1}\,\sigma_{\mu_{\underline{C}}}(Az,Az') - \sqrt{-1}\,\sigma_{\mu_{\underline{C}}}(Bz,Bz')$$

$$= -\sqrt{-1}\,\mu_{2,\underline{C}}(Az,(J_2)_{\underline{C}}Az') - \sqrt{-1}\,\mu_{2,\underline{C}}(Bz,(J_2)_{\underline{C}}Bz')$$

$$= \mu_{2,\underline{C}}(Az,Az') - \mu_{2,\underline{C}}(Bz,Bz')$$

$$= \mu_{1,\underline{C}}(A^*Az,z') - \mu_{1,\underline{C}}(B^*Bz,z')$$

$$\Rightarrow$$

$$A^*A - B^*B = I \text{ in } \mathcal{B}(H_1^+,H_1^+).$$

Analogously,

$$D^*D - C^*C = I \text{ in } \mathcal{B}(H_1^-,H_1^-).$$

Finally, if $z \in H_1^+, z' \in H_1^-$, then

$$0 = \mu_{1,\underline{C}}(z,z')$$

$$= \mu_{1,\underline{C}}(z, \sqrt{-1}\ (J_1)_{\underline{C}}z')$$

$$= \sqrt{-1}\ \mu_{1,\underline{C}}(z,(J_1)_{\underline{C}}z')$$

$$= \sqrt{-1}\ \sigma_{\mu_{\underline{C}}}(z,z')$$

$$= \sqrt{-1}\ \sigma_{\mu_{\underline{C}}}(P_2^+z,P_2^+z') + \sqrt{-1}\ \sigma_{\mu_{\underline{C}}}(P_2^-z,P_2^-z')$$

$$= \sqrt{-1}\ \sigma_{\mu_{\underline{C}}}(Az,Cz') + \sqrt{-1}\ \sigma_{\mu_{\underline{C}}}(Bz,Dz')$$

$$= \sqrt{-1}\ \mu_{2,\underline{C}}(Az,(J_2)_{\underline{C}}Cz') + \sqrt{-1}\ \mu_{2,\underline{C}}(Bz,(J_2)_{\underline{C}}Dz')$$

$$= -\ \mu_{2,\underline{C}}(Az,Cz') + \mu_{2,\underline{C}}(Bz,Dz')$$

$$= - \mu_{1,\underline{C}}(z,A^*Cz') + \mu_{2,\underline{C}}(z,B^*Dz')$$

$$\Rightarrow$$

$$B^*D - A^*C = 0 \text{ in } \mathcal{B}(H_1^-,H_1^+).$$

21.15 LEMMA We have

$$\begin{cases} AA^* - CC^* = I \text{ in } \mathcal{B}(H_2^+,H_2^+) \\ DD^* - BB^* = I \text{ in } \mathcal{B}(H_2^-,H_2^-) \\ AB^* - CD^* = 0 \text{ in } \mathcal{B}(H_2^-,H_2^+). \end{cases}$$

21.16 REMARK The matrix

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix}$$

is invertible, its inverse being

$$\begin{bmatrix} A^* & -B^* \\ -C^* & D^* \end{bmatrix}.$$

[Note: Observe that

$$A^* = P_1^+|H_2^+, \ -C^* = P_1^-|H_2^+, \ -B^* = P_1^+|H_2^-, \ D^* = P_1^-|H_2^-.]$$

21.17 LEMMA A (respec. D) is injective and A^{-1} (respec. D^{-1}) extends to a bounded linear operator $H_2^+ \to H_1^+$ (respec. $H_2^- \to H_1^-$).

PROOF It suffices to deal with A. On the basis of the foregoing, it is clear that $A^*A \geq I$ on H_1^+ and $AA^* \geq I$ on H_2^+, thus A and A* are injective. But $\{0\} = \mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp$, so the range of A is dense. If $Az \in \mathrm{Ran}(A)$, then

$$\begin{aligned} ||A^{-1}(Az)||_{H_1^+}^2 &= ||z||_{H_1^+}^2 \\ &\leq < z, A^*A >_{H_1^+} \\ &= ||Az||_{H_2^+}^2 . \end{aligned}$$

Therefore A^{-1} is bounded, hence can be extended to all of H_2^+.

From the definitions,

$$\begin{cases} S_1 = P_1^- \\ S_2 = P_2^- . \end{cases}$$

Accordingly,

$$\begin{cases} S_1 \longleftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \text{per } H_{\mu_{\underline{C}}} = H_1^+ \oplus H_2^- \\ S_2 \longleftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \text{per } H_{\mu_{\underline{C}}} = H_2^+ \oplus H_2^- . \end{cases}$$

To compute T_2, write

$$\mu_{1,\underline{C}}(z,T_2z')$$

$$= \frac{1}{2}\,(\mu_{2,\underline{C}}(z,z') + \sqrt{-1}\;\sigma_{\mu_{\underline{C}}}(z,z'))$$

$$= \mu_{2,\underline{C}}(z,S_2z')$$

$$= \mu_{2,\underline{C}}\left(\begin{bmatrix} P_2^+z \\ \\ P_2^-z \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_2^+z' \\ \\ P_2^-z' \end{bmatrix}\right)$$

$$= \mu_{2,\underline{C}}\left(\begin{bmatrix} A & C \\ \\ B & D \end{bmatrix} \begin{bmatrix} P_1^+z \\ \\ P_1^-z \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & C \\ \\ B & D \end{bmatrix} \begin{bmatrix} P_1^+z' \\ \\ P_1^-z' \end{bmatrix}\right)$$

$$= \mu_{1,\underline{C}}\left(\begin{bmatrix} P_1^+z \\ \\ P_1^-z \end{bmatrix}, \begin{bmatrix} A^* & B^* \\ \\ C^* & D^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & C \\ \\ B & D \end{bmatrix} \begin{bmatrix} P_1^+z' \\ \\ P_1^-z' \end{bmatrix}\right)$$

$\Rightarrow$

$$T_2 = \begin{bmatrix} A^* & B^* \\ \\ C^* & D^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & C \\ \\ B & D \end{bmatrix}$$

$$= \begin{bmatrix} B^*B & B^*D \\ D^*B & D^*D \end{bmatrix}.$$

Therefore

$$\sqrt{S_1} - \sqrt{T_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{1/2} - \begin{bmatrix} B^*B & B^*D \\ D^*B & D^*D \end{bmatrix}^{1/2}.$$

Put

$$Z = BB^* + DD^*.$$

Then (cf. 21.15),

$$DD^* = I + BB^*$$

$$\Rightarrow$$

$$Z = I + 2BB^*.$$

Consequently, $Z \geq I$ is a positive selfadjoint operator on $\mathcal{H}_2^-$, hence has a bounded inverse.

21.18 <u>LEMMA</u> We have

$$\begin{bmatrix} B^*B & B^*D \\ D^*B & D^*D \end{bmatrix}^{1/2}$$

$$= \begin{bmatrix} B^*Z^{-1/2}B & B^*Z^{-1/2}D \\ D^*Z^{-1/2}B & D^*Z^{-1/2}D \end{bmatrix}.$$

[E.g.:

$$(B^*Z^{-1/2}B)(B^*Z^{-1/2}B) + (B^*Z^{-1/2}D)(D^*Z^{-1/2}B)$$

$$= B^*Z^{-1/2}(BB^* + DD^*)Z^{-1/2}B$$

$$= B^*Z^{-1/2}ZZ^{-1/2}B$$

$$= B^*B.]$$

Let

$$X = \sqrt{S_1} - \sqrt{T_2}.$$

Then

$$X = \begin{bmatrix} -B^*Z^{-1/2}B & -B^*Z^{-1/2}D \\ -D^*Z^{-1/2}B & I - D^*Z^{-1/2}D \end{bmatrix} \quad \begin{matrix} \mathcal{H}_1^+ & & \mathcal{H}_1^+ \\ \oplus & \longrightarrow & \oplus \\ \mathcal{H}_1^- & & \mathcal{H}_1^- \end{matrix} .$$

Restated, the claim is that

$$J_2 - J_1 \text{ Hilbert-Schmidt} \Rightarrow X \text{ Hilbert-Schmidt}$$

or still, that

$$J_2 - J_1 \text{ Hilbert-Schmidt} \Rightarrow X^*X \text{ trace class.}$$

2.19 <u>LEMMA</u> Let $\mathcal{H}_i, \mathcal{K}_i$ $(i = 1,2)$ be Hilbert spaces. Suppose that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathcal{K}_1 \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_2 \\ \oplus \\ \mathcal{K}_2 \end{matrix}$$

is a bounded linear operator -- then A is trace class iff $A_{k\ell}$ is trace class $(k,\ell = 1,2)$.

In view of this, we need only check that each of the entries of the operator

$$X^*X = \begin{bmatrix} B^*B & B^*(I - Z^{-1/2})D \\ D^*(I - Z^{-1/2})B & I + D^*(I - 2Z^{-1/2})D \end{bmatrix}$$

is trace class.

By definition, $B = P_2^-|\mathcal{H}_1^+$, so B is Hilbert-Schmidt (cf. 21.13), thus B*B is trace class (as is BB*).

Next

$$Z - I = 2BB^*,$$

hence Z - I is trace class. On the other hand,

$$Z - I = (I - Z^{-1/2})(Z + Z^{1/2}).$$

But $Z + Z^{1/2}$ is a bounded linear operator on $\mathcal{H}_2^-$ with a bounded inverse. Therefore $I - Z^{-1/2}$ is trace class. Consequently,

$$\begin{bmatrix} B^*(I - Z^{-1/2})D \\ D^*(I - Z^{-1/2})B \end{bmatrix}$$

are trace class.

This leaves

$$I + D^*(I - 2Z^{-1/2})D.$$

Note first that

$$DD^* + DD^*(I - 2Z^{-1/2})DD^*$$

$$= D(I + D^*(I - 2Z^{-1/2})D)D^*.$$

Since D and D* are invertible (cf. 21.17), it will be enough to show that

$$DD^* + DD^*(I - 2Z^{-1/2})DD^*$$

is trace class. Write

$$DD^* = \frac{I + Z}{2}.$$

Then

$$\frac{I + Z}{2} + \frac{I + Z}{2}(I - 2Z^{-1/2})\frac{I + Z}{2}$$

$$= \frac{I + Z}{2}(I + (I - 2Z^{-1/2})\frac{I + Z}{2})$$

$$= \frac{I + Z}{4}(I - Z^{-1/2})(2 - Z^{1/2} + Z)$$

is trace class ($I - Z^{-1/2}$ being trace class).

To recapitulate:

$$J_2 - J_1 \text{ Hilbert-Schmidt} \Rightarrow X^*X \text{ trace class},$$

as claimed.

The condition that

$$\sqrt{S_1} - \sqrt{T_2}$$

be Hilbert-Schmidt is taken per $\mu_{1,\underline{C}}$. Of course, one could consider its analog per $\mu_{2,\underline{C}}$, namely

$$\sqrt{S_2} - \sqrt{T_1},$$

where S_2 and T_1 are defined in the obvious way.

This raises another question: Is it true that the conditions

$$\left[\begin{array}{l} \sqrt{S_1} - \sqrt{T_2} \text{ Hilbert-Schmidt} \\ \\ \sqrt{S_2} - \sqrt{T_1} \text{ Hilbert-Schmidt} \end{array}\right.$$

are equivalent? Because of the square roots, the issue is more subtle than might first appear.

[Note: The preceding discussion renders matters trivial if both μ_1 and μ_2 are pure.]

Fix an invertible bounded linear operator $R: H_{\mu_{\underline{C}}} \to H_{\mu_{\underline{C}}}$ such that $\forall\ z, z' \in H_{\mu_{\underline{C}}}$,

$$\mu_{1,\underline{C}}(z, z') = \mu_{2,\underline{C}}(Rz, Rz').$$

[Note: R is positive and selfadjoint per $\mu_{1,\underline{C}}$ or $\mu_{2,\underline{C}}$ (see the Appendix to §1).]

From the definitions:

- $\mu_{1,\underline{C}}(z,z') + \sqrt{-1}\,\sigma_{\mu_{\underline{C}}}(z,z')$

$$= \begin{cases} 2\mu_{1,\underline{C}}(z,S_1 z') \\ \\ 2\mu_{2,\underline{C}}(z,T_1 z') \end{cases}$$

=>

$$\mu_{1,\underline{C}}(z,S_1 z') = \mu_{2,\underline{C}}(Rz,RS_1 z')$$

$$= \mu_{2,\underline{C}}(z,R^2 S_1 z')$$

=>

$$T_1 = R^2 S_1.$$

- $\mu_{2,\underline{C}}(z,z') + \sqrt{-1}\,\sigma_{\mu_{\underline{C}}}(z,z')$

$$= \begin{cases} 2\mu_{2,\underline{C}}(z,S_2 z') \\ \\ 2\mu_{1,\underline{C}}(z,T_2 z') \end{cases}$$

=>

$$\mu_{1,\underline{C}}(z,T_2 z') = \mu_{2,\underline{C}}(Rz,RT_2 z')$$

$$= \mu_{2,\underline{C}}(z,R^2 T_2 z')$$

=>

$$S_2 = R^2 T_2.$$

Therefore

$$\sqrt{S_2} - \sqrt{T_1} = (R^2T_2)^{1/2} - (R^2S_1)^{1/2},$$

the square roots taken per $\mu_{2,\underline{C}}$.

21.20 <u>LEMMA</u> $\sqrt{S_1} - \sqrt{T_2}$ is Hilbert-Schmidt per $\mu_{1,\underline{C}}$ iff $(R^2S_1)^{1/2} - (R^2T_2)^{1/2}$ is Hilbert-Schmidt per $\mu_{2,\underline{C}}$.

It will be simplest to formalize the situation.

21.21 <u>LEMMA</u> Let $\mathcal{H}$ be a Hilbert space — then $\forall$ A,B $\in \mathcal{B}(\mathcal{H})$,

$$|| \ |A| - |B| \ ||_2 \leq \sqrt{2}||A - B||_2.$$

Let $\mathcal{H}$ be a Hilbert space equipped with inner products < , > , < , >'. Fix an invertible bounded linear operator $T:\mathcal{H} \to \mathcal{H}$ such that $\forall$ x,y $\in \mathcal{H}$,

$$< x,y > \ = \ < Tx,Ty >'.$$

[Note: T is positive and selfadjoint per < , > or < , >' (see the Appendix to §1).]

Suppose that $A \in \mathcal{B}(\mathcal{H})$ is nonnegative and selfadjoint — then $\forall$ x $\in \mathcal{H}$,

$$< x,TATx > \ = \ < Tx,ATx > \ \geq 0.$$

Therefore $TAT \in \mathcal{B}(\mathcal{H})$ is nonnegative per < , >, hence $(TAT)^{1/2}$ exists. Next, $\forall$ x $\in \mathcal{H}$,

$$< x,T^2Ax >' \ = \ < Tx,TAx >'$$

$$= < x,Ax > \ge 0.$$

Therefore $T^2A \in \mathcal{B}(\mathcal{H})$ is nonnegative per $<\ ,\ >'$, hence $(T^2A)^{1/2}$ exists.

21.22 LEMMA We have

$$(T^2A)^{1/2} = T(TAT)^{1/2}T^{-1}.$$

PROOF First, if $x \in \mathcal{H}$, then

$$< x, T(TAT)^{1/2}T^{-1}x >'$$

$$= < TT^{-1}x, T(TAT)^{1/2}T^{-1}x >'$$

$$= < T^{-1}x, (TAT)^{1/2}T^{-1}x >$$

$$\ge 0,$$

thus $T(TAT)^{1/2}T^{-1}$ is nonnegative per $<\ ,\ >'$. And

$$T(TAT)^{1/2}T^{-1}T(TAT)^{1/2}T^{-1}$$

$$= T(TAT)^{1/2}(TAT)^{1/2}T^{-1}$$

$$= TTATT^{-1}$$

$$= T^2A.$$

21.23 LEMMA Let $\mathcal{H}$ be a Hilbert space. Suppose that $A,B \in \mathcal{B}(\mathcal{H})$ are nonnegative and selfadjoint. Put

$$\begin{cases} A' = T^2A \\ B' = T^2B. \end{cases}$$

Then

$$A^{1/2} - B^{1/2} \text{ is Hilbert-Schmidt per } <\ ,\ >$$

iff

$$(A')^{1/2} - (B')^{1/2} \text{ is Hilbert-Schmidt per } <\ ,\ >'.$$

<u>PROOF</u> Assume that $A^{1/2} - B^{1/2}$ is Hilbert-Schmidt per < , >. Since < , >' and < , > are equivalent,

$$(A')^{1/2} - (B')^{1/2} \text{ is Hilbert-Schmidt per } <\ ,\ >'$$

iff

$$(A')^{1/2} - (B')^{1/2} \text{ is Hilbert-Schmidt per } <\ ,\ >,$$

thus one can work exclusively with < , > during the course of the following estimate:

$$||(A')^{1/2} - (B')^{1/2}||_2$$

$$= ||T(TAT)^{1/2}T^{-1} - T(TBT)^{1/2}T^{-1}||_2$$

$$= ||T((TAT)^{1/2} - (TBT)^{1/2})T^{-1}||_2$$

$$\leq ||T||\ ||T^{-1}||\ ||(TAT)^{1/2} - (TBT)^{1/2}||_2$$

$$= ||T|| \; ||T^{-1}|| \; || \, |A^{1/2}T| - |B^{1/2}T| \, ||_2$$

$$\leq \sqrt{2} \; ||T|| \; ||T^{-1}|| \; ||A^{1/2}T - B^{1/2}T||_2 \quad \text{(cf. 21.21)}$$

$$\leq \sqrt{2} \; ||T||^2 ||T^{-1}|| \; ||A^{1/2} - B^{1/2}||_2 < \infty.$$

[Note: Work with T^{-1} to run the argument in the other direction.]

Specializing the data then gives 21.20.

§22. *FINITE DIMENSIONAL GAUSSIANS*

Let γ be a probability measure on $\text{Bor}(\underline{R})$ — then γ is said to be gaussian if it is either the Dirac measure δ_a at the point a or has density

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right) \quad (\sigma > 0)$$

w.r.t. Lebesgue measure.

One calls a the mean and σ^2 the variance of γ (take $\sigma = 0$ if γ is Dirac). Obviously,

$$a = \int_{\underline{R}} t d\gamma(t), \quad \sigma^2 = \int_{\underline{R}} (t-a)^2 d\gamma(t).$$

[Note: A mean zero gaussian measure on $\underline{R}$ is centered.]

22.1 RAPPEL Let μ be a finite Borel measure on $\underline{R}^n$ — then the Fourier transform $\hat{\mu}$ of μ is the function defined by the rule

$$\hat{\mu}(x) = \int_{\underline{R}^n} \exp(\sqrt{-1} < x,y >) d\mu(y).$$

[Note: As regards the sign, in probability theory, $\hat{\mu}$ is called the "characteristic function" of μ and by firm convention the plus sign is always chosen.]

22.2 EXAMPLE Suppose that $\gamma \longleftrightarrow (a,\sigma^2)$ — then

$$\hat{\gamma}(t) = \int_{\underline{R}} e^{\sqrt{-1}\, ts} d\gamma(s) = \exp(\sqrt{-1}\, at - \frac{1}{2}\sigma^2 t^2).$$

22.3 LEMMA If $\hat{\mu}_1 = \hat{\mu}_2$, then $\mu_1 = \mu_2$, i.e., finite Borel measures on $\underline{R}^n$ are uniquely determined by their Fourier transforms.

Let γ be a probability measure on $\text{Bor}(\underline{R}^n)$ — then γ is said to be gaussian if for every linear functional λ on $\underline{R}^n$, the induced measure $\gamma \circ \lambda^{-1}$ on $\underline{R}$ is gaussian.

22.4 THEOREM Let γ be a probability measure on $\text{Bor}(\underline{R}^n)$ — then γ is gaussian iff its Fourier transform has the form

$$\hat{\gamma}(x) = \exp(\sqrt{-1} < a,x > - \frac{1}{2} < x,Kx >),$$

where $a \in \underline{R}^n$ and K is nonnegative and symmetric.

PROOF Assume that $\hat{\gamma}$ has the stated form. Given a linear functional $\lambda:\underline{R}^n \to \underline{R}$, write $\lambda(x) = < \lambda,x >$ $(x \in \underline{R}^n)$ and put $\gamma_\lambda = \gamma \circ \lambda^{-1}$ — then

$$\hat{\gamma}_\lambda(t) = \int_{\underline{R}} e^{\sqrt{-1}\, ts} d\gamma_\lambda(s)$$

$$= \int_{\underline{R}^n} e^{\sqrt{-1}\, t <\lambda,x>} d\gamma(x)$$

$$= \hat{\gamma}(t\ )$$

$$= \exp(\sqrt{-1} < a,\lambda > t - \frac{1}{2} < \lambda,K\lambda > t^2).$$

But

$$\exp(\sqrt{-1} < a,\lambda > t - \frac{1}{2} < \lambda,K\lambda > t^2)$$

is the Fourier transform of a gaussian measure on $\underline{R}$ (cf. 22.2), hence by uniqueness (cf. 22.3), γ_λ is gaussian. Therefore γ is gaussian. Conversely, suppose that $\forall\ \lambda$, γ_λ is gaussian. Denote their means and variances by $a(\lambda)$ and $\sigma(\lambda)^2$, thus

$$a(\lambda) = \int_{\underline{R}} t d\gamma_\lambda(t) = \int_{\underline{R}^n} \lambda(x) d\gamma(x)$$

and

$$\sigma(\lambda)^2 = \int_{\underline{R}} (t-a(\lambda))^2 d\gamma_\lambda(t) = \int_{\underline{R}^n} (< \lambda,x > - a(\lambda))^2 d\gamma(x).$$

The function $\lambda \to a(\lambda)$ is linear, so $\exists\ a \in R^n: a(\lambda) = < a,\lambda >$, and the function $\lambda \to \sigma(\lambda)^2$ is a nonnegative quadratic form, so $\exists\ K:\sigma(\lambda)^2 = < \lambda,K\lambda >$, where K is nonnegative and symmetric. Accordingly,

$$\hat{\gamma}(\lambda) = \hat{\gamma}_\lambda(1)$$

$$= \exp(\sqrt{-1}\ a(\lambda) - \frac{1}{2}\sigma(\lambda)^2)$$

$$= \exp(\sqrt{-1} < a,\lambda > - \frac{1}{2} < \lambda,K\lambda >),$$

which is of the required form.

We have

$$a = \int_{\underline{R}^n} x d\gamma(x)$$

and

$$< u,Kv > = \int_{\underline{R}^n} < u,x-a > < v,x-a > d\gamma(x).$$

One calls a the <u>mean</u> and K the <u>covariance</u> of γ.

[Note: A mean zero gaussian measure on $\underline{R}^n$ is <u>centered</u>.]

22.5 <u>REMARK</u> If $K = 0$, then γ is the Dirac measure δ_a at the point a. If $K \neq 0$, then the support of γ is the k-dimensional affine space

$$L_\gamma = a + K\underline{R}^n \quad (k = \text{rank } K).$$

So, $\forall\ B \in \text{Bor}(\underline{R}^n)$,

$$\gamma(B) = \int_{B\cap L_\gamma} p_\gamma(x)dx,$$

where

$$p_\gamma(x) = \frac{1}{((2\pi)^k \det K)^{1/2}} \exp(-\frac{1}{2} < x-a,K^{-1}(x-a) >).$$

Here det K is the determinant of K regarded as an operator on $K\underline{R}^n$ and K^{-1} is the inverse of K on this subspace.

22.6 <u>LEMMA</u> Suppose that γ is a centered gaussian measure on $\underline{R}^n$. Let

$$T_\theta : \begin{cases} \underline{R}^n \times \underline{R}^n \rightarrow \underline{R}^n \\ \\ (x,y) \rightarrow x \sin\theta + y \cos\theta \end{cases} \quad (\theta \in \underline{R}).$$

Then the image of $\gamma \times \gamma$ under T_θ is γ.

<u>PROOF</u> Set $\mu = (\gamma \times \gamma) \circ T_\theta^{-1}$ — then

$$\hat{\mu}(x) = \int_{\underline{R}^n} \exp(\sqrt{-1} < x,y >) d\mu(y)$$

$$= \int_{\underline{R}^n} \int_{\underline{R}^n} \exp(\sqrt{-1} < x,u \sin\theta + v \cos\theta >) d\gamma(u) d\gamma(v)$$

$$= \int_{\underline{R}^n} \exp(\sqrt{-1} < x \sin\theta,u >) d\gamma(u) \times \int_{\underline{R}^n} \exp(\sqrt{-1} < x \cos\theta,v >) d\gamma(v)$$

$$= \hat{\gamma}(x \sin\theta)\hat{\gamma}(x \cos\theta)$$

$$= \exp(-\frac{1}{2} \sin^2\theta < x,Kx >)\exp(-\frac{1}{2} \cos^2\theta < x,Kx >)$$

$$= \exp(-\frac{1}{2} < x,Kx >)$$

$$= \hat{\gamma}(x)$$

=> (cf. 22.3)

$$\mu = \gamma.$$

By definition, the <u>standard</u> gaussian measure γ_n on $\underline{R}^n$ has density

$$\frac{1}{(2\pi)^{n/2}} e^{-x^2/2} = \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_k^2/2}$$

w.r.t. Lebesgue measure (cf. 6.12). Put

$$\underline{H}_{k_1,\ldots,k_n}(x_1,\ldots,x_n)$$

$$= \frac{H_{k_1}(x_1)}{\sqrt{k_1!}} \cdots \frac{H_{k_n}(x_n)}{\sqrt{k_n!}}.$$

Then the $\underline{H}_{k_1,\ldots,k_n}$ are an orthonormal basis for $L^2(\underline{R}^n,\gamma_n)$.

Let W_k denote the closed linear subspace of $L^2(\underline{R}^n,\gamma_n)$ generated by the $\underline{H}_{k_1,\ldots,k_n}$ with $k_1 + \cdots + k_n = k$ and let I_k denote the orthogonal projection of $L^2(\underline{R}^n,\gamma_n)$ onto W_k — then

$$L^2(\underline{R}^n,\gamma_n) = \bigoplus_{k=0}^{\infty} W_k$$

and $\forall\ f \in L^2(\underline{R}^n,\gamma_n)$,

$$f = \sum_{k=0}^{\infty} I_k(f).$$

22.7 <u>EXAMPLE</u> Take $n = 1$ and let $f \in S(\underline{R})$ -- then

$$I_k(f) = < \frac{H_k}{\sqrt{k!}}, f > \frac{H_k}{\sqrt{k!}}.$$

But for $k \geq 1$,

$$< H_k, f > = \int_{\underline{R}} H_k(x) f(x) d\gamma_1(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} H_k(x) e^{-x^2/2} f(x) d\gamma_1(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} (-1)^k (\frac{d^k}{dx^k} e^{-x^2/2}) f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} (-1)^k (-1)^k \left(\frac{d^k}{dx^k} f(x)\right) e^{-x^2/2}\, dx$$

$$= \int_{\underline{R}} f^{(k)}(x)\, d\gamma_1(x)$$

$$= \langle 1, f^{(k)} \rangle_{L^2(\gamma_1)}.$$

Therefore

$$I_k(f) = \frac{1}{k!} \langle 1, f^{(k)} \rangle_{L^2(\gamma_1)} H_k.$$

22.8 REMARK The real topological vector space underlying $\underline{C}^n$ is $\underline{R}^{2n}$. Take $K = L^2(\underline{R}^n, \gamma_n)$ and given $z = a + \sqrt{-1}\, b$ $(a,b \in \underline{R}^n)$, define a unitary operator $W(a,b)$ by

$$W(a,b)\psi\Big|_x$$

$$= \exp(\sqrt{-1}\, (\langle x,b \rangle - \langle a,b \rangle/2))\, [\exp(\langle x,a \rangle - a^2/2)]^{1/2} \psi(x - a).$$

Then W is a Weyl system over $\underline{C}^n$ which is unitarily equivalent to the Schrödinger system (cf. 10.4).

[Note: Given $a \in \underline{R}^n$, define

$$T_a : L^2(\underline{R}^n, \gamma_n) \to L^2(\underline{R}^n, \gamma_n)$$

by

$$T_a f(x) = f(x - a)\ [\exp(\langle x,a \rangle - a^2/2)]^{1/2}.$$

Then T_a is unitary with inverse T_{-a}. Indeed,

$$||T_a f||^2 = \int_{\underline{R}^n} |f(x - a)|^2 \exp(< x,a > - a^2/2)\,d\gamma_n(x)$$

$$= \int_{\underline{R}^n} |f(x - a)|^2 \frac{e^{-(x-a)^2/2}}{e^{-x^2/2}}\,d\gamma_n(x)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} |f(x - a)|^2 e^{-(x-a)^2/2}\,dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} |f(x)|^2 e^{-x^2/2}\,dx$$

$$= \int_{\underline{R}^n} |f(x)|^2 d\gamma_n(x) = ||f||^2.]$$

§23. *THE ORNSTEIN-UHLENBECK SEMIGROUP*

We shall begin with a review of certain standard definitions and facts.

Let X be a Banach space -- then a collection $\{T_t : t \geq 0\}$ of bounded linear operators on X is said to be a <u>strongly continuous semigroup</u> if $T_0 = I$, $T_{t+s} = T_t T_s$ $\forall$ $t \geq 0$ & $\forall$ $s \geq 0$, and $\forall$ $x \in X$, the map

$$\begin{cases} [0,\infty[\to X \\ \quad t \to T_t x \end{cases}$$

is continuous.

[Note: It suffices to check continuity at 0^+ only.]

Let Dom(L) be the set of all $x \in X$ for which

$$\lim_{t \to 0} \frac{T_t x - x}{t}$$

exists and define L on Dom(L) by the equality

$$Lx = \lim_{t \to 0} \frac{T_t x - x}{t}.$$

Then Dom(L) is a dense linear subspace of X and L is closed on Dom(L). Moreover,

$$x \in \text{Dom}(L) \Rightarrow T_t x \in \text{Dom}(L)$$

and

$$\frac{d}{dt} T_t x = LT_t x = T_t Lx.$$

[Note: L is called the <u>generator</u> of the semigroup $\{T_t : t \geq 0\}$.]

Now let γ be a centered gaussian measure on $\underline{R}^n$ — then in view of 22.6, $\forall\ t \geq 0$, γ is the image of $\gamma \times \gamma$ under the map

$$\begin{cases} \underline{R}^n \times \underline{R}^n \to \underline{R}^n \\ (x,y) \to e^{-t}x + (1 - e^{-2t})^{1/2}y. \end{cases}$$

This said, in the above take $X = L^p(\underline{R}^n,\gamma)$ $(p \geq 1)$ and define T_t $(t \geq 0)$ by

$$T_tf(x) = \int_{\underline{R}^n} f(e^{-t}x + (1 - e^{-2t})^{1/2}y)d\gamma(y).$$

Since

$$\int_{\underline{R}^n} |f(x)|^p d\gamma(x)$$

$$= \int_{\underline{R}^n} \int_{\underline{R}^n} |f(e^{-t}x + (1 - e^{-2t})^{1/2}y|^p d\gamma(x)d\gamma(y),$$

it follows that $T_tf \in L^p(\underline{R}^n,\gamma)$ and

$$||T_tf||_p \leq ||f||_p.$$

Therefore $||T_t|| \leq 1$. But $T_t1 = 1$, so that actually $||T_t|| = 1$.

[Note: $\forall\ f \in L^1(\underline{R}^n,\gamma)$,

$$\int_{\underline{R}^n} T_tf(x)d\gamma(x) = \int_{\underline{R}^n} f(x)d\gamma(x).]$$

23.1 <u>LEMMA</u> The collection $\{T_t:t \geq 0\}$ is a strongly continuous semigroup on $L^p(\underline{R}^n,\gamma)$.

PROOF From its very definition, $T_0 = I$. Noting that γ is the image of $\gamma \times \gamma$ under the map

$$(u,v) \to e^{-s} \frac{(1 - e^{-2t})^{1/2}}{(1 - e^{-2t-2s})^{1/2}} u + \frac{(1 - e^{-2s})^{1/2}}{(1 - e^{-2t-2s})^{1/2}} v,$$

we have

$$T_t(T_s f)(x) = \int_{\underline{R}^n} T_s f(e^{-t}x + (1 - e^{-2t})^{1/2}y)d\gamma(y)$$

$$= \int_{\underline{R}^n} \int_{\underline{R}^n} f(e^{-s}e^{-t}x + e^{-s}(1 - e^{-2t})^{1/2}y + (1 - e^{-2s})^{1/2}z)d\gamma(z)d\gamma(y)$$

$$= \int_{\underline{R}^n} f(e^{-t-s}x + (1 - e^{-2t-2s})^{1/2}w)d\gamma(w)$$

$$= T_{t+s}f(x).$$

The verification of strong continuity is left to the reader.

[Note: This is the Ornstein-Uhlenbeck semigroup.]

23.2 REMARK Take $p = 2$ -- then the T_t are nonnegative and symmetric. In addition, $\forall$ $f,g \in L^2(\underline{R}^n,\gamma)$,

$$[T_t(fg)]^2 \leq T_t(f^2)T_t(g^2) \quad (\text{a.e.}[\gamma]).$$

Assume henceforth that $\gamma = \gamma_n$, the standard gaussian measure on $\underline{R}^n$ -- then there is an orthogonal decomposition

$$L^2(\underline{R}^n,\gamma_n) = \bigoplus_{k=0}^{\infty} W_k$$

and $\forall\ f \in L^2(\underline{R}^n,\gamma_n)$,

$$f = \sum_{k=0}^{\infty} I_k(f).$$

23.3 LEMMA We have

$$T_t f = \sum_{k=0}^{\infty} e^{-kt} I_k(f).$$

[The RHS defines a bounded linear operator on $L^2(\underline{R}^n,\gamma_n)$, hence it suffices to establish equality on the $\underline{H}_{k_1,\ldots,k_n}$. This, however, is a one dimensional problem, where one can proceed by induction on k. It is clearly true if $k = 0$. Suppose it is true for $k - 1$ — then for $\ell < k$,

$$\begin{aligned} &< T_t\underline{H}_k,\underline{H}_\ell > \\ &\qquad = < \underline{H}_k,T_t\underline{H}_\ell > \\ &\qquad = < \underline{H}_k,e^{-\ell t}\underline{H}_\ell > \\ &\qquad = e^{-\ell t}\delta_{k\ell} = 0. \end{aligned}$$

But $T_t\underline{H}_k$ is a polynomial of degree k, thus $T_t\underline{H}_k = c\underline{H}_k$ for some constant c. Comparing coefficients of x^k, we conclude that $c = e^{-kt}$.]

Let L be the generator of the semigroup $\{T_t : t \geq 0\}$ on $L^2(\underline{R}^n, \gamma_n)$.

23.4 <u>LEMMA</u> The domain of definition Dom(L) of L is

$$\{f: \sum_{k=0}^{\infty} k^2 ||I_k(f)||^2_{L^2(\gamma_n)} < \infty\}.$$

And, on this domain,

$$Lf = -\sum_{k=0}^{\infty} kI_k(f).$$

[Suppose that $f \in \text{Dom}(L)$ -- then $t \to T_t f$ is differentiable at zero, hence (cf. 23.3)

$$I_k Lf = \frac{d}{dt} e^{-kt} I_k(f) \Big|_{t=0}$$

$$= - kI_k(f)$$

=>

$$\sum_{k=0}^{\infty} k^2 ||I_k(f)||^2_{L^2(\gamma_n)} = ||Lf||^2_{L^2(\gamma_n)} < \infty.$$

And

$$Lf = -\sum_{k=0}^{\infty} kI_k(f).$$

Turning to the converse, note first that

$$|t^{-1}(e^{-kt}-1)| \leq k.$$

Therefore, as $t \to 0$,

$$\left|\left| \frac{T_t f - f}{t} + \sum_{k=0}^{\infty} k I_k(f) \right|\right|^2_{L^2(\gamma_n)}$$

$$= \sum_{k=0}^{\infty} \left[\frac{e^{-kt}-1}{t} + k \right]^2 \quad ||I_k(f)||^2_{L^2(\gamma_n)} \to 0.$$

I.e.: $t \to T_t f$ is differentiable at $t = 0$.]

Sobolev spaces play an important role in gaussian analysis. However, instead of providing ad hoc definitions at this point, it will be more convenient to postpone the discussion and place matters into a more general context later on. Still, there is one important fact that emerges from the theory and can be mentioned now, namely

$$\text{Dom}(L) = W^{2,2}(\underline{R}^n, \gamma_n) \quad (\text{cf. } 30.15).$$

Thinking of L as the gaussian analog of the laplacian Δ, this parallels the characterization of $\text{Dom}(\Delta)$ as $W^{2,2}(\underline{R}^n)$ (cf. 1.15).

23.5 <u>REMARK</u> The $\underline{H}_{k_1,\ldots,k_n}$ are total in $W^{2,2}(\underline{R}^n, \gamma_n)$ and $\forall\ f \in W^{2,2}(\underline{R}^n, \gamma_n)$,

$$\sum_{k=0}^{K} I_k(f) \to f \quad (K \to \infty).$$

23.6 <u>LEMMA</u> We have

$$L = \Delta - \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k}.$$

[Start by checking that

$$Lf = \Delta f - \sum_{k=1}^{n} x_k \frac{\partial f}{\partial x_k}$$

when f is a finite linear combination of the $\underline{H}_{k_1,\ldots,k_n}$.]

Let N be the number operator on $BO(\underline{R}^n)$ and let

$$T: BO(\underline{R}^n) \rightarrow L^2(\underline{R}^n, \gamma_n)$$

be the canonical isometric isomorphism (cf. 6.12) -- then

$$TNT^{-1} = - L.$$

[Note: See §8 for the case n = 1, the point being that

$$L = \frac{d^2}{dx^2} - x \frac{d}{dx}$$

and

$$H_k'' - xH_k' = - kH_k.$$

The extension to arbitrary n is straightforward.]

§24. *MEASURE THEORY ON* $\underline{R}^\infty$

Let $\underline{R}^\infty$ stand for the set of all real sequences $x = \{x_k : k \geq 1\}$ -- then $\underline{R}^\infty$ is a separable Fréchet space, the metric being

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}.$$

Write $\underline{R}^{\infty-n}$ for the subset of $\underline{R}^\infty$ consisting of those x such that $x_k = 0$ $(k = 1,\ldots,n)$ and identify $\underline{R}^n$ with a subset of $\underline{R}^\infty$ by adding zeros after the first n positions -- then

$$\underline{R}^\infty = \underline{R}^n \oplus R^{\infty-n}$$

and, by definition, a <u>cylinder set</u> is a subset of $\underline{R}^\infty$ of the form

$$B \oplus \underline{R}^{\infty-n},$$

where $B \in \text{Bor}(\underline{R}^n)$.

24.1 <u>LEMMA</u> The σ-algebra generated by the cylinder sets is $\text{Bor}(\underline{R}^\infty)$.

[Note: The σ-algebra generated by the cylinder sets is the same as the σ-algebra generated by the coordinate functions $x \to x_k$, i.e., is the smallest σ-algebra containing all sets of the form $\{x : x_k < r\}$ $(r \in \underline{R})$.]

24.2 <u>EXTENSION PRINCIPLE</u> Let μ_k be probability measures on $\text{Bor}(\underline{R})$ $(k = 1,2,\ldots)$ -- then there exists a unique probability measure μ on $\text{Bor}(\underline{R}^\infty)$

such that

$$\mu(B \oplus \underline{R}^{\infty-n}) = (\mu_1 \times \cdots \times \mu_n)(B)$$

for all $B \in \text{Bor}(\underline{R}^n)$ $(n = 1,2,\ldots)$. One calls μ the product of the μ_k:

$$\mu = \prod_{k=1}^{\infty} \mu_k.$$

24.3 THEOREM (Kakutani) Suppose given two products

$$\left[\begin{array}{l} \mu = \prod_{k=1}^{\infty} \mu_k \\ \\ \nu = \prod_{k=1}^{\infty} \nu_k. \end{array}\right.$$

Assume: $\forall\, k$, $\mu_k \sim \nu_k$ -- then either $\mu \sim \nu$ or $\mu \perp \nu$.

[Note: In the event that $\mu \sim \nu$, one has

$$\left[\begin{array}{l} \dfrac{d\mu}{d\nu} = \prod_{k=1}^{\infty} \dfrac{d\mu_k}{d\nu_k} \\ \\ \dfrac{d\nu}{d\mu} = \prod_{k=1}^{\infty} \dfrac{d\nu_k}{d\mu_k} \end{array}\right. \qquad \text{a.e.}[\mu \text{ or } \nu].]$$

24.4 THEOREM (Kakutani) Suppose given two products

$$\begin{cases} \mu = \prod_{k=1}^{\infty} \mu_k \\ \nu = \prod_{k=1}^{\infty} \nu_k. \end{cases}$$

Assume: $\forall$ k,

$$\exists \begin{cases} f_k > 0 \\ g_k > 0 \end{cases} : \begin{cases} d\mu_k = f_k(x_k)dx_k \\ d\nu_k = g_k(x_k)dx_k. \end{cases}$$

Then $\mu \sim \nu$ iff the infinite product

$$\prod_{k=1}^{\infty} \int_{\underline{R}} \sqrt{f_k} \sqrt{g_k}\, dx_k$$

is convergent.

[Note: Each term of the infinite product

$$\prod_{k=1}^{\infty} \int_{\underline{R}} \sqrt{f_k} \sqrt{g_k}\, dx_k$$

is ≤ 1, thus

$$\prod_{k=1}^{\infty} \int_{\underline{R}} \sqrt{f_k} \sqrt{g_k}\, dx_k$$

cannot diverge to infinity (but it might diverge to zero).]

24.5 <u>EXAMPLE</u> Suppose that $f > 0$, $g > 0$ are continuous and

$$\begin{cases} \int_{\underline{R}} f(x)dx = 1 \\ \int_{\underline{R}} g(x)dx = 1. \end{cases}$$

Take $f_k = f$, $g_k = g$, so

$$\prod_{k=1}^{\infty} \int_{\underline{R}} \sqrt{f_k}\,\sqrt{g_k}\,dx_k = \prod_{k=1}^{\infty} \int_{\underline{R}} \sqrt{f}\,\sqrt{g}\,dx$$

is convergent iff

$$\int_{\underline{R}} \sqrt{f}\,\sqrt{g}\,dx = 1.$$

But

$$< \sqrt{f},\sqrt{g} > \ \leq (\int_{\underline{R}} f dx)^{1/2} (\int_{\underline{R}} g dx)^{1/2} = 1.$$

Therefore

$$\int_{\underline{R}} \sqrt{f}\,\sqrt{g}\,dx = 1$$

iff $f = g$.

24.6 <u>LEMMA</u> $\forall\, t > 0$, $\forall\, a \in \underline{R}$,

$$\frac{1}{\sqrt{2\pi t}} \int_{\underline{R}} \exp(ax - \frac{x^2}{2t})\,dx = \exp(\frac{ta^2}{2}).$$

24.7 <u>EXAMPLE</u> Let

$$\begin{cases} d\mu_k = \dfrac{1}{\sqrt{2\pi t}} \exp(- \dfrac{x_k^2}{2t})\,dx_k \\[2ex] d\nu_k = \dfrac{1}{\sqrt{2\pi t}} \exp(- \dfrac{(x_k+a_k)^2}{2t})\,dx_k. \end{cases}$$

Then

$$\int_{\underline{R}} \sqrt{f_k}\,\sqrt{g_k}\,dx_k$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{\underline{R}} \exp\left(- \frac{x_k^2}{4t} - \frac{(x_k+a_k)^2}{4t}\right) dx_k$$

$$= \exp\left(- \frac{a_k^2}{4t}\right) \frac{1}{\sqrt{2\pi t}} \int_{\underline{R}} \exp\left(- \frac{a_k x_k}{2t} - \frac{x_k^2}{2t}\right) dx_k$$

$$= \exp\left(- \frac{a_k^2}{4t}\right) \exp\left(\frac{a_k^2}{8t}\right)$$

$$= \exp\left(- \frac{a_k^2}{8t}\right).$$

Since

$$\prod_{k=1}^{\infty} \exp\left(- \frac{a_k^2}{8t}\right)$$

is convergent iff $\sum_{k=1}^{\infty} a_k^2 < \infty$, it follows that $\mu \sim \nu$ iff $\sum_{k=1}^{\infty} a_k^2 < \infty$.

[Note: If $\mu \sim \nu$, then up to a set of measure 0, the relevant Radon-Nikodym derivatives are the functions

$$x \rightarrow \exp\left(\pm \frac{1}{2t} \sum_{k=1}^{\infty} a_k^2 \pm \frac{1}{t} \sum_{k=1}^{\infty} a_k x_k\right).$$

But is it really obvious that the set of $x \in \underline{R}$ for which the series $\sum_{k=1}^{\infty} a_k x_k$ is convergent constitutes a set of full measure? This point will be dealt with in 24.20.]

24.8 <u>EXAMPLE</u> Let

$$\begin{cases} d\mu_k = \dfrac{1}{\sqrt{2\pi t}} \exp\left(- \dfrac{x_k^2}{2t}\right) dx_k \\ d\nu_k = \dfrac{1}{\sqrt{2\pi t}} \exp\left(- \dfrac{(x_k+a_k)^2}{2s}\right) dx_k. \end{cases}$$

Then

$$\int_{\underline{R}} \sqrt{f_k}\,\sqrt{g_k}\,dx_k$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{\underline{R}} \exp(-\frac{x_k^2}{4t} - \frac{(x_k+a_k)^2}{4s})\,dx_k$$

$$= \exp(-\frac{a_k^2}{4s}) \frac{1}{\sqrt{2\pi t}} \int_{\underline{R}} \exp(-\frac{a_k x_k}{2s} - \frac{t+s}{4st} x_k^2)\,dx_k$$

$$= \exp(-\frac{a_k^2}{4s}) \frac{1}{\sqrt{2\pi t}} (\frac{4\pi st}{t+s})^{1/2} (\frac{t+s}{4\pi st})^{1/2}$$

$$\times \int_{\underline{R}} \exp(-\frac{a_k x_k}{2s} - \frac{t+s}{4st} x_k^2)\,dx_k$$

$$= \exp(-\frac{a_k^2}{4s}) (\frac{2s}{t+s})^{1/2} \exp(\frac{(2st)a_k^2}{(t+s)8s^2})$$

$$= (\frac{2s}{t+s})^{1/2} \exp(-\frac{a_k^2}{4(t+s)}).$$

So, if $t \neq s$, then no matter what the choice of the a_k, the infinite product

$$\prod_{k=1}^{\infty} (\frac{2s}{t+s})^{1/2} \exp(-\frac{a_k^2}{4(t+s)})$$

is divergent, hence $\mu \perp \nu$.

Fix $\sigma > 0$ -- then $\exists$ a unique probability measure γ_σ on $\mathrm{Bor}(\underline{R}^\infty)$ such that $\forall\ B \in \mathrm{Bor}(\underline{R}^n)$,

$$\gamma_\sigma(B \oplus \underline{R}^{\infty-n})$$

$$= \frac{1}{\sigma^n (2\pi)^{n/2}} \int_B \exp(- \frac{x_1^2 + \cdots + x_n^2}{2\sigma^2}) dx_1 \ldots dx_n.$$

24.9 LEMMA If $\sigma \neq \sigma'$, then $\gamma_\sigma \perp \gamma_{\sigma'}$.

[This is a special case of 24.5.]

In what follows, we shall take $\sigma = 1$ and write γ in place of γ_1.

24.10 REMARK We have (cf. 14.13)

$$BO(\ell^2(\underline{N})) = \bigotimes_1^\infty BO(\underline{C})$$

or still (cf. 14.14),

$$BO(\ell^2(\underline{N})) = \bigotimes_1^\infty L^2(R,\gamma_1)$$

$$= L^2(\underline{R}^\infty,\gamma).$$

[Here γ_1 refers to the standard gaussian measure on $\underline{R}$.]

Given a sequence $a = \{a_k : k \geq 1\}$ of positive real numbers, let

$$H_a = \{x \in \underline{R}^\infty : \sum_{k=1}^\infty a_k x_k^2 < \infty\}.$$

Then H_a is a real Hilbert space:

$$< x,y > = \sum_{k=1}^{\infty} a_k x_k y_k.$$

[Note: Take $a_k = 1 \ \forall \ k$ — then $H_a = \ell^2$, the real analog of $\ell^2(\underline{N})$.]

24.11 LEMMA $H_a \in \text{Bor}(\underline{R}^\infty)$ and

$$\begin{cases} \gamma(H_a) = 1 & \text{if } \sum_{k=1}^{\infty} a_k < \infty \\ \gamma(H_a) = 0 & \text{if } \sum_{k=1}^{\infty} a_k = \infty. \end{cases}$$

PROOF Define

$$f_\lambda : \underline{R}^\infty \to \underline{R} \quad (\lambda > 0)$$

by

$$f_\lambda(x) = \exp(-\lambda \sum_{k=1}^{\infty} a_k x_k^2).$$

Then

$$f_\lambda \to \chi_{H_a}$$

pointwise as $\lambda \downarrow 0$, hence $H_a \in \text{Bor}(\underline{R}^\infty)$. The functions

$$f_{\lambda,n}(x) = \exp(-\lambda \sum_{k=1}^{n} a_k x_k^2)$$

are in $L^1(\underline{R}^\infty, \gamma)$ and, for fixed λ, form a decreasing sequence. Therefore

$$\int_{\underline{R}^\infty} f_\lambda d\gamma = \int_{\underline{R}^\infty} \lim_{n \to \infty} f_{\lambda,n} d\gamma$$

$$= \lim_{n \to \infty} \int_{\underline{R}^\infty} f_{\lambda,n} d\gamma.$$

From the definitions,

$$\int_{\underline{R}^\infty} f_{\lambda,n} d\gamma$$

$$= (2\pi)^{-n/2} \int_{\underline{R}^n} \exp(-\lambda \sum_{k=1}^{n} a_k x_k^2) \exp(-\frac{1}{2} \sum_{k=1}^{n} x_k^2) dx_1 \ldots dx_n$$

$$= (2\pi)^{-n/2} \prod_{k=1}^{n} \int_{\underline{R}} \exp((-\lambda a_k - \frac{1}{2}) x_k^2) dx_k$$

$$= (2\pi)^{-n/2} \prod_{k=1}^{n} (1 + 2\lambda a_k)^{-1/2} \int_{\underline{R}} \exp(-\frac{t^2}{2}) dt$$

$$= \prod_{k=1}^{n} (1 + 2\lambda a_k)^{-1/2}$$

=>

$$\int_{\underline{R}^\infty} f_\lambda d\gamma = \begin{cases} \prod_{k=1}^{\infty} (1 + 2\lambda a_k)^{-1/2} & \text{if } \sum_{k=1}^{\infty} a_k < \infty \\ 0 & \text{if } \sum_{k=1}^{\infty} a_k = \infty. \end{cases}$$

Finally,

$$\gamma(H_a) = \int_{\underline{R}^\infty} \chi_{H_a} d\gamma$$

$$= \lim_{\lambda \downarrow 0} \int_{\underline{R}^\infty} f_\lambda d\gamma$$

$$= \begin{cases} 1 \text{ if } \sum_{k=1}^{\infty} a_k < \infty \\ \\ 0 \text{ if } \sum_{k=1}^{\infty} a_k = \infty, \end{cases}$$

which concludes the proof.

In particular:

$$\gamma(\ell^2) = 0.$$

24.12 REMARK Let a run through the sequences of positive real numbers such that $\gamma(H_a) = 1$ -- then

$$\bigcap_a H_a = \ell^\infty$$

and

$$\gamma(\ell^\infty) = 0.$$

Let $\pi_n:\underline{R}^\infty \to \underline{R}^n$ be the canonical projection -- then a Borel function f on $\underline{R}^\infty$ is said to be projectable if $\exists$ $n:f = \phi \circ \pi_n$ for some Borel function ϕ on $\underline{R}^n$.

[Note: Every Borel function on $\underline{R}^n$ determines a projectable function on $\underline{R}^\infty$.]

24.13 LEMMA The projectable functions are dense in $L^1(\underline{R}^\infty,\gamma)$.

PROOF The characteristic functions of cylinder sets are projectable.

Let γ_n be the standard gaussian measure on $\underline{R}^n$. Identify $\underline{R}^n \oplus \underline{R}^{\infty-n}$ with $\underline{R}^n \times \underline{R}^{\infty-n}$ and denote by $\gamma_{\infty-n}$ the measure on $\text{Bor}(\underline{R}^{\infty-n})$ constructed in the same way as the measure γ on $\text{Bor}(\underline{R}^{\infty})$ -- then γ can be regarded as the product $\gamma_n \times \gamma_{\infty-n}$.

Let $f \in L^1(\underline{R}^{\infty},\gamma)$. Given $x \in \underline{R}^n$, put

$$(E_n f)(x) = \int_{\underline{R}^{\infty-n}} f(x+y)\,d\gamma_{\infty-n}(y).$$

Then $E_n f \in L^1(\underline{R}^n,\gamma_n)$ and

$$||E_n f||_1 \leq ||f||_1.$$

24.14 <u>LEMMA</u> $\forall\ f \in L^1(\underline{R}^{\infty},\gamma)$, we have

$$E_n f \to f \quad (n \to \infty)$$

in $L^1(\underline{R}^{\infty},\gamma)$.

<u>PROOF</u> Fix $\varepsilon > 0$. Choose a projectable function $g: ||f - g||_1 < \varepsilon/2$ (cf. 24.13). Fix $N: n \geq N \Rightarrow E_n g = g$ — then

$$\begin{aligned} ||E_n f - f||_1 &\leq ||E_n(f-g)||_1 + ||f - E_n g||_1 \\ &= ||E_n(f-g)||_1 + ||f - g||_1 \\ &\leq 2||f - g||_1 < \varepsilon. \end{aligned}$$

Let $f:\underline{R}^{\infty} \to \underline{R}$ be Borel — then f is said to satisfy <u>condition K</u> if $\forall\ n$,

$$f(x,y) = f(x',y) \quad (x,x' \in \underline{R}^n, y \in \underline{R}^{\infty-n}).$$

24.15 LEMMA If f satisfies condition K, then f is constant a.e..

PROOF By taking the Arc Tan of f, it can be assumed that $f \in L^1(\underline{R}^\infty,\gamma)$, thus $f = \lim E_n f$ (cf. 24.14). But, since f satisfies condition K, $E_n f$ is a constant independent of n.

24.16 THE ZERO-ONE LAW Let $B \in \mathrm{Bor}(\underline{R}^\infty)$. Suppose that χ_B satisfies condition K -- then B is either of measure 0 or of measure 1.

PROOF In view of 24.15, χ_B is constant a.e., thus $\chi_B = 0$ a.e. or $\chi_B = 1$ a.e..

24.17 EXAMPLE Take for B the set of $x \in \underline{R}^\infty$: $\lim x_k$ exists -- then χ_B satisfies condition K, hence $\gamma(B) = 0$ or 1, and, in fact $\gamma(B) = 0$. For otherwise, the function which sends x to its limit would be defined a.e., hence would be a constant a.e..

Fix an element $a = \{a_k : k \geq 1\}$ in $\underline{R}^\infty$. Given $n \in \underline{N}$, define $s_n : \underline{R}^\infty \to \underline{R}$ by

$$s_n(x) = \sum_{k=1}^{n} a_k x_k.$$

24.18 LEMMA We have

$$\sum_{k=1}^{n} a_k^2 = \int_{\underline{R}^\infty} s_n^2 \, d\gamma.$$

PROOF In fact,

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} a_k^2 \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} x_k^2 \exp(-\frac{1}{2} x_k^2) dx_k$$

$$= \sum_{k=1}^{n} \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} a_k^2 x_k^2 \exp(-\frac{1}{2} x_k^2) dx_k$$

$$= \sum_{k=1}^{n} \int_{\underline{R}^\infty} a_k^2 x_k^2 d\gamma(x)$$

$$= \int_{\underline{R}^\infty} (\sum_{k=1}^{n} a_k x_k)^2 \, d\gamma(x)$$

$$= \int_{\underline{R}^\infty} s_n^2 d\gamma.$$

24.19 LEMMA $\forall\ \varepsilon > 0$,

$$\gamma\{x \in \underline{R}^\infty: \sup_{m \le n} |s_m(x)| > \varepsilon\} \le \frac{1}{\varepsilon^2} \sum_{k=1}^{n} a_k^2.$$

PROOF Define $f:\underline{R}^\infty \to \underline{R}$ by

$$f(x) = \inf \{n \in \underline{N}: |s_n(x)| > \varepsilon\}.$$

Put

$$B_m = f^{-1}(m) \quad (m = 1,\ldots,n).$$

Then from 24.18,

$$\sum_{k=1}^{n} a_k^2 = \int_{\underline{R}^\infty} s_n^2 d\gamma$$

$$\geq \sum_{m=1}^{n} \int_{B_m} s_n^2 d\gamma$$

$$= \sum_{m=1}^{n} \int_{B_m} (s_m^2 + 2s_m(s_n - s_m) + (s_n - s_m)^2) d\gamma$$

$$\geq \sum_{m=1}^{n} \int_{B_m} (s_m^2 + 2s_m(s_n - s_m)) d\gamma.$$

But

$$\int_{B_m} s_m(s_n - s_m) d\gamma = \int_{\underline{R}^\infty} \chi_{B_m} s_m(s_n - s_m) d\gamma$$

$$= \int_{\underline{R}^\infty} \chi_{B_m} s_m d\gamma \int_{\underline{R}^\infty} (s_n - s_m) d\gamma$$

$$= 0,$$

$s_n - s_m$ being linear in the variables $x_{m+1}, \ldots, x_n$. This leaves

$$\sum_{k=1}^{n} a_k^2 \geq \sum_{m=1}^{n} \int_{B_m} s_m^2 d\gamma$$

$$\geq \sum_{m=1}^{n} \varepsilon^2 \gamma(B_m) = \varepsilon^2 \gamma\left(\bigcup_{m=1}^{n} B_m\right).$$

And

$$\bigcup_{m=1}^{n} B_m$$

is precisely

$$\{x \in \underline{R}^{\infty}: \sup_{m \le n} |s_m(x)| > \varepsilon\}.$$

Consequently, $\forall\ \varepsilon > 0$,

$$\gamma\{x \in \underline{R}^{\infty}: \sup_{k \le n} |s_{m+k}(x) - s_m(x)| > \varepsilon\} \le \frac{1}{\varepsilon^2} \sum_{k=1}^{n} a_{m+k}^2$$

=>

$$\gamma\{x \in \underline{R}^{\infty}: \sup_{k \ge 1} |s_{m+k}(x) - s_m(x)| > \varepsilon\} \le \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} a_{m+k}^2$$

or still,

$$\gamma\{x \in \underline{R}^{\infty}: \sup_{k \ge 1} |s_{m+k}(x) - s_m(x)| > \varepsilon\} \le \frac{1}{\varepsilon^2} \sum_{k=m+1}^{\infty} a_k^2$$

=>

$$\lim_{m \to \infty} \gamma\{x \in \underline{R}^{\infty}: \sup_{k \ge 1} |s_{m+k}(x) - s_m(x)| > \varepsilon\} = 0.$$

24.20 THEOREM (Kolmogorov) Fix an element $a = \{a_k : k \ge 1\}$ in $\underline{R}^{\infty}$. Assume: $\sum_{k=1}^{\infty} a_k^2 < \infty$ -- then for almost every $x \in \underline{R}^{\infty}$, the series $\sum_{k=1}^{\infty} a_k x_k$ is convergent.

PROOF Put

$$\begin{cases} \bar{s}(x) = \limsup s_n(x) \\ \underline{s}(x) = \liminf s_n(x). \end{cases}$$

Then

$$\{x: |\bar{s}(x) - \underline{s}(x)| > 0\}$$

$$= \bigcup_{\varepsilon} \{x: |\bar{s}(x) - \underline{s}(x)| > 2\varepsilon\},$$

the union running over all positive rational ε. We claim that

$$\gamma\{x: |\bar{s}(x) - \underline{s}(x)| > 2\varepsilon\} = 0.$$

To see this, note that $\forall$ m,

$$|\bar{s}(x) - \underline{s}(x)| \leq 2 \sup_{k \geq 1} |s_{m+k}(x) - s_m(x)|.$$

Therefore

$$\gamma\{x: |\bar{s}(x) - \underline{s}(x)| > 2\varepsilon\}$$

$$\leq \gamma\{x: \sup_{k \geq 1} |s_{m+k}(x) - s_m(x)| > \varepsilon\},$$

so the claim follows upon letting $m \to \infty$, hence

$$\gamma\{x: |\bar{s}(x) - \underline{s}(x)| > 0\} = 0.$$

24.21 <u>EXAMPLE</u> Take $a_k = \frac{1}{k}$ -- then for almost every $x \in \underline{R}^{\infty}$, the series $\sum_{k=1}^{\infty} \frac{x_k}{k}$ is convergent, thus

$$\gamma\{x \in \underline{R}^{\infty}: \frac{x_k}{k} \to 0\} = 1.$$

Write $\underline{R}_0^{\infty}$ for the subspace of $\underline{R}^{\infty}$ consisting of those x such that $x_k = 0$

for all but a finite number of k.

24.22 LEMMA Let $B \in \text{Bor}(\underline{R}^{\infty})$. Assume: $\gamma(B) = 0$ -- then $\forall\, x_0 \in \underline{R}_0^{\infty}$, $\gamma(x_0 + B) = 0$.

A linear measurable functional (LMF) on $\underline{R}^{\infty}$ is a function

$$\lambda : E \to \underline{R}$$

whose domain E is a linear subspace of $\underline{R}^{\infty}$ of measure 1 such that λ is linear and measurable.

24.23 EXAMPLE Let $a = \{a_k : k \geq 1\}$ be a sequence of real numbers: $\sum_{k=1}^{\infty} a_k^2 < \infty$ -- then for almost every $x \in \underline{R}^{\infty}$, the series $\sum_{k=1}^{\infty} a_k x_k$ is convergent (cf. 24.20). Since this set is a linear subspace E_a of $\underline{R}^{\infty}$ of measure 1, the prescription

$$\lambda(x) = \sum_{k-1}^{\infty} a_k x_k \quad (x \in E_a)$$

defines a LMF on $\underline{R}^{\infty}$.

[Note: Observe that $\ell^2 \subset E_a$.]

24.24 REMARK Suppose that

$$\begin{bmatrix} \lambda_1 : E_1 \to \underline{R} \\ \\ \lambda_2 : E_2 \to \underline{R} \end{bmatrix}$$

are LMFs -- then the domain of $\lambda_1 + \lambda_2$ is $E_1 \cap E_2$, which is a set of measure 1. In fact,

$$\gamma(E_1 \cup E_2) + \gamma(E_1 \cap E_2) = \gamma(E_1) + \gamma(E_2)$$

$$= 2.$$

But

$$1 = \begin{bmatrix} \gamma(E_1) \\ \\ \gamma(E_2) \end{bmatrix} \leq \gamma(E_1 \cup E_2) \leq \gamma(\underline{R}^\infty) = 1$$

$$\Rightarrow$$

$$\gamma(E_1 \cap E_2) = 1.$$

Therefore $\lambda_1 + \lambda_2$ is a LMF.

24.25 <u>LEMMA</u> Let $\lambda:E \rightarrow \underline{R}$ be a LMF -- then $\underline{R}_0^\infty \subset E$.

<u>PROOF</u> Proceed by contradiction and assume that $\exists\ x_0 \in \underline{R}_0^\infty - E$. Put

$$E_t = tx_0 + E \quad (t > 0).$$

Then $\gamma(E_t) > 0$ (cf. 24.22). On the other hand, $t_1 \neq t_2 \Rightarrow E_{t_1} \cap E_{t_2} = \emptyset$. Accordingly, $\{E_t\}$ is an uncountable collection of pairwise disjoint sets of positive measure, an impossibility ($\gamma(\underline{R}^\infty) = 1 \ldots$).

[Note: This argument shows that any linear subspace of $\underline{R}^\infty$ of measure 1 necessarily contains $\underline{R}_0^\infty$.]

Let

$$e_k = (0,\ldots,0,1,0,\ldots),$$

where 1 is in the k^{th} position -- then $e_k \in \underline{R}_0^\infty$ and there is the evaluation

$$< e_k, x > = x_k.$$

24.26 LEMMA Let $\lambda: E \to \underline{R}$ be a LMF. Assume:

$$\lambda(e_k) = 0 \ \forall \ k.$$

Then $\lambda = 0$ a.e..

[Write

$$\begin{cases} E_{\geq 0} = E_{>0} \cup E_{=0} \\ E_{\leq 0} = E_{<0} \cup E_{=0}, \end{cases}$$

where

$$\begin{cases} E_{>0} = \{x \in E : \lambda(x) > 0\} \\ E_{<0} = \{x \in E : \lambda(x) < 0\} \end{cases}$$

and

$$E_{=0} = \{x \in E : \lambda(x) = 0\}.$$

Then

$$E_{\geq 0} = - E_{\leq 0}$$

=>

$$\gamma(E_{\geq 0}) = \gamma(E_{\leq 0}).$$

But

$$\chi_{E_{\geq 0}} \ \& \ \chi_{E_{\leq 0}}$$

satisfy condition K, thus

$$\begin{cases} 1 = \gamma(E_{\geq 0}) = \gamma(E_{>0}) + \gamma(E_{=0}) \\ 1 = \gamma(E_{\leq 0}) = \gamma(E_{<0}) + \gamma(E_{=0}). \end{cases}$$

And

$$1 = \gamma(E) = \gamma(E_{<0}) + \gamma(E_{>0}) + \gamma(E_{=0}).$$

Therefore

$$\begin{aligned} 1 = 2 - 1 &= \gamma(E_{>0}) + \gamma(E_{<0}) + 2\gamma(E_{=0}) \\ &\quad - \gamma(E_{<0}) - \gamma(E_{<0}) - \gamma(E_{=0}) \\ &= \gamma(E_{=0}) \end{aligned}$$

=>

$$\gamma\{x \in E:\lambda(x) = 0\} = 1.$$

24.27 <u>LEMMA</u> Let $\lambda:E \to \underline{R}$ be a LMF -- then

$$\sum_{k=1}^{\infty} |\lambda(e_k)|^2 < \infty.$$

<u>PROOF</u> Given $x \in E$, write

$$x = \sum_{k=1}^{n} x_k e_k + x_{(n)},$$

where

$$x_{(n)} = (0,\ldots,0,x_{n+1},x_{n+2},\ldots),$$

thus

$$\lambda(x) = \sum_{k=1}^{n} x_k\lambda(e_k) + \lambda_{(n)}(x)\,(\equiv \lambda(x_{(n)})).$$

Then $\forall\ a > 0$, we have

$$\int_{\underline{R}^\infty} \exp(\sqrt{-1}\ a\lambda(x))d\gamma(x)$$

$$= \int_{\underline{R}^\infty} \exp(\sqrt{-1}\ a \sum_{k=1}^{n} x_k\lambda(e_k))\exp(\sqrt{-1}\ a\lambda_{(n)}(x))d\gamma(x)$$

$$= \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \exp(\sqrt{-1}\ a\lambda(e_k)t - \frac{t^2}{2})dt \times \int_{\underline{R}^\infty} \exp(\sqrt{-1}\ a\lambda_{(n)}(x))d\gamma(x)$$

$$= \prod_{k=1}^{n} \exp(-\frac{a^2}{2}|\lambda(e_k)|^2) \times \int_{\underline{R}^\infty} \exp(\sqrt{-1}\ a\lambda_{(n)}(x))d\gamma(x)$$

=>

$$\left|\int_{\underline{R}^\infty} \exp(\sqrt{-1}\ a\lambda(x))d\gamma(x)\right|$$

$$\leq \exp(-\frac{a^2}{2} \sum_{k=1}^{n} |\lambda(e_k)|^2).$$

So:

$$\sum_{k=1}^{\infty} |\lambda(e_k)|^2 = \infty$$

=>

$$\int_{\underline{R}^\infty} \exp(\sqrt{-1}\ a\lambda(x))d\gamma(x) = 0$$

=>

$$0 = \lim_{n \to \infty} \int_{\underline{R}^\infty} \exp(\sqrt{-1}\,\frac{1}{n}\,\lambda(x))d\gamma(x)$$

$$= \int_{\underline{R}^\infty} \lim_{n \to \infty} \exp(\sqrt{-1}\,\frac{1}{n}\,\lambda(x))d\gamma(x)$$

$$= \int_{\underline{R}^\infty} 1d\gamma(x) = 1.$$

Contradiction.

Suppose that $\lambda:E \to \underline{R}$ is a LMF -- then 24.27, in conjunction with 24.20, implies that the series

$$\sum_{k=1}^{\infty} \lambda(e_k) < e_k,x >$$

converges a.e., thus defines a LMF Λ (cf. 24.23). Obviously,

$$\Lambda(e_k) = \lambda(e_k).$$

But the difference $\Lambda - \lambda$ is a LMF (cf. 24.24), hence $\Lambda = \lambda$ a.e. (cf. 24.26).

Given two LMFs λ_1 and λ_2, write $\lambda_1 \simeq \lambda_2$ if $\lambda_1 = \lambda_2$ a.e. -- then $\simeq$ is an equivalence relation, so the set of all LMFs is partitioned into equivalence classes $[\lambda]$.

N.B. Suppose that $\lambda_1 \simeq \lambda_2$ -- then

$$\gamma\{x:\lambda_1(x) = \lambda_2(x)\} = 1$$

=> (cf. 24.25)

$$\underline{R}_0^\infty \subset \{x:\lambda_1(x) = \lambda_2(x)\}$$

=>

$$\lambda_1(e_k) = \lambda_2(e_k) \ \forall \ k.$$

Denote by L^2 the set of all LMFs modulo $\simeq$.

24.28 LEMMA The map

$$\begin{cases} L^2 \to \ell^2 \\ [\lambda] \to \{\lambda(e_k):k \geq 1\} \end{cases}$$

is bijective.

PROOF Thanks to the preceding comment, our map is welldefined. That it is surjective is guaranteed by 24.23 and that it is injective is guaranteed by 24.26.

24.29 REMARK Two LMFs are either equal a.e. or not equal a.e..

§25. *RADON MEASURES*

We shall first agree that:

1. The term "measure" means a nonnegative finite countably additive set function whose domain is a σ-algebra.

2. The term "topological vector space" means an infinite dimensional real locally convex topological vector space which is Hausdorff.

If X is a topological vector space, then X* stands for its topological dual (the set of continuous linear functionals $\lambda:X \to \underline{R}$) and $X^\#$ stands for its algebraic dual (the set of linear functionals $\lambda:X \to \underline{R}$).

25.1 <u>EXAMPLE</u> Let $\underline{R}^T$ be the set of real valued functions on a nonempty set T. Equip $\underline{R}^T$ with the topology of pointwise convergence, i.e., with the topology generated by the seminorms

$$p_t(x) = |x(t)| \quad (t \in T).$$

Then $\underline{R}^T$ is a topological vector space. Its topological dual is spanned by the δ_t, where

$$\delta_t(x) = x(t) \quad (t \in T).$$

In particular: Take $T = \underline{N}$ -- then $\underline{R}^T = \underline{R}^\infty$ and the topological dual of $\underline{R}^\infty$ is $\underline{R}_0^\infty$.

Let X be a topological vector space -- then the <u>cylindrical σ-algebra</u> Cyl(X) is the σ-algebra generated by the sets of the form $\{x \in X:\lambda(x) < r\}$, where $\lambda \in X^*$ and $r \in \underline{R}$.

Obviously,

$$\text{Cyl}(X) \subset \text{Bor}(X),$$

the inclusion being strict in general.

25.2 <u>LEMMA</u> A set C belongs to Cyl(X) iff it has the form

$$C = \{x \in X: (\lambda_1(x),\ldots,\lambda_k(x),\ldots) \in B\},$$

where the $\lambda_k \in X^*$ and $B \in \text{Bor}(\underline{R}^\infty)$.

25.3 <u>EXAMPLE</u> Suppose that T is an uncountable set and let $X = \underline{R}^T$ — then $\forall\, x \in X$, $\{x\} \notin \text{Cyl}(X)$, hence in this situation, Cyl(X) is a proper subset of Bor(X).

25.4 <u>RAPPEL</u> X is a <u>separable LF-space</u> if it contains an increasing sequence of linear subspaces $X_n: X = \underset{n}{\cup} X_n$ subject to

(i) $\forall\, n$, X_n in the relative topology is a separable, metrizable, complete topological vector space, i.e., $\forall\, n$, X_n is a separable Fréchet space.

(ii) If U is a convex subset of X such that $\forall\, n$, $U \cap X_n$ is a neighborhood of 0 in X_n, then U is a neighborhood of 0 in X.

[Note: X is complete and admits a sequence $\{\lambda_k : k \geq 1\} \subset X^*$ that separates points.]

25.5 <u>LEMMA</u> If X is a separable LF-space, then

$$\text{Cyl}(X) = \text{Bor}(X).$$

Given a measure μ on $\mathrm{Cyl}(X)$, denote by $\mathrm{Cyl}(X)_\mu$ the completion of $\mathrm{Cyl}(X)$ w.r.t. μ.

[Note: Spelled out, $A \in \mathrm{Cyl}(X)_\mu$ iff $\exists\ C_1, C_2 \in \mathrm{Cyl}(X)$:

$$C_1 \subset A \subset C_2 \ \&\ \mu(C_2 - C_1) = 0.]$$

25.6 <u>REMARK</u> In general, $\mathrm{Cyl}(X)_\mu$ need not contain $\mathrm{Bor}(X)$. For example, let T be an uncountable set and take $X = \underline{R}^T$. Define μ on $\mathrm{Bor}(X)$ by

$$\begin{cases} \mu(B) = 1 \text{ if } 0 \in B \\ \mu(B) = 0 \text{ if } 0 \notin B. \end{cases}$$

Let $B = \underline{R}^T - \{0\}$ -- then $\mu(B) = 0$. But $\underline{R}^T - \{0\} \notin \mathrm{Cyl}(X)$ (since $\{0\} \notin \mathrm{Cyl}(X)$), thus the only element of $\mathrm{Cyl}(X)$ containing B is $\underline{R}^T$ and it has μ-measure 1.

25.7 <u>LEMMA</u> Let μ be a measure on $\mathrm{Cyl}(X)$. Suppose that $A \in \mathrm{Cyl}(X)$ -- then its convex hull and linear span belong to $\mathrm{Cyl}(X)_\mu$.

A Borel measure μ on X is said to be a <u>Radon measure</u> if $\forall\ B \in \mathrm{Bor}(X)$ and $\forall\ \varepsilon > 0$, $\exists$ a compact set $K \subset B: \mu(B-K) < \varepsilon$.

25.8 <u>REMARK</u> It is not necessarily true that every Borel measure on X is Radon but this will be the case if X is a separable LF-space.

25.9 <u>LEMMA</u> Let μ, ν be Radon measures on X. Assume:

$$\mu|\text{Cyl}(X) = \nu|\text{Cyl}(X).$$

Then $\mu = \nu$.

25.10 REMARK If Cyl(X) is a proper subset of Bor(X), then a measure on Cyl(X) need not admit a Radon extension to Bor(X). For a specific instance of this, take $X = \underline{R}^{[0,1]}$ and let $K \subset X$ be compact and nonempty -- then $K \not\in \text{Cyl}(X)$ (cf. 25.2). On the other hand, $K \subset X_1 \times X_2$, where X_1 is the product of countably many copies of $[-a,a]$ (some $a > 0$) and X_2 is the product of the real lines corresponding to the remaining coordinates ($\Rightarrow X_1 \times X_2 \in \text{Cyl}(X)$). Now let γ be the [0,1]-product of the standard gaussian measure on $\underline{R}$ and suppose that $\tilde{\gamma}$ is an extension of γ to a Radon measure on Bor(X) -- then $\forall$ $B \in \text{Bor}(X)$ & $\forall$ compact $K \subset B$,

$$\tilde{\gamma}(B-K) = \tilde{\gamma}(B) - \tilde{\gamma}(K).$$

But

$$\tilde{\gamma}(K) \leq \tilde{\gamma}(X_1 \times X_2) = \gamma(X_1 \times X_2) = 0,$$

meaning that $\tilde{\gamma}$ does not exist after all.

25.11 RAPPEL Let μ be a measure on Cyl(X) -- then the Fourier transform of μ is the function $\hat{\mu}:X^* \to \underline{C}$ defined by the rule

$$\hat{\mu}(\lambda) = \int_X e^{\sqrt{-1}\,\lambda(x)}\, d\mu(x).$$

[Note: $\hat{\mu}$ is sequentially continuous on X^* in the topology of pointwise convergence, i.e., if $\lambda_n \to \lambda$ pointwise, then $\hat{\mu}(\lambda_n) \to \hat{\mu}(\lambda)$ (dominated convergence).

Nevertheless, it is false in general that $\hat{\mu}$ is continuous on X* in the topology of pointwise convergence (a.k.a. the weak topology).]

25.12 UNIQUENESS PRINCIPLE If μ,ν are measures on Cyl(X) and if $\hat{\mu} = \hat{\nu}$, then $\mu = \nu$.

[Note: Suppose that μ,ν are Radon and

$$\hat{\mu}|\text{Cyl}(X) = \hat{\nu}|\text{Cyl}(X).$$

Then

$$\mu|\text{Cyl}(X) = \nu|\text{Cyl}(X)$$

$$\Rightarrow$$

$$\mu = \nu.]$$

25.13 LEMMA Let μ be a Radon measure on X -- then the linear span of functions of the form $e^{\sqrt{-1}\,\lambda}$ $(\lambda \in X^*)$ is dense in $L^p(X,\mu)$ $(1 \leq p < \infty)$.

If X and Y are topological vector spaces, then

$$\text{Bor}(X) \times \text{Bor}(Y) \subset \text{Bor}(X \times Y),$$

the inclusion being strict in general.

Suppose that

$$\begin{cases} \mu \text{ is a Borel measure on } X \\ \nu \text{ is a Borel measure on } Y. \end{cases}$$

Then $\mu \times \nu$ is defined on $\text{Bor}(X) \times \text{Bor}(Y)$.

25.14 <u>LEMMA</u> If μ,ν are Radon, then $\mu \times \nu$ admits a unique extension $\overline{\mu \times \nu}$ to a Radon measure on $\text{Bor}(X \times Y)$.

Take $X = Y$ and assume that μ,ν are Radon -- then the image of $\overline{\mu \times \nu}$ under the map

$$\begin{cases} X \times X \to X \\ (x,y) \to x + y \end{cases}$$

is called the <u>convolution</u> of μ,ν, written $\mu*\nu$.

25.15 <u>LEMMA</u> The convolution $\mu*\nu$ is a Radon measure on X.

<u>N.B.</u> $\forall\ B \in \text{Bor}(X)$,

$$(\mu*\nu)(B) = \int_X \mu(B-x)\,d\nu(x).$$

25.16 <u>REMARK</u> Suppose that

$$\begin{cases} \mu \text{ is a measure on } \text{Cyl}(X) \\ \nu \text{ is a measure on } \text{Cyl}(Y). \end{cases}$$

Then

$$(X \times Y)^* = X^* \times Y^*$$

=>

$$\text{Cyl}(X) \times \text{Cyl}(Y) = \text{Cyl}(X \times Y).$$

Therefore $\mu \times \nu$ is defined on $\text{Cyl}(X \times Y)$. Now take $X = Y$ and define $\mu*\nu$ to be the image of $\mu \times \nu$ under the map

$$\begin{cases} X \times X \to X \\ (x,y) \to x + y. \end{cases}$$

Then

$$\widehat{(\mu * \nu)}(\lambda) = \int_X e^{\sqrt{-1}\,\lambda(x)}\, d(\mu*\nu)(x)$$

$$= \int_{X \times X} e^{\sqrt{-1}\,\lambda(x+y)}\, d\mu(x) d\nu(y)$$

$$= \int_X e^{\sqrt{-1}\,\lambda(x)}\, d\mu(x) \int_X e^{\sqrt{-1}\,\lambda(y)}\, d\nu(y)$$

$$= \hat{\mu}(\lambda)\hat{\nu}(\lambda)$$

=>

$$\widehat{\mu * \nu} = \hat{\mu}\hat{\nu}.$$

25.17 <u>LEMMA</u> If X and Y are separable LF-spaces, then so is $X \times Y$.

Accordingly, under these circumstances (cf. 25.5),

$$\text{Bor}(X \times Y) = \text{Cyl}(X \times Y)$$

$$= \text{Cyl}(X) \times \text{Cyl}(Y)$$

$$= \text{Bor}(X) \times \text{Bor}(Y).$$

Let T be a Hausdorff topological space -- then T is <u>lusinien</u> if $\exists$ a complete separable metric space P and a continuous bijection $f:P \to T$.

25.18 EXAMPLE Every separable LF-space is lusinien but the Banach space ℓ^∞ is not lusinien.

If X is lusinien, then every Borel measure μ on X is Radon. In fact, $\forall$ B $\in$ Bor(X) and $\forall\ \varepsilon > 0$, $\exists$ a metrizable compact set $K \subset B:\mu(B-K) < \varepsilon$.

25.19 LEMMA If X and Y are lusinien and if $f:X \to Y$ is a continuous injection, then

$$B \in \text{Bor}(X) \Rightarrow f(B) \in \text{Bor}(Y).$$

25.20 LEMMA If X and Y are lusinien and if $f:X \to Y$ is sequentially continuous, then f is Borel.

25.21 EXAMPLE Let X be a separable LF-space. Equip X* with the weak topology -- then X* is lusinien. If now μ is Radon, then

$$\hat{\mu}:X^* \to \underline{C}$$

is sequentially continuous (cf. 25.11), hence is Borel (cf. 25.20).

§26. *INFINITE DIMENSIONAL GAUSSIANS*

Let X be a topological vector space (dim X = ∞), X* its topological dual. Let γ be a probability measure on Cyl(X) -- then γ is said to be gaussian if for every $\lambda \in X^*$, the induced measure $\gamma \circ \lambda^{-1}$ ($\equiv \gamma_\lambda$) on $\underline{R}$ is gaussian.

26.1 EXAMPLE Take $X = \underline{R}^\infty$ -- then X is a separable Fréchet space, hence Cyl(X) = Bor(X) (cf. 25.5) and $X^* = \underline{R}_0^\infty$ (cf. 25.1). Suppose that γ is the countable product of the standard gaussian measure on $\underline{R}$ (cf. §24) -- then γ is gaussian. Thus given $\lambda \in X^*$, write

$$\lambda = \sum_{k=1}^{n} r_k \delta_k \quad (\delta_k(x) = x_k).$$

Then

$$\hat{\gamma}_\lambda(t) = \int_{\underline{R}} e^{\sqrt{-1}\, ts}\, d\gamma_\lambda(s)$$

$$= \int_{\underline{R}^\infty} e^{\sqrt{-1}\, t\lambda(x)}\, d\gamma(x)$$

$$= \int_{\underline{R}^n} \exp(\sqrt{-1}\, t \sum_{k=1}^{n} r_k x_k) \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_k^2/2}\, dx_k$$

$$= \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \exp(\sqrt{-1}\, t r_k x_k) e^{-x_k^2/2}\, dx_k$$

$$= \prod_{k=1}^{n} \exp(-\frac{1}{2} t^2 r_k^2)$$

$$= \exp(-\frac{1}{2}(\sum_{k=1}^{n} r_k^2)t^2).$$

Therefore γ_λ is the centered gaussian measure on $\underline{R}$ with variance $\sigma^2 = \sum_{k=1}^{n} r_k^2$ (cf. 22.2).

[Note: In the sequel, we shall refer to γ as the <u>standard</u> gaussian measure on $\underline{R}^\infty$.]

26.2 <u>LEMMA</u> Suppose that γ is a gaussian measure on X -- then

$$\lambda \in X^* \Rightarrow \lambda \in L^2(X,\gamma),$$

thus

$$\lambda \in X^* \Rightarrow \lambda \in L^1(X,\gamma).$$

<u>PROOF</u> In fact,

$$\int_X \lambda(x)^2 \, d\gamma(x) = \int_{\underline{R}} t^2 \, d\gamma_\lambda(t) < \infty.$$

26.3 <u>THEOREM</u> Let γ be a probability measure on Cyl(X) -- then γ is gaussian if its Fourier transform has the form

$$\hat{\gamma}(\lambda) = \exp(\sqrt{-1}\, L(\lambda) - \frac{1}{2} Q(\lambda,\lambda)),$$

where L is a linear function on X* and Q is a symmetric bilinear function on X* such that $\forall\ \lambda$, $Q(\lambda,\lambda) \geq 0$.

<u>PROOF</u> If $\hat{\gamma}$ has the stated form then $\forall\ t \in \underline{R}$

$$\hat{\gamma}_\lambda(t) = \int_{\underline{R}} e^{\sqrt{-1}\, ts} \, d\gamma_\lambda(s)$$

$$= \int_X e^{\sqrt{-1}\, t\lambda(x)}\, d\gamma(x)$$

$$= \hat{\gamma}(t\lambda)$$

$$= \exp(\sqrt{-1}\, tL(\lambda) - \frac{1}{2} t^2 Q(\lambda,\lambda)),$$

from which it follows that γ is gaussian (cf. 22.2). The converse is also immediate: Thus, taking into account 26.2, put

$$L(\lambda) = \int_X \lambda(x) d\gamma(x)$$

and

$$Q(\lambda,\lambda') = \int_X (\lambda(x) - L(\lambda))(\lambda'(x) - L(\lambda'))d\gamma(x).$$

One calls L the <u>mean</u> and Q the <u>covariance</u> of γ.

A gaussian measure γ on X is <u>centered</u> provided this is the case of the $\gamma \circ \lambda^{-1}$. Since the Fourier transform of the measure $C \to \gamma(-C)$ $(C \in \text{Cyl}(X))$ is $\bar{\hat{\gamma}}$, it follows that γ is centered iff $\gamma(C) = \gamma(-C)$ $\forall\, C \in \text{Cyl}(X)$ or still, iff $L = 0$.

26.4 <u>EXAMPLE</u> Take $X = \underline{R}^{\infty}$ and let γ be the standard gaussian measure on X (cf. 26.1) -- then γ is centered and here

$$Q(\lambda,\lambda') = \sum_{k=1}^{\infty} r_k r_k' \quad (\lambda,\lambda' \in \underline{R}_0^{\infty}).$$

Given a gaussian measure γ on X and an element $h \in X$, let γ_h be the image of γ under the map $x \to x + h$.

[Note:

$$C \in \text{Cyl}(X) \Rightarrow C + h \in \text{Cyl}(X) \quad (\text{cf. } 25.2).]$$

26.5 <u>LEMMA</u> $\forall\, h \in X$, γ_h is gaussian.

<u>PROOF</u> Bearing in mind 26.3, one has only to observe that

$$\hat{\gamma}_h(\lambda) = \int_X e^{\sqrt{-1}\,\lambda(x)}\, d\gamma_h(x)$$

$$= \int_X e^{\sqrt{-1}\,\lambda(x+h)}\, d\gamma(x)$$

$$= e^{\sqrt{-1}\,\lambda(h)}\, \hat{\gamma}(\lambda).$$

26.6 <u>LEMMA</u> If

$$\left[\begin{array}{l} \gamma_1 \text{ is a gaussian measure on } X_1 \\ \\ \gamma_2 \text{ is a gaussian measure on } X_2, \end{array}\right.$$

then $\gamma_1 \times \gamma_2$ is a gaussian measure on $X_1 \times X_2$.

<u>PROOF</u> The conventions are those of 25.16:

$$\gamma_1 \hat{\times} \gamma_2\ (\lambda_1, \lambda_2) = \hat{\gamma}_1(\lambda_1)\hat{\gamma}_2(\lambda_2),$$

so 26.3 is applicable.

[Note: Take $X_1 = X_2 = X$ and conclude that $\gamma_1 * \gamma_2$ is gaussian as well.]

26.7 EXAMPLE The symmetrization γ_s of a gaussian measure γ is the convolution:

$$\gamma_s(C) = (\gamma_1 * \gamma_2)(\sqrt{2}\, C),$$

where

$$\begin{cases} \gamma_1(C) = \gamma(C) \\ \gamma_2(C) = \gamma(-C) \end{cases} \qquad (C \in \text{Cyl}(X)).$$

Thus γ_s is the image of $\gamma_1 * \gamma_2$ under the map $x \to x/\sqrt{2}$ and we have

$$\hat{\gamma}_s(\lambda) = (\gamma_1 \hat{*} \gamma_2)(\lambda/\sqrt{2})$$

$$= \hat{\gamma}_1(\lambda/\sqrt{2})\hat{\gamma}_2(\lambda/\sqrt{2})$$

$$= \hat{\gamma}(\lambda/\sqrt{2})\overline{\hat{\gamma}}(\lambda/\sqrt{2})$$

$$= \exp(\sqrt{-1}\, L(\lambda/\sqrt{2}) - \frac{1}{2} Q(\lambda/\sqrt{2},\lambda/\sqrt{2}))\exp(-\sqrt{-1}\, L(\lambda/2) - \frac{1}{2} Q(\lambda/\sqrt{2},\lambda/\sqrt{2}))$$

$$= \exp(-\frac{1}{4} Q(\lambda,\lambda))\exp(-\frac{1}{4} Q(\lambda,\lambda))$$

$$= \exp(-\frac{1}{2} Q(\lambda,\lambda)) = |\hat{\gamma}(\lambda)|.$$

To simplify the exposition, we shall assume henceforth that X is a separable LF-space (cf. 25.4), hence Cyl(X) = Bor(X) (cf. 25.5) and every Borel measure on X is Radon (cf. 25.8) (in particular, every gaussian measure on X is Radon).

26.8 LEMMA Let μ be a Borel measure on X — then $L^2(X,\mu)$ is separable.

[For $\exists$ a sequence of Borel functions that separates the points of X (cf. 25.4), hence Bor(X) is countably generated.]

Given a centered gaussian measure γ on X, write X^*_γ for the closure of the set

$$X^* \subset L^2(X,\gamma) \quad (\text{cf. } 26.2).$$

Then X^*_γ is a separable real Hilbert space and has an orthonormal basis consisting of continuous linear functionals $\lambda_k \in X^*$ $(k \geq 1)$.

26.9 LEMMA $\forall\ f \in X^*_\gamma$, $\gamma \circ f^{-1}$ $(\equiv \gamma_f)$ is a centered gaussian measure on $\underline{R}$ with variance

$$\sigma(f)^2 = ||f||^2_{L^2(\gamma)}.$$

PROOF Fix $f \in X^*_\gamma$ and choose a sequence $\{\lambda_k : k \geq 1\} \subset X^*$ such that $\lambda_k \to f$ in $L^2(X,\gamma)$ -- then $\lambda_k \to f$ in $L^1(X,\gamma)$, thus $\lambda_k \to f$ in measure and so, thanks to a wellknown lemma in probability theory (see below), $\gamma_{\lambda_k} \to \gamma_f$ weakly. Therefore $\hat{\gamma}_{\lambda_k} \to \hat{\gamma}_f$ pointwise, i.e.,

$$\hat{\gamma}_{\lambda_k}(t) = \int_{\underline{R}} e^{\sqrt{-1}\, ts}\, d\gamma_{\lambda_k}(s)$$

$$\rightarrow \int_{\underline{R}} e^{\sqrt{-1}\, ts}\, d\gamma_f(s) = \hat{\gamma}_f(t).$$

But

$$\hat{\gamma}_{\lambda_k}(t) = \exp(-\frac{1}{2} t^2 ||\lambda_k||^2_{L^2(\gamma)})$$

and this has limit

$$\exp(-\frac{1}{2} t^2 ||f||^2_{L^2(\gamma)}).$$

[Note: It is a corollary that

$$f \in X^*_\gamma \Rightarrow e^{|f|} \in L^1(X,\gamma).]$$

<u>N.B.</u> Let $\{\xi_k : k \geq 1\}$ be a sequence of random variables on a probability space (Ω, A, μ). Assume: $\xi_k \rightarrow \xi$ "in probability" (i.e., in measure) — then $\xi_k \rightarrow \xi$ "in distribution" (or "in law"), which is equivalent to saying that $P_{\xi_k} \rightarrow P_\xi$ weakly (here, $P_{\xi_k} = \mu \circ \xi_k^{-1}$, $P_\xi = \mu \circ \xi^{-1}$).

26.10 <u>REMARK</u> For the most part, the elements of X^*_γ can be treated as though they were functions rather than equivalence classes of functions but there are occasions when this distinction has to be taken into account.

E.g.: Every $f \in X_\gamma^*$ admits a <u>linear model</u> f_0. Thus choose a sequence $\{\lambda_k : k \geq 1\} \subset X^*$ such that $\lambda_k \to f$ a.e.. The set $\{x : \lambda_k(x) \to f(x)\}$ is certainly Borel but it is not a priori clear that it is linear. To remedy this, let E_0 be the set of $x : \{\lambda_k(x)\}$ is convergent -- then E_0 is Borel, linear, and $\gamma(E_0) = 1$. Define f_0 as follows:

$$\begin{cases} f_0(x) = \lim \lambda_k(x) & (x \in E_0) \\ f_0(x) = 0 & (x \in X - E_0). \end{cases}$$

Then f_0 is Borel. Moreover, $f_0|E_0$ is linear and $f_0 = f$ a.e..

26.11 <u>RAPPEL</u> The Mackey topology on X^* is the topology of uniform convergence on the weakly compact convex balanced subsets of X. Every linear functional $\Lambda : X^* \to \underline{R}$ which is continuous in the Mackey topology is representable, i.e., $\exists\, x_\Lambda \in X$:

$$\forall\, \lambda \in X^*,\ \Lambda(\lambda) = \lambda(x_\Lambda).$$

[Note: Let X_M^* stand for X^* equipped with the Mackey topology -- then the canonical arrow

$$X \to (X_M^*)^*$$

is bijective.]

Suppose that X is a separable LF-space. Given a centered gaussian measure

γ on X, define

$$R_\gamma : X^*_\gamma \to \text{Hom}(X^*, \underline{R})$$

by

$$R_\gamma(f)(\lambda) = \int_X f(x)\lambda(x)d\gamma(x).$$

26.12 LEMMA $\forall f \in X^*_\gamma$, the linear functional $R_\gamma(f):X^* \to \underline{R}$ is continuous in the Mackey topology, hence is representable, so $\exists\ x_f \in X$:

$$\forall \lambda \in X^*, R_\gamma(f)(\lambda) = \lambda(x_f).$$

PROOF Fix $\varepsilon > 0$ (& $\varepsilon < 1$). Choose $n \in \underline{N}$: $-\log(1 - \frac{1}{n}) < \varepsilon^2/2$ and choose $\delta > 0 : 3\delta < \frac{1}{n}$. Fix a compact set $K \subset X : \gamma(K) > 1 - \delta$ and let $< K >$ be the closed convex balanced hull of K -- then $< K >$ is compact (X being complete (cf. 25.4)) (hence a fortiori, is weakly compact) and $\forall \lambda \in X^*$,

$$|1 - \hat{\gamma}(\lambda)| = |\int_X (1 - e^{\sqrt{-1}\ \lambda(x)})d\gamma(x)|$$

$$\leq \int_{< K >} |1 - e^{\sqrt{-1}\ \lambda(x)}|d\gamma(x) + 2\delta.$$

Since

$$|1 - e^{\sqrt{-1}\ \lambda(x)}| = 2|\sin(\lambda(x)/2)|$$

$$\leq 2|\lambda(x)/2| = |\lambda(x)|,$$

it follows that

$$\sup_{< K >} |\lambda| \leq \delta \Rightarrow |1 - \hat{\gamma}(\lambda)| \leq 3\delta < \frac{1}{n}$$

$$\Rightarrow |1 - \exp(-\frac{1}{2} Q(\lambda,\lambda))| < \frac{1}{n}$$

$$\Rightarrow \exp(-\frac{1}{2} Q(\lambda,\lambda)) > 1 - \frac{1}{n}$$

$$\Rightarrow \frac{Q(\lambda,\lambda)}{2} < -\log(1 - \frac{1}{n}) < \frac{\varepsilon^2}{2}$$

$$\Rightarrow Q(\lambda,\lambda) < \varepsilon^2$$

$$\Rightarrow ||\lambda||^2_{L^2(\gamma)} < \varepsilon^2$$

$$\Rightarrow |R_\gamma(f)(\lambda)| \leq ||f||_{L^2(\gamma)} ||\lambda||_{L^2(\gamma)}$$

$$\leq ||f||_{L^2(\gamma)} \varepsilon,$$

from which the lemma.

[Note: Take $f \neq 0$ and let $\{\lambda_i : i \in I\}$ be a net in X^* such that $\lim \lambda_i = 0$. Given $\varepsilon > 0$, choose n and δ as above — then

$$\exists\ i_0 = i_0(\varepsilon) : i \geq i_0$$

$$\Rightarrow \sup_{< K >} |\lambda_i| \leq \delta \Rightarrow |R_\gamma(f)(\lambda_i)| \leq ||f||_{L^2(\gamma)} \varepsilon.]$$

26.13 <u>REMARK</u> If γ is not centered, then its mean

$$L \in \text{Hom}(X^*, \underline{R})$$

is representable:

$$L(\lambda) = \lambda(a_\gamma) \quad (\exists\ a_\gamma \in X).$$

[Note: The symmetrization γ_s of γ is centered (cf. 26.7) and $\gamma = (\gamma_s)_{a_\gamma}$.
In fact,

$$(\hat{\gamma}_s)_{a_\gamma}(\lambda) = e^{\sqrt{-1}\ \lambda(a_\gamma)}\hat{\gamma}_s(\lambda) \quad \text{(cf. 26.5)}$$

$$= \exp(\sqrt{-1}\ \lambda(a_\gamma))\exp(-\frac{1}{2}Q(\lambda,\lambda))$$

$$= \exp(\sqrt{-1}\ L(\lambda) - \frac{1}{2}Q(\lambda,\lambda))$$

$$= \hat{\gamma}(\lambda).$$

Because of this, the bottom line is that for most purposes, it suffices to consider centered gaussian measures and their translates.]

Suppose that X is a separable LF-space. Given a centered gaussian measure γ on X, put $H(\gamma) = R_\gamma(X_\gamma^*)$ -- then $H(\gamma)$ is called the Cameron-Martin space of γ.

26.14 EXAMPLE Take $X = \underline{R}^\infty$ and let γ be the standard gaussian measure on X (cf. 26.1) -- then the elements $f \in X_\gamma^*$ are of the form

$$f(x) = \sum_{k=1}^{\infty} a_k x_k,$$

where $\sum_{k=1}^{\infty} a_k^2 < \infty$ (cf. 24.20). And $\forall\ \lambda \in X^*$ $(= \underline{R}_0^\infty)$,

$$R_\gamma(f)(\lambda) = \int_{\underline{R}^\infty} f(x)\lambda(x)d\gamma(x)$$

$$= \sum_{k=1}^{\infty} a_k r_k.$$

Therefore $R_\gamma(f)$ is represented by $a_f = \{a_k : k \geq 1\}$ and $H(\gamma) = \ell^2$.

The prescription

$$< x_f, x_g >_{H(\gamma)} = \int_X f(x)g(x)d\gamma(x)$$

equips $H(\gamma)$ with the structure of a separable real Hilbert space. Its closed unit ball $B_{H(\gamma)}$ is compact in X and $\forall\ \lambda \in X^*$,

$$Q(\lambda,\lambda) = \int_X \lambda(x)^2 d\gamma(x) = \sup_{h \in B_{H(\gamma)}} \lambda(h)^2.$$

[Note: By construction, the arrow

$$R_\gamma : X_\gamma^* \to H(\gamma)$$

is an isometric isomorphism.]

26.15 LEMMA Let γ_1, γ_2 be centered gaussian measures on X. Assume: $H(\gamma_1) = H(\gamma_2)$ and $||\cdot||_{H(\gamma_1)} = ||\cdot||_{H(\gamma_2)}$ — then $\gamma_1 = \gamma_2$.

PROOF $\forall\ \lambda \in X^*$,

$$\begin{cases} Q_1(\lambda,\lambda) = \sup\limits_{h \in B_{H(\gamma_1)}} \lambda(h)^2 \\ Q_2(\lambda,\lambda) = \sup\limits_{h \in B_{H(\gamma_2)}} \lambda(h)^2 \end{cases}$$

=>

$$Q_1(\lambda,\lambda) = Q_2(\lambda,\lambda)$$

=>

$$\hat{\gamma}_1 = \hat{\gamma}_2 \Rightarrow \gamma_1 = \gamma_2.$$

Maintaining the assumption that γ is centered, suppose that $h \in H(\gamma):h = R_\gamma(f)$ $(f \in X_\gamma^*)$ -- then $\gamma_h << \gamma$ and

$$\frac{d\gamma_h}{d\gamma}(x) = \exp(f(x) - \frac{1}{2}||h||_{H(\gamma)}^2)$$

or still,

$$\frac{d\gamma_h}{d\gamma}(x) = \exp(f(x) - \frac{1}{2}||f||_{L^2(\gamma)}^2).$$

To see this, let ρ_h be the density on the right hand side. Consider $\mu = \rho_h\gamma$, a Borel measure on X with Fourier transform

$$\hat{\mu}(\lambda) = \exp(\sqrt{-1}\, R_\gamma(f)(\lambda) - \frac{1}{2} Q(\lambda,\lambda))$$

$$= \exp(\sqrt{-1}\, \lambda(h) - \frac{1}{2} Q(\lambda,\lambda))$$

$$= \hat{\gamma}_h(\lambda).$$

26.16 <u>EXAMPLE</u> Let $\phi \in L^p(X,\gamma)$ $(p > 1)$ -- then the function $\Phi:H(\gamma) \to \underline{R}$

defined by

$$\Phi(h) = \int_X \phi(x+h)d\gamma(x)$$

is continuous.

[We have

$$\int_X \phi(x+h)d\gamma(x) = \int_X \phi(x)d\gamma_h(x)$$

$$= \int_X \phi(x) \frac{d\gamma_h}{d\gamma}(x)d\gamma(x)$$

$$= \int_X \phi(x)\exp(f(x) - \frac{1}{2}||h||^2_{H(\gamma)})d\gamma(x).$$

Determine $q > 1$ by $\frac{1}{p} + \frac{1}{q} = 1$ -- then the function

$$h \rightarrow \exp(f - \frac{1}{2}||h||^2_{H(\gamma)})$$

from $H(\gamma)$ to $L^q(X,\gamma)$ is continuous on bounded open sets and this implies the continuity of Φ.]

26.17 <u>EXAMPLE</u> Let $1 < p < r$ and suppose that $\phi \in L^r(X,\gamma)$ -- then $\forall$ $h \in H(\gamma)$, $\phi(\cdot+h) \in L^p(X,r)$.

[Choose $t,s > 1: tp = r$ & $t^{-1} + s^{-1} = 1$. Determine $f \in X^*_\gamma: R_\gamma(f) = h$. An application of Hölder's inequality then gives

$$\int_X |\phi(x+h)|^p d\gamma(x)$$

$$= \int_X |\phi(x+h)|^{r/t} d\gamma(x)$$

$$= \int_X |\phi(x)|^{r/t} \exp(f(x) - \frac{1}{2}||f||^2_{L^2(\gamma)}) d\gamma(x)$$

$$\leq (\int_X |\phi(x)|^r \, d\gamma(x))^{1/t} (\int_X \exp(sf(x) - \frac{s}{2}||f||^2_{L^2(\gamma)}) d\gamma(x))^{1/s}$$

$$= (\int_X |\phi(x)|^r \, d\gamma(x))^{1/t} \exp(\frac{s-1}{2}||f||^2_{L^2(\gamma)}),$$

which is finite.]

[Note: Thanks to 26.9,

$$\int_X e^{sf(x)} \, d\gamma(x) = \int_{\underline{R}} e^{sy} \, d(\gamma \circ f^{-1})(y)$$

$$= \frac{1}{||f||_{L^2(\gamma)} \sqrt{2\pi}} \int_{\underline{R}} e^{sy} \exp(- \frac{y^2}{2||f||^2_{L^2(\gamma)}}) dy$$

$$= \exp(\frac{s^2}{2}||f||^2_{L^2(\gamma)}) \quad \text{(cf. 24.6)}.$$

Therefore

$$(\int_X \exp(sf(x) - \frac{s}{2}||f||^2_{L^2(\gamma)}) d\gamma(x))^{1/s}$$

$$= (\int_X e^{sf(x)} \, d\gamma(x))^{1/s} \, (\exp(- \frac{s}{2}||f||^2_{L^2(\gamma)}))^{1/s}$$

$$= \exp(\frac{s-1}{2}||f||^2_{L^2(\gamma)}).]$$

26.18 <u>REMARK</u> The function

$$\left[\begin{array}{l} H(\gamma) \to L^p(X,\gamma) \\ \quad h \to \phi(\cdot + h) \end{array}\right.$$

is continuous.

26.19 LEMMA Let γ be a centered gaussian measure on X -- then

$$H(\gamma) = \{h \in X : \gamma_h \sim \gamma\}.$$

What we know so far is that $H(\gamma)$ is contained in $\{h \in X : \gamma_h \sim \gamma\}$, thus it remains to be shown that

$$\gamma_h \sim \gamma \Rightarrow h \in H(\gamma),$$

a fact whose proof depends on some auxilliary considerations.

26.20 REDUCTION PRINCIPLE Let γ be a centered gaussian measure on X. Fix an orthonormal basis $\{\lambda_k : k \geq 1\}$ for X_γ^* consisting of continuous linear functionals which separate the points of X. Define $T : X \to \underline{R}^\infty$ by

$$Tx = \{\lambda_k(x) : k \geq 1\}.$$

Then the induced measure $\gamma \circ T^{-1}$ on $\underline{R}^\infty$ is the standard gaussian measure on $\underline{R}^\infty$ (cf. 26.1). Indeed, $\forall\ \lambda \in \underline{R}_0^\infty$,

$$(\gamma \hat{\circ} T^{-1})(\lambda) = \int_{\underline{R}^\infty} e^{\sqrt{-1}\,\lambda(x)}\, d(\gamma \circ T^{-1})(x)$$

$$= \int_X e^{\sqrt{-1}\ \lambda(Tx)}\ d\gamma(x)$$

$$= \exp(-\tfrac{1}{2}\ Q(\lambda \circ T, \lambda \circ T))$$

$$= \exp(-\tfrac{1}{2}\ Q(\sum_{k=1}^{n} r_k\lambda_k, \sum_{\ell=1}^{n} r_\ell\lambda_\ell))$$

$$= \exp(-\tfrac{1}{2} \sum_{k,\ell=1}^{n} r_k r_\ell Q(\lambda_k,\lambda_\ell))$$

$$= \exp(-\tfrac{1}{2} \sum_{k=1}^{n} r_k^2)$$

$$= \hat{\gamma}(\lambda)$$

=>

$$\gamma \circ T^{-1} = \gamma$$

in the obvious abuse of notation... .

N.B. To establish the existence of the λ_k, fix a sequence $\{\lambda_k':k \geq 1\} \subset X^*$ that separates the points of X and fix a sequence $\{\lambda_k'':k \geq 1\} \subset X^*$ which is dense in X_γ^*. Consider $\lambda_1,\lambda_1',\lambda_2,\lambda_2',\ldots$. Proceed recursively and throw out any element in the span of its predecessors. Apply Gran-Schmidt to what remains -- then the result is an orthonormal basis $\{\lambda_k:k \geq 1\}$ for X_γ^* consisting of continuous linear functionals which separate the points of X.

Given $h \in X$, denote by A_h the map $x \to x + h$ -- then by definition, $\gamma_h = \gamma \circ A_h^{-1}$.

26.21 LEMMA $\forall\ h \in X$,

$$\gamma_h \circ T^{-1} = (\gamma \circ T^{-1})_{Th}.$$

PROOF $\forall\ B \in \text{Bor}(\underline{R})$,

$$(\gamma_h \circ T^{-1})(B) = (\gamma \circ A_h^{-1} \circ T^{-1})(B)$$

$$= \gamma(T^{-1}(B) - h).$$

On the other hand,

$$(\gamma \circ T^{-1})_{Th}(B) = (\gamma \circ T^{-1})(B - Th)$$

$$= \gamma(T^{-1}(B) - T^{-1}Th)$$

$$= \gamma(T^{-1}(B) - h),$$

T being one-to-one.

26.22 LEMMA The image under T of $H(\gamma)$ is ℓ^2.

PROOF Let $f \in X_\gamma^*$ -- then $Tx_f = \{\lambda_k(x_f) : k \geq 1\}$. But

$$\lambda_k(x_f) = \int_X f(x)\lambda_k(x)d\gamma(x).$$

And

$$f = \sum_{k=1}^{\infty} < f, \lambda_k > \lambda_k$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} |< f,\lambda_k >|^2 < \infty$$

$$\Rightarrow \quad TH(\gamma) \subset \ell^2.$$

To go the other way, let $\{a_k : k \geq 1\} \in \ell^2$ and define $f \in X_\gamma^*$ by

$$f = \sum_{k=1}^{\infty} a_k \lambda_k.$$

Then $x_f \in H(\gamma)$ and

$$Tx_f = \{\lambda_k(x_f) : k \geq 1\}$$

$$= \{< f,\lambda_k > : k \geq 1\}$$

$$= \{a_k : k \geq 1\}.$$

[Note: It follows from this that if $Tx \in \ell^2$, then $x \in H(\gamma)$. For $\exists\, h \in H(\gamma)$: $Th = Tx \Rightarrow h = x$.]

To complete the proof of 26.19, suppose that $\gamma_h \sim \gamma$ -- then

$$\gamma_h \circ T^{-1} \sim \gamma \circ T^{-1}$$

or still,

$$(\gamma \circ T^{-1})_{Th} \sim \gamma \circ T^{-1}.$$

But, as will be shown below, $Th \in \ell^2$, hence $h \in H(\gamma)$, as desired.

Thus take $X = \underline{R}^{\infty}$ and let γ be the standard gaussian measure on X (cf. 26.1) -- then $\forall\ h \in \underline{R}^{\infty}$,

$$\gamma_h = \prod_{k=1}^{\infty} \gamma_{h,k},$$

where

$$d\gamma_{h,k} = f_{h,k}(x_k)\,dx_k$$

and

$$f_{h,k}(x_k) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(x_k - h_k)^2).$$

So, $\forall\ h',h'' \in \underline{R}^{\infty}$,

$$\prod_{k=1}^{\infty} \int_{\underline{R}} \sqrt{f_{h',k}}\ \sqrt{f_{h'',k}}\ dx_k$$

$$= \prod_{k=1}^{\infty} \exp(-\frac{1}{8}(h_k' - h_k'')^2)$$

which is convergent iff

$$h' - h'' \in \ell^2.$$

Consequently (cf. 24.4),

$$\gamma_{h'} \sim \gamma_{h''} \iff h' - h'' \in \ell^2.$$

In particular:

$$\gamma_h \sim \gamma \iff h \in \ell^2.$$

[Note: If $h \notin \ell^2$, then $\gamma_h \not\sim \gamma$, hence $\gamma_h \perp \gamma$ (cf. 24.3).]

26.23 <u>REMARK</u> If $h \notin H(\gamma)$, then $\gamma_h \not\approx \gamma$ but more is true: $\gamma_h \perp \gamma$ (as was noted above in the case when $X = \underline{R}^\infty$). To see this, fix a Lebesgue decomposition of γ_h w.r.t. γ:

$$\gamma_h = \rho + \sigma \quad (\rho << \gamma, \ \sigma \perp \gamma).$$

Then the claim is that $\rho = 0$.

- $\forall \lambda \in X^*$,

$$\int_{\underline{R}} \hat{\gamma}_h(t\lambda) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

$$= \int_X \left(\int_{\underline{R}} \exp(\sqrt{-1}\, t\lambda(x)) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt\right) d\gamma_h(x)$$

$$= \int_X \exp\left(-\frac{1}{2}\lambda(x)^2\right) d\gamma_h(x) \quad \text{(cf. 24.6)}$$

$$\geq \int_X \exp\left(-\frac{1}{2}\lambda(x)^2\right) d\rho(x).$$

- $\forall \lambda \in X^*$,

$$\int_{\underline{R}} \hat{\gamma}_h(t\lambda) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

$$= \int_{\underline{R}} \left(\int_X \exp(\sqrt{-1}\, t\lambda(x+h)) d\gamma(x)\right) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

$$= \int_{\underline{R}} \exp\left(\sqrt{-1}\, t\lambda(h) - \frac{t^2}{2} ||\lambda||^2_{L^2(\gamma)}\right) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

$$= \frac{1}{(||\lambda||^2_{L^2(\gamma)} + 1)^{1/2}} \exp\left(-\frac{\lambda(h)^2}{2(||\lambda||^2_{L^2(\gamma)} + 1)}\right) \quad \text{(cf. 24.6)}.$$

Since $h \notin H(\gamma)$, $\forall\ k \in \underline{N}$, $\exists\ \lambda_k \in X^*$:

$$||\lambda_k||_{L^2(\gamma)} = 1\ \&\ \lambda_k(h) > k.$$

Put

$$\eta_k = \lambda_k/\sqrt{\lambda_k(h)}.$$

Then $\eta_k \to 0$ in $L^2(X,\gamma)$, hence it can be assumed that $\eta_k \to 0$ a.e. $[\gamma]$. But $\rho << \gamma$, so $\eta_k \to 0$ a.e. $[\rho]$ as well. Therefore

$$0 = \lim_{k \to \infty} \frac{1}{(||\eta_k||^2_{L^2(\gamma)} + 1)^{1/2}} \exp\left(-\frac{\eta_k(h)^2}{2(||\eta_k||^2_{L^2(\gamma)} + 1)}\right)$$

$$\geq \lim_{k \to \infty} \int_X \exp\left(-\frac{1}{2}\eta_k(x)^2\right) d\rho(x)$$

$$= \int_X 1 d\rho(x)$$

=>

$$\rho(x) = 0 \Rightarrow \rho = 0.$$

26.24 <u>LEMMA</u> Let γ be a centered gaussian measure on X. Suppose that $E \subset X$ is a linear subspace of measure 1 -- then $\forall\ h \in H(\gamma)$, $\gamma_h(E) = 1$.

<u>PROOF</u> According to 26.19, $\gamma_h \sim \gamma$. This said, write

$$1 = \gamma_h(X) = \gamma_h(E) + \gamma_h(X - E).$$

Then

$$\gamma(X - E) = 0 \Rightarrow \gamma_h(X - E) = 0 \Rightarrow \gamma_h(E) = 1.$$

26.25 <u>LEMMA</u> Let γ be a centered gaussian measure on X. Suppose that $E \subset X$ is a linear subspace of measure 1 -- then $H(\gamma) \subset E$.

<u>PROOF</u> Take an $h \in H(\gamma)$ and assume that $h \notin E$ -- then

$$E \cap (E - h) = \emptyset$$

=>

$$\gamma(E \cup (E - h)) + \gamma(E \cap (E - h))$$

$$= \gamma(E) + \gamma(E - h)$$

$$= \gamma(E) + \gamma_h(E)$$

=>

$$1 + 0 = 1 + 1 \quad (\text{cf. } 26.24),$$

an impossibility.

26.26 <u>REMARK</u> Actually

$$H(\gamma) = \bigcap_E E,$$

where $E \subset X$ runs through the linear subspaces of measure 1.

[If $h \notin H(\gamma)$, then $\forall\ k \in \underline{N}$, $\exists\ \lambda_k \in X^*$:

$$||\lambda_k||_{L^2(\gamma)} = 1 \ \&\ \lambda_k(h) > k \quad (\text{cf. } 26.23).$$

Denote by E the set of all $x \in X$ such that the series

$$\sum_{k=1}^{\infty} k^{-2}\lambda_k(x)$$

is convergent -- then E is a linear subspace of measure 1 but $h \notin E$.]

26.27 LEMMA Let γ be a centered gaussian measure on X -- then $\gamma(H(\gamma)) = 0$.

PROOF In the notation introduced above, $T^{-1}(\ell^2) = H(\gamma)$. Therefore

$$(\gamma \circ T^{-1})(\ell^2) = \gamma(T^{-1}(\ell^2)) = \gamma(H(\gamma)).$$

But (cf. 24.11)

$$(\gamma \circ T^{-1})(\ell^2) = 0.$$

26.28 LEMMA Let γ be a centered gaussian measure on X. Suppose that $\iota:X \to Y$ is a continuous linear embedding, where Y is a separable LF-space -- then

$$\iota H(\gamma) = H(\gamma \circ \iota^{-1}).$$

PROOF $\forall\ h \in H(\gamma)$,

$$\gamma_h \sim \gamma \qquad \text{(cf. 26.19)}$$

=>

$$(\gamma \circ \iota^{-1})_{\iota(h)} \sim \gamma \circ \iota^{-1}$$

=>

$$\iota(h) \in H(\gamma \circ \iota^{-1}) \qquad \text{(cf. 26.19)}$$

=>

$$\iota H(\gamma) \subset H(\gamma \circ \iota^{-1}).$$

Turning to the converse, note first that $(\gamma \circ \iota^{-1})(\iota X) = 1$, hence

$$\iota X \supset H(\gamma \circ \iota^{-1}) \quad \text{(cf. 26.25)}.$$

Now take $h' \in H(\gamma \circ \iota^{-1})$ and write $h' = \iota(h)$ $(h \in X)$ -- then

$$(\gamma \circ \iota^{-1})_{h'} \sim \gamma \circ \iota^{-1}$$

$\Rightarrow$

$$\gamma_h \sim \gamma \Rightarrow h \in H(\gamma) \quad \text{(cf. 26.19)}.$$

Let γ be a centered gaussian measure on X. Denote by spt γ the intersection of all closed subsets $F \subset X$ with $\gamma(F) = 1$ -- then

$$\text{spt } \gamma = \{x \in X: \forall \text{ open } U \supset \{x\}, \gamma(U) > 0\}$$

and

$$\gamma(\text{spt } \gamma) = 1.$$

[Note: spt γ is called the <u>topological support</u> of γ.]

26.29 <u>LEMMA</u> We have

$$\text{spt } \gamma = \overline{H(\gamma)}.$$

<u>PROOF</u> To begin with, if $\lambda \in X^*$ and if $||\lambda||_{L^2(\gamma)} = 0$, then $\lambda = 0$ a.e., thus $\gamma(\text{Ker } \lambda) = 1$. Let D be the set of all such λ -- then

$$\text{spt } \gamma \subset \bigcap_{\lambda \in D} \text{Ker } \lambda$$

and we claim that

$$\bigcap_{\lambda \in D} \text{Ker } \lambda \subset \overline{H(\gamma)}.$$

This is obvious if $\overline{H(\gamma)} = X$, so assume that $\overline{H(\gamma)} \neq X$ and, to get a contradiction, choose

$$x_0 \in \bigcap_{\lambda \in D} \text{Ker } \lambda, \ x_0 \notin \overline{H(\gamma)}.$$

By Hahn-Banach, $\exists \ \lambda \in X^*$:

$$\lambda(x_0 = 1, \ \lambda|\overline{H(\gamma)} = 0.$$

But

$$< \lambda, \lambda > = \lambda(x_\lambda) = 0 \quad (x_\lambda \in H(\gamma))$$

$\Rightarrow$

$$\lambda \in D \Rightarrow \lambda(x_0) = 0.$$

Take now any $x \in \text{spt } \gamma$ -- then

$$x + H(\gamma) \subset \text{spt } \gamma$$

$\Rightarrow$

$$x + \overline{H(\gamma)} \subset \text{spt } \gamma \subset \overline{H(\gamma)}$$

$\Rightarrow$

$$x + \overline{H(\gamma)} = \overline{H(\gamma)}$$

$\Rightarrow$

$$\overline{H(\gamma)} \subset \text{spt } \gamma.$$

In summary:

$$\text{spt } \gamma \subset \bigcap_{\lambda \in D} \text{Ker } \lambda \subset \overline{H(\gamma)} \subset \text{spt } \gamma.$$

Therefore

$$\text{spt } \gamma = \overline{H(\gamma)}.$$

Let γ be a centered gaussian measure on X -- then γ is said to be <u>nondegenerate</u> if spt $\gamma = X$. So, in view of 26.29, γ is nondegenerate iff its Cameron-Martin space $H(\gamma)$ is dense in X.

[Note: If γ is nondegenerate, then $\lambda \neq 0 \Rightarrow Q(\lambda,\lambda) > 0$ $(\lambda \in X^*)$. Proof: $Q(\lambda,\lambda) = 0$ $(\lambda \neq 0) \Rightarrow ||\lambda||_{L^2(\gamma)} = 0 \Rightarrow \gamma(\text{Ker } \lambda) = 1 \Rightarrow \text{Ker } \lambda \supset H(\gamma) \Rightarrow \text{Ker } \lambda = \overline{\text{Ker } \lambda} \supset \overline{H(\gamma)} = X \Rightarrow \lambda = 0.$]

26.30 <u>EXAMPLE</u> Take $X = \underline{R}^\infty$ and let γ be the standard gaussian measure on X (cf. 26.1) — then $H(\gamma) = \ell^2$ (cf. 26.14). But $\ell^2 \supset \underline{R}_0^\infty$ and $\underline{R}_0^\infty$ is dense in $\underline{R}^\infty$. Therefore γ is nondegenerate.

26.31 <u>LEMMA</u> Let γ be a centered gaussian measure on X. Suppose that $B \in \text{Bor}(X)$ and $\gamma(B) > 0$ -- then $\exists\, r > 0$:

$$rB_{H(\gamma)} \subset B - B,$$

where $B_{H(\gamma)}$ is the closed unit ball in $H(\gamma)$.

<u>PROOF</u> The function

$$\begin{cases} H(\gamma) \to \underline{R} \\ h \to \gamma((B + h) \cap B) \end{cases}$$

is positive at zero and continuous (cf. 26.16 and 26.17) (observe that

$$\gamma((B + h) \cap B) = \int_X \chi_B(x - h)\chi_B(x)d\gamma(x)).$$

So $\exists$ r > 0:

$$h \in rB_{H(\gamma)} \Rightarrow \gamma((B + h) \cap B) > 0$$

$$\Rightarrow h \in B - B.$$

Here is a corollary. Let E be a linear subspace of X of positive measure -- then

$$H(\gamma) \subset E.$$

26.32 <u>LEMMA</u> Let γ be a centered gaussian measure on X -- then the set of functions of the form

$$\frac{H_{k_1}(\lambda_1)}{\sqrt{k_1!}} \cdots \frac{H_{k_n}(\lambda_n)}{\sqrt{k_n!}},$$

where the

$$\lambda_i \in X^* \;\&\; \langle \lambda_i, \lambda_j \rangle = \delta_{ij},$$

is total in $L^2(X,\gamma)$.

26.33 <u>THE ZERO-ONE LAW</u> Suppose that $B \in \text{Bor}(X)$ and satisfies the condition

$$\gamma_h(B) = \gamma(B) \quad \forall\, h \in H(\gamma).$$

Then either $\gamma(B) = 0$ or $\gamma(B) = 1$.

<u>PROOF</u> Let $\lambda_1,\ldots,\lambda_n \in X^*$ and assume that the λ_i are orthonormal in $L^2(X,\gamma)$. Put $h_1 = R_\gamma(\lambda_1),\ldots,h_n = R_\gamma(\lambda_n)$ and consider the function

$$F(t_1,\ldots,t_n) = \lambda(B - t_1h_1 - \cdots - t_nh_n).$$

Since $\forall\ h \in H(\gamma)$,

$$\begin{aligned}\gamma(B) &= \gamma(B - h)\\ &= \int_X \chi_{B-h}(x)\,d\gamma(x)\\ &= \int_X \chi_B(x + h)\,d\gamma(x)\\ &= \int_X \chi_B(x)\exp(f(x) - \tfrac{1}{2}||h||^2_{H(\gamma)})\,d\gamma(x),\end{aligned}$$

it follows that

$$\begin{aligned}&F(t_1,\ldots,t_n)\\ &= \int_X \chi_B(x)\exp(\sum_{i=1}^n t_i\lambda_i(x) - \tfrac{1}{2}||\sum_{i=1}^n t_ih_i||^2_{H(\gamma)})\,d\gamma(x)\end{aligned}$$

is constant. So, for any collection $k_1,\ldots,k_n$ of nonnegative integers, not all of which are zero, we have

$$\frac{\partial^{k_1+\cdots+k_n}}{\partial t_1^{k_1}\cdots\partial t_n^{k_n}} F\Big|_{(0,\ldots,0)} = 0.$$

But, from our assumptions,

$$||\sum_{i=1}^n t_ih_i||^2_{H(\gamma)} = \sum_{i=1}^n t_i^2.$$

And

$$\frac{\partial^{k_1+\cdots+k_n}}{\partial t_1^{k_1}\cdots\partial t_n^{k_n}} \exp\left(\sum_{i=1}^{n} t_i\lambda_i(x) - \frac{1}{2}\sum_{i=1}^{n} t_i^2\right)\Bigg|_{(0,\ldots,0)}$$

$$= H_{k_1}(\lambda_1(x))\cdots H_{k_n}(\lambda_n(x)).$$

Therefore

$$\int_X \chi_B(x) H_{k_1}(\lambda_1(x))\cdots H_{k_n}(\lambda_n(x))\,d\gamma(x)$$

$$= 0.$$

Owing now to 26.32, χ_B is necessarily a constant and the only possibilities are 0 and 1.

Consequently, if E is a linear subspace of X of positive measure, then $\gamma(E) = 1$. In fact,

$$\gamma(E) > 0 \Rightarrow H(\gamma) \subset E \Rightarrow \gamma(E) = 1.$$

26.34 LEMMA Suppose that $L \in \mathrm{Bor}(X)$ is affine — then either $\gamma(L) = 0$ or $\gamma(L) = 1$.

[Note: If E is linear and if $L = E + h$, where $h \notin E$, then $\gamma(L) = 0$. For otherwise, $\gamma(E + h) = 1$. But γ is centered, hence $\gamma(E + h) = \gamma(E - h)$. Therefore

$$\gamma((E + h) \cup (E - h)) + \gamma((E + h) \cap (E - h)) = \gamma(E + h) + \gamma(E - h)$$

$\Rightarrow$

$$1 + \gamma(\emptyset) = 1 + 1,$$

which is nonsense.]

Let γ be a centered gaussian measure on X -- then a Borel function $p:X \to \underline{R}_{\geq 0}$ is said to be a <u>measurable seminorm</u> if $\exists$ a linear subspace E of X of measure 1 such that the restriction $p|E$ is a seminorm.

[Note: In view of 26.25, $H(\gamma) \subset E$.]

26.35 <u>EXAMPLE</u> Take $X = \underline{R}^\infty$ and let γ be the standard gaussian measure on X (cf. 26.1). Set

$$p_n(x) = (\frac{1}{n} \sum_{k=1}^{n} x_k^2)^{1/2}.$$

Then

$$p(x) = \limsup p_n(x)$$

is a measurable seminorm such that $p = 1$ a.e..

26.36 <u>LEMMA</u> Let γ be a centered gaussian measure on X. Suppose that p is a measurable seminorm -- then $p|H(\gamma)$ is continuous.

<u>PROOF</u> Fix $n:\gamma(B_n) > 0$, where

$$B_n = \{x:p(x) \leq n\}.$$

Fix $r > 0$:

$$rB_{H(\gamma)} \subset B_n - B_n \quad \text{(cf. 26.31)}.$$

Then $p|B_{H(\gamma)}$ is bounded, hence $p|H(\gamma)$ is continuous.

26.37 <u>THEOREM</u> (Fernique) Let γ be a centered gaussian measure on X. Suppose that p is a measurable seminorm -- then $\exists\ \alpha > 0$:

$$\int_X \exp(\alpha p^2(x))\, d\gamma(x) < \infty.$$

PROOF In order not to obscure the overall structure of the argument with measure theoretic technicalities, it will be convenient to assume from the outset that p is a seminorm. This done, $\forall\ t,t' \in \underline{R}_{\geq 0}$, we have

$$\gamma(p \leq t)\gamma(p > t')$$

$$= \iint_{p(x) \leq t,\, p(y) > t'} d\gamma(x)\, d\gamma(y)$$

$$= \iint_{p(\frac{u-v}{\sqrt{2}}) \leq t,\ p(\frac{u+v}{\sqrt{2}}) > t'} d\gamma(u)\, d\gamma(v)$$

$$\leq \iint_{p(u) > \frac{t'-t}{\sqrt{2}},\ p(v) > \frac{t'-t}{\sqrt{2}}} d\gamma(u)\, d\gamma(v)$$

=>

$$\gamma(p \leq t)\gamma(p > t') \leq \left(\gamma\left(p > \frac{t'-t}{\sqrt{2}}\right)\right)^2.$$

Choose $t_0 > 0$:

$$r = \gamma(p \leq t_0) > \frac{1}{2}.$$

The assertion of the theorem is trivial if $r = 1$, so take $r < 1$. Define $t_n\,(n > 0)$ recursively by the prescription

$$t_n = t_0 + t_{n-1}\sqrt{2}.$$

Then

$$t_n = t_0(1 + \sqrt{2})((\sqrt{2})^{n+1} - 1).$$

Put

$$\begin{cases} r_0 = \gamma(p > t_0)/r \\ r_n = \gamma(p > t_n)/r. \end{cases}$$

By the above,

$$r_n = \frac{\gamma(p > t_n)}{\gamma(p \le t_0)}$$

$$= \frac{\gamma(p \le t_0)\gamma(p > t_n)}{\gamma(p \le t_0)^2}$$

$$\le \left[\frac{\gamma(p > \frac{t_n - t_0}{\sqrt{2}})}{\gamma(p \le t_0)} \right]^2$$

$$= \left[\frac{\gamma(p > t_{n-1})}{\gamma(p \le t_0)} \right]^2$$

$$= (\gamma(p > t_{n-1})/r)^2$$

$$= (r_{n-1})^2$$

=>

$$\gamma(p > t_n) \le r(\frac{1-r}{r})^{2^n}.$$

Let

$$\alpha = \frac{1}{24\ t_0^2} \log \frac{r}{1-r}\ .$$

Then

$$\int_X \exp(\alpha p^2(x))d\gamma(x)$$

$$\leq \int_{p\leq t_0} \exp(\alpha p^2(x))d\gamma(x) + \sum_{n=0}^{\infty} \exp(\alpha t_{n+1}^2)\gamma(t_n < p \leq t_{n+1})$$

$$\leq r \exp(\alpha t_0^2) + \sum_{n=0}^{\infty} \exp(\alpha t_{n+1}^2)\gamma(p > t_n)$$

$$\leq r \exp(\alpha t_0^2) + \sum_{n=0}^{\infty} r(\frac{1-r}{r})^{2^n} \exp(4\alpha t_0^2(1 + \sqrt{2})^2 2^n)$$

$$\leq r \exp(\alpha t_0^2) + r \sum_{n=0}^{\infty} \exp(2^n(\log \frac{1-r}{r} + 4\alpha t_0^2(1 + \sqrt{2})^2))$$

$$\leq r \exp(\alpha t_0^2) + r \sum_{n=0}^{\infty} \exp(2^n(\log \frac{1-r}{r} + C \log \frac{r}{1-r}))$$

$$\leq r \exp(\alpha t_0^2) + r \sum_{n=0}^{\infty} \exp(2^n(1-C)\log \frac{1-r}{r})$$

$$\leq r \exp(\alpha t_0^2) + r \sum_{n=0}^{\infty} ((\frac{1-r}{r})^{1-C})^{2^n}$$

$$< \infty.$$

[Note: Here

$$C = \frac{4(1+\sqrt{2})^2}{24} < 1$$

=>

$$1 - C > 0$$

and

$$\frac{1}{2} < r < 1 \Rightarrow 0 < \frac{1-r}{r} < 1.$$

Therefore

$$0 < \left(\frac{1-r}{r}\right)^{1-C} < 1.]$$

26.38 <u>REMARK</u> Because

$$\exp(\alpha p^2(x)) \geq 1 + \alpha p^2(x) \quad (\alpha > 0),$$

it follows from 26.37 that $p \in L^2(X,\gamma)$.

26.39 <u>EXAMPLE</u> Let $f:X \rightarrow \underline{R}$ be Borel. Assume: $\exists$ a linear subspace E of X of measure 1 such that the restriction $f|E$ is linear -- then $f \in L^2(X,\gamma)$.

Take an $f \in X_\gamma^*$ and let f_0 be a linear model for f (cf. 26.10) -- then $H(\gamma) \subset E_0$ (cf. 26.25), $f_0|H(\gamma)$ is continuous (cf. 26.36), and by construction, $\forall\ h \in H(\gamma)$,

$$f_0(h) = \lim_{k \to \infty} \lambda_k(h)$$

$$= \lim_{k \to \infty} \langle g,\lambda_k \rangle_{L^2(\gamma)} \quad (h = R_\gamma(g))$$

$$= \langle g,f_0 \rangle_{L^2(\gamma)}$$

$$= \langle h,R_\gamma(f_0) \rangle_{H(\gamma)}.$$

§27. DICHOTOMIES

Let X be a separable LF-space.

27.1 <u>THEOREM</u> (Feldman-Hajeck) Let γ_1, γ_2 be gaussian measures on X -- then either $\gamma_1 \sim \gamma_2$ or $\gamma_1 \perp \gamma_2$.

Our primary objective in the present § is to give a proof of this result. To begin with, there are two possibilities:

$$\begin{cases} \dim X < \infty \\ \dim X = \infty. \end{cases}$$

The finite dimensional case can be treated directly sans machinery (cf. infra). The infinite dimensional case is, of course, more complicated but the introduction of certain measure theoretic generalities will help smooth the way. Before getting involved with this, however, we shall first make some preliminary reductions.

27.2 <u>EXAMPLE</u> Suppose that γ is centered -- then $\forall\ h \in X$,

$$\begin{cases} \gamma \sim \gamma_h \text{ if } h \in H(\gamma) \quad \text{(cf. 26.19)} \\ \gamma \perp \gamma_h \text{ if } h \notin H(\gamma) \quad \text{(cf. 26.23)}. \end{cases}$$

Therefore $\forall\ h_1, h_2 \in X$,

$$\begin{cases} \gamma_{h_1} \sim \gamma_{h_2} \text{ if } h_1 - h_2 \in H(\gamma) \\ \gamma_{h_1} \perp \gamma_{h_2} \text{ if } h_1 - h_2 \notin H(\gamma). \end{cases}$$

27.3 <u>LEMMA</u> If γ_1, γ_2 are centered and if $\gamma_1 \perp \gamma_2$, then $\forall\ h_1, h_2 \in X$,

$$(\gamma_1)_{h_1} \perp (\gamma_2)_{h_2}.$$

<u>PROOF</u> Assume, as we may, that $h_2 = 0$. If $h_1 \in H(\gamma_1)$, then $(\gamma_1)_{h_1} \sim \gamma_1$, hence $(\gamma_1)_{h_1} \perp \gamma_2$. So suppose that $h_1 \notin H(\gamma_1)$. Fix a linear subspace E_1: $\gamma_1(E_1) = 1$ and $h_1 \notin E_1$ (cf. 26.26) -- then

$$(\gamma_1)_{h_1}(E_1 + h_1) = 1 \text{ and } \gamma_2(E_1 + h_1) = 0,$$

thus $(\gamma_1)_{h_1} \perp \gamma_2$.

Admit for the time being that 27.1 is true in the centered situation. Write

$$\left[\begin{array}{ll} \gamma_1 = ((\gamma_1)_s)_{a_1} & (a_1 = a_{\gamma_1}) \\ \\ \gamma_2 = ((\gamma_2)_s)_{a_2} & (a_2 = a_{\gamma_2}) \end{array}\right. \qquad \text{(cf. 26.13).}$$

Assume that $\gamma_1 \not\perp \gamma_2$ -- then we claim that $\gamma_1 \sim \gamma_2$.

<u>Step 1</u>: $(\gamma_1)_s \not\perp (\gamma_2)_s$ (cf. 27.3).

<u>Step 2</u>: $(\gamma_1)_s \sim (\gamma_2)_s$ (by hypothesis)(symmetrizations are centered).

<u>Step 3</u>: $((\gamma_1)_s)_{a_2} \sim ((\gamma_2)_s)_{a_2}$ (obvious).

<u>Step 4</u>: $((\gamma_1)_s)_{a_1} \not\perp ((\gamma_1)_s)_{a_2}$ (use Step 3).

Step 5: $((\gamma_1)_s)_{a_1} \sim ((\gamma_1)_s)_{a_2}$ (cf. 27.2).

Step 6: $((\gamma_1)_s)_{a_1} \sim ((\gamma_2)_s)_{a_2}$ (use Step 3).

Therefore

$$\gamma_1 \not\perp \gamma_2 \Rightarrow \gamma_1 \sim \gamma_2.$$

In other words, the centered case implies the general case.

27.4 LEMMA Let γ_1, γ_2 be centered gaussian measures on $\underline{R}^n$ -- then either $\gamma_1 \sim \gamma_2$ or $\gamma_1 \perp \gamma_2$.

PROOF It can be assumed outright that $\gamma_1 \neq \delta_0$, $\gamma_2 \neq \delta_0$. This said, put

$$\begin{cases} L_1 = K_1\underline{R}^n \\ L_2 = K_2\underline{R}^n \end{cases} \quad \text{(cf. 22.5).}$$

If $L_1 \cap L_2$ is a proper subspace of L_1 or L_2, then $\gamma_1 \perp \gamma_2$. E.g.: Say $L_1 \cap L_2$ is strictly contained in L_1, so $\gamma_1(L_1 \cap L_2) = 0$. Let $A = L_1 - L_1 \cap L_2$ ($\Rightarrow \underline{R}^n - A \supset L_2$) -- then

$$\begin{cases} \gamma_1(A) = 1 \\ \gamma_2(\underline{R}^n - A) = 1 \end{cases} \Rightarrow \gamma_1 \perp \gamma_2.$$

Thus the upshot is that $\gamma_1 \perp \gamma_2$ unless $L_1 = L_2$. Accordingly, there is no loss of generality in supposing that $L_1 = L_2 = \underline{R}^n$ and both γ_1, γ_2 are nondegenerate with densities

$$\begin{cases} p_{\gamma_1}(x) = \frac{1}{((2\pi)^n \det K_1)^{1/2}} \exp(-\frac{1}{2} < x, K_1^{-1} x >) \\ \\ p_{\gamma_2}(x) = \frac{1}{((2\pi)^n \det K_2)^{1/2}} \exp(-\frac{1}{2} < x, K_2^{-1} x >). \end{cases}$$

But then $\gamma_1 \sim \gamma_2$.

Assume henceforth that $\dim X = \infty$. Let γ_1, γ_2 be centered gaussian measures on X. Define $T: X \to \underline{R}^\infty$ per γ_1 as in 26.20 -- then T is a continuous injection. But X is a separable LF-space, thus X is lusinien (cf. 25.18) and so T sends Borel sets to Borel sets (cf. 25.19).

27.5 <u>LEMMA</u> Let μ, ν be Borel measures on X -- then

$$\begin{cases} \mu \sim \nu \iff \mu \circ T^{-1} \sim \nu \circ T^{-1} \\ \\ \mu \perp \nu \iff \mu \circ T^{-1} \perp \nu \circ T^{-1}. \end{cases}$$

[This is immediate.]

Put

$$\begin{cases} P_1 = \gamma_1 \circ T^{-1} \\ \\ P_2 = \gamma_2 \circ T^{-1}. \end{cases}$$

Then P_1 is the standard gaussian measure on $\underline{R}^\infty$ (cf. 26.20), while P_2 is a centered gaussian measure on $\underline{R}^\infty$.

27.6 <u>LEMMA</u> Either $P_1 \sim P_2$ or $P_1 \perp P_2$.

Since

$$\begin{cases} P_1 \sim P_2 \Rightarrow \gamma_1 \sim \gamma_2 \\ P_1 \perp P_2 \Rightarrow \gamma_1 \perp \gamma_2, \end{cases}$$

27.6 serves to complete the proof of 27.1.

27.7 <u>LEMMA</u> If $H(P_1) \cap H(P_2)$ is a proper subspace of either $H(P_1)$ or $H(P_2)$, then $P_1 \perp P_2$.

<u>PROOF</u> Assume $\exists\, h \in \underline{R}^\infty : h \in H(P_1)\ \&\ h \notin H(P_2)$. Choose a linear subspace $E: P_2(E) = 1\ \&\ h \notin E$ (cf. 26.26). Since $h \notin E$, $P_1(E + h) = 0$ (cf. 26.34), i.e., $(P_1)_{-h}(E) = 0$. But $-h \in H(P_1)$, which implies that $P_1 \sim (P_1)_{-h}$ (cf. 26.19), so

$$(P_1)_{-h}(E) = 0 \Rightarrow P_1(E) = 0.$$

Therefore $P_1 \perp P_2$.

Consequently, $P_1 \perp P_2$ unless $H(P_1) = H(P_2)$, a condition that we shall assume to be in force from this point on.

[Note: Recall that P_1 is nondegenerate (cf. 26.30), hence the same is true of P_2.]

Let us now turn to the results from measure theory that will be needed to complete the proof (details can be found in any sufficiently enlightened text on probability).

Fix a measurable space $(\Omega,\mathcal{A})$ (i.e., Ω is a nonempty set and $\mathcal{A}$ is a σ-algebra of subsets of Ω). Given a pair of probability measures P_1,P_2 on $(\Omega,\mathcal{A})$, let p_1,p_2 be the Radon-Nikodym derivative of P_1,P_2 w.r.t. $P_1 + P_2$ -- then the Lebesgue decomposition of P_2 w.r.t. P_1 can be written as

$$P_2(A) = \int_A \left(\frac{p_2}{p_1}\right) dP_1 + P_2(A \cap (p_1 = 0)) \quad (A \in \mathcal{A}).$$

27.8 <u>LEMMA</u> We have

$$\begin{bmatrix} (\perp)\ P_1 \perp P_2 \iff \int_\Omega \left(\frac{p_2}{p_1}\right)^{1/2} dP_1 = 0 \\ \\ (\ll)\ P_1 \ll P_2 \iff \lim_{\alpha \downarrow 0} \int_\Omega \left(\frac{p_2}{p_1}\right)^{\alpha} dP_1 = 1. \end{bmatrix}$$

Suppose that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$ is an increasing sequence of sub σ-algebras of $\mathcal{A}$ such that $\mathcal{A} = \sigma(\bigcup_{n=1}^{\infty} \mathcal{A}_n)$. Let ρ_n denote the Radon-Nikodym derivative of the absolutely continuous part of $P_{2,n} = P_2|\mathcal{A}_n$ w.r.t. $P_{1,n} = P_1|\mathcal{A}_n$ -- then $\forall\ \alpha \in\ ]0,1[$,

$$\int_\Omega \left(\frac{P_2}{P_1}\right)^\alpha dP_1 = \inf_n \int_\Omega (\rho_n)^\alpha \, dP_{1,n}.$$

27.9 <u>LEMMA</u> We have

$$\begin{cases} (\perp) \; P_1 \perp P_2 \iff \inf_n \int_\Omega (\rho_n)^{1/2} \, dP_{1,n} = 0 \\ (\ll) \; P_1 \ll P_2 \iff \lim_{\alpha \downarrow 0} \inf_n \int_\Omega (\rho_n)^\alpha \, dP_{1,n} = 1. \end{cases}$$

Specialize and take $\Omega = \underline{R}^\infty$, $\mathcal{A} = \text{Bor}(\underline{R}^\infty)$,

$$\begin{cases} P_1 = \gamma_1 \circ T^{-1} \\ P_2 = \gamma_2 \circ T^{-1}, \end{cases}$$

and let $\mathcal{A}_n$ be the σ-algebra generated by the coordinate functions δ_k $(k = 1,\ldots,n)$ $(\delta_k(x) = x_k)$.

27.10 <u>LEMMA</u> If $P_1 \not\perp P_2$, then $P_1 \sim P_2$.

[Note: Obviously,

$$27.10 \Rightarrow 27.6 \; (\Rightarrow 27.1).]$$

It will be enough to show that $P_1 \ll P_2$ and for this, we shall employ 27.9.

27.11 LEMMA Suppose that γ_1,γ_2 are two nondegenerate centered gaussian measures on $\underline{R}^n$ with densities

$$\begin{cases} p_{\gamma_1}(x) = \frac{1}{((2\pi)^n \det K_1)^{1/2}} \exp(-\frac{1}{2} < x,K_1^{-1}x >) \\ \\ p_{\gamma_2}(x) = \frac{1}{((2\pi)^n \det K_2)^{1/2}} \exp(-\frac{1}{2} < x,K_2^{-1}x >). \end{cases}$$

Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of $K_1^{1/2}\ K_2^{-1}\ K_1^{1/2}$ -- then $\forall\ \alpha \in\ ]0,1[$,

$$\int_{\underline{R}^n} (\frac{d\gamma_2}{d\gamma_1})^\alpha\ d\gamma_1$$

$$= \int_{\underline{R}^n} (p_{\gamma_1})^{1-\alpha}\ (p_{\gamma_2})^\alpha\ dx$$

$$= \prod_{k=1}^{n} \left[\frac{\lambda_k^\alpha}{\alpha\lambda_k + (1-\alpha)} \right]^{1/2}.$$

Define

$$T_n:\underline{R}^\infty \to \underline{R}^n$$

by

$$T_n(x) = (\delta_1(x),\ldots,\delta_n(x))\ (= (x_1,\ldots,x_n)).$$

Then 27.11 is applicable to

$$\begin{cases} P_1 \circ T_n^{-1} \\ P_2 \circ T_n^{-1}. \end{cases}$$

Let

$$\begin{cases} p_{1,n} = p_{P_1 \circ T_n^{-1}} \\ p_{2,n} = p_{P_2 \circ T_n^{-1}}. \end{cases}$$

Then

$$\rho_n(x) = \frac{p_{2,n}}{p_{1,n}} (T_n(x)) \quad (x \in \underline{R}^\infty).$$

And

$$\int_{\underline{R}^\infty} (\rho_n)^\alpha \, dP_{1,n}$$

$$= \int_{\underline{R}^\infty} \left(\frac{p_{2,n}}{p_{1,n}} \circ T_n\right)^\alpha dP_{1,n}$$

$$= \int_{\underline{R}^n} \left(\frac{p_{2,n}}{p_{1,n}}\right)^\alpha d(P_1 \circ T_n^{-1})$$

$$= \prod_{k=1}^{n} \left[\frac{\lambda_k(n)^\alpha}{\alpha\lambda_k(n) + (1-\alpha)} \right]^{1/2}.$$

With this preparation, we are ready to proceed to the proof of 27.10. If $P_1 \not\perp P_2$, then

$$\inf_n \int_{\underline{R}^\infty} (\rho_n)^{1/2} \, dP_{1,n} > 0 \quad \text{(cf. 27.9)}$$

or still,

$$\inf_n \prod_{k=1}^{n} \left[\frac{2\sqrt{\lambda_k(n)}}{\lambda_k(n)+1} \right]^{1/2} > 0$$

or still,

$$\sup_n \prod_{k=1}^{n} \left[\frac{\lambda_k(n)+1}{2\sqrt{\lambda_k(n)}} \right] < \infty$$

or still,

$$\sup_n \sum_{k=1}^{n} \left[\frac{\lambda_k(n)+1}{2\sqrt{\lambda_k(n)}} - 1 \right] < \infty .$$

27.12 LEMMA Let $f(x) = \frac{x+1}{2\sqrt{x}}$ $(x > 0)$ -- then for $M > 1$,

$$\{x : 1 \leq f(x) \leq M\} = [x_1, x_2] \quad (0 < x_1 < x_2 < \infty)$$

and $\exists\, r_1, r_2$ $(0 < r_1 < r_2 < \infty)$ such that for $x_1 \leq x \leq x_2$,

$$r_1(1-x)^2 \leq f(x) - 1 \leq r_2(1-x)^2.$$

Therefore

$$\sup_n \sum_{k=1}^{n} (\lambda_k(n)-1)^2 < \infty$$

and $\exists$ positive constants C_1, C_2: $\forall$ k & $\forall$ n,

$$C_1 \leq \lambda_k(n) \leq C_2.$$

Using these facts, we shall now prove that

$$\lim_{\alpha \downarrow 0} \inf_n \int_{\underline{R}^\infty} (\rho_n)^\alpha \, dP_{1,n} = 1,$$

from which $P_1 << P_2$ (cf. 27.9).

Rephrased, the claim is that

$$\lim_{\alpha \downarrow 0} \inf_n \prod_{k=1}^{n} \left[\frac{\lambda_k(n)^\alpha}{\alpha\lambda_k(n) + (1-\alpha)} \right]^{1/2} = 1.$$

I.e.: $\forall$ $\varepsilon > 0$ (& $\varepsilon < 1$), $\exists$ $\alpha(\varepsilon) \in]0,1[$:

$$\prod_{k=1}^{n} \left[\frac{\lambda_k(n)^\alpha}{\alpha\lambda_k(n) + (1-\alpha)} \right]^{1/2} > 1 - \varepsilon$$

for all $\alpha \in]0,\alpha(\varepsilon)[$ and for all $n \in \underline{N}$.

Take logs on both sides:

$$\frac{1}{2} \sum_{k=1}^{n} (\alpha \log \lambda_k(n) - \log(\alpha\lambda_k(n) + (1-\alpha)) > \log(1-\varepsilon)$$

or, as is more convenient,

$$\sum_{k=1}^{n} (\log(\alpha\lambda_k(n) + (1-\alpha)) - \alpha \log \lambda_k(n)) < - 2\log(1-\varepsilon).$$

27.13 <u>LEMMA</u> If $-1 < x_1 \le x \le x_2$ and $0 < \alpha < 1$, then $\exists\, C > 0$ (depending on x_1, x_2 but independent of α) such that

$$\log(1 + \alpha x) - \alpha \log(1+x) \le \alpha C x^2.$$

To apply this in our situation, note that

$$C_1 \le \lambda_k(n) \le C_2$$

$\Rightarrow$

$$-1 < C_1 - 1 \le \lambda_k(n) - 1 \le C_2 - 1$$

$\Rightarrow$

$$\log(1 + \alpha(\lambda_k(n)-1)) - \alpha \log(1 + \lambda_k(n)-1) \le \alpha C(\lambda_k(n)-1)^2$$

$\Rightarrow$

$$\log(\alpha\lambda_k(n) + (1-\alpha)) - \alpha \log(\lambda_k(n)) \le \alpha C(\lambda_k(n)-1)^2.$$

Fix $M > 0$:

$$\sup_n \sum_{k=1}^{n} (\lambda_k(n)-1)^2 < M < \infty.$$

Then

$$\sum_{k=1}^{n} (\log(\alpha\lambda_k(n) + (1-\alpha)) - \alpha \log \lambda_k(n))$$

$$\le \alpha C \sum_{k=1}^{n} (\lambda_k(n)-1)^2 < \alpha CM.$$

It remains only to choose $\alpha(\varepsilon)$:

$$\alpha(\varepsilon)CM < -2\log(1-\varepsilon).$$

Having finally dispatched 27.1, suppose again that X is a separable LF-space ($\dim X = \infty$).

27.14 <u>LEMMA</u> Let γ_1, γ_2 be centered gaussian measures on X — then $H(\gamma_1) \neq H(\gamma_2) \Rightarrow \gamma_1 \perp \gamma_2$.

[Argue as in 27.7.]

27.15 <u>LEMMA</u> If $H(\gamma_1) = H(\gamma_2)$ but the norms

$$\begin{cases} ||\cdot||_{H(\gamma_1)} \\ ||\cdot||_{H(\gamma_2)} \end{cases}$$

are not equivalent, then $\gamma_1 \perp \gamma_2$.

<u>PROOF</u> Choose a sequence $\{\lambda_k : k \geq 1\} \subset X^*$:

$$\begin{cases} ||\lambda_k||_{L^2(\gamma_1)} \to 0 \quad (k \to \infty) \\ ||\lambda_k||_{L^2(\gamma_2)} = 1 \quad (\forall\ k) \end{cases}$$

and assume that $\lambda_k \to 0$ a.e. $[\gamma_1]$. Let

$$E = \{x:\lambda_k(x) \to 0\}.$$

Then $\gamma_1(E) = 1$. On the other hand, either $\gamma_2(E) = 0$ or $\gamma_2(E) = 1$ (cf. 26.34). But $\gamma_2(E) = 1$ is untenable, hence $\gamma_2(E) = 0$, so $\gamma_1 \perp \gamma_2$.

<u>N.B.</u> Let $\{\xi_k : k \geq 1\}$ be a sequence of random variables on a probability space (Ω, A, μ). Assume: The ξ_k are centered gaussian and converge in measure to a random variable ξ — then ξ is centered gaussian and $\xi_k \to \xi$ in $L^2(\mu)$.

Assume that $H(\gamma_1) = H(\gamma_2)$. Assume further that the norms

$$\begin{cases} ||\cdot||_{H(\gamma_1)} \\ \\ ||\cdot||_{H(\gamma_2)} \end{cases}$$

are equivalent. Put

$$H = \begin{cases} H(\gamma_1) \\ \\ H(\gamma_2). \end{cases}$$

Fix an invertible bounded linear operator $T:H \to H$ such that $\forall\ h,h' \in H$,

$$< h,h' >_{H(\gamma_1)} = < Th,Th' >_{H(\gamma_2)}.$$

[Note: T is positive and selfadjoint per $< , >_{H(\gamma_1)}$ or $< , >_{H(\gamma_2)}$ (see the Appendix to §1).]

27.16 THEOREM (Segal) $\gamma_1 \sim \gamma_2$ iff $T - I$ is Hilbert-Schmidt.

27.17 EXAMPLE Suppose that γ is a centered gaussian measure on X. Given $r > 0$, define γ^r by the rule $\gamma^r(B) = \gamma(rB)$ $(B \in \text{Bor}(X))$ -- then $H(\gamma) = H(\gamma^r)$ and the corresponding norms are equivalent. But $\gamma \perp \gamma^r$ unless $r = 1$.

[Note: More generally, if $r_1 > 0$, $r_2 > 0$ and if $r_1 \neq r_2$, then $\gamma^{r_1} \perp \gamma^{r_2}$. Proof: $(\gamma^{r_1})^{r_2/r_1} = \gamma^{r_2}$.]

§28. *CHAOS*

Let X be a separable LF-space (dim X = ∞). Suppose that γ is a centered gaussian measure on X — then X_γ^* is a separable real Hilbert space and $\forall\ f \in X_\gamma^*$, $\gamma \circ f^{-1}$ ($\equiv \gamma_f$) is a centered gaussian measure on $\underline{R}$ with variance

$$\sigma(f)^2 = ||f||_{L^2(\gamma)}^2 \qquad \text{(cf. 26.9)}.$$

28.1 LEMMA We have

$$X_\gamma^* \subset \bigcap_{0<p<\infty} L^p(X,\gamma)$$

and $\forall\ f \in X_\gamma^*$,

$$||f||_p = \sqrt{2}\ (\Gamma(\tfrac{p+1}{2})/\sqrt{\pi})^{1/p}||f||_2.$$

In addition, X_γ^* is a closed subspace of $L^p(X,\gamma)$ and is closed w.r.t. convergence in measure.

[Note: The topology of convergence in measure on X_γ^* coincides with the L^2-topology, hence with the L^p-topologies.]

28.2 REMARK It follows from 28.1 that a finite product $f_1\cdots f_n$ ($f_i \in X_\gamma^*$, $i = 1,\ldots,n$) is in $L^p(X,\gamma)$ $(0 < p < \infty)$.

28.3 LEMMA Let $f_1,\ldots,f_n \in X_\gamma^*$.

<u>n odd</u>:

$$\int_X f_1 \cdots f_n d\gamma = 0.$$

<u>n even</u>:

$$\int_X f_1 \cdots f_n d\gamma = \Sigma \prod_{k=1}^{n/2} \int_X f_{i_k} f_{j_k} d\gamma,$$

where the sum is over all partitions $\{P_1, \ldots, P_{n/2}\}$ of $\{1, \ldots, n\}$ such that $P_k = \{i_k, j_k\}$ with $i_k < j_k$ $(k = 1, \ldots, n/2)$.

28.4 <u>EXAMPLE</u> Suppose that $f_i = f$ $(i = 1, \ldots, n)$ -- then

$$\int_X f^n d\gamma = \begin{cases} 0 & (n \text{ odd}) \\ (n-1)!! \ \sigma(f)^n & (n \text{ even}). \end{cases}$$

[Note: Here

$$(n-1)!! = 1 \cdot 3 \cdots (n-1).]$$

28.5 <u>RAPPEL</u>

$$BO(X_\gamma^*) = \bigoplus_{n=0}^{\infty} BO_n(X_\gamma^*)$$

is the bosonic Fock space over X_γ^*.

[Note: The fact that we are working over $\underline{R}$ rather than $\underline{C}$ is of no importance.]

Let $f_1, f_2, \ldots$ be an orthonormal basis for X_γ^*. Take $n > 0$ and consider any sequence $\kappa = \{k_j\}$ of nonnegative integers, almost all of whose terms are zero, with $\sum_j k_j = n$. Let

$$f_n(\kappa) = \left[\frac{n!}{k_1!k_2!\cdots}\right]^{1/2} P_n(f_1^{k_1} \otimes f_2^{k_2} \otimes \cdots).$$

Then the collection $\{f_n(\kappa)\}$ is an orthonormal basis for $BO_n(X_\gamma^*)$ (cf. 6.4).

28.6 <u>LEMMA</u> Let $\{f_j\}$ be an orthonormal basis for X_γ^* -- then the functions

$$\prod_{j=1}^{\infty} \frac{H_{k_j}(f_j)}{\sqrt{k_j!}}$$

constitute an orthonormal basis for $L^2(X,\gamma)$.

Let W_n denote the closed linear subspace of $L^2(X,\gamma)$ generated by the

$$\prod_{j=1}^{\infty} \frac{H_{k_j}(f_j)}{\sqrt{k_j!}},$$

where $\sum_j k_j = n$, and let I_n denote the orthogonal projection of $L^2(X,\gamma)$ onto W_n -- then

$$L^2(X,\gamma) = \bigoplus_{n=0}^{\infty} W_n$$

and $\forall\, f \in L^2(X,\gamma)$,

$$f = \sum_{n=0}^{\infty} I_n(f).$$

[Note: Obviously, $W_0 = \underline{R}$ and $W_1 = X_\gamma^*$.]

28.7 REMARK The chaos decomposition of $L^2(X,\gamma)$ is, by definition, the splitting $\bigoplus_{n=0}^{\infty} W_n$.

[Note: The chaos decomposition is independent of the choice of the orthonormal basis in X_γ^*.]

Define now

$$T_n : BO_n(X_\gamma^*) \to W_n$$

by

$$T_n f_n(\kappa) = \prod_{j=1}^{\infty} \frac{H_{k_j}(f_j)}{\sqrt{k_j!}} \quad (\sum_j k_j = n).$$

Then

$$T : BO(X_\gamma^*) \to L^2(X,\gamma)$$

is an isometric isomorphism.

In particular: $\forall\, f \in X_\gamma^*$ $(f \neq 0)$,

$$Tf^{\otimes n} = \frac{1}{\sqrt{n!}} \, ||f||_2^n \, H_n\left(\frac{f}{||f||_2}\right)$$

$$= \frac{1}{\sqrt{n!}} \, I_n(f^n).$$

Therefore

$$T\ \underline{\exp}(f) = T\Big(\sum_{n=0}^{\infty} \frac{f^{\otimes n}}{\sqrt{n!}}\Big)$$

$$= \sum_{n=0}^{\infty} \frac{||f||_2^n}{n!} H_n\Big(\frac{f}{||f||_2}\Big).$$

Put

$$\Lambda_f = \exp(f - \tfrac{1}{2}\,||f||_2^2).$$

Then

$$\Lambda_f = \exp(f - \tfrac{1}{2}\,||f||_2^2)$$

$$= \exp\Big(||f||_2 \frac{f}{||f||_2} - \tfrac{1}{2}\,||f||_2^2\Big)$$

$$= \sum_{n=0}^{\infty} \frac{||f||_2^n}{n!} H_n\Big(\frac{f}{||f||_2}\Big)$$

=>

$$T\ \underline{\exp}(f) = \Lambda_f.$$

And

$$\Lambda_f = \sum_{n=0}^{\infty} I_n(\Lambda_f) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^n).$$

28.8 <u>LEMMA</u> The $\Lambda_f (f \in X_\gamma^*)$ are linearly independent and total in $L^2(X,\gamma)$ (cf. 6.8 and 6.9).

Therefore

$$T\ \underline{\exp}(f) = T\left(\sum_{n=0}^{\infty} \frac{f^{\otimes n}}{\sqrt{n!}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{||f||_2^n}{n!} H_n\left(\frac{f}{||f||_2}\right).$$

Put

$$\Lambda_f = \exp(f - \tfrac{1}{2}\, ||f||_2^2).$$

Then

$$\Lambda_f = \exp(f - \tfrac{1}{2}\, ||f||_2^2)$$

$$= \exp\left(||f||_2 \frac{f}{||f||_2} - \tfrac{1}{2}\, ||f||_2^2\right)$$

$$= \sum_{n=0}^{\infty} \frac{||f||_2^n}{n!} H_n\left(\frac{f}{||f||_2}\right)$$

=>

$$T\ \underline{\exp}(f) = \Lambda_f.$$

And

$$\Lambda_f = \sum_{n=0}^{\infty} I_n(\Lambda_f) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^n).$$

28.8 <u>LEMMA</u> The $\Lambda_f (f \in X_\gamma^*)$ are linearly independent and total in $L^2(X,\gamma)$ (cf. 6.8 and 6.9).

[Note: Conventionally, $\Lambda_0 = 1$.]

28.9 LEMMA $\forall$ $f,g \in X_\gamma^*$, we have

$$\int_X \Lambda_f \Lambda_g d\gamma = e^{<f,g>}.$$

PROOF In fact,

$$\int_X \Lambda_f \Lambda_g d\gamma = < \Lambda_f, \Lambda_g >$$

$$= < T\, \underline{\exp}\, f, T\, \underline{\exp}\, g >$$

$$= < \underline{\exp}\, f, \underline{\exp}\, g >$$

$$= e^{<f,g>} \text{ (cf. 6.6).]}$$

28.10 REMARK The preceding considerations generalize to the infinite dimensional case what has been already seen in the finite dimensional case. Thus take $X = \underline{R}^n$ and let

$$d\gamma(x) = \frac{1}{(2\pi)^{n/2}} e^{-x^2/2} dx$$

$$= \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_k^2/2} dx_k.$$

Here $X^* = X_\gamma^* = \underline{R}^n$. Let $f \in X_\gamma^*$, say

$$f(x) = a_1x_1 + \cdots + a_nx_n.$$

Then

$$||f||^2_{L^2(\gamma)} = \int_{\underline{R}^n} (\sum_{k=1}^{n} a_kx_k)^2 \, d\gamma(x)$$

$$= \prod_{k=1}^{n} a_k^2 \cdot \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} x_k^2 e^{-x_k^2/2} \, dx_k$$

$$= \prod_{k=1}^{n} a_k^2,$$

the square of the euclidean norm of f. Moreover, the arrow

$$\underline{\exp}(f) = \underline{\exp}(a_1,\ldots,a_n)$$

$$\rightarrow \exp(\sum_{k=1}^{n} a_kx_k - \frac{1}{2} \sum_{k=1}^{n} a_k^2) = \Lambda_f$$

identifies $BO(\underline{R}^n)$ with $L^2(\underline{R}^n,\gamma)$.

Let $f_1,\ldots,f_n \in X_\gamma^*$ — then by construction,

$$T_n(P_n(f_1 \otimes \cdots \otimes f_n)) = \frac{1}{\sqrt{n!}} I_n(f_1\ldots f_n).$$

[Note: Bear in mind that

$$f_1 \cdots f_n \in L^2(X,\gamma) \quad (\text{cf. } 28.2).]$$

28.11 LEMMA We have

$$\int_X I_n(f_1' \cdots f_n') I_n(f_1'' \cdots f_n'') d\gamma(x)$$

$$= \sum_{\sigma \in S_n} < f_{\sigma(1)}', f_1'' > \cdots < f_{\sigma(n)}', f_n'' >.$$

[This is clear (T being isometric).]

28.12 LEMMA Let $f \in X_\gamma^*$ $(f \neq 0)$ -- then

$$I_n(f^n)$$

$$= \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} \left(-\frac{1}{2} <f,f>\right)^k f^{n-2k}.$$

PROOF For

$$I_n(f^n) = ||f||_2^n H_n\left(\frac{f}{||f||_2}\right).$$

And

$$H_n\left(\frac{f}{||f||_2}\right) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^k k!(n-2k)!} \left(\frac{f}{||f||_2}\right)^{n-2k}.$$

[Note: The linear span of the $I_n(f^n)$ $(f \in X_\gamma^*)$ is dense in W_n (cf. 6.5).]

The final result of this § is the generalization of 28.1 from $n = 1$ to $n > 1$, thus taking us full circle.

28.13 LEMMA We have

$$W_n \subset \bigcap_{0<p<\infty} L^p(X,\gamma)$$

and $\forall\ p,q < \infty$, $\exists\ C_n(p,q) > 0$: $\forall\ f \in W_n$,

$$||f||_q \leq C_n(p,q)\,||f||_p.$$

In addition, W_n is a closed subspace of $L^p(X,\gamma)$ and is closed w.r.t. convergence in measure.

[Note: The topology of convergence in measure on W_n coincides with the L^2-topology, hence with the L^p-topologies.]

The first step is to prove that

$$||f||_q \leq C_n(p,q)\,||f||_p$$

when $2 \leq p < q$. Since this is a simple corollary to the generalities outlined in the next §, details will be postponed until then. However, it is perfectly possible to proceed in an elementary (albeit tedious) manner, starting with $p = 2$, $q = 4$, and from there by induction to $p = 2$, $q = 2k$, which suffices. Indeed, given $2 < p < q$, choose $2k > q$ -- then

$$\begin{aligned} ||f||_p &\geq ||f||_2 \\ &\geq C_n(2,2k)^{-1}\,||f||_{2k} \\ &\geq C_n(2,2k)^{-1}\,||f||_q \end{aligned}$$

=>

$$||f||_q \le C_n(2,2k)\,||f||_p.$$

Suppose next that $0 < p < 2 \le q$. Choose $r > q$ and define $s \in]0,1[$ by $\frac{1}{q} = \frac{s}{p} + \frac{1-s}{r}$ -- then

$$\int_X |f|^q \, d\gamma = \int_X |f|^{sq}|f|^{(1-s)q} \, d\gamma$$

$$\le ||\ |f|^{sq}||_{p/sq}||\ |f|^{(1-s)q}||_{r/(1-s)q}$$

$$= (\int_X |f|^p \, d\gamma)^{sq/p}(\int_X |f|^r \, d\gamma)^{(1-s)q/r}$$

$$= ||f||_p^{sq}||f||_r^{(1-s)q}$$

=>

$$||f||_q \le ||f||_p^s \, ||f||_r^{1-s}$$

$$\le ||f||_p^s(C_n(q,r))^{1-s}||f||_q^{1-s}$$

=>

$$||f||_q^s \le (C_n(q,r))^{1-s}||f||_p^s$$

=>

$$||f||_q \le (C_n(q,r))^{(1-s)/s}||f||_p.$$

This leaves two possibilities:

1. $0 < p < q < 2$:

$$||f||_q \leq ||f||_2 \leq C_n(p,2)||f||_p.$$

2. $0 < q \leq p$:

$$||f||_q \leq ||f||_p.$$

28.14 <u>RAPPEL</u> Let $\{\xi_k : k \geq 1\}$ be a sequence of random variables on a probability space (Ω, A, μ). Fix $p: 0 < p < \infty$.

- If $\xi_k, \xi \in L^p(\Omega,\mu)$ and if $\xi_k \underset{L^p}{\longrightarrow} \xi$, then $\xi_k \to \xi$ in measure.

- If $\xi_k \to \xi$ in measure and if the $|\xi_k|^p$ are uniformly integrable, then $\xi \in L^p(X,\mu)$ and $\xi_k \underset{L^p}{\rightarrow} \xi$.

<u>N.B.</u> If the $\xi_k \in L^1(\Omega,\mu)$ and if $\exists\ p > 1$, $M > 0$ such that

$$\int_\Omega |\xi_k|^p \, d\mu \leq M \ \forall\ k,$$

then the $|\xi_k|$ are uniformly integrable.

Returning to 28.13, suppose that $\{f_k : k \geq 1\}$ is a sequence in $W_n : f_k \to f$ in measure -- then we claim that $\{f_k : k \geq 1\}$ is L^2-Cauchy. For if not, then $\exists$ increasing sequences $u(k)$, $v(k)$ and $\varepsilon > 0$:

$$||f_{u(k)} - f_{v(k)}||_2 \geq \varepsilon > 0.$$

Let

$$F_k = \frac{f_{u(k)} - f_{v(k)}}{||f_{u(k)} - f_{v(k)}||_2}.$$

Then $F_k \to 0$ in measure. On the other hand, $||F_k||_2 = 1$, thus the $|F_k|$ are uniformly integrable. Therefore $||F_k||_1 \to 0$, contradicting

$$1 = ||F_k||_2 \leq C_n(1,2)||F_k||_1.$$

So $\{f_k : k \geq 1\}$ is L^2-Cauchy, hence $f \in L^2(X,\gamma)$ and $f_k \underset{L^2}{\longrightarrow} f$. The earlier discussion then implies that $f_k \underset{L^p}{\longrightarrow} f$ $(0 < p < \infty)$. And the rest is now obvious.

§29. CONTRACTION THEORY

Let X be a separable LF-space. Suppose that γ is a centered gaussian measure on X -- then as we have seen in §28, there is a canonical isometric isomorphism

$$T:BO(X_\gamma^*) \to L^2(X,\gamma)$$

such that

$$T\,\underline{\exp}(f) = \Lambda_f \quad (f \in X_\gamma^*).$$

Let $A:X_\gamma^* \to X_\gamma^*$ be a bounded linear operator with $||A|| \leq 1$. Define

$$\Gamma(A):BO(X_\gamma^*) \to BO(X_\gamma^*)$$

as in 6.14. Put

$$\Gamma_T(A) = T\Gamma(A)T^{-1}.$$

Then

$$\Gamma_T(A):L^2(X,\gamma) \to L^2(X,\gamma)$$

is a bounded linear operator such that

$$\Gamma_T(A)\Lambda_f = \Lambda_{Af}.$$

29.1 <u>LEMMA</u> $\Gamma_T(A)$ admits a unique extension to a bounded linear operator

$$\Gamma_T(A):L^1(X,\gamma) \to L^1(X,\gamma)$$

such that $\forall$ $f \in L^p(X,\gamma)$,

$$||\Gamma_T(A)f||_p \leq ||f||_p \quad (1 \leq p < \infty).$$

29.2 <u>EXAMPLE</u> If $|r| \leq 1$ ($r \in \underset{\sim}{R}$), then

$$\Gamma_T(rI)\Lambda_f = \Lambda_{rf}.$$

[Note: As a special case,

$$\Gamma(e^{-t}I) = e^{-tN} \quad (t \geq 0)$$

=>

$$\Gamma_T(e^{-t}I) = Te^{-tN}T^{-1},$$

which is precisely the Ornstein-Uhlenbeck semigroup (see §30).]

29.3 <u>REMARK</u> Fix $r:|r| < 1$ — then $\forall\ f \in L^2(X,\gamma)$,

$$\Gamma_T(rI)f\Big|_x = \int_X f(rx + (1-r^2)^{1/2}y)d\gamma(y).$$

29.4 <u>THEOREM</u> (Nelson) If $1 \leq p \leq q < \infty$ and if

$$||A|| \leq \left[\frac{p-1}{q-1}\right]^{1/2} \qquad (\tfrac{0}{0} = 1),$$

then $\Gamma_T(A)$ maps $L^p(X,\gamma)$ into $L^q(X,\gamma)$ with

$$||\Gamma_T(A)||_{p,q} = 1.$$

Although we shall not stop to give the proof of this result (it can be approached in a number of ways), note that

$$\Gamma(A) = \Gamma(||A||\ I)\Gamma(A/||A||) \quad (A \neq 0),$$

thus it suffices to consider the case when A = rI subject to

$$0 \leq r \leq \left[\frac{p-1}{q-1}\right]^{1/2}.$$

29.5 LEMMA $\forall\ f \in X_\gamma^*$,

$$||\Lambda_f||_p = \exp(\frac{p-1}{2}\ ||f||_2^2) \quad (0 < p < \infty).$$

PROOF In fact,

$$\int_X \Lambda_f^p d\gamma$$

$$= \int_X \exp(f - \frac{1}{2}\ ||f||_2^2)^p d\gamma$$

$$= \exp(-\frac{p}{2}\ ||f||_2^2) \int_X \exp(pf)d\gamma$$

$$= \exp(-\frac{p}{2}\ ||f||_2^2)\exp(\frac{p^2}{2}\ ||f||_2^2) \quad \text{(cf. 26.9 (and 26.17))}$$

$$= \exp(\frac{p^2-p}{2}\ ||f||_2^2)$$

=>

$$||\Lambda_f||_p = \exp(\frac{p-1}{2}\ ||f||_2^2).$$

29.6 <u>REMARK</u> If

$$||A|| > \left[\frac{p-1}{q-1} \right]^{1/2},$$

then $\Gamma_T(A)$ does not map $L^p(X,\gamma)$ into $L^q(X,\gamma)$.

[If $\Gamma_T(A)$ maps $L^p(X,\gamma)$ into $L^q(X,\gamma)$, then it is bounded (closed graph theorem), so $\exists\ C > 0$: $\forall\ f \in X_\gamma^*$ & $\forall\ t \in \underline{R}$,

$$||\Lambda_{tAf}||_q \leq C||\Lambda_{tf}||_p$$

or still (cf. 29.5),

$$\exp(\frac{q-1}{2}\, t^2||Af||_2^2) \leq C\, \exp(\frac{p-1}{2}\, t^2||f||_2^2)$$

$\Rightarrow$

$$(q-1)\,||Af||_2^2 \leq (p-1)\,||f||_2^2$$

$\Rightarrow$

$$||A||^2 \leq \frac{p-1}{q-1}\,.]$$

We are now in a position to tie up the loose end in 28.13 which, as will be recalled, is the assertion that $\forall\ f \in W_n$,

$$||f||_q \leq C_n(p,q)\,||f||_p$$

when $2 \leq p < q$.

To begin with, it is clear that $\forall\ f \in W_n$,

$$\Gamma_T(rI)f = r^n f.$$

This said, assume that $2 \leq p < q$. Write

$$f = \Gamma_T((q-1)^{-1/2}I)(q-1)^{n/2}f.$$

Then

$$\left[\frac{p-1}{q-1}\right]^{1/2} \leq \left[\frac{2-1}{q-1}\right]^{1/2}$$

$$\leq \frac{1}{(q-1)^{1/2}}$$

$$= ||(q-1)^{-1/2}I||,$$

so by 29.4,

$$||f||_q = ||\Gamma_T((q-1)^{-1/2}I)(q-1)^{n/2}f||_q$$

$$\leq ||\Gamma_T((q-1)^{-1/2}I)||_{p,q}||(q-1)^{n/2}f||_p$$

$$= (q-1)^{n/2}||f||_p,$$

as desired.

§30. *SOBOLEV SPACES*

Let X be a separable LF-space. Suppose that γ is a centered gaussian measure on X -- then there is a canonical isometric isomorphism

$$T:BO(X_\gamma^*) \to L^2(X,\gamma)$$

such that $\forall\ f \in X_\gamma^*$,

$$T\ \underline{\exp}(f) = \Lambda_f \quad (\text{cf. §28}).$$

Put

$$T_t = \Gamma_T(e^{-t}I) = Te^{-tN}T^{-1} \quad (t \geq 0).$$

Then the collection $\{T_t:t \geq 0\}$ is a strongly continuous semigroup on $L^2(X,\gamma)$ with $||T_t|| = 1\ \forall\ t$, the Ornstein-Uhlenbeck semigroup.

30.1 LEMMA $\forall\ f \in W_n$,

$$T_t f = e^{-tn} f$$

and $\forall\ f \in L^2(X,\gamma)$,

$$T_t f = \sum_{n=0}^{\infty} e^{-tn} I_n(f).$$

30.2 EXAMPLE Let $f \in X_\gamma^*$ $(f \neq 0)$ -- then

$$\Lambda_{e^{-t}f} = \exp(e^{-t}f - \frac{1}{2}||e^{-t}f||_2^2)$$

$$= \sum_{n=0}^{\infty} \frac{||e^{-t}f||_2^n}{n!} H_n\left(\frac{e^{-t}f}{||e^{-t}f||_2}\right)$$

$$= \sum_{n=0}^{\infty} e^{-tn} \frac{||f||_2^n}{n!} H_n\left(\frac{f}{||f||_2}\right)$$

$$= \sum_{n=0}^{\infty} e^{-tn} I_n(\Lambda_f).$$

On the other hand,

$$T_t\Lambda_f = Te^{-tN}T^{-1}\Lambda_f$$

$$= Te^{-tN}\underline{\exp}(f)$$

$$= Te^{-tN} \sum_{n=0}^{\infty} \frac{f^{\otimes n}}{\sqrt{n!}}$$

$$= \sum_{n=0}^{\infty} e^{-tn} \frac{Tf^{\otimes n}}{\sqrt{n!}}$$

$$= \sum_{n=0}^{\infty} e^{-tn} \frac{1}{n!} I_n(f^n)$$

$$= \sum_{n=0}^{\infty} e^{-tn} I_n(\Lambda_f).$$

Therefore

$$T_t\Lambda_f = \Lambda_{e^{-t}f}.$$

30.3 REMARK In view of 29.1, T_t admits a unique extension to a bounded linear operator

$$T_t : L^1(X,\gamma) \to L^1(X,\gamma)$$

such that $\forall\ f \in L^p(X,\gamma)$,

$$||T_t f||_p \le ||f||_p \quad (1 \le p < \infty).$$

[Note: If $1 \le p \le q < \infty$ and if

$$e^{-t} \le \left[\frac{p-1}{q-1} \right]^{1/2} \qquad (\tfrac{0}{0} = 1),$$

then T_t maps $L^p(X,\gamma)$ into $L^q(X,\gamma)$ with

$$||T_t||_{p,q} = 1 \quad \text{(cf. 29.4).]}$$

30.4 LEMMA If $1 < p < 2$, then $L^p(X,\gamma) \supset L^2(X,\gamma)$ and I_n extends to a bounded linear operator on $L^p(X,\gamma)$.

PROOF Choose $t > 0 : 2 = e^{2t}(p-1) + 1$ -- then $\forall\ f \in L^2(X,\gamma)$,

$$||e^{-nt} I_n(f)||_p$$

$$= ||T_t I_n(f)||_p$$

$$\le ||T_t I_n(f)||_2$$

$$= ||I_n(T_t f)||_2$$

$$\leq ||T_t f||_2$$

$$\leq ||f||_p$$

=>

$$||I_n(f)||_p \leq e^{nt}||f||_p.$$

[Note: This fails for $p = 1$.]

30.5 <u>LEMMA</u> If $p > 2$, then $L^p(X,\gamma) \subset L^2(X,\gamma)$ and I_n restricts to a bounded linear operator on $L^p(X,\gamma)$.

<u>PROOF</u> Choose $t > 0 : p = e^{2t} + 1$ -- then $\forall\, f \in L^p(X,\gamma)$,

$$||e^{-nt} I_n(f)||_p$$

$$= ||T_t I_n(f)||_p$$

$$\leq ||I_n(f)||_2$$

$$\leq ||f||_2$$

$$\leq ||f||_p$$

=>

$$||I_n(f)||_p \leq e^{nt}||f||_p.$$

30.6 <u>REMARK</u> If $1 < p < 2$ or $2 < p$, then $\exists\, f \in L^p(X,\gamma)$ such that

$$\sum_{n=0}^{N} I_n(f) \not\to f \quad (N \to \infty)$$

in $L^p(X,\gamma)$ and

$$\sup_n ||I_n(f)||_p = \infty.$$

Define L by the relation

$$TNT^{-1} = -L.$$

Then L is selfadjoint and is the generator of the semigroup $\{T_t : t \geq 0\}$ on $L^2(X,\gamma)$.

30.7 <u>LEMMA</u> The domain of definition Dom(L) of L is

$$\{f: \sum_{n=0}^{\infty} n^2 ||I_n(f)||_2^2 < \infty\}.$$

And on this domain

$$Lf = -\sum_{n=0}^{\infty} nI_n(f).$$

[Note:

$$\text{Dom}(L) = T\,\text{Dom}(N) \quad (\text{cf. } 6.17).]$$

30.8 <u>EXAMPLE</u> Let $f \in X_\gamma^*$ $(f \neq 0)$ — then $\Lambda_f \in \text{Dom}(L)$ and

$$L\Lambda_f = (||f||_2^2 - f)\Lambda_f.$$

In fact,

$$L\Lambda_f = \frac{d}{dt} T_t \Lambda_f \Big|_{t=0}$$

$$= \frac{d}{dt} \Lambda_{e^{-t}f} \Big|_{t=0} \qquad \text{(cf. 30.2)}$$

$$= \frac{d}{dt} \exp(e^{-t}f - \tfrac{1}{2}||e^{-t}f||^2) \Big|_{t=0}$$

$$= (||f||_2^2 - f)\Lambda_f.$$

When specialized to the finite dimensional case, it is clear that the preceding considerations are equivalent to those of §23, where it was pointed out that Dom(L) is a Sobolev space, L being realized as

$$\Delta - x\cdot\nabla \quad (X = \underline{R}^n, \ \gamma = \gamma_n) \quad \text{(cf. 31.1)}.$$

How does one extend this set of circumstances to the infinite dimensional case?

Using the spectral theorem, write

$$- L = \int_0^\infty \lambda \, dE_\lambda.$$

Then

$$T_t = \int_0^\infty e^{-t\lambda} \, dE_\lambda.$$

30.9 <u>LEMMA</u> Given $r > 0$, we have

$$(1 - L)^{-r/2} = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} T_t dt.$$

PROOF Work from the LHS to the RHS:

$$(1 - L)^{-r/2} = \int_0^\infty (1 + \lambda)^{-r/2} \, dE_\lambda$$

$$= \frac{1}{\Gamma(r/2)} \int_0^\infty (1 + \lambda)^{-r/2} \, dE_\lambda \int_0^\infty u^{r/2-1} e^{-u} \, du$$

$$= \frac{1}{\Gamma(r/2)} \int_0^\infty dE_\lambda \int_0^\infty e^{-u} \left[\frac{u}{1+\lambda}\right]^{r/2-1} \frac{du}{1+\lambda}$$

$$= \frac{1}{\Gamma(r/2)} \int_0^\infty dE_\lambda \int_0^\infty t^{r/2-1} e^{-(\lambda+1)t} \, dt$$

$$= \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} \, dt \int_0^\infty e^{-t\lambda} \, dE_\lambda$$

$$= \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} \, T_t \, dt.$$

30.10 LEMMA $\forall\, r > 0$ & $\forall\, p \geq 1$, $(1 - L)^{-r/2}$ is a bounded linear operator on $L^p(X,\gamma)$ of norm 1.

PROOF $\forall\, f \in L^p(X,\gamma)$,

$$||(1 - L)^{-r/2} f||_p$$

$$\leq \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} ||T_t f||_p \, dt$$

$$\leq \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} ||f||_p \, dt$$

$$= ||f||_p.$$

Since constants are preserved, the norm of $(1 - L)^{-r/2}$ is exactly one.

30.11 LEMMA $\forall\ r,s > 0,$

$$(1 - L)^{-r/2}\ (1 - L)^{-s/2} = (1 - L)^{-(r+s)/2}$$

as bounded linear operators on $L^p(X,\gamma)$ $(p \geq 1)$.

PROOF Write

$$(1 - L)^{-r/2}\ (1 - L)^{-s/2}$$

$$= \frac{1}{\Gamma(r/2)\Gamma(s/2)} \int_0^\infty \int_0^\infty t^{r/2-1}\ u^{s/2-1}\ e^{-t}\ e^{-u}\ T_t T_u\ dt\ du$$

$$= \frac{1}{\Gamma(r/2)\Gamma(s/2)} \int_0^\infty \int_0^\infty t^{r/2-1}\ u^{s/2-1}\ e^{-(t+u)}\ T_{t+u}\ dt\ du$$

$$= \frac{1}{\Gamma(r/2)\Gamma(s/2)} \int_0^\infty e^{-w}\ T_w dw \int_0^w v^{r/2-1}\ (w-v)^{s/2-1}\ dv$$

$$= \frac{1}{\Gamma(r/2)\Gamma(s/2)} \int_0^\infty w^{(r+s)/2-1}\ e^{-w}\ T_w dw \int_0^1 x^{r/2-1}\ (1-x)^{s/2-1}\ dx$$

$$= \frac{B(r/2,s/2)}{\Gamma(r/2)\Gamma(s/2)} \int_0^\infty w^{(r+s)/2-1}\ e^{-w}\ T_w dw$$

$$= \frac{1}{\Gamma((r+s)/2)} \int_0^\infty w^{(r+s)/2-1}\ e^{-w}\ T_w dw$$

$$= (1 - L)^{-(r+s)/2}.$$

30.12 REMARK Put $(1 - L)^0 = 1$ — then the collection $\{(1 - L)^{-r/2} : r \geq 0\}$ is a strongly continuous semigroup on $L^p(X,\gamma)$ $(p \geq 1)$.

[Note: L^p-continuity follows from L^2-continuity and the latter is immediate.]

30.13 LEMMA $\forall\ r > 0$, $(1 - L)^{-r/2}$ is injective.

PROOF $(1 - L)^{-1}$ is certainly injective. To establish injectivity in the range $0 < r < 2$, write

$$(1 - L)^{-(2-r)/2}\ (1 - L)^{-r/2} = (1 - L)^{-1} \quad \text{(cf. 30.11)}.$$

To establish injectivity in the range $r > 2$, bootstrap back to the case $0 < r \leq 2$.

30.14 LEMMA $\forall\ r > 0$,

$$(1 - L)^{-r/2}\ L^p(X,\gamma)$$

is dense in $L^p(X,\gamma)$ $(p \geq 1)$.

PROOF $\forall\ f \in W_n$,

$$(1 - L)^{-r/2} f = (n + 1)^{-r/2} f$$

$$\Rightarrow$$

$$f = (n + 1)^{r/2}\ (1 - L)^{-r/2} f$$

$$\Rightarrow$$

$$(1 - L)^{-r/2}\ L^p(X,\gamma) \supset W_n.$$

[Note: Recall that $\forall\ n$,

$$W_n \subset L^p(X,\gamma) \quad \text{(cf. 28.13)}.]$$

Put

$$\begin{cases} W^{p,r}(X,\gamma) = (1 - L)^{-r/2} L^p(X,\gamma) \\ ||f||_{p,r} = ||(1 - L)^{r/2} f||_p. \end{cases}$$

Then $W^{p,r}(X,\gamma)$ is complete and will be termed the <u>Sobolev space</u> per the pair (p,r) $(p \geq 1,\ r \geq 0)$.

[Note: When $r = 0$,

$$W^{p,0}(X,\gamma) = L^p(X,\gamma).]$$

30.15 <u>LEMMA</u> The domain of definition $\text{Dom}(L)$ of L is $W^{2,2}(X,\gamma)$.

<u>PROOF</u> Suppose that $f \in W^{2,2}(X,\gamma)$:

$$f = (1 - L)^{-1}g \quad (g \in L^2(X,\gamma)).$$

Then

$$I_n(f) = I_n((1 - L)^{-1}g)$$

$$= (n + 1)^{-1} I_n(g)$$

=>

$$\sum_{n=0}^{\infty} n^2||I_n(f)||_2^2$$

$$= \sum_{n=0}^{\infty} \frac{n^2}{(n+1)^2} ||I_n(g)||_2^2$$

$$\leq \sum_{n=0}^{\infty} ||I_n(g)||_2^2$$

$$\leq ||g||_2^2 < \infty.$$

Conversely, given $f \in \text{Dom}(L)$, put $g = f - Lf$ — then

$$(1 - L)^{-1}g = (1 - L)^{-1}(1 - L)f$$

$$= f.$$

30.16 LEMMA Suppose that $1 \leq p \leq p'$ and $r \leq r'$ — then

$$W^{p',r'}(X,\gamma) \subset W^{p,r}(X,\gamma)$$

and

$$||f||_{p,r} \leq ||f||_{p',r'} \ \forall \ f \in W^{p',r'}(X,\gamma).$$

PROOF For

$$||f||_{p,r} = ||(1 - L)^{r/2}f||_p$$

$$= ||(1 - L)^{-(r'-r)/2}\ (1 - L)^{r'/2}f||_p \quad \text{(cf. 30.11)}$$

$$\leq ||(1 - L)^{r'/2}f||_p \quad \text{(cf. 30.10)}$$

$$= ||f||_{p,r'}$$

$$\leq ||f||_{p',r'}.$$

We have defined $W^{p,r}(X,\gamma)$ if $p \geq 1$, $r \geq 0$ and by construction

$$(1 - L)^{-r/2}:L^p(X,\gamma) \to W^{p,r}(X,\gamma)$$

is an isometric isomorphism. Given $f \in L^p(X,\gamma)$, put

$$||f||_{p,-r} = ||(1 - L)^{-r/2}f||_p.$$

Denote by $W^{p,-r}(X,\gamma)$ the completion of $L^p(X,\gamma)$ w.r.t. $||\cdot||_{p,-r}$ -- then

$$(1 - L)^{-r/2}:L^p(X,\gamma) \to L^p(X,\gamma)$$

extends to an isometric isomorphism

$$(1 - L)^{-r/2}:W^{p,-r}(X,\gamma) \to L^p(X,\gamma).$$

[Note: In general, the elements of $W^{p,-r}(X,\gamma)$ are not functions.]

30.17 <u>LEMMA</u> Fix $p,q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$ and $r \geq 0$ -- then the dual of $W^{p,r}(X,\gamma)$ is $W^{q,-r}(X,\gamma)$.

<u>PROOF</u> Denote the arrows

$$\begin{cases} (1 - L)^{-r/2}:L^p(X,\gamma) \to W^{p,r}(X,\gamma) \\ (1 - L)^{-r/2}:W^{q,-r}(X,\gamma) \to L^q(X,\gamma) \end{cases}$$

by

$$\begin{cases} A_{p,r} \\ A_{q,-r}. \end{cases}$$

Then the composite

$$W^{q,-r}(X,\gamma) \xrightarrow{A_{q,-r}} L^q(X,\gamma) \approx L^p(X,\gamma)^* \xrightarrow{(A^*_{p,r})^{-1}} W^{p,r}(X,\gamma)^*$$

identifies $W^{p,r}(X,\gamma)^*$ with $W^{q,-r}(X,\gamma)$.

[Note: If $f \in W^{p,r}(X,\gamma)$ $(\subset L^p(X,\gamma))$ and if $g \in L^q(X,\gamma)$ $(\subset W^{q,-r}(X,\gamma))$, then

$$_{p,r}\langle f,g \rangle_{q,-r} = \int_X (1 - L)^{r/2} f \ (1 - L)^{-r/2} g \ d\gamma(x)$$

$$= {}_p\langle f,g \rangle_q.]$$

30.18 <u>REMARK</u> Let E be a separable real Hilbert space -- then the spaces

$$\begin{cases} W^{p,r}(X,\gamma;E) \\ W^{q,-r}(X,\gamma;E) \end{cases}$$

can be defined in the obvious way and it is still the case that

$$W^{p,r}(X,\gamma;E)^* = W^{q,-r}(X,\gamma;E) \quad (p,q > 1: \frac{1}{p} + \frac{1}{q} = 1 \text{ and } r \geq 0).$$

§31. DERIVATIVES

Let $\Phi:\underline{R}^n \to \underline{R}$ — then Φ is said to be slowly increasing if Φ is C^∞ and it and all its partial derivates are of polynomial growth.

[Note: In particular, every polynomial is slowly increasing.]

Write $\mathcal{O}(\underline{R}^n)$ for the set of slowly increasing functions on $\underline{R}^n$ — then each $\Phi \in \mathcal{O}(\underline{R}^n)$ has a gradient $\nabla\Phi$ and $\forall\ x,h \in \underline{R}^n$, we have

$$\frac{d}{dt}\,\Phi(x + th)\Big|_{t=0} = \langle h, \nabla\Phi(x) \rangle.$$

Here

$$\nabla\Phi(x) = (\partial_1\Phi(x),\ldots,\partial_n\Phi(x)).$$

[Note: Obviously,

$$\nabla\Phi \in \mathcal{O}(R^n;R^n),\ \nabla^2\Phi \in \mathcal{O}(\underline{R}^n;\underline{R}^n \otimes \underline{R}^n),\ldots\ .]$$

31.1 LEMMA Let γ_n be the standard gaussian measure on $\underline{R}^n$ — then $\mathcal{O}(\underline{R}^n) \subset \text{Dom}(L)$ and $\forall\ \Phi \in \mathcal{O}(\underline{R}^n)$,

$$L\Phi(x) = \Delta\Phi(x) - \sum_{i=1}^{n} x_i\partial_i\Phi(x).$$

PROOF For $t > 0$,

$$\frac{d}{dt}\,T_t\Phi(x)$$

$$= \frac{d}{dt} \int_{\underline{R}^n} \Phi(e^{-t}x + (1 - e^{-2t})^{1/2}y)\, d\gamma_n(y)$$

$$= - \int_{\underline{R}^n} \sum_{i=1}^{n} e^{-t}x_i \partial_i \Phi(e^{-t}x + (1 - e^{-2t})^{1/2}y)\, d\gamma_n(y)$$

$$+ \int_{\underline{R}^n} \sum_{i=1}^{n} \partial_i \Phi(e^{-t}x + (1 - e^{-2t})^{1/2}y) \frac{y_i e^{-2t}}{(1-e^{-2t})^{1/2}}\, d\gamma_n(y)$$

$$= - \int_{\underline{R}^n} \sum_{i=1}^{n} e^{-t}x_i \partial_i \Phi(e^{-t}x + (1 - e^{-2t})^{1/2}y)\, d\gamma_n(y)$$

$$- \int_{\underline{R}^n} \sum_{i=1}^{n} \partial_i \Phi(e^{-t}x + (1 - e^{-2t})^{1/2}y) \frac{e^{-2t}}{(1-e^{-2t})^{1/2}}$$

$$\times (2\pi)^{-n/2} \partial_i (e^{-|y|^2/2})\, dy$$

$$= - e^{-t} \sum_{i=1}^{n} x_i \int_{\underline{R}^n} \partial_i \Phi(e^{-t}x + (1 - e^{-2t})^{1/2}y)\, d\gamma_n(y)$$

$$+ e^{-2t} \int_{\underline{R}^n} \Delta\Phi(e^{-t}x + (1 - e^{-2t})^{1/2}y)\, d\gamma_n(y)$$

$$= - e^{-t} \sum_{i=1}^{n} x_i T_t(\partial_i \Phi)(x) + e^{-2t}\, T_t(\Delta\Phi)(x)$$

$$\Rightarrow$$

$$L\Phi(x) = \lim_{t \to 0} \frac{d}{dt} T_t \Phi(x)$$

$$= \Delta\Phi(x) - \sum_{i=1}^{n} x_i \partial_i \Phi(x).$$

[Note: Strictly speaking the differentiation is pointwise but by dominated convergence, it takes place in $L^2(\underline{R}^n,\gamma)$.]

Let X be a separable LF-space — then a function $\alpha: X \to \underline{R}$ is <u>slowly increasing</u> if it has the form

$$\alpha(x) = \Phi(\lambda_1(x),\ldots,\lambda_n(x)),$$

where $\lambda_i \in X^*$ $(i = 1,\ldots,n)$ and $\Phi: \underline{R}^n \to \underline{R}$ is slowly increasing.

Write $\mathcal{O}(X)$ for the set of slowly increasing functions on X — then each $\alpha \in \mathcal{O}(X)$ has a gradient $\nabla\alpha$ and $\forall\ x,h \in X$, we have

$$\frac{d}{dt}\,\alpha(x + th)\Big|_{t=0} = \langle h,\nabla\alpha(x)\rangle.$$

Here

$$\nabla\alpha(x) = \sum_{i=1}^{n} \partial_i\Phi(\lambda_1(x),\ldots,\lambda_n(x))\lambda_i.$$

Suppose that γ is a centered gaussian measure on X — then $H(\gamma)$ is a separable real Hilbert space and the injection $H(\gamma) \to X$ is continuous, hence $X^* \to H(\gamma)^*$ under the arrow of restriction.

[Note: If

$$\alpha(x) = \Phi(\lambda_1(x),\ldots,\lambda_n(x))$$

is slowly increasing, then one can always arrange that the λ_i are orthonormal

(Gram-Schmidt the data).]

To reflect this additional structure, we shall say that a function $F:X \to \underline{R}$ is <u>differentiable along $H(\gamma)$</u> if $\forall\, x \in X$, $\exists$ an element

$$\nabla_\gamma F(x) \in H(\gamma)$$

such that

$$\partial_h F(x) = \frac{d}{dt} F(x + th)\Big|_{t=0} = \langle h, \nabla_\gamma F(x) \rangle \ \forall\, h \in H(\gamma).$$

[Note: If F is differentiable along $H(\gamma)$, then $\nabla_\gamma F$ is a map from X to $H(\gamma)$.]

31.2 <u>LEMMA</u> If α is slowly increasing, then α is differentiable along $H(\gamma)$ and $\forall\, x \in X$,

$$\nabla_\gamma \alpha(x) = \sum_{i=1}^{n} \partial_i \Phi(\lambda_1(x), \ldots, \lambda_n(x)) \lambda_i | H(\gamma).$$

[Note: Obviously,

$$\nabla_\gamma \alpha \in \mathcal{O}(X;H(\gamma)),\ \nabla_\gamma^2 \alpha \in \mathcal{O}(X;H(\gamma) \hat{\otimes} H(\gamma)), \ldots .]$$

31.3 <u>LEMMA</u> (Integration by Parts) Let $\alpha \in \mathcal{O}(X)$ — then $\forall\, h \in H(\gamma)$,

$$\int_X \partial_h \alpha(x) d\gamma(x) = \int_X \alpha(x) f(x) d\gamma(x) \quad (R_\gamma(f) = h).$$

<u>PROOF</u> We have

$$\int_X \partial_h \alpha(x) d\gamma(x) = \int_X \lim_{t \to 0} \frac{\alpha(x+th) - \alpha(x)}{t} d\gamma(x)$$

$$= \int_X \alpha(x) \frac{d}{dt} \exp(tf(x) - \frac{t^2}{2} ||h||^2_{H(\gamma)})\Big|_{t=0} d\gamma(x)$$

$$= \int_X \alpha(x) f(x) d\gamma(x).$$

31.4 <u>EXAMPLE</u> $\forall\ \lambda \in X^*$,

$$\int_X e^{\sqrt{-1}\ \lambda}(\partial_h \alpha) d\gamma$$

$$= -\ \sqrt{-1}\ \lambda(h) \int_X e^{\sqrt{-1}\ \lambda} \alpha d\gamma + \int_X e^{\sqrt{-1}\ \lambda}(\alpha f) d\gamma.$$

Fix $p > 1$ and define a norm $N_{p,1}$ on $\mathcal{O}(X)$ by

$$N_{p,1}(\alpha) = ||\alpha||_{L^p(\gamma)} + ||\nabla_\gamma \alpha||_{L^p(\gamma;H(\gamma))}).$$

31.5 <u>LEMMA</u> Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $\mathcal{O}(X)$ which are fundamental in the norm $N_{p,1}$ and converge in $L^p(X,\gamma)$ to ϕ -- then the sequences $\{\nabla_\gamma \alpha_n\}$ and $\{\nabla_\gamma \beta_n\}$ have the same limit in $L^p(X,\gamma;H(\gamma))$, denoted by $\nabla_\gamma \phi$ and called the <u>Sobolev derivative</u> of ϕ.

<u>PROOF</u> Given any $\lambda \in X^*$,

$$\int_X e^{\sqrt{-1}\ \lambda}(\partial_h \alpha_n) d\gamma$$

$$= -\sqrt{-1}\,\lambda(h)\int_X e^{\sqrt{-1}\,\lambda}\alpha_n d\gamma + \int_X e^{\sqrt{-1}\,\lambda}(\alpha_n f)d\gamma \qquad \text{(cf. 31.4)}$$

$$\longrightarrow$$

$$-\sqrt{-1}\,\lambda(h)\int_X e^{\sqrt{-1}\,\lambda}\phi d\gamma + \int_X e^{\sqrt{-1}\,\lambda}(\phi f)d\gamma.$$

Ditto for β_n. Since the $e^{\sqrt{-1}\,\lambda}$ are dense in $L^p(X,\gamma)$, it follows that $\partial_h\alpha_n$ and $\partial_h\beta_n$ have the same limits in $L^p(X,\gamma)$, hence

$$\lim \nabla_\gamma\alpha_n = \lim \nabla_\gamma\beta_n$$

in $L^p(X,\gamma;H(\gamma))$.

31.6 <u>THEOREM</u> (Meyer) Fix $p > 1$ — then on $\mathcal{O}(X)$, the norms $N_{p,1}$ and $||\cdot||_{p,1}$ are equivalent.

This result implies that the completion of $\mathcal{O}(X)$ w.r.t. $N_{p,1}$ can be identified with $W^{p,1}(X,\gamma)$ (up to equivalence of norms). In particular: Each element of $W^{p,1}(X,\gamma)$ admits a Sobolev derivative.

31.7 <u>REMARK</u> The entire procedure can be iterated, i.e., extended from $k = 1$ to $k > 1$.

31.8 <u>LEMMA</u> Fix $p > 1$ and $r \in \underline{R}$ — then

$$\nabla_\gamma : \mathcal{O}(X) \to \mathcal{O}(X;H(\gamma))$$

admits a unique extension to a bounded linear operator

$$\nabla_\gamma : W^{p,r+1}(X,\gamma) \to W^{p,r}(X,\gamma;H(\gamma)).$$

Fix $p,q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$ -- then by definition,

$$\nabla_\gamma^* : W^{q,-r}(X,\gamma;H(\gamma)) \to W^{q,-r-1}(X,\gamma)$$

is the dual to

$$\nabla_\gamma : W^{p,r+1}(X,\gamma) \to W^{p,r}(X,\gamma;H(\gamma)) \qquad \text{(cf. 30.17).}$$

N.B. It therefore makes sense to form $\pm \nabla_\gamma^* \nabla_\gamma$, where

$$\nabla_\gamma : W^{p,r+1}(X,\gamma) \to W^{p,r}(X,\gamma;H(\gamma))$$

and

$$\nabla_\gamma^* : W^{p,r}(X,\gamma;H(\gamma)) \to W^{p,r-1}(X,\gamma).$$

31.9 LEMMA Let

$$\left[\begin{array}{l} \nabla_\gamma : W^{2,1}(X,\gamma) \to L^2(X,\gamma;H(\gamma)) \\ \\ \nabla_\gamma^* : W^{2,1}(X,\gamma;H(\gamma)) \to L^2(X,\gamma). \end{array}\right.$$

Then $\forall\ \phi \in W^{2,1}(X,\gamma)\ \&\ \forall\ A \in W^{2,1}(X,\gamma;H(\gamma))$,

$$\int_X \langle \nabla_\gamma \phi(x), A(x) \rangle_{H(\gamma)} d\gamma(x)$$

$$= \int_X \phi(x)\nabla_\gamma^* A(x)\,d\gamma(x).$$

31.10 EXAMPLE Recall that $W^{2,2}(X,\gamma)$ is the domain of L (cf. 30.15). This said, we claim that

$$L = -\nabla_\gamma^*\nabla_\gamma,$$

where

$$\nabla_\gamma : W^{2,2}(X,\gamma) \to W^{2,1}(X,\gamma;H(\gamma))$$

and

$$\nabla_\gamma^* : W^{2,1}(X,\gamma;H(\gamma)) \to L^2(X,\gamma).$$

Thus let $\alpha,\beta \in \mathcal{O}(X)$ -- then

$$< L\alpha,\beta >_{L^2(\gamma)} = -\int_X < \nabla_\gamma\alpha,\nabla_\gamma\beta >_{H(\gamma)} d\gamma$$

$$= -\int_X < \nabla_\gamma\beta,\nabla_\gamma\alpha >_{H(\gamma)} d\gamma$$

$$= -\int_X \beta(\nabla_\gamma^*\nabla_\gamma\alpha)\,d\gamma$$

$$= \int_X (-\nabla_\gamma^*\nabla_\gamma\alpha)\,\beta d\gamma$$

=>

$$L\alpha = -\nabla_\gamma^*\nabla_\gamma\alpha.$$

[Note: To check that

$$< L\alpha,\beta >_{L^2(\gamma)} = - \int_X < \nabla_\gamma\alpha,\nabla_\gamma\beta >_{H(\gamma)} d\gamma,$$

take $X = \underline{R}^n$, $\gamma = \gamma_n$, and apply 31.1.]

The divergence of an element $A \in W^{2,1}(X,\gamma;H(\gamma))$, written div A, is $- \nabla_\gamma^* A$. Accordingly, with this convention,

$$L = \text{div } \nabla_\gamma.$$

[Note: In $\underline{R}^n$, the laplacian is the divergence of the gradient.]

31.11 LEMMA Fix an orthonormal basis $\{h_j : j \geq 1\}$ for $H(\gamma)$. Given $A \in W^{2,1}(X,\gamma;H(\gamma))$, write

$$A = \sum_{j=1}^{\infty} A_j h_j \quad (A_j \in W^{2,1}(X,\gamma)).$$

Then

$$\text{div } A = \sum_{j=1}^{\infty} (\partial_{h_j} A_j - A_j f_j) \quad (R_\gamma(f_j) = h_j),$$

the series being convergent in $L^2(X,\gamma)$.

[Note: In general, the series

$$\left[\begin{array}{l} \sum_{j=1}^{\infty} \partial_{h_j} A_j \\ \\ \sum_{j=1}^{\infty} A_j f_j \end{array}\right.$$

do not converge on their own.]

31.12 EXAMPLE Let $\alpha \in \mathcal{O}(X)$ — then

$$L\alpha(x) = \sum_{j=1}^{\infty} \partial_{h_j}^2 \alpha(x) - \sum_{j=1}^{\infty} f_j(x)\partial_{h_j}\alpha(x).$$

Compare this with 31.1: The role of $\Delta\alpha(x)$ is played by

$$\sum_{j=1}^{\infty} \partial_{h_j}^2 \alpha(x)$$

and the role of $x\cdot\nabla\alpha(x)$ is played by

$$\sum_{j=1}^{\infty} f_j(x)\partial_{h_j}\alpha(x).$$

§32. *THE H-DERIVATIVE*

Let X,Y be Banach spaces over $\underline{R}$ -- then a function $F:X \to Y$ is said to be <u>differentiable at $x \in X$</u> if $\exists$ a continuous linear map $DF(x):X \to Y$ such that

$$\lim_{\Delta x \to 0} \frac{||F(x+\Delta x) - F(x) - DF(x)\Delta x||}{||\Delta x||} = 0,$$

F being called <u>differentiable</u> if F is differentiable at each $x \in X$.

[Note: A differentiable function is necessarily continuous.]

The derivative of a differentiable function F is thus a map

$$DF:X \to \mathcal{B}(X,Y).$$

32.1 <u>EXAMPLE</u> Take $Y = \underline{R}$ — then $DF:X \to X^*$ and F admits a gradient, viz. $\nabla F(x) = DF(x)$.

Equip $\mathcal{B}(X,Y)$ with the operator norm. Suppose that $F:X \to Y$ is differentiable — then it makes sense to consider the derivative of DF, the second derivative of F:

$$D^2F:X \to \mathcal{B}(X,\mathcal{B}(X,Y))$$

or still,

$$D^2F:X \to \mathcal{B}_2(X,Y),$$

where $\mathcal{B}_2(X,Y)$ is the Banach space of continuous bilinear maps of $X \times X$ into Y.

[Note: This process can, of course, be iterated.]

32.2 REMARK By definition, F is continuously differentiable if

$$DF:X \to \mathcal{B}(X,Y)$$

is continuous (which is implied by the existence of D^2F).

Suppose that H is a linear subspace of X equipped with a stronger Banach space topology (so that the injection $H \to X$ is continuous) — then a function $F:X \to Y$ is said to be H-differentiable if $\forall\ x \in X$, the function $h \to F(x+h)$ is differentiable at $h = 0$. The H-derivative of F, written D_HF, thus gives rise to a map

$$D_HF:X \to \mathcal{B}(H,Y).$$

The construction can then be iterated. In particular:

$$D_H^2F:X \to \mathcal{B}_2(H,Y).$$

A differentiable function is necessarily H-differentiable (but not conversely).

32.3 EXAMPLE Assume that X is a Hilbert space and let H be a proper subspace. Fix $h_0 \in H$ ($h_0 \neq 0$) and define $F:X \to \underline{R}$ by

$$F(x) = \begin{cases} \langle x,h_0 \rangle & (x \in H) \\ 0 & (x \notin H). \end{cases}$$

Then

$$D_HF(x) = \begin{cases} h_0 & (x \in H) \\ 0 & (x \notin H). \end{cases}$$

In fact,

$$x \in H \Rightarrow x + h \in H$$

$$\Rightarrow$$

$$F(x+h) - F(x) - \langle h,h_0 \rangle$$

$$= \langle x+h,h_0 \rangle - \langle x,h_0 \rangle - \langle h,h_0 \rangle$$

$$= 0.$$

On the other hand,

$$x \notin H \Rightarrow x + h \notin H$$

$$\Rightarrow$$

$$F(x+h) - F(x) - \langle h,0 \rangle$$

$$= 0.$$

[Note: This function is infinitely H-differentiable but is not continuous.]

32.4 EXAMPLE Take $X = L^2[0,1]$, $Y = L^2[0,1]$ and define $F:X \to Y$ by

$$F(f)(t) = \sin(f(t)).$$

Then F is nowhere differentiable. On the other hand, F is H-differentiable ($H = C[0,1]$): $\forall\, h \in H$,

$$D_H F(f)(h)(t) = \cos(f(t))h(t).$$

In fact,

$$||\sin (f(\cdot) + h(\cdot)) - \sin (f(\cdot)) - \cos (f(\cdot))h(\cdot)||_{L^2[0,1]}$$

$$\leq \frac{1}{2} \sup_{0\leq t\leq 1} |h(t)|^2.$$

Given separable Hilbert spaces H_1 and H_2, let $\underline{L}_2(H_1,H_2)$ stand for the set of Hilbert-Schmidt operators from H_1 to H_2 -- then $\underline{L}_2(H_1,H_2)$ is a separable Hilbert space when equipped with the Hilbert-Schmidt inner product.

[Note: In general, the set $\underline{L}_2^n(H_1,H_2)$ of n-multilinear Hilbert-Schmidt operators from H_1 to H_2 is a separable Hilbert space.]

32.5 <u>REMARK</u> Let $H_1 = H$, $H_2 = \underline{R}$ and put $\underline{H}_n = \underline{L}_2^n(H,\underline{R})$ — then $\underline{H}_n$ is canonically isomorphic to $\underline{L}_2(H,\underline{H}_{n-1})$.

In practice, H and Y are separable Hilbert spaces and $D_H F(x) \in \underline{L}_2(H,Y)$. Therefore $D_H F$ is a Hilbert space valued map, hence all higher derivatives $D_H^n F$ also take values in a Hilbert space.

Assume now that X is a separable Banach space and let γ be a centered gaussian measure on X -- then in what follows, the role of $H \subset X$ will be played by $H(\gamma)$ and we shall abbreviate $D_{H(\gamma)}$ to D_γ.

32.6 LEMMA Fix $p > 1$. Put

$$\rho(h,\cdot) = \exp(f - \frac{1}{2}||h||^2_{H(\gamma)}) \quad (R_\gamma(f) = h).$$

Then the function

$$\begin{cases} H(\gamma) \to L^p(X,\gamma) \\ h \to \rho(h,\cdot) \end{cases} \qquad \text{(cf. 29.5)}$$

is infinitely differentiable and

$$D^n_\gamma \rho(0,\cdot)(h_1,\ldots,h_n) = I_n(f_1 \ldots f_n),$$

where

$$R_\gamma(f_1) = h_1,\ldots,R_\gamma(f_n) = h_n.$$

32.7 EXAMPLE Let $\phi:X \to \underline{R}$ be bounded and Borel. Put

$$\Phi(x) = \int_X \phi(x+y)d\gamma(y).$$

Then Φ is infinitely H-differentiable and

$$\partial_h\Phi(x) = \int_X \phi(x+y)f(y)d\gamma(y) \quad (R_\gamma(f) = h).$$

Now fix an orthonormal basis $\{h_j:j \geq 1\}$ for $H(\gamma)$ and apply Bessel's inequality to get

$$\sum_{j=1}^{\infty} |\partial_{h_j}\Phi(x)|^2 \leq \int_X |\phi(x+y)|^2 d\gamma(y)$$

$$\leq \sup |\phi|^2 < \infty.$$

Therefore $D_\gamma \Phi(x)$ is Hilbert-Schmidt and

$$||D_\gamma \Phi(x)||_{\underline{L}_2(H(\gamma),\underline{R})} \leq ||\phi||_\infty.$$

Higher derivatives can be dealt with analogously.

32.8 LEMMA Fix $t > 0$ and $p > 1$. Put

$$\rho(t,h,\cdot)$$

$$= \exp(\frac{e^{-t}}{(1-e^{-2t})^{1/2}} f - \frac{e^{-2t}}{2(1-e^{-2t})} ||h||^2_{H(\gamma)}) \quad (R_\gamma(f) = h).$$

Then the function

$$\begin{cases} H(\gamma) \rightarrow L^p(X,\gamma) \\ h \rightarrow \rho(t,h,\cdot) \end{cases} \qquad \text{(cf. 29.5)}$$

is infinitely differentiable and

$$D^n_\gamma \rho(t,0,\cdot)(h_1,\ldots,h_n)$$

$$= P(\frac{e^{-t}}{(1-e^{-2t})^{1/2}} f_1,\ldots, \frac{e^{-t}}{(1-e^{-2t})^{1/2}} f_n),$$

where P is a polynomial on $\underline{R}^n$ whose coefficients are polynomials in the

$$\frac{e^{-2t}}{1-e^{-2t}} < h_i,h_j >_{H(\gamma)} \qquad (i,j = 1,\ldots,n)$$

and

$$R_\gamma(f_1) = h_1, \ldots, R_\gamma(f_n) = h_n.$$

32.9 EXAMPLE Let $\phi:X \to \underline{R}$ be bounded and Borel -- then $\forall\ t > 0$, the function $T_t\phi:X \to \underline{R}$ is infinitely H-differentiable and $\forall\ h \in H(\gamma)$,

$$\partial_h T_t \phi(x)$$

$$= \frac{e^{-t}}{(1-e^{-2t})^{1/2}} \int_X \phi(e^{-t}x + (1-e^{-2t})^{1/2}y) f(y) d\gamma(y) \quad (R_\gamma(f) = h).$$

Now fix an orthonormal basis $\{h_j : j \geq 1\}$ for $H(\gamma)$ and apply Bessel's inequality to get

$$\sum_{j=1}^{\infty} |\partial_{h_j} T_t \phi(x)|^2$$

$$= \frac{e^{-2t}}{1-e^{-2t}} \sum_{j=1}^{\infty} |\int_X \phi(e^{-t}x + (1-e^{-2t})^{1/2}y) f_j(y) d\gamma(y)|^2$$

$$\leq \frac{e^{-2t}}{1-e^{-2t}} \int_X |\phi(e^{-t}x + (1-e^{-2t})^{1/2}y)|^2 d\gamma(y)$$

$$< \infty .$$

Therefore $D_\gamma T_t \phi(x)$ is Hilbert-Schmidt and

$$||D_\gamma T_t \phi(x)||_{\underline{L}_2(H(\gamma),\underline{R})} \leq \frac{e^{-t}}{(1-e^{-2t})^{1/2}} ||\phi||_\infty.$$

Higher derivatives can be dealt with analogously.

Suppose that $\phi:X \to \underline{R}$ is bounded and Borel. Given $t > 0$, define $P_t\phi:X \to \underline{R}$ by

$$P_t\phi(x) = \int_X \phi(x + \sqrt{t}\, y)d\gamma(y)$$

and make the convention that $P_0\phi = \phi$. Then

$$P_t\phi(x + h)$$

$$= \int_X \phi(x + h + \sqrt{t}\, y)d\gamma(y)$$

$$= \int_X \phi(x + \sqrt{t}\, y)\exp(\frac{1}{t} f(\sqrt{t}\, y) - \frac{1}{2t}||h||^2_{H(\gamma)})d\gamma(y).$$

<u>N.B.</u> Here $R_\gamma(f) = h$, hence $R_\gamma(f_0) = h$, where f_0 is a linear model for f (cf. 26.10), and by construction, $f_0(\sqrt{t}\, y) = \sqrt{t}\, f_0(y)$ (f_0 is linear on E_0 and identically zero on $X - E_0$). So, without loss of generality, it can and will be assumed that f has this property as well, thus

$$P_t\phi(x + \sqrt{t}\, h)$$

$$= \int_X \phi(x + \sqrt{t}\, y)\exp(f(y) - \frac{1}{2}||h||^2_{H(\gamma)})d\gamma(y)$$

or still,

$$P_t\phi(x + \sqrt{t}\, h)$$

$$= \int_X \phi(x + \sqrt{t}\, y)\rho(h,y)d\gamma(y).$$

32.10 LEMMA $P_t\phi$ is infinitely H-differentiable and

$$D_\gamma^n P_t \phi(x)(h_1,\ldots,h_n)$$

$$= \frac{1}{t^{n/2}} \int_X \phi(x + \sqrt{t}\, y) I_n(f_1 \ \ldots \ f_n)(y) d\gamma(y) \qquad \text{(cf. 32.6)}.$$

[Note: It follows from this that

$$D_\gamma^n P_t \phi(x) \in \underline{L}_2^n (H(\gamma), \underline{R}).]$$

Denote by $bC_u(X)$ the Banach space of bounded uniformly continuous functions on X endowed with the supremum norm.

[Note:

$$\phi \in bC_u(X) \Rightarrow P_t\phi \in bC_u(X).$$

Moreover,

$$||P_t\phi||_\infty \le ||\phi||_\infty \Rightarrow ||P_t|| \le 1.]$$

32.11 LEMMA The collection $\{P_t : t \ge 0\}$ is a strongly continuous semigroup on $bC_u(X)$.

PROOF From its very definition, $P_0 = I$. Noting that γ is the image of $\gamma \times \gamma$ under the map

$$(u,v) \to \frac{t^{1/2}}{(t+s)^{1/2}} u + \frac{s^{1/2}}{(t+s)^{1/2}} v,$$

we have

$$P_t(P_s\phi)(x) = \int_X P_s\phi(x + \sqrt{t}\, y)d\gamma(y)$$

$$= \int_X \int_X \phi(x + \sqrt{t}\, y + \sqrt{s}\, z)d\gamma(y)d\gamma(z)$$

$$= \int_X \phi(x + (t+s)^{1/2}w)d\gamma(w)$$

$$= P_{t+s}\phi(x).$$

There remains the verification of strong continuity. Fix $\varepsilon > 0$ — then $\exists\ \delta > 0$:

$$||x - y|| < \delta \Rightarrow |\phi(x) - \phi(y)| < \varepsilon.$$

So

$$|P_t\phi(x) - \phi(x)|$$

$$= |\int_X (\phi(x + \sqrt{t}\, y) - \phi(x))d\gamma(y)|$$

$$\leq \int_X |\phi(x + \sqrt{t}\, y) - \phi(x)|d\gamma(y)$$

$$\leq \int_{||\sqrt{t}\, y|| < \delta} \varepsilon d\gamma(y) + \int_{||\sqrt{t}\, y|| \geq \delta} |\phi(x + \sqrt{t}\, y) - \phi(x)|d\gamma(y)$$

$$\leq \varepsilon + 2||\phi||_\infty\ \gamma\{y: ||\sqrt{t}\, y|| \geq \delta\}.$$

But

$$\gamma\{y: ||\sqrt{t}\, y|| \geq \delta\} = \gamma\{y: ||y|| \geq \delta/\sqrt{t}\} \to 0 \text{ as } t \to 0.$$

Accordingly,

$$\lim_{t \to 0} ||P_t\phi - \phi||_\infty = 0.$$

32.12 REMARK The story for the Ornstein-Uhlenbeck semigroup is a little bit different. Indeed,

$$\phi \in bC_u(X) \Rightarrow T_t\phi \in bC_u(X)$$

but the collection $\{T_t : t \geq 0\}$ is not strongly continuous on $bC_u(X)$.

Some of the formulas appearing above implicitly assume that the data is infinite dimensional but this is not necessary. E.g.: Take $X = \underline{R}^n$, $\gamma = \gamma_n$ — then under suitable regularity hypotheses,

$$P_t\phi(x) = \int_{\underline{R}^n} \phi(x + \sqrt{t}\, y) d\gamma_n(y)$$

$$= \frac{1}{(2\pi t)^{n/2}} \int_{\underline{R}^n} e^{-(x-y)^2/2t} f(y) dy$$

$$= e^{t\Delta/2} \phi(x).$$

It is for this reason that, in general, the collection $\{P_t : t \geq 0\}$ is called the heat semigroup.

§33. *POSITIVE DEFINITE FUNCTIONS*

Let G be an additive group. Given a function $\chi:G \to \underline{C}$, put

$$K_\chi(\sigma,\tau) = \chi(\tau-\sigma) \quad (\sigma,\tau \in G).$$

Then χ is said to be positive definite if K_χ is a kernel on G, i.e., if for all

$$\left[\begin{array}{l} \sigma_1,\ldots,\sigma_n \in G \\ \\ c_1,\ldots,c_n \in \underline{C}, \end{array}\right.$$

we have

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \chi(\sigma_j - \sigma_i) \geq 0.$$

33.1 EXAMPLE Let X be a topological vector space. Let μ be a probability measure on Cyl(X) -- then its Fourier transform $\hat{\mu}$ is a positive definite function on $G = X^*$. In fact,

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \hat{\mu}(\lambda_j - \lambda_i)$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j \int_X e^{\sqrt{-1}(\lambda_j-\lambda_i)(x)} \, d\mu(x)$$

$$= \int_X \Big(\sum_{i=1}^{n} \bar{c}_i e^{-\sqrt{-1}\,\lambda_i(x)}\Big)\Big(\sum_{j=1}^{n} c_j e^{\sqrt{-1}\,\lambda_j(x)}\Big) d\mu(x)$$

$$= \int_X \left| \sum_{i=1}^{n} c_i e^{\sqrt{-1}\,\lambda_i(x)} \right|^2 d\mu(x)$$

$$\geq 0.$$

33.2 <u>EXAMPLE</u> Let X be a separable real Hilbert space -- then the function

$$x \to \exp\left(-\frac{1}{2}||x||^2\right)$$

is positive definite on $G = X$. In fact,

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \exp(-||x_j - x_i||^2)$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j e^{-||x_i||^2} e^{-||x_j||^2} e^{2<x_i,x_j>}$$

$$= \sum_{i,j=1}^{n} (\bar{c}_i e^{-||x_i||^2})(c_j e^{-||x_j||^2}) e^{2<x_i,x_j>}$$

$$\geq 0.$$

[Note: Recall that $< , >$ and $e^{<,>}$ are kernels on X (see §14).]

33.3 <u>THEOREM</u> (Bochner) In order that a function $\chi:\underline{R}^n \to \underline{C}$ be the Fourier transform of a probability measure μ on $\text{Bor}(\underline{R}^n)$, it is necessary and sufficient that χ be positive definite, continuous, and equal to one at zero.

[Note: The characteristic function of $\underline{Z}^n$ is positive definite and equal to one at zero but it is not continuous.]

33.4 EXAMPLE Let X be a separable real Hilbert space. Assume: $\dim X = \infty$ -- then the function $x \to \exp(-\frac{1}{2}||x||^2)$ cannot be the Fourier transform of a probability measure on Bor(X). Proof: It is not weakly sequentially continuous.

[Note: One can also argue directly. Fix an orthonormal basis $\{e_n\}$ for X and assume that

$$\exp(-\frac{1}{2}||x||^2) = \int_X \exp(\sqrt{-1}\,\langle x,y\rangle)d\mu(y)$$

for some probability measure μ on Bor(X) -- then $\forall$ n,

$$e^{-\frac{1}{2}} = \int_X \exp(\sqrt{-1}\,\langle e_n,y\rangle)d\mu(y).$$

But $\forall$ y, $\lim\limits_{n \to \infty} \langle e_n,y\rangle = 0$, hence by dominated convergence,

$$\lim_{n \to \infty} \int_X \exp(\sqrt{-1}\,\langle e_n,y\rangle)d\mu(y) = 1.]$$

33.5 REMARK Therefore 33.3 is false in the context of infinite dimensional separable real Hilbert spaces and one of the objectives of the present § is to address this issue (cf. 33.10).

Let E be a vector space over $\underline{R}$. Per §17, take $\sigma = 0$ and write

$$\begin{cases} \mathcal{PD}(E) \text{ for } \mathcal{PD}(E,0) \\ \mathcal{S}(\mathcal{W}(E)) \text{ for } \mathcal{S}(\mathcal{W}(E,0)). \end{cases}$$

Then there is a canonical one-to-one correspondence

$$\mathcal{PD}(E) \longleftrightarrow \mathcal{S}(\mathcal{W}(E)) \quad \text{(cf. 17.16).}$$

Now equip E with the finite topology (cf. 18.2) -- then the elements of $\mathcal{PD}(E)$ which are continuous in the finite topology are precisely the characteristic functions of the nonsingular states on $\mathcal{W}(E)$ or still, the elements of the folium $\mathcal{F}_{ns}$ (cf. 18.7).

Let $E^{\#}$ be the algebraic dual of E -- then $\forall\ \lambda \in E^{\#}$ and any finite dimensional linear subspace $F \subset E$, the restriction $\lambda|F$ is continuous, thus by the very definition of the finite topology, $\lambda: E \to \underline{R}$ is continuous.

Given $e \in E$, define $\hat{e}: E^{\#} \to \underline{R}$ by $\hat{e}(\lambda) = \langle e,\lambda\rangle$ and let $\text{Cyl}(E^{\#})$ be the σ-algebra generated by the $\hat{e}$. If μ is a probability measure on $\text{Cyl}(E^{\#})$, then its Fourier transform $\hat{\mu}$ is the function

$$\hat{\mu}(e) = \int_{E^{\#}} \exp(\sqrt{-1}\ \langle e,\lambda\rangle) d\mu(\lambda).$$

33.6 LEMMA $\hat{\mu}$ is positive definite, continuous in the finite topology, and equal to one at zero.

PROOF To verify the continuity of $\hat{\mu}$ in the finite topology, fix F and let $\pi_F: E^{\#} \to F^{\#}$ be the arrow of restriction -- then

$$\hat{\mu}|F = \hat{\mu}_F,$$

where $\mu_F = \mu \circ \pi_F^{-1}$.

33.7 THEOREM (Kolmogorov) Suppose that $\chi: E \to \underline{C}$ is positive definite, continuous in the finite topology, and equal to one a zero -- then χ is the

Fourier transform of a unique probability measure μ on $\text{Cyl}(E^{\#})$.

PROOF Let Λ be a Hamel basis for E -- then $E^{\#}$ can be identified with $\underline{R}^{\Lambda}$. Let F be the family of finite nonempty subsets of Λ. Attach to each $\alpha \in F$ a function $\chi_{\alpha}:\underline{R}^{\alpha} \rightarrow \underline{C}$ by

$$\chi_{\alpha}(t) = \chi\left(\sum_{e \in \alpha} t(e)e\right) \quad (t \in \underline{R}^{\alpha}).$$

Then χ_{α} is positive definite, continuous, and equal to one at zero, so, by Bochner's theorem, $\exists$ a unique probability measure μ_{α} on $\text{Bor}(\underline{R}^{\alpha})$ such that $\hat{\mu}_{\alpha} = \chi_{\alpha}$. The collection of measures $\{\mu_{\alpha}:\alpha \in F\}$ is consistent in the sense that $\mu_{\alpha} = \mu_{\beta} \circ \pi_{\beta\alpha}^{-1}$ whenever $\alpha \subset \beta$ ($\pi_{\beta\alpha}:\underline{R}^{\beta} \rightarrow \underline{R}^{\alpha}$ the projection). Therefore $\exists$ a unique probability measure μ on $\text{Cyl}(\underline{R}^{\Lambda})$ such that $\forall\ \alpha$, $\mu_{\alpha} = \mu \circ \pi_{\alpha}^{-1}$ ($\pi_{\alpha}:\underline{R}^{\Lambda} \rightarrow \underline{R}^{\alpha}$ the projection). But

$$\left[\begin{array}{l} \underline{R}^{\Lambda} \longleftrightarrow E^{\#} \\ \\ \text{Cyl}(\underline{R}^{\Lambda}) \longleftrightarrow \text{Cyl}(E^{\#}). \end{array}\right.$$

Accordingly, μ can be interpreted as a probability measure on $\text{Cyl}(E^{\#})$ and it is then easy to check that $\hat{\mu} = \chi$.

33.8 EXAMPLE Take $E = \underline{R}_0^{\infty}$ and equip E with the finite topology -- then the set of positive definite, continuous functions on E which are equal to one at zero coincides with the set of Fourier transforms of probability measures on $\text{Cyl}(E^{\#})$. Since $E^{\#}$ can be identified with $\underline{R}^{\infty}$ and since under this identification,

$\text{Cyl}(E^{\#})$ becomes $\text{Cyl}(\underline{R}^{\infty})$, it follows that the set of Fourier transforms of probability measures on $\text{Cyl}(\underline{R}^{\infty})$ is the same as the set of positive definite, continuous functions on $\underline{R}_0^{\infty}$ which are equal to one at zero.

[Note: Let us also bear in mind that $\underline{R}^{\infty}$ is a separable LF-space and $\text{Cyl}(\underline{R}^{\infty}) = \text{Bor}(\underline{R}^{\infty})$.]

33.9 LEMMA Let X be an infinite dimensional separable real Hilbert space. Suppose that μ is a finite Borel measure on X -- then

$$\int_X ||x||^2 \, d\mu(x) < \infty$$

iff $\exists$ a nonnegative, symmetric, trace class operator K_μ such that $\forall$ $u,v \in X$,

$$\langle u, K_\mu v\rangle = \int_X \langle u,x\rangle\langle v,x\rangle \, d\mu(x),$$

in which case

$$\text{tr}(K_\mu) = \int_X ||x||^2 \, d\mu(x).$$

Given an infinite dimensional separable real Hilbert space X, write $\mathcal{K}$ for the set of nonnegative symmetric operators on X which are of the trace class.

33.10 THEOREM (Prokhorov) Let $\chi : X \to \underline{C}$ -- then χ is the Fourier transform of a probability measure μ on Bor(X) iff χ is positive definite, equal to one at zero, and

(P) $\forall$ $\varepsilon > 0$, $\exists$ $K_\varepsilon \in \mathcal{K}$:

$$1 - \text{Re}\,\chi(x) \leq \langle x, K_\varepsilon x\rangle + \varepsilon \;\; \forall\, x \in X.$$

We shall first consider the necessity. So suppose that $\chi = \hat{\mu}$, where μ is a probability measure on Bor(X). Fix $\varepsilon > 0$ and choose $r > 0$:

$$\mu\{x: ||x|| \leq r\} > 1 - \frac{\varepsilon}{2}.$$

Then

$$\chi(x) = \int_{||y|| \leq r} e^{\sqrt{-1}\langle x,y\rangle} d\mu(y) + \int_{||y|| > r} e^{\sqrt{-1}\langle x,y\rangle} d\mu(y).$$

Since

$$\left| \int_{||y|| > r} e^{\sqrt{-1}\langle x,y\rangle} d\mu(y) \right| < \frac{\varepsilon}{2},$$

it will be enough to produce a $K_\varepsilon \in \mathcal{K}$ such that

$$1 - \mathrm{Re} \int_{||y|| \leq r} e^{\sqrt{-1}\langle x,y\rangle} d\mu(y) \leq \langle x, K_\varepsilon x\rangle + \frac{\varepsilon}{2}.$$

To this end, write

$$1 - \mathrm{Re} \int_{||y|| \leq r} e^{\sqrt{-1}\langle x,y\rangle} d\mu(y)$$

$$\leq \int_{||y|| \leq r} (1 - \cos \langle x,y\rangle) d\mu(y) + \frac{\varepsilon}{2}$$

$$\leq \int_{||y|| \leq r} 2 \sin^2 \frac{\langle x,y\rangle}{2} d\mu(y) + \frac{\varepsilon}{2}$$

$$\leq \frac{1}{2} \int_{||y|| \leq r} \langle x,y\rangle^2 d\mu(y) + \frac{\varepsilon}{2}.$$

Apply 33.9 to the measure

$$B \to \frac{1}{2}\,\mu(B \cap \{y: ||y|| \le r\}) \quad (B \in \text{Bor}(X))$$

to get $K_\varepsilon \in \mathcal{K}$:

$$\langle u, K_\varepsilon v\rangle = \frac{1}{2}\int_{||y|| \le r} \langle u,y\rangle\langle v,y\rangle d\mu(y),$$

from which

$$\langle x, K_\varepsilon x\rangle = \frac{1}{2}\int_{||y|| \le r} \langle x,y\rangle^2\, d\mu(y),$$

as desired.

Turning to the sufficiency, observe first that condition P implies that Re χ is continuous at the origin. But χ is positive definite, hence

$$|1 - \chi(x)| \le \sqrt{2}\,(1 - \text{Re}\,\chi(x))^{1/2}.$$

So χ is continuous at the origin, thus everywhere. Now fix an orthonormal basis $\{e_j\}$ for X. Put

$$\chi_{j_1,\ldots,j_n} = \chi(\omega_1 e_{j_1} + \cdots + \omega_n e_{j_n}) \quad (\omega_j \in \underline{R},\ 1 \le j \le n).$$

Then $\chi_{j_1,\ldots,j_n}$ satisfies the conditions of 33.3. Therefore

$$\chi_{j_1,\ldots,j_n} = \hat{\mu}_{j_1,\ldots,j_n},$$

where $\mu_{j_1,\ldots,j_n}$ is a probability measure on $\text{Bor}(\underline{R}^n)$. It is clear that the collection $\{\mu_{j_1,\ldots,j_n}\}$ is consistent, thus $\exists$ a unique probability measure ν on

$\text{Bor}(\underline{R}^\infty)$ such that

$$\mu_{j_1,\ldots,j_n} = \nu \circ (\xi_{j_1},\ldots,\xi_{j_n})^{-1}.$$

Here

$$\xi_j(\omega) = \omega_j \quad (\omega = (\omega_1,\omega_2,\ldots) \in \underline{R}^\infty).$$

33.11 <u>LEMMA</u> $\sum_{j=1}^{\infty} \xi_j^2 < \infty$ a.e. $[\nu]$.

<u>PROOF</u> By hypothesis, $\forall\ \varepsilon > 0,\ \exists\ K_\varepsilon \in \mathcal{K}$:

$$1 - \text{Re}\ \chi(x) \leq \langle x, K_\varepsilon x\rangle + \varepsilon\ \forall\ x \in X.$$

This said, we have

$$1 - \int_{\underline{R}^\infty} \exp(-\frac{1}{2}\sum_{j=1}^{n} \xi_{k+j}^2)d\nu$$

$$= 1 - \int_{\underline{R}^\infty} (\int_{\underline{R}^n} \exp(\sqrt{-1}\sum_{j=1}^{n} t_j\xi_{k+j})d\gamma_n(t))d\nu$$

$$= 1 - \int_{\underline{R}^n} \chi(\sum_{j=1}^{n} t_j e_{k+j})d\gamma_n(t)$$

$$= \int_{\underline{R}^n} (1 - \text{Re}\ \chi(\sum_{j=1}^{n} t_j e_{k+j}))d\gamma_n(t)$$

$$\leq \int_{\underline{R}^n} \langle\sum_{j=1}^{n} t_j e_{k+j}, K_\varepsilon \sum_{j=1}^{n} t_j e_{k+j}\rangle d\gamma_n(t) + \varepsilon$$

$$= \sum_{j=1}^{n} \langle e_{k+j}, K_\varepsilon e_{k+j}\rangle + \varepsilon$$

=>

$$1 - \int_{\underline{R}^\infty} \exp(-\frac{1}{2} \sum_{j=k+1}^{\infty} \xi_j^2) d\nu$$

$$\leq \sum_{j=k+1}^{\infty} \langle e_j, K_\varepsilon e_j \rangle + \varepsilon$$

=>

$$1 - \int_{\underline{R}^\infty} \exp(-\frac{1}{2} \sum_{j=k+1}^{\infty} \xi_j^2) d\nu$$

$$\leq 2\varepsilon \quad (k \geq k(\varepsilon))$$

=>

$$\int_{\underline{R}^\infty} \exp(-\frac{1}{2} \sum_{j=k+1}^{\infty} \xi_j^2) d\nu \geq 1 - 2\varepsilon \quad (k \geq k(\varepsilon)).$$

But

$$\nu\{\omega: \sum_{j=1}^{\infty} \xi_j^2(\omega) < \infty\}$$

$$\geq \int_{\underline{R}^\infty} \exp(-\frac{1}{2} \sum_{j=k+1}^{\infty} \xi_j^2) d\nu$$

$$\geq 1 - 2\varepsilon \quad (k \geq k(\varepsilon))$$

=>

$$\nu\{\omega: \sum_{j=1}^{\infty} \xi_j^2(\omega) < \infty\} = 1.$$

To finish the proof of the sufficiency, let

$$\xi(\omega) = \sum_{j=1}^{\infty} \xi_j(\omega) e_j.$$

Thus ξ is defined on $\underline{R}^{\infty}$ a.e.[ν] and is an X-valued Borel measurable function. Put $\mu = \nu \circ \xi^{-1}$ -- then μ is a probability measure on Bor(X) and $\forall\ n \geq 1$,

$$\hat{\mu}\left(\sum_{j=1}^{n} \langle e_j, x\rangle e_j\right)$$

$$= \chi_{1,\ldots,n}(\langle e_1, x\rangle, \ldots, \langle e_n, x\rangle)$$

$$= \chi\left(\sum_{j=1}^{n} \langle e_j, x\rangle e_j\right),$$

hence $\hat{\mu} = \chi$.

33.11 REMARK Assign to each $K \in \mathcal{K}$ a seminorm $p_K : X \to \underline{R}$ by writing

$$p_K(x) = \langle x, Kx\rangle \quad (x \in X).$$

Then these seminorms generate a topology on X, the Sazonov topology. Suppose that $\chi : X \to \underline{C}$ is positive definite with $\chi(0) = 1$. Assume: χ is continuous in the Sazonov topology — then χ satisfies condition P. To see this, fix $\varepsilon > 0$. Owing to the continuity of χ in the Sazonov topology, $\exists\ K_\varepsilon \in \mathcal{K}$:

$$\langle x, K_\varepsilon x\rangle < 1 \Rightarrow 1 - \mathrm{Re}\,\chi(x) < \varepsilon.$$

But

$$\langle x, K_\varepsilon x\rangle \geq 1 \Rightarrow 1 - \mathrm{Re}\,\chi(x) \leq 2\langle x, K_\varepsilon x\rangle.$$

So, for all $x \in X$,

$$1 - \mathrm{Re}\,\chi(x) \le 2\langle x, K_\varepsilon x\rangle + \varepsilon.$$

33.12 EXAMPLE Fix $a \in X$, $K \in \mathcal{K}$ -- then the function

$$\chi(x) = \exp(\sqrt{-1}\,\langle a,x\rangle - \frac{1}{2}\langle x,Kx\rangle) \quad (x \in X)$$

is the Fourier transform of a probability measure on Bor(X) (which is necessarily gaussian (cf. 26.3)).

[It is clear that χ is positive definite with $\chi(0) = 1$. Now take $a = 0$ (cf. 26.5) and note that condition P is satisfied. Proof:

$$\begin{aligned} 1 - \mathrm{Re}\,\chi(x) &= 1 - e^{-\frac{1}{2}\langle x,Kx\rangle} \\ &\le \frac{1}{2}\langle x,Kx\rangle \end{aligned}$$

since $1 - e^{-t} \le t$ $(t \ge 0)$.]

33.13 THEOREM (Mourier) Let X be an infinite dimensional separable real Hilbert space. Suppose that γ is a gaussian measure on X, hence

$$\hat{\gamma}(x) = \exp(\sqrt{-1}\,\langle a_\gamma,x\rangle - \frac{1}{2}\langle x,K_\gamma x\rangle) \quad (x \in X),$$

where $a_\gamma \in X$ and K_γ is nonnegative and symmetric (cf. 26.3) -- then K_γ is trace class.

PROOF Take $a_\gamma = 0$, thus

$$1 - \mathrm{Re}\,\chi(x) = 1 - \exp(-\frac{1}{2}\langle x,K_\gamma x\rangle).$$

In condition P, choose

$$\varepsilon = \frac{1 - e^{-\frac{1}{2}}}{2}$$

and put

$$T = \frac{1}{\varepsilon} K_\varepsilon .$$

Then

$$1 - \mathrm{Re}\, \chi(x) \le \langle x, K_\varepsilon x\rangle + \varepsilon$$

$$= \varepsilon\langle x, \frac{1}{\varepsilon} K_\varepsilon x\rangle + \varepsilon$$

$$= \frac{1 - e^{-\frac{1}{2}}}{2} \langle x, Tx\rangle + \frac{1 - e^{-\frac{1}{2}}}{2} .$$

Therefore

$$\langle x, Tx\rangle < 1$$

$$\Rightarrow$$

$$1 - \mathrm{Re}\, \chi(x) < 1 - e^{-\frac{1}{2}} .$$

I.e.:

$$\langle x, Tx\rangle < 1$$

$$\Rightarrow$$

$$\langle x, K_\gamma x\rangle < 1.$$

But this implies that

$$\langle x, K_\gamma x\rangle \leq \langle x, Tx\rangle$$

for all $x \in X$, so K_γ is trace class.

[Note: If $\exists\ x \in X$ such that

$$\langle x, K_\gamma x\rangle > \langle x, Tx\rangle,$$

then $\langle x, K_\gamma x\rangle \neq 0$ and

$$< \frac{x}{\langle x, K_\gamma x\rangle^{1/2}}, T \frac{x}{\langle x, K_\gamma x\rangle^{1/2}} >$$

$$= \frac{\langle x, Tx\rangle}{\langle x, K_\gamma x\rangle} < 1$$

=>

$$< \frac{x}{\langle x, K_\gamma x\rangle^{1/2}}, K_\gamma \frac{x}{\langle x, K_\gamma x\rangle^{1/2}} > < 1$$

=>

$$\frac{\langle x, K_\gamma x\rangle}{\langle x, K_\gamma x\rangle} < 1 \ \ldots\ .]$$

33.14 <u>REMARK</u> Take $a_\gamma = 0$ — then $\exists\ \alpha > 0$:

$$\int_X e^{\alpha||x||^2}\, d\gamma(x) < \infty \quad \text{(cf. 26.37)}.$$

Therefore

$$\int_X ||x||^2\, d\gamma(x) < \infty.$$

But $\forall\, u,v \in X$,

$$<u,K_\gamma v> = \int_X <u,x><v,x>d\gamma(x) \qquad \text{(cf. 26.3)}.$$

The fact that K_γ is trace class thus follows from 33.9.

Keeping to the supposition that X is an infinite dimensional separable real Hilbert space, let γ be a centered gaussian measure on X. Identify X and X* and assume that $K_\gamma > 0$. Fix an orthonormal basis $\{e_n\}$ for X consisting of eigenvectors for K_γ: $K_\gamma e_n = \lambda_n e_n$ ($\lambda_n > 0$: $\sum_{n=1}^{\infty} \lambda_n < \infty$) -- then $\sqrt{K_\gamma} > 0$ and is Hilbert-Schmidt.

33.15 <u>REMARK</u> The closure of $X = X^*$ in $L^2(X,\gamma)$ is the completion of X w.r.t. the norm $x \to <\sqrt{K_\gamma}x, \sqrt{K_\gamma}x>$. In fact,

$$\begin{aligned} &<\sqrt{K_\gamma}x, \sqrt{K_\gamma}x> \\ &\quad = \int_X <x,y>^2 \, d\gamma(y) \\ &\quad = ||<x,\text{—}>||^2_{L^2(\gamma)}. \end{aligned}$$

Therefore X_γ^* can be identified with the Hilbert space of real sequences $\{a_n : n \geq 1\}$: $\sum_{n=1}^{\infty} \lambda_n a_n^2 < \infty$.

33.16 <u>LEMMA</u> The Cameron-Martin space $H(\gamma)$ of γ is $\sqrt{K_\gamma}X$, hence is dense in X.

[Note: Here

$$<\sqrt{K_\gamma}x,\sqrt{K_\gamma}y>_{H(\gamma)} = <x,y> \quad (x,y \in X).]$$

33.17 <u>REMARK</u> We have

$$R_\gamma(<x,\text{—}>) = K_\gamma x.$$

Indeed

$$R_\gamma(<x,\text{—}>)(y)$$

$$= \int_X <x,z><y,z>d\gamma(z)$$

$$= <x,K_\gamma y>$$

$$= <K_\gamma x,y>$$

$$\Rightarrow$$

$$x_{<x,\text{—}>} = K_\gamma x \quad (= \sqrt{K_\gamma}\ \sqrt{K_\gamma}x).$$

To run a reality check, write

$$||K_\gamma x||^2_{H(\gamma)} = <K_\gamma x,K_\gamma x>_{H(\gamma)}$$

$$= <\sqrt{K_\gamma}\ \sqrt{K_\gamma}x,\sqrt{K_\gamma}\ \sqrt{K_\gamma}x>_{H(\gamma)}$$

$$= <\sqrt{K_\gamma}x,\sqrt{K_\gamma}x>$$

$$= ||<x,\text{—}>||^2_{L^2(\gamma)}.$$

[Note: In terms of the expansion

$$x = \sum_{n=1}^{\infty} \langle e_n, x\rangle e_n,$$

$x \in H(\gamma)$ iff

$$\sum_{n=1}^{\infty} \frac{\langle e_n, x\rangle^2}{\lambda_n} < \infty.]$$

Let γ_1, γ_2 be centered gaussian measures on X. Suppose that $H(\gamma_1) = H(\gamma_2)$ and that the norms

$$\begin{cases} ||\cdot||_{H(\gamma_1)} \\ \\ ||\cdot||_{H(\gamma_2)} \end{cases}$$

are equivalent. Put

$$H = \begin{cases} H(\gamma_1) \\ \\ H(\gamma_2) \end{cases}$$

and

$$T = \sqrt{K_{\gamma_2}} \sqrt{K_{\gamma_1}}^{-1}.$$

Then $T:H \to H$ is an invertible bounded linear operator. Moreover, $\forall\, h,h' \in H$,

- $\langle h,h'\rangle_{H(\gamma_1)}$

$$= \langle \sqrt{K_{\gamma_1}}x, \sqrt{K_{\gamma_1}}x' \rangle_{H(\gamma_1)}$$

$$= \langle x, x' \rangle.$$

- $\langle Th, Th' \rangle_{H(\gamma_2)}$

$$= \langle \sqrt{K_{\gamma_2}}\ \sqrt{K_{\gamma_1}}^{-1}h, \sqrt{K_{\gamma_2}}\ \sqrt{K_{\gamma_1}}^{-1}h' \rangle_{H(\gamma_2)}$$

$$= \langle \sqrt{K_{\gamma_2}}x, \sqrt{K_{\gamma_2}}x' \rangle_{H(\gamma_2)}$$

$$= \langle x, x' \rangle.$$

Therefore $\gamma_1 \sim \gamma_2$ iff $\sqrt{K_{\gamma_2}}\ \sqrt{K_{\gamma_1}}^{-1} - I$ is Hilbert-Schmidt (cf. 27.16).

§34. *INTEGRATION ON THE DUAL*

Let X be a separable LF-space with sequence of definition $\{X_n\}$.

34.1 LEMMA Suppose that $\chi:X \rightarrow \underline{C}$ is positive definite, $\chi(0) = 1$, and $\forall\ n$, $\chi|X_n$ is continuous — then χ is continuous.

PROOF It suffices to prove that χ is continuous at 0. Fix $\varepsilon > 0$. For each n, choose an open convex neighborhood U_n of 0 in X_n:

$$|\chi(x_n) - 1|^{1/2} \leq \frac{\varepsilon}{2^{n+1}}\ (x_n \in U_n).$$

Let U be the subset of X consisting of all elements of the form $x = x_1 + \cdots + x_n$, where $x_i \in U_i$ $(i = 1,\ldots,n)$ (n variable) — then U is a neighborhood of 0 in X. Since

$$|\chi(x+y) - \chi(x)| \leq \sqrt{2}\ |\chi(y) - 1|^{1/2}$$

it follows that in U:

$$|\chi(x) - 1| = |\chi(x_1 + \cdots + x_n) - 1|$$

$$\leq |\chi(x_1) - 1| + \sum_{1<i\leq n} |\chi(x_1 + \cdots + x_i) - \chi(x_1 + \cdots + x_{i-1})|$$

$$\leq \sqrt{2} \sum_{1\leq i\leq n} |\chi(x_i) - 1|^{1/2} \leq \varepsilon.$$

Write X_w^* for X^* equipped with the weak topology (i.e., with the topology of pointwise convergence: $\lambda_i \rightarrow \lambda$ iff $\forall\ x \in X$, $\lambda_i(x) \rightarrow \lambda(x)$) — then X_w^* is lusinien (cf. 25.21), thus every Borel measure μ on X_w^* is Radon.

Given $x \in X$, define $\hat{x} \in (X_w^*)^*$ by $\hat{x}(\lambda) = \lambda(x)$ -- then the arrow

$$\begin{cases} X \to (X_w^*)^* \\ x \to \hat{x} \end{cases}$$

is bijective, hence X can be regarded as the dual of its weak dual.

34.2 <u>LEMMA</u> Let $\text{Cyl}(X^*)$ be the σ-algebra generated by the $\hat{x}(x \in X)$ -- then

$$\text{Cyl}(X^*) = \text{Bor}(X_w^*).$$

Let μ be a probability measure on $\text{Bor}(X_w^*)$ -- then the <u>Fourier transform</u> of μ is the function $\hat{\mu}:X \to \underline{C}$ defined by the rule

$$\hat{\mu}(x) = \int_{X^*} e^{\sqrt{-1}\,\hat{x}(\lambda)}\, d\mu(\lambda).$$

It is clear that $\hat{\mu}$ is positive definite. Moreover, $\hat{\mu}$ is continuous. In fact, the restriction $\hat{\mu}|X_n$ is continuous $\forall$ n (dominated convergence), from which the assertion (cf. 34.1).

34.3 <u>REMARK</u> Let μ be a probability measure on $\text{Bor}(X_w^*)$ -- then $\hat{\mu}:X \to \underline{C}$ is positive definite, continuous, and equal to one at zero, so on abstract grounds (cf. 14.10) $\exists$ a complex Hilbert space $H_{\hat{\mu}}$, a unitary representation $U_{\hat{\mu}}$ of X on $H_{\hat{\mu}}$, and a cyclic unit vector $x_{\hat{\mu}} \in H_{\hat{\mu}}$ such that

$$\hat{\mu}(x) = \langle x_{\hat{\mu}}, U_{\hat{\mu}}(x)x_{\hat{\mu}}\rangle \quad (x \in X).$$

Explicitly, this data can be realized as follows:

$$\begin{cases} \mathcal{H}_{\hat{\mu}} = L^2(X^*,\mu) \\ x_{\hat{\mu}} = 1 \\ U_{\hat{\mu}}(x) = \text{multiplication by } e^{\sqrt{-1}\,\hat{x}}. \end{cases}$$

Indeed, the function 1 is cyclic and

$$\begin{aligned} \langle 1, U_{\hat{\mu}}(x)1\rangle &= \langle 1, e^{\sqrt{-1}\,\hat{x}}\,1\rangle \\ &= \int_{X^*} e^{\sqrt{-1}\,\hat{x}(\lambda)}\,d\mu(\lambda) \\ &= \hat{\mu}(x). \end{aligned}$$

The map $\mu \to \hat{\mu}$ from the probability measures on $\text{Bor}(X_w^*)$ to the continuous positive definite functions on X is one-to-one but, in general, is not onto but this will be the case if X is nuclear.

34.4 <u>THEOREM</u> (Minlos) Suppose that X is nuclear — then a function $\chi:X \to \underline{C}$ is the Fourier transform of a probability measure μ on $\text{Bor}(X_w^*)$ iff χ is positive definite, continuous, and equal to one at zero.

The proof rests on some preliminaries which are probabilistic in nature (nuclearity plays no role in these considerations).

By definition, a <u>linear process</u> L on X is the assignment of a probability measure $\Lambda_{x_1 \ldots x_p}$ on $\text{Bor}(\underline{R}^p)$ to each finite sequence $x_1, \ldots, x_p$ of elements in X subject to the assumption:

(A) If $x_1, \ldots, x_p$ and $y_1, \ldots, y_q$ are two finite sequences of elements of X that are connected by linear relations

$$x_i = \sum_{j=1}^{q} a_{ij} y_j \quad (i = 1, \ldots, p),$$

then $\forall\ B \in \text{Bor}(\underline{R}^p)$, we have

$$\Lambda_{x_1 \ldots x_p}(B) = \Lambda_{y_1 \ldots y_q}(f^{-1}(B)),$$

where $f: \underline{R}^q \to \underline{R}^p$ is the linear map with matrix $[a_{ij}]$.

[Note: The $\Lambda_{x_1 \ldots x_p}$ are called the <u>marginals</u> of L.]

34.5 <u>EXAMPLE</u> Let $\chi: X \to \underline{C}$ be positive definite, continuous, and equal to one at zero -- then χ gives rise to a linear process on X. Thus let $x_1, \ldots, x_p$ be a finite sequence of elements in X -- then the function from $\underline{R}^p$ to $\underline{C}$ defined by

$$(t_1, \ldots, t_p) \to \chi(t_1 x_1 + \cdots + t_p x_p)$$

satisfies the conditions of 33.3, hence $\exists$ a probability measure $\Lambda_{x_1 \ldots x_p}$ on on $\text{Bor}(\underline{R}^p)$ such that

$$\chi(t_1x_1 + \cdots + t_px_p) = \int_{\underline{R}^p} \exp(\sqrt{-1} \sum_{k=1}^{p} t_k\tau_k) d\Lambda_{x_1\ldots x_p}(\tau).$$

And here, the requirements of condition (A) are clearly met.

34.6 REMARK Every probability measure μ on $\text{Bor}(X_w^*)$ determines a linear process on X: Given a finite sequence $x_1,\ldots,x_p$ of elements in X, define a probability measure $\mu_{x_1\ldots x_p}$ on $\text{Bor}(\underline{R}^p)$ by specifying that

$$\mu_{x_1\ldots x_p}(B) = \mu\{\lambda:(\hat{x}_1(\lambda),\ldots,\hat{x}_p(\lambda)) \in B\}.$$

Then condition (A) is automatic.

[Note: The $\mu_{x_1\ldots x_p}$ are called the marginals of μ.]

34.7 LEMMA Suppose given a linear process L on X -- then $\exists$ a probability measure μ on $\text{Bor}(X_w^*)$ whose marginals are those of L iff $\forall\ \varepsilon > 0\ \&\ \forall\ n$, $\exists$ a neighborhood $U_n(L)$ of zero in X_n such that $\forall\ p$,

$$\Lambda_{x_1\ldots x_p}(I^p) \geq 1 - \varepsilon\ \forall\ x_1,\ldots,x_p \in U_n(L),$$

where

$$I^p = \{(t_1,\ldots,t_p) \in \underline{R}^p : |t_i| \leq 1 \quad (1 \leq i \leq p)\}.$$

[Note: This is a variant on Prokhorov's wellknown "(ε,K)-condition".]

We shall now pass to the proof of 34.4, it being enough to deal with the sufficiency.

34.8 RAPPEL Let E be a vector space over $\underline{R}$ -- then a seminorm $||\cdot||$ on E is said to be hilbertian if it is induced by some nonnegative symmetric bilinear form B on E × E, i.e., if $||\cdot|| = \sqrt{Q}$, where Q is the quadratic form associated with B.

[Note: It is not assumed that $||e|| = 0 \Rightarrow e = 0$, thus B is not necessarily an inner product.]

Since X is nuclear, the same is true of each X_n, so for every neighborhood U_n of zero in X_n, $\exists$ continuous hilbertian seminorms

$$\begin{cases} ||\cdot||_1 = \sqrt{Q_1} & (Q_1 \longleftrightarrow B_1) \\ ||\cdot||_2 = \sqrt{Q_2} & (Q_2 \longleftrightarrow B_2) \end{cases}$$

on X_n such that

$$\{x : Q_1(x) \leq 1\} \subset U_n$$

and

$$B_2(u_i, u_j) = \delta_{ij} \quad (1 \leq i, j \leq q)$$

$$\Rightarrow$$

$$Q_1(u_1) + \cdots + Q_1(u_q) \leq 1.$$

Consider the linear process on X canonically attached to χ (cf. 34.5). If its marginals satisfy the criterion set down in 34.7, then $\exists$ a probability measure μ on $\mathrm{Bor}(X_w^*)$:

$$\mu_{x_1 \ldots x_p} = \Lambda_{x_1 \ldots x_p}.$$

And this implies that $\chi = \hat{\mu}$.

<u>Step 1</u>: Fix $\varepsilon > 0$. Recalling that χ is continuous, let U_n be the neighborhood of 0 in X_n consisting of those x:

$$|\chi(\sqrt{2/\varepsilon}\, x) - 1| \le \varepsilon.$$

Then $\forall\, y \in X_n$:

$$|\chi(y) - 1| \le \varepsilon(1 + Q_1(y)).$$

To see this, write $y = \sqrt{2/\varepsilon}\, x$.

Case (i): $Q_1(x) \le 1 \Rightarrow x \in U_n \Rightarrow |\chi(y) - 1| \le \varepsilon \le \varepsilon(1 + Q_1(y))$.

Case (ii): $Q_1(x) > 1 \Rightarrow Q_1(y) = \frac{2}{\varepsilon} Q_1(x) > \frac{2}{\varepsilon} \Rightarrow \varepsilon(1 + Q_1(y)) > \varepsilon + \varepsilon \cdot \frac{2}{\varepsilon} =$

$\varepsilon + 2 \ge \varepsilon + |\chi(y)| + 1 \ge \varepsilon + |\chi(y) - 1| > |\chi(y) - 1|$.

<u>Step 2</u>: Since Q_2 is continuous, the set of $x \in X_n : Q_2(x) \le 1$ is a neighborhood of 0 in X_n, call it $U_n(L)$. Let $x_1, \ldots, x_p \in U_n(L)$, let $u_1, \ldots, u_q$ be an orthonormal basis per B_2 for the subspace of X_n generated by $x_1, \ldots, x_p$ -- then $\exists$ real numbers $r_{ij}\,(1 \le i \le p,\ 1 \le j \le q)$:

$$x_i = \sum_j r_{ij} u_j,$$

where $\sum_j r_{ij}^2 = Q_2(x_i) \le 1$ $(1 \le i \le p)$. Let

$$\left[\begin{aligned} S &= \{\xi \in \underline{R}^q : \sum_j \xi_j^2 \leq 1\} \\ T &= \{\xi \in \underline{R}^q : |\sum_j r_{ij}\xi_j| \leq 1 \ (1 \leq i \leq p)\}. \end{aligned}\right.$$

Then $\forall$ i = 1,...,p,

$$|\sum_j r_{ij}\xi_j| \leq (\sum_j r_{ij}^2)^{1/2}(\sum_j \xi_j^2)^{1/2}$$

$$\leq (\sum_j \xi_j^2)^{1/2}$$

$$\Rightarrow$$

$$S \subset T.$$

But condition (A) gives:

$$\Lambda_{x_1 \ldots x_p}(I^p) = \Lambda_{u_1 \ldots u_q}(T).$$

Therefore

$$\Lambda_{x_1 \ldots x_p}(I^p) \geq \Lambda_{u_1 \ldots u_q}(S).$$

<u>Step 3:</u> Let S' be the complement of S in $\underline{R}^q$. Since

$$1 - e^{-\langle\xi,\xi\rangle/2} \geq 1 - e^{-1/2} \geq \frac{1}{3} \quad (\xi \in S'),$$

it follows that

$$\Lambda_{u_1 \ldots u_q}(S')/3$$

$$\leq \int_{S'} (1 - e^{-\langle\xi,\xi\rangle/2}) d\Lambda_{u_1 \ldots u_q}(\xi)$$

$$\leq \int_{\underline{R}^q} (1 - e^{-\langle\xi,\xi\rangle/2}) d\Lambda_{u_1 \ldots u_q}(\xi),$$

call the last integral I.

Step 4: We have

$$I = \frac{1}{(2\pi)^{q/2}} \int_{\underline{R}^q} (1 - \chi(\sum_j \eta_j u_j)) e^{-\langle\eta,\eta\rangle/2} d\eta_1 \ldots d\eta_q.$$

But

$$|1 - \chi(\sum_j \eta_j u_j)|$$

$$\leq \varepsilon(1 + Q_1(\sum_j \eta_j u_j))$$

$$= \varepsilon(1 + \sum_j \eta_j^2 Q_1(u_j)).$$

Therefore

$$|I| \leq \varepsilon(1 + \frac{1}{(2\pi)^{q/2}} \int_{\underline{R}^q} \sum_j \eta_j^2 Q_1(u_j) e^{-\langle\eta,\eta\rangle/2} d\eta_1 \ldots d\eta_q$$

$$= \varepsilon(1 + \sum_j Q_1(u_j) \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \eta_j^2 e^{-\eta_j^2/2} d\eta_j)$$

$$= \varepsilon(1 + \sum_j Q_1(u_j))$$

$$\leq \varepsilon(1 + 1) = 2\varepsilon.$$

Step 5:

$$\Lambda_{u_1 \ldots u_q}(S') \leq 6\varepsilon$$

$$\Rightarrow$$

$$\Lambda_{u_1 \ldots u_q}(S) \geq 1 - 6\varepsilon$$

$$\Rightarrow$$

$$\Lambda_{x_1 \ldots x_p}(I^p) \geq 1 - 6\varepsilon.$$

Thus the conditions of 34.7 are fulfilled by the marginals $\Lambda_{x_1 \ldots x_p}$.

34.9 REMARK Let X be an infinite dimensional separable real Hilbert space — then its Sazonov topology (cf. 33.11) is not nuclear, so in this context, 34.4 is not applicable.

Write X_s^* for X^* equipped with the strong topology (i.e., with the topology of uniform convergence on bounded subsets of X).

If X is nuclear, then X is Montel (being complete and barreled), as is X_s^* (the strong dual of a Montel space is Montel). In addition, X_s^* is nuclear (the strong dual of a nuclear Fréchet space is nuclear and X_s^* is the projective limit of such duals).

34.10 LEMMA Suppose that X is nuclear — then

$$\mathrm{Bor}(X_w^*) = \mathrm{Bor}(X_s^*)$$

and X_s^* is lusinien.

34.11 EXAMPLE $C_c^\infty(\underline{R}^n)$ is a nuclear separable LF-space.

[Note: $C_c^\infty(\underline{R}^n)^*$ (the space of distributions), when equipped with the strong topology, is nuclear.]

34.12 EXAMPLE $\mathcal{S}(\underline{R}^n)$ is a nuclear separable Fréchet space.

[Note: $\mathcal{S}(\underline{R}^n)^*$ (the space of tempered distributions), when equipped with the strong topology, is nuclear.]

34.13 REMARK If X is nuclear, then X is reflexive (being Montel). Therefore the canonical arrow $X \rightarrow (X_s^*)^*$ is an isomorphism of topological vector spaces.

Suppose that X is a nuclear separable LF-space. Fix a continuous quadratic form Q on $X: x \neq 0 \Rightarrow Q(x) > 0$ -- then the function

$$x \rightarrow \exp(-\frac{1}{2} Q(x))$$

is positive definite (cf. 33.2), continuous, and equal to one at zero, thus by 34.4, $\exists$ a unique probability measure γ on $\mathrm{Bor}(X_s^*)$ (= $\mathrm{Cyl}(X^*)$) such that

$$\hat{\gamma}(x) = \exp(-\frac{1}{2} Q(x)).$$

N.B. γ is gaussian (cf. 26.3).

The induced measure $\gamma \circ (\hat{x})^{-1}$ on $\underline{R}$ is centered gaussian with variance $\sigma^2 = Q(x)$. And

$$Q(x) = \int_{X^*} \hat{x}(\lambda)^2 \, d\gamma(\lambda) = ||\hat{x}||^2_{L^2(\gamma)}.$$

Denote by X_γ the completion of X per Q -- then X_γ can be regarded as the closure of $\hat{X}$ in $L^2(X^*,\gamma)$.

34.14 LEMMA There exists an isometric isomorphism

$$T:BO(X_\gamma) \to L^2(X^*,\gamma)$$

characterized by the relation

$$T \, \underline{\exp}(f) = \exp(f - \frac{1}{2} ||f||_2^2) \qquad (\text{cf. } \S 28).$$

[Note: $\forall \ x \in X \ (x \neq 0)$,

$$T(\hat{x}^{\otimes n}) = \frac{1}{\sqrt{n!}} Q(x)^n \, H_n(\frac{\hat{x}}{Q(x)}).]$$

34.15 EXAMPLE Take $X = S(\underline{R}^n)$ in its usual topology as a Fréchet space. Put

$$Q(f) = \langle f,(-\Delta + m^2)^{-1} f\rangle_{L^2(\underline{R}^n)} \quad (m > 0).$$

Because X is nuclear, $e^{-Q/2}$ is the Fourier transform of a unique gaussian measure γ_m on X_s^*, the free scalar field of mass m.

[Note: The <u>white noise space</u> is the pair $(S(\underline{R}^n)^*,\gamma_S)$, where γ_S is determined by

$$Q(f) = \exp(-\frac{1}{2} ||f||^2_{L^2(\underline{R}^n)}).$$

Here the theory implies that

$$BO(L^2(\underline{R}^n))$$

can be identified with

$$L^2(S(\underline{R}^n)^*,\gamma_S).]$$

34.16 <u>REMARK</u> Take m = 1 — then

$$\sqrt{Q(f)} = ||(1 - \Delta)^{-1/2} f||_{L^2(\underline{R}^n)},$$

so the relevant completion is the Sobolev space $W^{2,-1}(\underline{R}^n)$ and we have

$$BO(W^{2,-1}(\underline{R}^n)) \approx L^2(S(\underline{R}^n)^*,\gamma_1).$$

§35. *THE WIENER MEASURE*

The setting for the construction is either C[0,1] (which is a separable Banach space in the topology of uniform convergence) or $C[0,\infty[$ (which is a separable Fréchet space in the topology of uniform convergence on compacta). While the details in both cases are similar, the situation for C[0,1] is somewhat simpler so we shall start with it.

35.1 <u>REMARK</u> There are various roads that lead to the Wiener measure on C[0,1] but no matter what route is followed, the conclusion is that its topological support is the hyperplane

$$C_0[0,1] = \{f \in C[0,1]:f(0) = 0\}.$$

To avoid certain technicalities, it will be best to proceed directly and deal with $C_0[0,1]$ from the outset.

Consider the collection $\mathfrak{C}$ of subsets of $C_0[0,1]$ which have the form

$$C = \{f:(f(t_1),\ldots,f(t_n)) \in B\},$$

where $0 < t_1 < t_2 < \cdots < t_n \leq 1$ and $B \in \text{Bor}(\underline{R}^n)$ -- then $\mathfrak{C}$ is an algebra and the σ-algebra generated by $\mathfrak{C}$ is

$$\text{Cyl}(C_0[0,1]) = \text{Bor}(C_0[0,1]).$$

Define a set function $w:\mathfrak{C} \rightarrow [0,1]$ by

$$w(C) = w_n(\vec{t}) \int_B \exp(-\frac{1}{2} W_n(\vec{t},\vec{u}))d\vec{u},$$

where

$$w_n(\vec{t}) = [(2\pi)^n t_1(t_2-t_1)\ldots(t_n-t_{n-1})]^{-1/2}$$

and

$$W_n(\vec{t},\vec{u})$$

$$= \frac{u_1^2}{t_1} + \frac{(u_2-u_1)^2}{t_2-t_1} + \cdots + \frac{(u_n-u_{n-1})^2}{t_n-t_{n-1}} .$$

Then it is clear that w is finitely additive on $\mathfrak{C}$.

35.2 <u>EXAMPLE</u> Fix $t: 0 < t \leq 1$ -- then

$$w\{f: a \leq f(t) \leq b\} = \frac{1}{\sqrt{2\pi t}} \int_a^b \exp(-\frac{u^2}{2t})du.$$

35.3 <u>THEOREM</u> (Wiener) w is countably additive on $\mathfrak{C}$.

Therefore w can be extended to a probability measure P^W on the σ-algebra generated by $\mathfrak{C}$, i.e., to $\mathrm{Bor}(C_0[0,1])$, and P^W is, by definition, the <u>Wiener measure</u>.

35.4 <u>LEMMA</u> Suppose that $T: \underline{R}^n \to \underline{R}$ is Borel -- then

$$\int_{C_0[0,1]} T(f(t_1),\ldots,f(t_n))dP^W(f)$$

$$= w_n(\vec{t}) \int_{\underline{R}^n} T(\vec{u})\exp(-\frac{1}{2} W_n(\vec{t},\vec{u}))d\vec{u}.$$

PROOF Define

$$F_{t_1\cdots t_n}:C_0[0,1] \rightarrow \underline{R}^n$$

by

$$F_{t_1\cdots t_n}(f) = (f(t_1),\ldots,f(t_n)).$$

Then $F_{t_1\cdots t_n}$ is continuous, hence Borel. And

$$\int_{C_0[0,1]} T(f(t_1),\ldots,f(t_n))dP^W(f)$$

$$= \int_{C_0[0,1]} T \circ F_{t_1\cdots t_n}(f)dP^W(f)$$

$$= \int_{\underline{R}^n} T(\vec{u})d(P^W \circ F^{-1}_{t_1\cdots t_n})(\vec{u})$$

$$= w_n(\vec{t}) \int_{\underline{R}^n} T(\vec{u})\exp(-\frac{1}{2} W_n(\vec{t},\vec{u}))d\vec{u}.$$

35.5 EXAMPLE We have

$$\int_{C_0[0,1]} f(t)dP^W(f) = 0 \quad (0 < t \leq 1).$$

[In fact, $f(t) = T(f(t))$ $(Tu = u)$, hence

$$\int_{C_0[0,1]} f(t)dP^W(f)$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{\underline{R}} u \exp(-\frac{u^2}{2t}) du = 0.]$$

35.6 <u>EXAMPLE</u> We have

$$\int_{C_0[0,1]} f(t_1)f(t_2)dP^W(f) = \min(t_1,t_2) \quad (t_1 \neq t_2).$$

[Suppose that $0 < t_1 < t_2 \leq 1$. Let $T(u_1,u_2) = u_1u_2$ — then

$$\int_{C_0[0,1]} f(t_1)f(t_2)dP^W(f)$$

$$= \frac{1}{((2\pi)^2 t_1(t_2-t_1))^{1/2}}$$

$$\times \int_{\underline{R}^2} u_1u_2 \exp(-\frac{1}{2}\frac{u_1^2}{t_1} - \frac{1}{2}\frac{(u_2-u_1)^2}{t_2-t_1}) du_1du_2.$$

Let

$$\begin{cases} v_1 = \dfrac{u_1}{\sqrt{2t_1}} \\ v_2 = \dfrac{u_2-u_1}{\sqrt{2(t_2-t_1)}} \end{cases}$$

or still,

$$\begin{cases} u_1 = \sqrt{2t_1}\, v_1 \\ u_2 = \sqrt{2t_1}\, v_1 + \sqrt{2(t_2-t_1)}\, v_2. \end{cases}$$

Then

$$\begin{vmatrix} \sqrt{2t_1} & 0 \\ \sqrt{2t_1} & \sqrt{2(t_2-t_1)} \end{vmatrix} = 2\sqrt{t_1(t_2-t_1)}$$

=>

$$\int_{C_0[0,1]} f(t_1)f(t_2)dP^W(f)$$

$$= \frac{1}{\pi}\int_{\underline{R}} 2t_1v_1^2 e^{-v_1^2}\left[\int_{\underline{R}} e^{-v_2^2} dv_2\right]dv_1$$

$$= \frac{2t_1}{\pi}\int_{\underline{R}} v_1^2 e^{-v_1^2}(\sqrt{\pi})dv_1$$

$$= \frac{2t_1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = t_1.]$$

[Note: A similar but easier calculation gives

$$\int_{C_0[0,1]} f^2(t)dP^W(f) = t.$$

Therefore

$$\int_{C_0[0,1]} ||f||_2^2 \, dP^W(f)$$

$$= \int_0^1 \left(\int_{C_0[0,1]} f^2(t)dP^W(f) \right) dt$$

$$= \int_0^1 tdt = \frac{1}{2} .]$$

35.7 REMARK Consider the one parameter family of random variables $\{\delta_t : 0 \le t \le 1\}$ $(\delta_t(f) = f(t), \delta_0 = 0)$. From the above,

- $\int_{C_0[0,1]} (\delta_t - \delta_{t'}) dP^W = 0.$

- $\int_{C_0[0,1]} (\delta_t - \delta_{t'})^2 \, dP^W = |t-t'|.$

Furthermore, if $0 \le t_1 < \cdots < t_n \le 1$, then $\delta_{t_2} - \delta_{t_1}, \ldots, \delta_{t_n} - \delta_{t_{n-1}}$ are independent.

[Note: The distribution of the random variables $\pm (\delta_t - \delta_{t'})$ is gaussian of mean 0 and variance $|t-t'|$.

The dual of $C_0[0,1]$ is the space of all Borel signed measures on $[0,1]$ modulo the scalar multiples of the Dirac measure δ_0.

35.8 LEMMA $\forall \lambda \in C_0[0,1]^*$,

$$\hat{P}^W(\lambda) = \exp(-\frac{1}{2} \int_0^1 \int_0^1 \min(u,v) d\lambda(u) d\lambda(v)).$$

PROOF Suppose first that $\lambda = \delta_t$ $(0 < t \le 1)$ — then

$$\hat{P}^W(\delta_t) = \int_{C_0[0,1]} e^{\sqrt{-1}\, \delta_t(f)} \, dP^W(f)$$

$$= \int_{C_0[0,1]} e^{\sqrt{-1}\, f(t)} \, dP^W(f)$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{\underline{R}} e^{\sqrt{-1}\, u} \, d\exp(-\frac{u^2}{2t}) du \quad \text{(cf. 35.4)}$$

$$= \exp(-\frac{t}{2}) \quad \text{(cf. 22.2)}.$$

On the other hand,

$$\exp(-\frac{1}{2}\int_0^1\int_0^1 \min(u,v)\,d\delta_t(u)\,d\delta_t(v))$$

$$= \exp(-\frac{1}{2}\int_0^1 \min(t,v)\,d\delta_t(v))$$

$$= \exp(-\frac{t}{2}).$$

Therefore the claimed relation is valid if $\lambda = \delta_t$ $(0 \le t \le 1)$ (matters are obvious at $t = 0$), hence if λ is a finite linear combination of Dirac measures. If λ is arbitrary, then there is a sequence of Borel signed measures λ_k which are finite linear combinations of Dirac measures and which converge weakly to λ, i.e., $\forall\ f \in C[0,1]$, $\int_0^1 f d\lambda_k \to \int_0^1 f d\lambda$.

35.9 <u>LEMMA</u> P^W is a centered gaussian measure on $C_0[0,1]$.

<u>PROOF</u> This follows from 35.8 (cf. 26.3).

Our next objective will be to determine the Cameron-Martin space $H(P^W)$, a space which is independent of whether P^W is considered on $C_0[0,1]$ or $L^2[0,1]$ (cf. 26.28), it being more convenient to work with the latter.

Let K_{P^W} be the nonnegative, symmetric, trace class operator canonically associated with P^W (regarded now as a centered gaussian measure on $L^2[0,1]$), so $\forall\ f \in L^2[0,1]$,

$$\hat{P}^W(f) = \exp(-\frac{1}{2} \langle f, K_{P^W} f\rangle) \qquad \text{(cf. 33.13)}.$$

35.10 <u>LEMMA</u> K_{P^W} is an integral operator on $L^2[0,1]$ with kernel $\min(u,v)$:

$$K_{P^W} f(u) = \int_0^1 \min(u,v) f(v)\, dv \qquad (f \in L^2[0,1]).$$

35.11 <u>LEMMA</u> Put $\lambda_n = \pi^{-2}(n - \frac{1}{2})^{-2}$ — then the functions

$$f_n(u) = \sqrt{2}\,\sin(\pi(n - \frac{1}{2})u)$$

are an orthonormal basis for $L^2[0,1]$ with

$$K_{P^W} f_n = \lambda_n f_n \qquad (n = 1,2,\ldots).$$

<u>PROOF</u> Fix $\lambda > 0$ and consider the relation

$$\int_0^1 \min(u,v) f_\lambda(v)\, dv = \lambda f_\lambda(u)$$

or still,

$$\int_0^u v f_\lambda(v)\, dv + u \int_u^1 f_\lambda(v)\, dv = \lambda f_\lambda(u).$$

Since $K_{P^W} f_\lambda$ is continuous, the same must be true of $f_\lambda = K_{P^W} f_\lambda / \lambda$, hence $K_{P^W} f_\lambda$ is differentiable. Therefore

$$\begin{aligned} \lambda f_\lambda'(u) &= u f_\lambda(u) + \int_u^1 f_\lambda(v)\, dv - u f_\lambda(u) \\ &= \int_u^1 f_\lambda(v)\, dv. \end{aligned}$$

But this implies that f_λ' is differentiable and $\lambda f_\lambda'' = - f_\lambda$. As for the initial conditions, they are $f_\lambda(0) = 0$ and $f_\lambda'(1) = 0$. The solutions are then as stated.

[Note: Analogously, when $\lambda = 0$, one concludes that $f_0 = 0$, so $K_{P^W} > 0$.]

Let $W_0^{2,1}[0,1]$ denote the set of functions f on [0,1] such that f is absolutely continuous, $f' \in L^2[0,1]$, and $f(0) = 0$.

35.12 LEMMA We have

$$H(P^W) = W_0^{2,1}[0,1].$$

PROOF Take an $f \in L^2[0,1]$ and write $f = \sum_{n=1}^{\infty} \langle f_n, f\rangle f_n$ — then $f \in H(P^W)$ iff

$$\sum_{n=1}^{\infty} \frac{\langle f_n, f\rangle^2}{\lambda_n} < \infty \quad \text{(cf. 33.17)}$$

or still, iff

$$\sum_{n=1}^{\infty} \langle f_n, f\rangle^2 \, \pi^2 (n - \tfrac{1}{2})^2 < \infty \quad \text{(cf. 35.11)}.$$

The latter is equivalent to the existence of a function $g \in L^2[0,1]$:

$$\langle g_n, g\rangle = \langle f_n, f\rangle \pi (n - \tfrac{1}{2}),$$

where

$$g_n(u) = \sqrt{2} \cos(\pi (n - \tfrac{1}{2}) u).$$

But then

$$\int_0^1 f_n(x) (\int_0^x g) dx$$

$$= -\sqrt{\lambda_n}\int_0^1 g_n'(x)\left(\int_0^x g\right)dx$$

$$= \sqrt{\lambda_n}\int_0^1 g_n(x)g(x)dx$$

$$= \sqrt{\lambda_n}\,\langle g_n,g\rangle$$

$$= \sqrt{\lambda_n}\,\langle f_n,f\rangle\,\frac{1}{\sqrt{\lambda_n}}$$

$$= \langle f_n,f\rangle.$$

Therefore f is absolutely continuous, $f' \in L^2[0,1]$, and $f(0) = 0$. Conversely, if f has these properties, then

$$\pi(n - \tfrac{1}{2})\ \langle f_n,f\rangle = -\int_0^1 g_n'f$$

$$= \int_0^1 g_nf' = \langle g_n,f'\rangle.$$

And

$$\sum_{n=1}^{\infty} \langle g_n,f'\rangle^2 < \infty$$

=>

$$\sum_{n=1}^{\infty} \frac{\langle f_n,f\rangle^2}{\lambda_n} < \infty$$

=>

$$f \in H(P^W) \quad \text{(cf. 33.17)}.$$

Define $T:L^2[0,1] \to L^2[0,1]$ by

$$Tf(x) = \int_0^x f = F(x).$$

Then

$$\int_0^1 (Tf(x))^2 dx = \int_0^1 (\int_0^x f)^2 dx$$

$$= \int_0^1 |\int_0^x f|^2 dx$$

$$\leq \int_0^1 (\int_0^x |f|)^2 dx$$

$$\leq \int_0^1 (\int_0^1 |f|)^2 dx$$

$$= (\int_0^1 |f|)^2$$

$$\leq \int_0^1 |f|^2 \leq ||f||^2_{L^2[0,1]}.$$

Therefore T is bounded.

35.13 <u>LEMMA</u> We have

$$T^*f(x) = \int_x^1 f.$$

<u>PROOF</u> Let

$$\begin{cases} F(x) = \int_0^x f & (\Rightarrow F(0) = 0) \\ G(x) = \int_0^x g & (\Rightarrow G(0) = 0). \end{cases}$$

Then F,G are absolutely continuous, so integration by parts is permissible, thus

$$\int_0^1 (\int_x^1 f)g$$

$$= \int_0^1 [\int_0^1 f - \int_0^x f]g$$

$$= (\int_0^1 f)(\int_0^1 g) - \int_0^1 Fg$$

$$= F(1)G(1) - \int_0^1 FG'$$

$$= F(1)G(1) - [FG|_0^1 - \int_0^1 F'G]$$

$$= \int_0^1 F'G$$

$$= \int_0^1 f(\int_0^x g)$$

$$= \langle f,Tg\rangle.$$

I.e.:

$$T^*f(x) = \int_x^1 f.$$

35.14 LEMMA There is a factorization

$$K_{P^W} = TT^*.$$

PROOF $\forall\ f \in L^2[0,1]$,

$$TT^*f(u) = \int_0^u \left(\int_v^1 f\right) dv$$

$$= \int_0^u \left(\int_0^1 f - \int_0^v f\right) dv$$

$$= \left(\int_0^1 f\right) \int_0^u dv - \int_0^u \left(\int_0^v f\right) dv$$

$$= u\int_0^1 f - \int_0^u F(v) dv$$

$$= u\int_0^1 f - \left[vF(v)\Big|_0^u - \int_0^u f(v) v dv\right]$$

$$= u\int_0^1 f - u\int_0^u f + \int_0^u f(v) v dv.$$

Meanwhile

$$K_{P^W} f(u) = \int_0^1 \min(u,v) f(v) dv$$

$$= \int_0^u v f(v) dv + u\int_u^1 f(v) dv$$

$$= \int_0^u v f(v) dv + u\left[\int_0^1 f - \int_0^u f\right]$$

$$= u\int_0^1 f - u\int_0^u f + \int_0^u f(v) v dv.$$

35.15 <u>RAPPEL</u> T is injective.

[From real variable theory, if $f \in L^1[0,1]$ and if $\int_0^x f = 0$ for all x ($0 \le x \le 1$), then $f = 0$ almost everywhere.]

Therefore

$$\{0\} = \mathrm{Ker}(T) = \overline{\mathrm{Ran}(T^*)}^{\perp},$$

which means that the range of T^* is dense.

Bearing in mind that $\sqrt{K_{P^W}}$ is injective, put

$$\zeta(\sqrt{K_{P^W}}\, f) = T^*f.$$

Then

$$||\zeta(\sqrt{K_{P^W}}\, f)||^2 = ||T^*f||^2$$

$$= \langle T^*f, T^*f\rangle$$

$$= \langle f, TT^*f\rangle$$

$$= \langle f, K_{P^W} f\rangle \qquad (\text{cf. } 35.14)$$

$$= \langle \sqrt{K_{P^W}}\, f, \sqrt{K_{P^W}}\, f\rangle.$$

Therefore

$$\zeta : \sqrt{K_{P^W}}\, L^2[0,1] \rightarrow T^*L^2[0,1]$$

is isometric. Since

$$\left[\begin{array}{l} \sqrt{K_{P^W}}\, L^2[0,1] \\ \\ T^*L^2[0,1] \end{array}\right.$$

are both dense in $L^2[0,1]$, ζ can be extended to an isometric isomorphism $L^2[0,1] \to L^2[0,1]$ (denoted still by ζ).

N.B.

$$\zeta \circ \sqrt{K_{P^W}} = T^* \Rightarrow \sqrt{K_{P^W}} \circ \zeta^* = T.$$

Given $f,g \in W_0^{2,1}[0,1]$, put

$$<f,g>' = \int_0^1 f'g'.$$

Then under this inner product, $W_0^{2,1}[0,1]$ is a separable real Hilbert space.

[Note: Recall that if the derivative of an absolutely continuous function is zero almost everywhere, then this function is a constant C and in our case, C = 0.]

35.16 LEMMA $\forall$ $f,g \in W_0^{2,1}[0,1]$,

$$<f,g>' = <f,g>_{H(P^W)}.$$

PROOF On general grounds,

$$H(P^W) = \sqrt{K_{P^W}}\, L^2[0,1] \quad \text{(cf. 33.16)}.$$

And here, according to 35.12,

$$H(P^W) = W_0^{2,1}[0,1].$$

This said, take $f,g \in W_0^{2,1}[0,1]$ and write $f = \sqrt{K_{P^W}}\,\phi$, $g = \sqrt{K_{P^W}}\,\psi$ — then

$$\langle f,g\rangle_{H(P^W)} = \langle \phi,\psi\rangle_{L^2[0,1]} \qquad \text{(cf. 33.16)}.$$

But

$$\begin{bmatrix} Tf' = f \\ \\ Tg' = g \end{bmatrix} \Rightarrow \begin{bmatrix} Tf' = \sqrt{K_{P^W}}\,\phi \\ \\ Tg' = \sqrt{K_{P^W}}\,\psi \end{bmatrix}$$

$$\Rightarrow$$

$$\begin{bmatrix} (\sqrt{K_{P^W}} \circ \zeta^*)f' = \sqrt{K_{P^W}}\,\phi \\ \\ (\sqrt{K_{P^W}} \circ \zeta^*)g' = \sqrt{K_{P^W}}\,\psi \end{bmatrix}$$

$$\Rightarrow$$

$$\begin{bmatrix} \zeta^* f' = \phi \\ \zeta^* g' = \psi, \end{bmatrix}$$

$\sqrt{K_{P^W}}$ being injective. Finally,

$$\begin{aligned} \langle f,g\rangle' &= \langle f',g'\rangle_{L^2[0,1]} \\ &= \langle \zeta^* f',\zeta^* g'\rangle_{L^2[0,1]} \\ &= \langle \phi,\psi\rangle_{L^2[0,1]} \\ &= \langle f,g\rangle_{H(P^W)}. \end{aligned}$$

Given $0 \le t,\ t' \le 1$ and $M > 0$, let

$$C_0[0,1](t,t';M) = \{f \in C_0[0,1] : |f(t) - f(t')| \le M|t - t'|\}$$

and put

$$C_0[0,1](t;M) = \bigcap_{0 \le t' \le 1} C_0[0,1](t,t';M).$$

Then $C_0[0,1](t,t';M)$ is a closed subset of $C_0[0,1]$, hence the same is true of $C_0[0,1](t;M)$.

35.17 LEMMA For $t \ne t'$, we have

$$P^W(C_0[0,1](t,t';M)) \le \sqrt{2/\pi}\, M|t - t'|^{1/2}.$$

PROOF Take $t' < t$ — then there are two possibilities: $t' = 0$ or $t' > 0$. As the second is slightly more involved than the first, we shall deal with it. From the definitions,

$$P^W(C_0[0,1](t,t';M))$$

$$= \frac{1}{((2\pi)^2\ t'(t-t'))^{1/2}}$$

$$\times \int_B \exp(-\frac{1}{2}\ (\frac{u_1^2}{t'} + \frac{(u_2-u_1)^2}{2(t-t')}\ ))du_1du_2,$$

where

$$B = \{(u_1,u_2) \in \underline{R}^2 : |u_2 - u_1| \le M|t - t'|\}.$$

To estimate this integral, let

$$\begin{cases} v_1 = \dfrac{u_1}{\sqrt{t'}} \\[2ex] v_2 = \dfrac{u_2-u_1}{\sqrt{t-t'}}. \end{cases}$$

Then

$$P^W(C_0[0,1](t,t';M))$$

$$= \frac{1}{2\pi} \int_{\underline{R}} \exp(-\frac{v_1^2}{2}) \ [\int_{-M|t-t'|^{1/2}}^{M|t-t'|^{1/2}} \exp(-\frac{v_2^2}{2}) dv_2] dv_1$$

$$\leq \frac{1}{2\pi} \int_{\underline{R}} \exp(-\frac{v_1^2}{2}) \ [\int_{-M|t-t'|^{1/2}}^{M|t-t'|^{1/2}} dv_2] dv_1$$

$$= \frac{M|t-t'|^{1/2}}{\pi} \int_{\underline{R}} \exp(-\frac{v_1^2}{2}) dv_1$$

$$= \sqrt{2/\pi}\, M|t-t'|^{1/2}.$$

35.18 <u>LEMMA</u> $\forall\ t \in [0,1]$,

$$P^W(C_0[0,1](t;M)) = 0.$$

<u>PROOF</u> Choose a sequence of points $t_k (k = 1,2,\ldots)$ in $[0,1]: t_k \neq t$, $t_k \to t\ (k \to \infty)$ — then

$$P^W(C_0[0,1](t;M))$$

$$\leq P^W(C_0[0,1](t,t_k;M))$$

$$\leq \sqrt{2/\pi}\, M|t-t_k|^{1/2} \to 0 \quad (k \to \infty).$$

Given $0 \leq t \leq 1$, let D_t be the set of $f \in C_0[0,1]:f'(t)$ exists (use a one sided derivative at the endpoints) -- then

$$D_t \subset \bigcup_{m=1}^{\infty} C_0[0,1](t;m).$$

To see this, just note that for any $f \in D_t$, $\exists$ a positive integer m_f with the property that

$$|f(t) - f(t')| \leq m_f|t-t'| \quad (0 \leq t' \leq 1).$$

I.e.:

$$f \in C_0[0,1](t;m_f).$$

So, thanks to 35.18,

$$P^W(\bigcup_{m=1}^{\infty} C_0[0,1](t;m))$$

$$\leq \sum_{m=1}^{\infty} P^W(C_0[0,1](t;m))$$

$$0.$$

Therefore D_t lies in the domain of the completion $\overline{P^W}$ of P^W and

$$\overline{P^W}(D_t) = 0.$$

N.B. It is not claimed that D_t is Borel.

35.19 REMARK Introducing $\overline{P^W}$ is not a big deal and avoids thorny measurability issues. E.g.: Let S be the subset of $C_0[0,1]$ consisting of those f whose derivative exists on a set of positive Lebesgue measure -- then it can be shown that $\overline{P^W}(S) = 0$. Thus, as a corollary, if S_{bv} is the set of f in $C_0[0,1]$ which are of bounded variation on some subinterval of $[0,1]$, then $S_{bv} \subset S$, so $\overline{P^W}(S_{bv}) = 0$.

[Note: Here is a sketch of the argument. Define $F:C_0[0,1] \times [0,1]$ by: $F(f,t) = 1$ if $f'(t)$ exists and $F(f,t) = 0$ otherwise — then F is measurable w.r.t. the completion of $\overline{P^W} \times \text{Leb}$ (which is not totally obvious) and

$$\int_{C_0[0,1]} [\int_0^1 F(f,t)dt]d\overline{P^W}(f)$$

$$= \int_0^1 [\int_{C_0[0,1]} F(f,t)d\overline{P^W}(f)]dt$$

$$= \int_0^1 [\int_{C_0[0,1]} \chi_{D_t}(f)d\overline{P^W}(f)]dt$$

$$= \int_0^1 \overline{P^W}(D_t)dt$$

$$= \int_0^1 0\ dt = 0$$

=>

$$\overline{P^W}\{f:\int_0^1 F(f,t)dt = 0\} = 1$$

=>

$$\overline{P^W}(C_0[0,1] - S) = 1.]$$

The theory of the Wiener measure P^W goes through with no essential changes when $C_0[0,1]$ is replaced by $C_0[0,\infty[$, where

$$C_0[0,\infty[= \{f \in C[0,\infty[: f(0) = 0\}.$$

There are, however, some additional features stemming from the fact that $[0,\infty[$ allows for asymptotics at infinity.

Fix $T > 0$ and $n \in \underline{N}$. Let

$$\xi_k = \delta_{kT/n} - \delta_{(k-1)t/n} \quad (k = 1,2,\ldots).$$

Then the ξ_k are independent (cf. 35.7). Note too that

$$S_k = \xi_1 + \cdots + \xi_k = \delta_{kT/n} \ (\delta_0 = 0),$$

so for $\ell \leq k$,

$$S_k - S_\ell = \delta_{kT/n} - \delta_{\ell T/n}.$$

<u>N.B.</u> The number 0 is a median for $S_k - S_\ell$ (the distribution of $\delta_{kT/n} - \delta_{\ell T/n}$ is gaussian of mean 0 and variance $|kT/n - \ell T/n|$ (cf. 35.7)).

35.20 <u>LEMMA</u> Fix $T > 0$ -- then

$$P^W\{f: \sup_{0 \leq t \leq T} |f(t)| \geq M\}$$

$$\leq 2\ \exp(-\frac{M^2}{2T}) \quad (M > 0).$$

<u>PROOF</u> $\forall\ n \in \underline{N}$,

$$P^W(\max_{1\leq k\leq n} |S_k| \geq M) \leq 2P^W(|S_n| \geq M) \quad \text{(Lévy)}$$

$$= 2P^W(|S_T| \geq M)$$

=>

$$P^W\{f: \sup_{0\leq t\leq T} |f(t)| \geq M\}$$

$$\leq 2P^W\{f: |f(T)| \geq M\}$$

$$= 2[\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{-M} \exp(-\frac{u^2}{2T})du$$

$$+ \frac{1}{\sqrt{2\pi T}} \int_M^{\infty} \exp(-\frac{u^2}{2T})du]$$

$$= \frac{4}{\sqrt{2\pi T}} \int_M^{\infty} \exp(-\frac{u^2}{2T})du$$

$$= \frac{4}{\sqrt{2\pi}} \int_{M/\sqrt{T}}^{\infty} \exp(-\frac{t^2}{2})dt$$

$$\leq \frac{4}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{2}} \exp(-\frac{M^2}{2T})$$

$$= 2\ \exp(-\frac{M^2}{2T}).$$

35.21 <u>LEMMA</u> Given $n \in \underset{\sim}{N}$, define

$$\Delta_n : C_0[0,\infty[\ \to C_0[0,\infty[$$

by

$$(\Delta_n f)(t) = f(t+n) - f(n).$$

Then

$$(\Delta_n)_* P^W = P^W.$$

35.22 <u>EXAMPLE</u> Fix $M > 0$ and let

$$B_M = \{f: \sup_{[0,1]} |f| \geq M\}.$$

Then

$$(\Delta_n)_* P^W(B_M) = P^W(B_M).$$

But

$$(\Delta_n)_* P^W(B_M) = P^W(\Delta_n^{-1} B_M)$$

and

$$\Delta_n^{-1} B_M = \{f: \Delta_n f \in B_M\}$$

$$= \{f: \sup_{[0,1]} |\Delta_n f| \geq M\}$$

$$= \{f: \sup_{0 \leq t \leq 1} |f(t+n) - f(n)| \geq M\}$$

$$= \{f: \sup_{n \leq t \leq n+1} |f(t) - f(n)| \geq M\}.$$

Therefore

$$P^W\{f: \sup_{n\le t\le n+1} |f(t) - f(n)| \ge M\}$$

$$= P^W\{f: \sup_{[0,1]} |f| \ge M\}$$

$$\le 2 \exp(-\frac{M^2}{2}) \quad (\text{cf. } 35.20).$$

One of the drawbacks to working with $C_0[0,\infty[$ is that it is a Fréchet space rather than a Banach space. This will now be rectified.

Let

$$X_0[0,\infty[= \{f \in C_0[0,\infty[: \lim_{t\to\infty} \frac{|f(t)|}{t} = 0\}.$$

35.23 <u>LEMMA</u> $X_0[0,\infty[$ is a Borel subset of $C_0[0,\infty[$.

<u>PROOF</u> Let

$$B(m,\frac{1}{n})$$

$$= \{f \in C_0[0,\infty[: |f(t)| \le \frac{1}{n} (1+t) \ \forall\ t \ge m\}.$$

Then $B(m,\frac{1}{n})$ is closed in $C_0[0,\infty[$ and

$$X_0[0,\infty[= \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} B(m,\frac{1}{n}).$$

35.24 <u>LEMMA</u> We have

$$P^W(X_0[0,\infty[) = 1.$$

<u>PROOF</u> Let

$$\xi_n = \delta_n - \delta_{n-1} \quad (n = 1,2,\ldots).$$

Then the ξ_n are independent square integrable random variables of mean 0 and variance $n - (n-1) = 1$ (cf. 35.7). Since

$$\delta_n = \xi_1 + \cdots + \xi_n \quad (\delta_0 = 0),$$

the strong law of large numbers implies that

$$\lim_{n \to \infty} \frac{\delta_n}{n} = 0 \quad \text{a.e.} [P^W].$$

Write

$$\left|\frac{\delta_t}{t}\right| = \left|\frac{\delta_t}{t} - \frac{\delta_n}{n} + \frac{\delta_n}{n}\right|$$

$$\leq \left|\frac{\delta_t}{t} - \frac{\delta_n}{n}\right| + \frac{|\delta_n|}{n}$$

and for $t \in [n,n+1]$, write

$$\left|\frac{\delta_t}{t} - \frac{\delta_n}{n}\right| = \left|\frac{\delta_t}{t} - \frac{\delta_n}{t} + \frac{\delta_n}{t} - \frac{\delta_n}{n}\right|$$

$$\leq \frac{|\delta_t - \delta_n|}{t} + \frac{|\delta_n|(t-n)}{nt}$$

$$\leq \frac{|\delta_t - \delta_n|}{n} + \frac{|\delta_n|}{nt} .$$

Then $\forall\ M > 0$,

$$\sum_{n=1}^{\infty} P^W\{f:\ \sup_{n\le t\le n+1} \frac{|f(t)-f(n)|}{n} \ge M\}$$

$$\le \sum_{n=1}^{\infty} 2\exp(-\frac{n^2M^2}{2}) \qquad (\text{cf. } 35.22)$$

$$< \infty.$$

So, by Borel-Cantelli,

$$\overline{\lim_{n\to\infty}} \sup_{n\le t\le n+1} \left|\frac{\delta_t}{t} - \frac{\delta_n}{n}\right| = 0 \quad \text{a.e.} [P^W].$$

Therefore

$$P^W(X_0[0,\infty[) = 1.$$

Let

$$C_\infty(\underline{R}) = \{\phi \in C(\underline{R}):\ \lim_{|t|\to\infty} |\phi(t)| = 0\}.$$

Then in the uniform norm, $C_\infty(\underline{R})$ is a separable Banach space, its dual being the space of all Borel signed measures on $\underline{R}$ of finite total variation.

Returning to $X_0[0,\infty[$, put

$$||f||_W = \sup_{0\le t<\infty} \frac{|f(t)|}{1+t}.$$

Then the pair $(X_0[0,\infty[, ||\cdot||_W)$ is a separable Banach space. In fact, given

$f \in X_0[0,\infty[$, define $\phi_f:\underline{R} \to \underline{R}$ by

$$\phi_f(t) = \frac{f(e^t)}{1+e^t}.$$

Then the arrow $f \to \phi_f$ is an isometric isomorphism

$$X_0[0,\infty[\to C_\infty(\underline{R}).$$

Therefore the dual of $X_0[0,\infty[$ consists of all Borel signed measures λ on $]0,\infty[$ such that

$$||\lambda|| = \int_{\underline{R}_{>0}} (1+t)d|\lambda|(t) < \infty.$$

35.25 LEMMA The arrow of inclusion

$$X_0[0,\infty[\to C_0[0,\infty[$$

is a continuous linear embedding.

To summarize, the upshot is that $X_0[0,\infty[$ is a separable Banach space which is a Borel subset of $C_0[0,\infty[$ of measure 1, thus P^W restricts to a probability measure on $\text{Bor}(X_0[0,\infty[)$ $(= \text{Bor}(C_0[0,\infty[) \cap X_0[0,\infty[)$.

[Note: Both $X_0[0,\infty[$ and $C_0[0,\infty[$ are lusinien, hence

$$B \in \text{Bor}(X_0[0,\infty[) \Rightarrow B \in \text{Bor}(C_0[0,\infty[) \qquad \text{(cf. 25.19).]}$$

Specializing the general theory to the case at hand leads to:

$$X_0[0,\infty[^* \subset X_0[0,\infty[^*_{P^W} \subset L^2(X_0[0,\infty[,P^W)$$

$$\Big\downarrow R_{P^W}$$

$$H(P^W) \subset X_0[0,\infty[.$$

35.26 <u>LEMMA</u> $\forall\ \lambda \in X_0[0,\infty[^*$,

$$\hat{P}^W(\lambda) = \exp(-\frac{1}{2}\int_{\underline{R}_{>0}}\int_{\underline{R}_{>0}} \min(u,v)d\lambda(u)d\lambda(v)).$$

[Argue as in 35.8. By the way, this confirms that P^W is centered gaussian.]

Let $W_0^{2,1}[0,\infty[$ denote the set of functions f on $[0,\infty[$ such that f is absolutely continuous, $f' \in L^2[0,\infty[$, and $f(0) = 0$ — then $W_0^{2,1}[0,\infty[$ is a separable real Hilbert space under the inner product

$$\langle f,g\rangle' = \int_0^\infty f'g'.$$

Moreover,

$$W_0^{2,1}[0,\infty[\subset X_0[0,\infty[.$$

Proof:

$$f \in W_0^{2,1}[0,\infty[$$

$$\Rightarrow$$

$$\frac{f(t)}{t} = \frac{1}{t}\int_0^t f'$$

=>

$$\frac{|f(t)|}{t} \leq \frac{1}{t} \int_0^t |f'|$$

$$\leq \frac{1}{t} (\int_0^t |f'|^2)^{1/2} \sqrt{t}$$

$$\leq \frac{1}{\sqrt{t}} (\int_0^\infty |f'|^2)^{1/2}$$

$$= \frac{1}{\sqrt{t}} ||f||' \rightarrow 0 \quad (t \rightarrow \infty)$$

=>

$$f \in X_0[0,\infty[.$$

In addition,

$$\frac{|f(t)|}{t+1} = \frac{t}{t+1} \frac{|f(t)|}{t}$$

$$\leq \frac{\sqrt{t}}{t+1} ||f||'$$

$$\leq ||f||'$$

=>

$$||f||_W \leq ||f||'.$$

[Note: $X_0[0,\infty[$ is the completion of $W_0^{2,1}[0,\infty[$ per $||\cdot||_W$.]

35.27 <u>LEMMA</u> $H(P^W) = W_0^{2,1}[0,\infty[$ as sets and as Hilbert spaces.

While a direct computational attack is feasible, there is little to be gained from it as a simple conceptual approach is available.

35.28 <u>LEMMA</u> $\exists$ an isometric isomorphism

$$I:L^2[0,\infty[\rightarrow X_0[0,\infty[^*_{P^W}$$

with the property that

$$I(\sum_{i=1}^{n} r_i \chi_{[0,t_i]}) = \sum_{i=1}^{n} r_i \delta_{t_i}.$$

[Note: In the same way, one can construct an isometric isomorphism

$$I:L^2[0,1] \rightarrow C_0[0,1]^*_{P^W}.]$$

35.29 <u>LEMMA</u> Let $\phi \in L^2[0,\infty[$ -- then

$$R_{P^W}(I(\phi))(t)$$

$$= \int_{X_0[0,1]} I(\phi)(f) I(\chi_{[0,1]})(f) dP^W(f).$$

By definition,

$$H(P^W) = R_{P^W}(X_0[0,\infty[^*_{P^W})$$

or still,

$$H(P^W) = \{R_{P^W}(I(\phi)) : \phi \in L^2[0,\infty[\}.$$

And

$$\begin{aligned} R_{P^W}(I(\phi))(t) &= \langle \phi, \chi_{[0,t]} \rangle_{L^2[0,\infty[} \\ &= \int_0^t \phi. \end{aligned}$$

Therefore

$$H(P^W) \subset W_0^{2,1}[0,\infty[.$$

But the containment is reversible: Take an $f \in W_0^{2,1}[0,\infty[$ and consider $I(f')$.

To check the equality of the inner products, let $f,g \in W_0^{2,1}[0,\infty[$ -- then

$$\begin{aligned} \langle f,g \rangle' &= \langle f',g' \rangle_{L^2[0,\infty[} \\ &= \langle I(f'),I(g') \rangle_{L^2(P^W)} \\ &= \langle R_{P^W}(I(f')),R_{P^W}(I(g')) \rangle_{H(P^W)} \\ &= \langle f,g \rangle_{H(P^W)}. \end{aligned}$$

35.30 <u>REMARK</u> Fix $\lambda \in X_0[0,\infty[^*$ and put $h_\lambda = R_{P^W}(\lambda)$ -- then $\forall\ h \in H(P^W)$,

$$\lambda(h) = \langle h_\lambda, h \rangle_{H(P^W)}.$$

Here

$$h_\lambda(u) = \int_0^u \lambda(]v,\infty[)dv.$$

As we know (see §28), there is an isometric isomorphism

$$T:BO(X_0[0,\infty[^*_{P^W}) \rightarrow L^2(X_0[0,\infty[,P^W)$$

characterized by the relation

$$T\ \underline{\exp}(f) = \Lambda_f.$$

Put

$$T(I) = T \circ \Gamma(I) \quad (\text{cf. } 6.14).$$

Then

$$T(I):BO(L^2[0,\infty[) \rightarrow L^2(X_0[0,\infty[,P^W)$$

is an isometric isomorphism such that

$$T(I)\ \underline{\exp}(\phi) = \Lambda_{I(\phi)}.$$

[Note: Put $h = R_{P^W}(I(\phi))$ -- then

$$\frac{dP_h^W}{dP^W} = \Lambda_{I(\phi)}.$$

Consequently,

$$\begin{aligned} 1 &= \int_{X_0[0,\infty[} dP_h^W \\ &= \int_{X_0[0,\infty[} \frac{dP_h^W}{dP^W}\, dP^W \end{aligned}$$

$$= \int_{X_0[0,\infty[} \Lambda_{I(\phi)} dP^W$$

$$= \int_{X_0[0,\infty[} \exp(I(\phi) - \frac{1}{2} ||\phi||^2_{L^2[0,\infty[}) dP^W$$

$$= \exp(- \frac{1}{2} ||\phi||^2_{L^2[0,\infty[}) \int_{X_0[0,\infty[} \exp(I(\phi)) dP^W$$

$\Rightarrow$

$$\int_{X_0[0,\infty[} \exp(I(\phi)) dP^W$$

$$= \exp(\frac{1}{2} ||\phi||^2_{L^2[0,\infty[}).]$$

§36. *ABSTRACT WIENER SPACES*

Let X be an infinite dimensional separable real Hilbert space. Denote by $\mathcal{P}_X$ the set of finite dimensional orthogonal projections P of X and let $\mathcal{C}_X$ be the set of subsets of X of the form

$$C = \{x \in X : Px \in B\},$$

where $P \in \mathcal{P}_X$ and $B \in \text{Bor}(PX)$ -- then $\mathcal{C}_X$ is an algebra.

36.1 <u>LEMMA</u> Given $P \in \mathcal{P}_X$, let

$$\mathcal{C}_P = \{P^{-1}(B) : B \in \text{Bor}(PX)\}.$$

Then $\mathcal{C}_P$ is a σ-algebra and

$$\mathcal{C}_X = \bigcup_P \mathcal{C}_P.$$

[Note: $\mathcal{C}_X$ is not a σ-algebra but the σ-algebra generated by $\mathcal{C}_X$ is Cyl(X) (= Bor(X)) (cf. 25.5).]

The <u>canonical measure</u> on X is the set function

$$\gamma_X : \mathcal{C}_X \to [0,1]$$

defined by the rule

$$\gamma_X(C) = \frac{1}{(2\pi)^{n/2}} \int_B \exp(-\frac{1}{2} ||x||^2) dx,$$

where n = dim PX.

36.2 <u>LEMMA</u> γ_X is finitely additive but γ_X is not countably additive.

<u>PROOF</u> It is obvious that γ_X is finitely additive. If γ_X were countably additive, then γ_X would admit an extension to a probability measure $\tilde{\gamma}_X$ on Bor(X). To derive a contradiction, fix an orthonormal basis $\{e_k\}$ for X -- then for all positive integers N and M, we have

$$\tilde{\gamma}_X\{x: \sum_{k=1}^{N} \langle e_k,x\rangle^2 \leq M\}$$

$$\leq \tilde{\gamma}_X\{x: |\langle e_k,x\rangle| \leq \sqrt{M}, 1 \leq k \leq N\}$$

$$= \prod_{k=1}^{N} \tilde{\gamma}_X\{x: |\langle e_k,x\rangle| \leq \sqrt{M}\}$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\sqrt{M}}^{\sqrt{M}} e^{-\frac{1}{2}t^2}\, dt\right]^N.$$

Since

$$0 < \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{M}}^{\sqrt{M}} e^{-\frac{1}{2}t^2}\, dt < 1,$$

it follows that

$$\lim_{N \to \infty} \tilde{\gamma}_X\{x: \sum_{k=1}^{N} \langle e_k,x\rangle^2 \leq M\} = 0.$$

But

$$\{x: ||x||^2 \leq M\}$$

$$= \{x: \sum_{k=1}^{\infty} \langle e_k, x\rangle^2 \leq M\}$$

$$= \bigcap_{N=1}^{\infty} \{x: \sum_{k=1}^{N} \langle e_k, x\rangle^2 \leq M\}.$$

And

$$\{x: \sum_{k=1}^{N+1} \langle e_k, x\rangle^2 \leq M\} \subset \{x: \sum_{k=1}^{N} \langle e_k, x\rangle^2 \leq M\}.$$

Therefore

$$\tilde{\gamma}_X\{x: ||x||^2 \leq M\}$$

$$= \lim_{N \to \infty} \tilde{\gamma}_X\{x: \sum_{k=1}^{N} \langle e_k, x\rangle^2 \leq M\}$$

$$= 0$$

=>

$$1 = \tilde{\gamma}_X(X)$$

$$= \tilde{\gamma}_X(\bigcup_{M=1}^{\infty} \{x: ||x||^2 \leq M\})$$

$$= \lim_{M \to \infty} \tilde{\gamma}_X\{x: ||x||^2 \leq M\}$$

$$= 0.$$

I.e.: $1 = 0\ldots$.

36.3 <u>REMARK</u> The restriction $\gamma_X|C_P$ of γ_X to C_P is a probability measure, thus it is meaningful to consider

$$\int_X \phi \circ P(x) d\gamma_X(x),$$

where $\phi: PX \to \underline{R}$ is Borel. E.g.: Fix $x_0 \neq 0$ in X — then

$$\int_X \langle x, x_0\rangle^2 \, d\gamma_X(x) = ||x_0||^2.$$

Let p be a seminorm on X — then p is said to be tight if $\forall \, \varepsilon > 0$, $\exists \, P_\varepsilon \in \mathcal{P}_X$:

$$\gamma_X\{x: p(Px) > \varepsilon\} < \varepsilon \ \forall \, P \in \mathcal{P}_X: P \perp P_\varepsilon.$$

36.4 EXAMPLE Let $||\cdot||$ be the norm on X — then $||\cdot||$ is not tight. For if the opposite were true, then we could find an increasing sequence $P_n \in \mathcal{P}_X$:

$$\gamma_X\{x: ||Px|| > \frac{1}{n}\} < \frac{1}{n} \ \forall \, P \in \mathcal{P}_X: P \perp P_n.$$

Take $m > n > 2$, thus $(P_m - P_n) \perp P_2$, so

$$\gamma_X\{x: ||(P_m - P_n)x|| > \frac{1}{2}\} < \frac{1}{2}$$

or still,

$$\gamma_X\{x: ||(P_m - P_n)x||^2 > \frac{1}{4}\} < \frac{1}{2}$$

or still,

$$1 - \gamma_X\{x: ||(P_m - P_n)x||^2 \leq \frac{1}{4}\} < \frac{1}{2} .$$

But as m & n tend to ∞,

$$\gamma_X\{x: ||(P_m - P_n)x||^2 \leq \frac{1}{4}\}$$

tends to 0.

36.5 LEMMA Suppose that $A \in \mathcal{B}(X)$ is Hilbert-Schmidt. Set $p_A(x) = ||Ax||$ $(x \in X)$ -- then p_A is tight.

PROOF Assuming that the range of A^*A is infinite dimensional, let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of A^*A and let $e_1, e_2, \ldots$ be the corresponding eigenvectors so that $\forall\, x \in X$,

$$A^*Ax = \sum_{k=1}^{\infty} \lambda_k <e_k, x> e_k.$$

Denote by P_n the orthogonal projection of X onto the span of $e_1, \ldots, e_n$ -- then for $P \perp P_n$ $(P \in \mathcal{P}_X)$, we have

$$\begin{aligned} p_A(Px)^2 &= ||APx||^2 \\ &= <APx, APx> \\ &= <Px, A^*APx> \\ &= \sum_{k=1}^{\infty} \lambda_k <e_k, Px>^2 \\ &= \sum_{k=n+1}^{\infty} \lambda_k <e_k, Px>^2. \end{aligned}$$

The function

$$x \to \sum_{k=n+1}^{\infty} \lambda_k <e_k, Px>^2$$

is positive and C_P-measurable, hence $\forall\ \varepsilon > 0$ (cf. 36.3),

$$\gamma_X\{x:p_A(Px) > \varepsilon\}$$

$$= \gamma_X\{x:p_A(Px)^2 > \varepsilon^2\}$$

$$= \gamma_X\{x: \sum_{k=n+1}^{\infty} \lambda_k\langle e_k,Px\rangle^2 > \varepsilon^2\}$$

$$\leq \frac{1}{\varepsilon^2} \int_X \sum_{k=n+1}^{\infty} \lambda_k\langle e_k,Px\rangle^2\ d\gamma_X(x)$$

$$= \frac{1}{\varepsilon^2} \sum_{k=n+1}^{\infty} \lambda_k \int_X \langle e_k,Px\rangle^2\ d\gamma_X(x)$$

$$= \frac{1}{\varepsilon^2} \sum_{k=n+1}^{\infty} \lambda_k \int_X \langle x,Pe_k\rangle^2\ d\gamma_X(x)$$

$$= \frac{1}{\varepsilon^2} \sum_{k=n+1}^{\infty} \lambda_k||Pe_k||^2$$

$$\leq \frac{1}{\varepsilon^2} \sum_{k=n+1}^{\infty} \lambda_k.$$

Now choose $n >> 0$:

$$\sum_{k=n+1}^{\infty} \lambda_k < \varepsilon^3.$$

36.6 EXAMPLE Suppose that $\tilde{X}$ is a separable real Hilbert space and $\iota:X \rightarrow \tilde{X}$ is a continuous linear injection with a dense range. Assume: ι is Hilbert-Schmidt and set $p_\iota(x) = ||\iota x||^{\sim}$ — then p_ι is tight.

[Fix a bounded linear operator $A:X \rightarrow X$ such that

$$<\iota x, \iota y>^{\sim} = <x,Ay> \quad (x,y \in X).$$

Then it is clear that A is positive and symmetric. Moreover A is trace class. To see this, consider any orthonormal basis $\{e_n\}$ for X. To say that $\iota:X \rightarrow \tilde{X}$ is Hilbert-Schmidt means:

$$\sum_{n=1}^{\infty} (||\iota e_n||^{\sim})^2 < \infty.$$

But

$$\sum_{n=1}^{\infty} <e_n,Ae_n> = \sum_{n=1}^{\infty} (||\iota e_n||^{\sim})^2,$$

thus A is trace class and $\sqrt{A}$ is Hilbert-Schmidt. Finally,

$$\begin{aligned} p_\iota(x) &= ||\iota x||^{\sim} \\ &= (<\iota x, \iota x>^{\sim})^{1/2} \\ &= (<x,Ax>)^{1/2} \\ &= (<\sqrt{A}\, x, \sqrt{A}\, x>)^{1/2} \\ &= ||\sqrt{A}\, x|| = p_{\sqrt{A}}(x), \end{aligned}$$

which implies that p_ι is tight (cf. 36.5).

[Note: $\tilde{X}$ is called a Hilbert-Schmidt enlargement of X. If $\tilde{X}_1$ and $\tilde{X}_2$ are two Hilbert-Schmidt enlargements of X, then $\exists$ a third Hilbert-Schmidt enlargement $\tilde{X}_3$ of X finer than $\tilde{X}_1$ and $\tilde{X}_2$.]

36.7 REMARK Consider the seminorms p_K $(K \in \mathcal{K})$ figuring in the definition of the Sazonov topology (cf. 33.11) — then each of them is tight (cf. 36.5).

36.8 LEMMA Let p be a tight seminorm on X — then $\exists\ C > 0 : p(x) \leq C||x||$ $\forall\ x \in X$, thus p is continuous.

PROOF Define a > 0 by

$$\frac{2}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{1}{2}t^2}\, dt = \frac{1}{2}\ .$$

Take $\varepsilon = \frac{1}{2}$ and choose $P_{1/2} \in \mathcal{P}_X$:

$$\gamma_X\{x \in X : p(Px) > \tfrac{1}{2}\} < \tfrac{1}{2}\ \forall\ P \in \mathcal{P}_X : P \perp P_{1/2}.$$

Since $P_{1/2}X$ is finite dimensional, $\exists\ M > 0 : p(y) \leq M||y||\ \forall\ y \in P_{1/2}X$. Given $z \neq 0$ in $(P_{1/2}X)^\perp$, define P_z by

$$P_z x = \langle x, \frac{z}{||z||}\rangle \frac{z}{||z||}\ .$$

Then $P_z \in \mathcal{P}_X$ and $P_z \perp P_{1/2}$, hence if $p(z) \neq 0$,

$$\gamma_X\{x \in X : p(P_z x) > \tfrac{1}{2}\} < \tfrac{1}{2}$$

=>

$$\gamma_X\{x \in X: |\langle x, \frac{z}{||z||}\rangle| > \frac{||z||}{2p(z)}\} < \frac{1}{2}$$

=>

$$\frac{2}{\sqrt{2\pi}} \int_{\frac{||z||}{2p(z)}}^{\infty} e^{-\frac{1}{2}t^2} dt < \frac{1}{2}$$

=>

$$\frac{||z||}{2p(z)} > a \Rightarrow p(z) < \frac{1}{2a} ||z||.$$

Any $x \in X$ admits a decomposition $x = y+z : y \in P_{1/2}X,\ z \in (P_{1/2}X)^{\perp}$. Therefore

$$\begin{aligned} p(x)^2 &\le (p(y) + p(z))^2 \\ &\le 2(p(y)^2 + p(z)^2) \\ &\le 2(M^2||y||^2 + \frac{1}{4a^2} ||z||^2) \\ &\le 2(M^2 + \frac{1}{4a^2})(||y||^2 + ||z||^2) \\ &= 2(M^2 + \frac{1}{4a^2}) ||x||^2 \end{aligned}$$

=>

$$p(x) \le C||x|| \quad (C = \sqrt{2}\ (M^2 + \frac{1}{4a^2})^{1/2}).$$

36.9 <u>REMARK</u> The preceding result can be sharpened since it is always possible to find a compact operator $A:X \to X$ such that

$$p(x) \leq ||Ax|| \ \forall \ x \in X.$$

But, in general, A is not Hilbert-Schmidt as this would mean that for any orthonormal basis $\{e_n\}$ for X, we would have

$$\sum_{n=1}^{\infty} p(e_n)^2 \leq \sum_{n=1}^{\infty} ||Ae_n||^2 < \infty,$$

which need not be true. To illustrate, let $X = \ell^2$ and define p by

$$p(x_1,x_2,\ldots) = \sup_n \frac{|x_n|}{\sqrt{n}} .$$

Then p is tight and

$$p(e_n) = \frac{1}{\sqrt{n}} \Rightarrow \sum_{n=1}^{\infty} p(e_n)^2 = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

A triple (X,Y,ι) is said to be an <u>abstract Wiener space</u> if

$$\begin{cases} X \text{ is a separable real Hilbert space } (\dim X = \infty) \\ Y \text{ is a separable real Banach space } (\dim Y = \infty) \end{cases}$$

and $\iota:X \to Y$ is a continuous linear injection with a dense range such that $||\cdot||_Y \circ \iota$ is tight, where $||\cdot||_Y$ is the norm on Y.

36.10 <u>EXAMPLE</u> Let p be a tight norm on X; let X_p be the completion of X per p -- then the triple (X,X_p,ι) is an abstract Wiener space.

[Note: X is not complete w.r.t. p. For if it were, then p would be equivalent to $||\cdot||$ (open mapping theorem), thus $||\cdot||$ would be tight, which it isn't (cf. 36.4).]

36.11 LEMMA Suppose that (X,Y,ι) is an abstract Wiener space -- then $\iota:X \to Y$ is a compact operator.

36.12 EXAMPLE The triple $(L^2[0,1],L^1[0,1],\iota)$ is not an abstract Wiener space.

[The inclusion $\iota:L^2[0,1] \to L^1[0,1]$ is not compact (the sequence $\{\cos(2n\pi x):n \geq 1\} \subset L^2[0,1]$ is bounded but does not have an L^1-convergent subsequence).]

36.13 LEMMA Let γ be a centered gaussian measure on a separable real Banach space X ($\dim X = \infty$). Suppose that $H(\gamma)$ is dense in X -- then the triple $(H(\gamma),X,\iota)$ is an abstract Wiener space.

[Note: Of course it is necessary that the inclusion $\iota:H(\gamma) \to X$ be a compact operator (cf. 36.11), which is indeed the case (the closed unit ball $B_{H(\gamma)}$ is compact in X).]

Before proceeding to the proof, we shall first consider the situation when X is a separable real Hilbert space (infinite dimensional as always) and $K_\gamma > 0$. For then $H(\gamma) = \sqrt{K_\gamma}\, X$ (cf. 33.16), the question being: Why is $||\cdot||_X|H(\gamma)$ tight? Put $A = \sqrt{K_\gamma} \circ \iota$ -- then $\forall\ h \in H(\gamma)$,

$$||Ah||_{H(\gamma)}$$

$$= ||(\sqrt{K_\gamma} \circ \iota)(h)||_{H(\gamma)}$$

$$= ||h||_X.$$

So, to finish the verification, one has only to show that A is Hilbert-Schmidt (cf. 36.5). To this end, fix an orthonormal basis $h_1, h_2, \ldots$ for $H(\gamma)$ and define $e_1, e_2, \ldots$ by the relation $h_n = \sqrt{K_\gamma}\, e_n$, thus

$$\langle h_i, h_j \rangle_{H(\gamma)} = \langle e_i, e_j \rangle_X = \delta_{ij}.$$

And

$$\sum_{n=1}^{\infty} ||Ah_n||^2_{H(\gamma)}$$

$$= \sum_{n=1}^{\infty} ||(\sqrt{K_\gamma} \circ \iota)h_n||^2_{H(\gamma)}$$

$$= \sum_{n=1}^{\infty} ||\sqrt{K_\gamma}\sqrt{K_\gamma}\, e_n||^2_X$$

$$= \sum_{n=1}^{\infty} ||K_\gamma e_n||^2_X < \infty.$$

[Note: K_γ is trace class (cf. 33.13), hence is Hilbert-Schmidt.]

Turning now to the proof of 36.13, recall the setup:

$$\begin{array}{ccc} X^* \subset X^*_\gamma \subset L^2(X,\gamma) \\ R_\gamma \big\downarrow \\ H(\gamma) \subset X. \end{array}$$

- Given $\lambda \in X^*$, put $h_\lambda = R_\gamma(\lambda)$.
- Given $h \in H(\gamma)$, put $\hat{h} = R_\gamma^{-1}(h)$.

Then

$$\lambda(h) = \langle h_\lambda, h\rangle_{H(\gamma)}$$

$$= \int_X \lambda(x)\hat{h}(x)d\gamma(x).$$

[Note: $\forall\ \lambda \in X^*$,

$$||\lambda||^2_{L^2(\gamma)} = ||R_\gamma(\lambda)||^2_{H(\gamma)} = ||h_\lambda||^2_{H(\gamma)}.]$$

Given $P \in \mathcal{P}_{H(\gamma)}$, let $h_1,\ldots,h_d$ be an orthonormal basis for $PH(\gamma)$ and define

$$\Xi_P : X \to X$$

by the prescription

$$\Xi_P(x) = \sum_{i=1}^{d} \hat{h}_i(x)h_i.$$

Then Ξ_P does not depend on the choice of the h_i.

36.14 <u>LEMMA</u> If the net $\{\Xi_P : P \in \mathcal{P}_{H(\gamma)}\}$ is fundamental in measure, then $||\cdot||_X|H(\gamma)$ is tight.

PROOF Fix $\varepsilon > 0$ and choose $P_\varepsilon \in \mathcal{P}_{H(\gamma)}$:

$$P_1, P_2 \geq P_\varepsilon$$

$$\Rightarrow$$

$$\gamma\{x: ||\Xi_{P_1}(x) - \Xi_{P_2}(x)||_X > \varepsilon\} < \varepsilon.$$

Suppose that $P \in \mathcal{P}_{H(\gamma)}: P \perp P_\varepsilon$ -- then $P = Q - P_\varepsilon$, where $Q \geq P_\varepsilon$. Take $P_1 = Q$, $P_2 = P_\varepsilon$:

$$\gamma\{x: ||\Xi_Q(x) - \Xi_{P_\varepsilon}(x)||_X > \varepsilon\} < \varepsilon$$

$$\Rightarrow$$

$$\gamma\{x: ||\Xi_P(x)||_X > \varepsilon\} < \varepsilon$$

$$\Rightarrow$$

$$\gamma_{H(\gamma)}\{h: ||Ph||_X > \varepsilon\} < \varepsilon.$$

Therefore $||\cdot||_X | H(\gamma)$ is tight.

[Note: Let $C \in \mathcal{C}_P$ -- then

$$\gamma\{x: \Xi_P(x) \in C\} = \gamma_{H(\gamma)}(C).$$

Specialize and take for $B \in \mathrm{Bor}(PH(\gamma))$ the subset of $PH(\gamma)$ consisting of those $h: ||h||_X > \varepsilon$ so that $C = P^{-1}(B)$ is the subset of $H(\gamma)$ consisting of those $h: ||Ph||_X > \varepsilon$, hence

$$\gamma_{H(\gamma)}(C) = \gamma_{H(\gamma)}\{h: ||Ph||_X > \varepsilon\}.$$

On the other hand,

$$\Xi_P(x) \in C \iff ||P\Xi_P(x)||_X > \varepsilon.$$

And

$$P\Xi_P(x) = P \sum_{i=1}^{d} \hat{h}_i(x) h_i$$

$$= \sum_{i=1}^{d} \hat{h}_i(x) Ph_i$$

$$= \sum_{i=1}^{d} \hat{h}_i(x) h_i$$

$$= \Xi_P(x).$$

Consequently,

$$\Xi_P(x) \in C \iff ||\Xi_P(x)||_X > \varepsilon.]$$

36.15 LEMMA Suppose that $P_n \in \mathcal{P}_{H(\gamma)}$ is an increasing sequence which converges strongly to the identity $I_{H(\gamma)}$ -- then Ξ_{P_n} converges in measure to the identity I_X.

[See the discussion following 36.17 below.]

36.16 LEMMA The net $\{\Xi_P : P \in \mathcal{P}_{H(\gamma)}\}$ converges in measure to the identity I_X.

PROOF If not, then $\exists\ \varepsilon > 0\ \&\ \delta > 0$ such that $\forall\ P \in \mathcal{P}_{H(\gamma)}$, $\exists\ P' \in \mathcal{P}_{H(\gamma)}$: $P' \geq P$ and

$$\gamma\{x: ||\Xi_{P'}(x) - x||_X > \varepsilon\} \geq \delta.$$

Fix an increasing sequence $P_n \in \mathcal{P}_{H(\gamma)}$ which converges strongly to the identity $I_{H(\gamma)}$. Choose $P_1' \geq P_1$ such that

$$\gamma\{x: ||\Xi_{P_1'}(x) - x||_X > \varepsilon\} \geq \delta.$$

Let $P_{1,2}'$ be the orthogonal projection of $H(\gamma)$ onto $P_1'H(\gamma) + P_2H(\gamma)$, thus $P_{1,2}' \geq P_1'$ and $P_{1,2}' \geq P_2$. Choose $P_2' \geq P_{1,2}'$ such that

$$\gamma\{x: ||\Xi_{P_2'}(x) - x||_X > \varepsilon\} \geq \delta.$$

Proceed from here by iteration to get an increasing sequence $P_n' \in \mathcal{P}_{H(\gamma)}$ which converges strongly to the identity $I_{H(\gamma)}$ subject to

$$\gamma\{x: ||\Xi_{P_n'}(x) - x||_X > \varepsilon\} \geq \delta.$$

But this means that $\Xi_{P_n'}$ does not converge in measure to the identity I_X, contradicting 36.15.

[Note: $\forall\, h \in H(\gamma)$,

$$P_n' \geq P_n \Rightarrow ||P_n'h - h||_{H(\gamma)} \leq ||P_nh - h||_{H(\gamma)}.]$$

It is therefore a corollary that the net $\{\Xi_P : P \in \mathcal{P}_{H(\gamma)}\}$ is fundamental in measure, hence $||\cdot||_X|H(\gamma)$ is tight (cf. 36.14).

To establish 36.15, we shall employ a classical criterion.

So assume that (Ω, A, μ) is a probability space. Given a random variable $\xi:\Omega \to X$, let $P_\xi = \gamma \circ \xi^{-1}$ be the distribution of ξ and call ξ symmetric if $P_\xi = P_{-\xi}$.

36.17 THEOREM (Ito-Nisio) Let $\xi_1, \xi_2, \ldots$ be a sequence of independent symmetric X-valued random variables on Ω and put $S_n = \sum_{k=1}^{n} \xi_k$. Suppose that $\forall\ \lambda \in X^*$,

$$\prod_{k=1}^{n} \hat{P}_{\xi_k}(\lambda) \to \hat{\gamma}(\lambda) \quad (n \to \infty).$$

Then the sequence $\{S_n\}$ converges a.e.$[\mu]$ to an X-valued random variable ξ.

Given an increasing sequence $P_n \in \mathcal{P}_X$ which converges strongly to the identity, let

$$\xi_1 = \Xi_{P_1}, \ \xi_n = \Xi_{P_n} - \Xi_{P_{n-1}} \quad (n > 1).$$

Then the ξ_n are independent symmetric X-valued random variables on the probability space $(X, \mathrm{Bor}(X), \gamma)$ and we have

$$\prod_{k=1}^{n} \hat{P}_{\xi_k}(\lambda)$$

$$= \prod_{k=1}^{n} \int_X e^{\sqrt{-1}\,\lambda(x)}\, dP_{\xi_k}(x)$$

$$= \prod_{k=1}^{n} \int_X e^{\sqrt{-1}\,\lambda(x)}\, d(\gamma \circ \xi_k^{-1})(x)$$

$$= \prod_{k=1}^{n} \int_X e^{\sqrt{-1}\ \lambda(\xi_k(x))}\, d\gamma(x)$$

$$= \int_X \prod_{k=1}^{n} e^{\sqrt{-1}\ \lambda(\xi_k(x))}\, d\gamma(x)$$

$$= \int_X \exp(\sqrt{-1}\ \lambda(\sum_{k=1}^{n} \xi_k(x)))d\gamma(x)$$

$$= \int_X \exp(\sqrt{-1}\ \lambda(\Xi_{P_n}(x)))d\gamma(x)$$

$$= \int_X \exp(\sqrt{-1}\ \sum_{i=1}^{d(n)} \hat{h}_i(x)\lambda(h_i))d\gamma(x)$$

$$= \int_X \exp(\sqrt{-1}\ \sum_{i=1}^{d(n)} \hat{h}_i(x)\langle h_\lambda, h_i\rangle_{H(\gamma)})d\gamma(x)$$

$$= \int_X \exp(\sqrt{-1}\ \sum_{i=1}^{d(n)} \hat{h}_i(x)\langle h_\lambda, P_n h_i\rangle_{H(\gamma)})d\gamma(x)$$

$$= \int_X \exp(\sqrt{-1}\ R_\gamma^{-1}(P_h h_\lambda)(x))d\gamma(x)$$

$$\to \int_X \exp(\sqrt{-1}\ R_\gamma^{-1}(h_\lambda)(x))d\gamma(x) \quad (n \to \infty)$$

$$= \int_X e^{\sqrt{-1}\ \lambda(x)}\, d\gamma(x)$$

$$= \hat{\gamma}(\lambda).$$

Therefore

$$S_n = \sum_{k=1}^{n} \xi_k = \Xi_{P_n}$$

converges a.e.[γ] (cf. 36.17), thus is convergent in measure ($\gamma(X) = 1 < \infty$).

N.B. Let $\Xi(x) = \lim_{n \to \infty} \Xi_{P_n}(x)$ -- then $\forall\ \lambda \in X^*$,

$$\lim_{n \to \infty} \lambda(\Xi_{P_n}(x)) = \lambda(x)$$

=>

$$\lambda(\Xi(x)) = \lambda(x)$$

=>

$$\Xi(x) = x \text{ a.e.}[\gamma].$$

Therefore Ξ_{P_n} converges in measure to the identity I_X.

36.18 EXAMPLE The triple

$$(W_0^{2,1}[0,1], C_0[0,1], \iota)$$

is an abstract Wiener space.

36.19 EXAMPLE The triple

$$(W_0^{2,1}[0,\infty[, X_0[0,\infty[, \iota)$$

is an abstract Wiener space.

Let Y be an infinite dimensional separable real Banach space. Denote by $\mathcal{C}_Y$ the collection of subsets of Y of the form

$$C = \{y \in Y: (\lambda_1(y),\ldots,\lambda_n(y)) \in B\},$$

where $\lambda_i \in Y^*$ $(i = 1,\ldots,n)$ and $B \in \mathrm{Bor}(\underline{R}^n)$ -- then $\mathcal{C}_Y$ is an algebra and the σ-algebra generated by $\mathcal{C}_Y$ is $\mathrm{Cyl}(Y)$ $(= \mathrm{Bor}(Y))$ (cf. 25.5).

Let (X,Y,ι) be an abstract Wiener space -- then ι induces a map $\mathcal{C}_Y \to \mathcal{C}_X$.

36.20 <u>THEOREM</u> (Gross) Let (X,Y,ι) be an abstract Wiener space -- then the set function $\gamma_X \circ \iota^{-1}$ is countably additive on $\mathcal{C}_Y$, hence can be extended to a centered gaussian measure γ_Y on $\mathrm{Bor}(Y)$.

[Note: It turns out that X can be identified with the Cameron-Martin space $H(\gamma_Y)$.]

We shall postpone the proof until §39 (cf. 39.1).

36.21 <u>EXAMPLE</u> Take $X = \ell^2$ and let p be defined by

$$p(x) = \Big(\sum_{n=1}^{\infty} \frac{1}{n^2} x_n^2\Big)^{1/2}.$$

Then p is a tight norm on X and in the notation of 36.10,

$X_p = \{x \in \underline{R}^{\infty}: \sum_{n=1}^{\infty} \frac{1}{n^2} x_n^2 < \infty\}$. Here, $\gamma_X \circ \iota^{-1}$, when extended to $\mathrm{Bor}(X_p)$, is the restriction $\gamma|X_p$, where γ is the standard gaussian measure on $\underline{R}^\infty$ (cf. 26.1) (recall that $X_p \in \text{Bor}(\underline{R}^\infty)$ and $\gamma(X_p) = 1$ (cf. 24.11)).

36.22 EXAMPLE Take $X = L^2[0,1]$ and let p be defined by

$$p(f) = \sup_{0\leq t\leq 1} |\int_0^t f(s)ds|.$$

Then p is a tight norm on X and in the notation of 36.10, $X_p = C_0[0,1]$. Here, $\gamma_X \circ \iota^{-1}$, when extended to $\text{Bor}(X_p)$, is the Wiener measure P^W.

36.23 LEMMA Let X be an infinite dimensional separable real Hilbert space. Let p be a tight norm on X. Assume: p is hilbertian (cf. 34.8) -- then $\exists$ a Hilbert-Schmidt operator K_p on X such that

$$p(x) = ||K_p x|| \quad (x \in X).$$

PROOF As an initial reduction, note that $\{x:p(x) = 0\}$ is a closed subspace of X (cf. 36.8), hence by passing to $\{x:p(x) = 0\}^\perp$ if necessary, it can be assumed that p is actually a norm, call it $||\cdot||_p$. Denote by X_p the associated completion. Identify X* with X itself — then X_p^* can be viewed as a dense linear subspace of X. Consider now the triple (X,X_p,ι). Put $\gamma_p = \gamma_{X_p}$ (cf. 36.20). By definition, the Fourier transform $\hat{\gamma}_p$ of γ_p lives on X_p^* which, for the purposes at hand, will not be identified with X_p. Accordingly, $\exists$ a nonnegative, symmetric operator

$K_p \in \underline{L}_2(X_p^*)$: $\forall \lambda \in X_p^*$,

$$\hat{\gamma}_p(\lambda) = \exp(-\frac{1}{2}\ ||K_p\lambda||_p^*) \quad \text{(cf. 33.13)},$$

where $||\cdot||_p^*$ is the norm on X_p^*. And (cf. 33.9),

$$(||K_p\lambda||_p^*)^2 = \int_{X_p} \lambda(x)^2\, d\gamma_p(x)$$

or still,

$$(||K_p\lambda||_p^*)^2 = \int_X <\lambda,x>^2\, d\gamma_X(x)$$

$$= ||\lambda||^2.$$

Therefore K_p is one-to-one. Let $\kappa_1,\kappa_2,\ldots$ be the eigenvalues of K_p and let $\lambda_1,\lambda_2,\ldots$ be the corresponding eigenvectors -- then

$$<\lambda_i,\lambda_j> = \int_X <\lambda_i,x><\lambda_j,x>d\gamma_X(x)$$

$$= \int_{X_p} \lambda_i(x)\lambda_j(x)d\bar{\gamma}_p(x)$$

$$= <K_p\lambda_i,K_p\lambda_j>_p^*$$

$$= \kappa_i\kappa_j\delta_{ij},$$

so $\{\frac{\lambda_k}{\kappa_k}:k = 1,2,\ldots\}$ is an orthonormal basis for X. But

$$\sum_{k=1}^{\infty} ||K_p(\frac{\lambda_k}{\kappa_k})||^2 = \sum_{k=1}^{\infty} (||K_p^2(\frac{\lambda_k}{\kappa_k})||_p^*)^2$$

$$= \sum_{k=1}^{\infty} (||\kappa_k \lambda_k||_p^*)^2$$

$$= \sum_{k=1}^{\infty} \kappa_k^2 < \infty,$$

which implies that K_p can be extended to a Hilbert-Schmidt operator on X (call it K_p again):

$$\begin{array}{ccc} X_p^* & \xrightarrow{K_p} & X_p^* \\ \downarrow & & \downarrow \\ X & \xrightarrow[K_p]{} & X. \end{array}$$

Finally, $\forall\ x \in X$,

$$p(x) = \sup_{\lambda:||\lambda||_p^* = 1} |\lambda(x)|$$

$$= \sup_{\lambda:||K_p\lambda||_p^* = 1} |(K_p\lambda)(x)|$$

$$= \sup_{\lambda:||\lambda|| = 1} |\langle K_p\lambda, x\rangle|$$

$$= \sup_{\lambda:||\lambda|| = 1} |\langle \lambda, K_p x\rangle|$$

$$= ||K_p x||.$$

36.24 <u>REMARK</u> Take X as above and given $A \in \mathcal{B}(X)$, put $p_A(x) = ||Ax||$ $(x \in X)$ --

then p_A is hilbertian. Moreover, p_A is tight iff A is Hilbert-Schmidt.

[That the condition is sufficient is the gist of 36.5. To ascertain necessity, use 36.23 to write

$$p_A(x) = ||K_p x|| \quad (x \in X).$$

Fix an orthonormal basis $\{e_n\}$ for X -- then

$$\sum_{n=1}^{\infty} ||Ae_n||^2 = \sum_{n=1}^{\infty} ||K_p e_n||^2 < \infty,$$

so A is Hilbert-Schmidt.]

§37. *INTEGRATION THEORY*

Let X be an infinite dimensional separable real Hilbert space -- then by definition, a cylinder measure on X is a finitely additive set function $\Pi: \mathcal{C}_X \to [0,1]$ with $\Pi(X) = 1$ such that $\forall$ $P \in \mathcal{P}_X$, the restriction $\Pi|\mathcal{C}_P$ is countably additive.

37.1 EXAMPLE The canonical measure γ_X on X is a cylinder measure.

37.2 REMARK Since the σ-algebra generated by $\mathcal{C}_X$ is Bor(X), it follows that every Borel probability measure on X determines by restriction a cylinder measure on X.

Let Π be a cylinder measure on X -- then the Fourier transform of Π is the function $\hat{\Pi}: X \to \underline{C}$ defined by the rule

$$\hat{\Pi}(x) = \int_X \exp(\sqrt{-1} \langle x,y\rangle) d\Pi(y).$$

[Note: This makes sense. In fact, the integrand is $\mathcal{C}_P$-measurable for any $P \in \mathcal{P}_X: x \in PX$ and Π is countably additive on $\mathcal{C}_P$.]

37.3 EXAMPLE We have

$$\hat{\gamma}_X(x) = \exp(-\frac{1}{2}||x||^2).$$

Let Π be a cylinder measure on X — then it is clear that $\hat{\Pi}$ is positive definite and equal to one at zero. Moreover, $\hat{\Pi}$ is continuous in the finite topology. For suppose that $F \subset X$ is a finite dimensional linear subspace of X. Let $P_F: X \to F$ be the orthogonal projection of X onto F and put $\hat{\Pi}_F = \hat{\Pi}|F$ -- then $\forall\ x \in F$

$$\begin{aligned}
\hat{\Pi}_F(x) &= \hat{\Pi}(x) \\
&= \int_X \exp(\sqrt{-1}\ \langle x,y\rangle)\, d\Pi(y) \\
&= \int_X \exp(\sqrt{-1}\ \langle x,y\rangle)\, d(\Pi|C_{P_F})(y) \\
&= \int_X \exp(\sqrt{-1}\ \langle P_F x,y\rangle)\, d(\Pi|C_{P_F})(y) \\
&= \int_X \exp(\sqrt{-1}\ \langle x,P_F y\rangle)\, d(\Pi|C_{P_F})(y) \\
&= \int_F \exp(\sqrt{-1}\ \langle x,y'\rangle)\, d(\Pi|C_{P_F} \circ P_F^{-1})(y').
\end{aligned}$$

Therefore $\hat{\Pi}_F$ is the Fourier transform of a probability measure on Bor(F), hence is a continuous function on F.

37.4 <u>LEMMA</u> Suppose that $\chi: X \to \underline{C}$ is positive definite, continuous in the finite topology, and equal to one at zero — then χ is the Fourier transform of a unique cylinder measure on X.

<u>PROOF</u> Given $P \in \mathcal{P}_X$, let $\chi_P = \chi|PX$ -- then by Bochner's theorem (cf. 33.3),

there exists a unique probability measure Π_P on Bor(PX): $\hat{\Pi}_P = \chi_P$. Define $\tilde{\Pi}_P$ on $\mathcal{C}_P$ by

$$\tilde{\Pi}_P(P^{-1}(B)) = \Pi_P(B) \quad (B \in \text{Bor}(PX)).$$

Then the collection $\{\tilde{\Pi}_P : P \in \mathcal{P}_X\}$ is consistent (i.e., $P_1 \le P_2 \Rightarrow \tilde{\Pi}_{P_1} = \tilde{\Pi}_{P_2} | \mathcal{C}_{P_1}$), so the prescription

$$\Pi(C) = \tilde{\Pi}_P(C) \quad (C \in \mathcal{C}_P)$$

defines a cylinder measure $\Pi : \mathcal{C}_X \to [0,1]$ on X having χ as its Fourier transform.

[Note: The hypotheses here are the same as those of 33.7, thus alternatively, χ is the Fourier transform of a unique probability measure on Cyl$(X^{\#})$.]

37.5 <u>REMARK</u> A cylinder measure Π on X admits an extension to a probability measure on Bor(X) iff $\hat{\Pi}$ is continuous in the Sazonov topology.

37.6 <u>EXAMPLE</u> Let K be a nonnegative symmetric operator. Define χ by

$$\chi(x) = \exp(-\frac{1}{2} \langle x, Kx \rangle).$$

Then there exists a unique cylinder measure Π on X: $\hat{\Pi} = \chi$.

[Note: When K = I, we recover γ_X and when K is trace class, Π extends to a centered gaussian measure on X.]

A function $f : X \to \underline{R}$ is a <u>cylinder function</u> if f is $\mathcal{C}_P$-measurable for some $P \in \mathcal{P}_X$.

[Note: Such a function is said to be <u>based</u> at P.]

37.7 <u>EXAMPLE</u> If

$$f(x) = \Phi(\langle x_1,x\rangle,\ldots,\langle x_n,x\rangle),$$

where $\Phi:\underline{R}^n \to \underline{R}$ is Borel, then f is a cylinder function based at P (the orthogonal projection onto the span of $x_1,\ldots,x_n$).

37.8 <u>LEMMA</u> The cylinder functions based at P are exactly those real valued functions on X of the form $f = \phi \circ P$, where $\phi:PX \to \underline{R}$ is Borel.

Let Π be a cylinder measure on X. Suppose that f is a cylinder function based at P -- then $\int_X |f(x)|d\Pi(x)$ is defined because $\Pi|C_P$ is countably additive. And if this integral is finite, then $\int_X f(x)d\Pi(x)$ is also defined.

[Note: If $P' \in \mathcal{P}_X$ and if $P' \geq P$, then f is based at P' as well and $\int_X f(x)d\Pi(x)$ is unchanged if P is replaced by P'.]

37.9 <u>REMARK</u> Fix P and set F = PX. Write Π_F in place of $\Pi \circ P^{-1}$ -- then

$$\int_F \phi d\Pi_F = \int_X \phi \circ Pd\Pi.$$

Therefore the arrow $\phi \to \phi \circ P$ is a unitary map from $L^2(F,\Pi_F)$ onto $L^2(X,C_P,\Pi)$, the space of square integrable cylinder functions based at P.

[Note: If $P' \in \mathcal{P}_X$ and if $P' \geq P$, then $L^2(X,\mathcal{C}_P,\Pi)$ is a closed subspace of $L^2(X,\mathcal{C}_{P'},\Pi)$.]

Let $M(X,\mathcal{C}_X,\Pi)$ be the set of Borel measurable functions $f:X \to \underline{R}$ such that $\forall\ \varepsilon > 0,\ \forall\ \delta > 0,\ \exists\ P_0 \in \mathcal{P}_X: P_1,P_2 \in \mathcal{P}_X\ \&\ P_1 \geq P_0, P_2 \geq P_0$

$$\Rightarrow$$

$$\Pi\{x:|f \circ P_1(x) - f \circ P_2(x)| > \varepsilon\} < \delta.$$

[Note: In other words, $M(X,\mathcal{C}_X,\Pi)$ is the set of Borel measurable functions $f:X \to \underline{R}$ such that the net $\{f \circ P:P \in \mathcal{P}_X\}$ of cylinder functions is fundamental in measure.]

Every cylinder function belongs to $M(X,\mathcal{C}_X,\Pi)$.

37.10 <u>EXAMPLE</u> Take $\Pi = \gamma_X$ and let p be a tight seminorm on X -- then

$$p \in M(X,\mathcal{C}_X,\gamma_X).$$

In fact, by definition, $\forall\ \varepsilon > 0,\ \exists\ P_\varepsilon \in \mathcal{P}_X$:

$$\gamma_X\{x:p(Px) > \varepsilon\} < \varepsilon\ \forall\ P \in \mathcal{P}_X:P \perp P_\varepsilon$$

or still,

$$\gamma_X\{x:p(Px - P_\varepsilon x) > \varepsilon\} < \varepsilon\ \forall\ P \in \mathcal{P}_X:P_\varepsilon \leq P.$$

Since

$$|p(Px) - p(P_\varepsilon x)| \leq p(Px - P_\varepsilon x),$$

it follows that

$$\gamma_X\{x: |p(Px) - p(P_\varepsilon x)| > \varepsilon\} < \varepsilon \ \forall \ P \in \mathcal{P}_X : P_\varepsilon \leq P.$$

So

$$P_1 \geq P_{\varepsilon/2}, \ P_2 \geq P_{\varepsilon/2}$$

$$\Rightarrow$$

$$\gamma_X\{x: |p \circ P_1(x) - p \circ P_2(x)| > \varepsilon\}$$

$$\leq \gamma_X\{x: |p(P_1 x) - p(P_{\varepsilon/2} x)| > \varepsilon/2\}$$

$$+ \gamma_X\{x: |p(P_2 x) - p(P_{\varepsilon/2} x)| > \varepsilon/2\}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Keeping $\varepsilon > 0$ fixed, introduce $\delta > 0$. If $\varepsilon < \delta$, take $P_0 = P_{\varepsilon/2}$ but if $\delta \leq \varepsilon$, take $P_0 = P_{\delta/2}$:

$$\gamma_X\{x: |p \circ P_1(x) - p \circ P_2(x)| > \varepsilon\}$$

$$\leq \gamma_X\{x: |p \circ P_1(x) - p \circ P_2(x)| > \delta\}$$

$$< \delta.$$

Therefore

$$p \in M(X, \mathcal{C}_X, \gamma_X).$$

[Note: Recall that p is continuous (cf. 36.8), hence is Borel.]

37.11 <u>LEMMA</u> Let $f_1,\ldots,f_n \in M(X,\mathcal{C}_X,\Pi)$ and suppose that $\Phi:\underline{R}^n \rightarrow \underline{R}$ is continuous -- then

$$\Phi(f_1,\ldots,f_n) \in M(X,\mathcal{C}_X,\Pi).$$

Consequently, $M(X,\mathcal{C}_X,\Pi)$ is closed under addition, multiplication, and the formation of maxima and minima.

Suppose that $f \in M(X,\mathcal{C}_X,\Pi)$ is bounded: $|f| \leq C$. Given $\varepsilon > 0$, choose $P_0 \in \mathcal{P}_X : P_1, P_2 \in \mathcal{P}_X \ \& \ P_1 \geq P_0,\ P_2 \geq P_0$

$$\Rightarrow$$

$$\Pi\{x:|f \circ P_1(x) - f \circ P_2(x)| > \varepsilon\} < \varepsilon.$$

Then

$$\int_X |f \circ P_1(x) - f \circ P_2(x)| d\Pi(x)$$

$$\leq \varepsilon + \int_X |f \circ P_1(x) - f \circ P_2(x)|$$

$$\cdot \chi_{\{|f \circ P_1 - f \circ P_2| > \varepsilon\}} d\Pi(x)$$

$$\leq \varepsilon + 2C\varepsilon.$$

Therefore the net

$$\{\int_X f \circ P d\Pi : P \in \mathcal{P}_X\}$$

of real numbers is Cauchy and by definition the <u>integral of f w.r.t. Π</u> is

$$\int_X f d\Pi = \lim_{P \in \mathcal{P}_X} \int_X f \circ P d\Pi.$$

The integral can be extended to nonnegative functions:

$$\left[\begin{array}{l} f \in M(X, \mathcal{C}_X, \Pi) \quad (f \geq 0) \\ \\ \int_X f d\Pi = \lim_{n \to \infty} \int_X \min(f,n) d\Pi. \end{array}\right.$$

[Note: It is possible, of course, that $\int_X f d\Pi$ is infinite.]

Let

$$L^1(X,\Pi) = \{f \in M(X,\mathcal{C}_X,\Pi) : \int_X |f| d\Pi < \infty\}.$$

Write

$$\int_X f d\Pi = \int_X f^+ d\Pi - \int_X f^- d\Pi \quad (f \in L^1(X,\Pi)).$$

Then the map $f \to \int_X f d\Pi$ from $L^1(X,\Pi)$ to $\underline{R}$ is linear and monotone, i.e.,

$$\int_X (a_1 f_1 + a_2 f_2) d\Pi = a_1 \int_X f_1 d\Pi + a_2 \int_X f_2 d\Pi$$

and

$$f_1 \leq f_2 \Rightarrow \int_X f_1 d\Pi \leq \int_X f_2 d\Pi.$$

37.12 <u>EXAMPLE</u> Suppose that $A \in \mathcal{B}(X)$ is Hilbert-Schmidt. Set $p_A(x) = ||Ax||$ $(x \in X)$ -- then p_A is tight (cf. 36.5), so

$$p_A \in M(X, \mathcal{C}_X, \gamma_X) \quad \text{(cf. 37.10)}$$

$$\Rightarrow$$

$$p_A^2 \in M(X, \mathcal{C}_X, \gamma_X).$$

But $\forall$ n,

$$\int_X \min(p_A^2, n)\, d\gamma_X \leq ||A||_2^2.$$

Therefore

$$p_A^2 \in L^1(X, \gamma_X).$$

[Note: One can say more, viz.

$$\int_X ||Ax||^2 \, d\gamma_X(x) = ||A||_2^2.]$$

37.13 LEMMA Let $f \in M(X, \mathcal{C}_X, \Pi)$ -- then the net $\{\Pi \circ (f \circ P)^{-1} : P \in \mathcal{P}_X\}$ of probability measures converges weakly to a probability measure $\Pi_f = \Pi \circ f^{-1}$ on $\text{Bor}(\underline{R})$. One has

$$f \in L^1(X, \Pi) \iff \int_{\underline{R}} |t| d\Pi_f(t) < \infty,$$

in which case

$$\int_X f d\Pi = \int_{\underline{R}} t d\Pi_f(t).$$

Let $f, g \in M(X, \mathcal{C}_X, \Pi)$ -- then f is said to be equal to g mod Π, written $f \equiv g \bmod \Pi$, if $\forall\ \varepsilon > 0$, $\exists\ P_0 \in \mathcal{P}_X : \forall\ P \geq P_0$,

$$\Pi\{x : |f \circ P(x) - g \circ P(x)| > \varepsilon\} < \varepsilon.$$

37.14 LEMMA Let $f \in M(X, C_X, \Pi)$ -- then $f \equiv 0 \bmod \Pi$ iff $\Pi_f = \delta_0$.

PROOF The condition $f \equiv 0 \bmod \Pi$ reads: $\forall\ \varepsilon > 0,\ \exists\ P_0 \in P_X: \forall\ P \geq P_0,$

$$\Pi \circ (f \circ P)^{-1}\{]-\infty,-\varepsilon[\cup]\varepsilon,\infty[\} < \varepsilon.$$

So, $\forall\ \varepsilon > 0$,

$$\Pi_f\{]-\infty,-\varepsilon[\cup]\varepsilon,\infty[\} \leq \varepsilon$$

$$\Rightarrow$$

$$\Pi_f = \delta_0.$$

The converse is equally obvious.

Suppose that $f \equiv 0 \bmod \Pi$ -- then

$$\int_X f d\Pi = 0.$$

Proof:

$$\int_X f d\Pi = \int_{\underline{R}} t d\Pi_f(t)$$

$$= \int_{\underline{R}} t d\delta_0(t) = 0.$$

37.15 REMARK Let $f \in L^1(X, \Pi)$ and suppose that $\int_C f d\Pi = 0\ \forall\ C \in C_X$ -- then $f \equiv 0 \bmod \Pi$ (cf. 38.15).

§38. *LINEAR STOCHASTIC PROCESSES*

Suppose that (Ω,A,μ) is a probability space. Let $f:\Omega \to \underline{R}$, $g:\Omega \to \underline{R}$ be Borel measurable functions. Write $f \sim g$ if $f = g$ almost everywhere -- then this relation is an equivalence relation, the corresponding equivalence classes being termed random variables.

[Note: When equipped with pointwise operations, the random variables are a commutative algebra over $\underline{R}$, call it $M(\Omega,A,\mu)$.]

Let X be an infinite dimensional separable real Hilbert space -- then a linear stochastic process (LSP) on X is a map L that assigns to each $x \in X$ a random variable L_x on a probability space (Ω,A,μ) such that $\forall\ a,b \in \underline{R}$ & $\forall\ x,y \in X$:

$$L_{ax+by} = aL_x + bL_y.$$

[Note: The reduction of L is the triple (Ω,A_L,μ_L), where $A_L \subset A$ is the σ-algebra generated by the L_x $(x \in X)$ and $\mu_L = \mu|A_L$.]

38.1 EXAMPLE Construct the isometric isomorphism

$$I:L^2[0,\infty[\to X_0[0,\infty[^*_{P^W}$$

as in 35.28. Let

$$\iota:X_0[0,\infty[^*_{P^W} \to L^2(X_0[0,\infty[,P^W)$$

be the inclusion -- then the assignment $f \to \iota I(f)$ is a LSP on $L^2[0,\infty[$.

38.2 <u>REMARK</u> Let γ be a centered gaussian measure on X -- then the inclusion $X = X^* \rightarrow L^2(X,\gamma)$ defines a LSP on X.

Suppose that $\mathcal{L}'$ and $\mathcal{L}''$ are LSPs on X -- then $\mathcal{L}'$ is said to be <u>equivalent</u> to $\mathcal{L}''$ if $\forall\, x \in X$,

$$\int_{\Omega'} e^{\sqrt{-1}\, \mathcal{L}'_x}\, d\mu' = \int_{\Omega''} e^{\sqrt{-1}\, \mathcal{L}''_x}\, d\mu''.$$

38.3 <u>LEMMA</u> Suppose that $\mathcal{L}', \mathcal{L}''$ are equivalent LSPs on X -- then $\exists$ an isomorphism

$$\phi: M(\Omega', A'_{\mathcal{L}'}, \mu'_{\mathcal{L}'}) \rightarrow M(\Omega'', A''_{\mathcal{L}''}, \mu_{\mathcal{L}''})$$

such that

$$\phi(\mathcal{L}'_x) = \mathcal{L}''_x \ \forall\, x \in X$$

and

$$\phi(bM(\Omega', A'_{\mathcal{L}'}, \mu'_{\mathcal{L}'})) = bM(\Omega'', A''_{\mathcal{L}''}, \mu''_{\mathcal{L}''})$$

with

$$E'(f') = E''(\phi(f'))$$

for all $f' \in bM(\Omega', A'_{\mathcal{L}'}, \mu'_{\mathcal{L}'})$.

[Note: The "b" stands for bounded while E' (respec. E") is the expectation per $\mu_{\mathcal{L}'}$ (respec. $\mu_{\mathcal{L}''}$).]

Let L be a LSP on X. Define $\chi_L: X \to \underline{C}$ by

$$\chi_L(x) = \int_\Omega e^{\sqrt{-1}\, L_x} d\mu.$$

Then χ_L is positive definite, continuous in the finite topology, and equal to one at zero, thus $\exists$ a unique cylinder measure Π_L on $X: \hat{\Pi}_L = \chi_L$ (cf. 37.4). Since Π_L depends only on $[L]$ (the equivalence class of L), it follows that we have a map $[L] \to \Pi_L$ from the set of LSPs on X modulo equivalence to the set of cylinder measures on X.

38.4 LEMMA Let Π be a cylinder measure on X -- then $\exists$ a LSP L on X such that

$$\hat{\Pi}(x) = \int_\Omega e^{\sqrt{-1}\, L_x} d\mu$$

for all $x \in X$.

PROOF Take $\Omega = \underline{R}^X$, $A = \underset{X}{\times} \text{Bor}(\underline{R})$, and let L_x be the coordinate map on Ω, i.e., $L_x(\omega) = \omega(x)$. Consider A_0, the subalgebra of A consisting of those sets of the form

$$\{\omega: (L_{x_1}(\omega), \ldots, L_{x_n}(\omega)) \in B\},$$

where $B \in \text{Bor}(\underline{R}^n)$. Define a set function μ_0 on A_0 by

$$\mu_0\{\omega: (L_{x_1}(\omega), \ldots, L_{x_n}(\omega)) \in B\}$$

$$= \Pi\{x: (\langle x_1, x\rangle, \ldots, \langle x_n, x\rangle) \in B\}.$$

Then there exists a unique probability measure μ on $A:\mu|A_0 = \mu_0$. To check linearity, one has to show that

$$\mu\{\omega: L_{ax+by}(\omega) = aL_x(\omega) + bL_y(\omega)\} = 1.$$

To this end, let

$$B = \{(t_1,t_2,t_3) \in \underline{R}^3 : at_1 + bt_2 = t_3\}.$$

Then

$$\mu\{\omega: (L_x(\omega), L_y(\omega), L_{ax+by}(\omega)) \in B\}$$

$$= \Pi\{z: (<x,z>, <y,z>, <ax+by,z>) \in B\}$$

$$= \Pi(X) = 1.$$

Finally, $\forall\ x \in X$ and $\forall\ B \in \mathrm{Bor}(\underline{R})$,

$$\mu \circ L_x^{-1}(B) = \mu\{\omega: L_x(\omega) \in B\}$$

$$= \Pi\{y: <x,y> \in B\}$$

$$= \Pi \circ <x,\underline{\quad}>^{-1}(B).$$

Therefore

$$\hat{\Pi}(x) = \int_\Omega e^{\sqrt{-1}\, L_x}\, d\mu.$$

Let Π be a cylinder measure on X — then a LSP L on X such that

$$\hat{\Pi}(x) = \int_\Omega e^{\sqrt{-1}\, L_x}\, d\mu$$

for all $x \in X$ is called a <u>model</u> of Π. E.g.: Take $X = L^2(\underline{R}^n)$, $\Pi = \gamma_X$ -- then a model for this data can be constructed from the white noise space (cf. 34.15).

38.5 <u>REMARK</u> If L' and L'' are models of Π, then $\forall\ B \in \text{Bor}(\underline{R}^n)$,

$$\Pi\{x:(<x_1,x>,\ldots,<x_n,x>) \in B\}$$

$$= \begin{cases} \mu'\{\omega':(L'_{x_1}(\omega'),\ldots,L'_{x_n}(\omega')) \in B\} \\ \\ \mu''\{\omega'':(L''_{x_1}(\omega''),\ldots,L''_{x_n}(\omega'')) \in B\}. \end{cases}$$

Write $\underline{A}_X$ $(b\underline{A}_X)$ for the algebra of cylinder functions (bounded cylinder functions) on X.

38.6 <u>LEMMA</u> Suppose that L is a model of Π. Let $f \in \underline{A}_X$, say

$$f(x) = \begin{cases} \Phi(<x_1,x>,\ldots,<x_n,x>) \\ \\ \Psi(<y_1,x>,\ldots,<y_m,x>), \end{cases}$$

where

$$\begin{cases} x_1,\ldots,x_n \\ \\ y_1,\ldots,y_m \end{cases} \in X$$

and

$$\left[\begin{array}{l} \Phi:\underline{R}^n \to \underline{R} \\ \\ \Psi:\underline{R}^m \to \underline{R} \end{array}\right.$$

are Borel measurable functions — then

$$\Phi(L_{x_1},\ldots,L_{x_n}) = \Psi(L_{y_1},\ldots,L_{y_m}) \text{ a.e. } [\mu].$$

PROOF Define $B \in \text{Bor}(\underline{R}^{n+m})$ by

$$B = \{(t_1,\ldots,t_{n+m}):\Phi(t_1,\ldots,t_n) \neq \Psi(t_{n+1},\ldots,t_{n+m})\}.$$

Then

$$\{x:(\langle x_1,x\rangle,\ldots,\langle x_n,x\rangle,\langle y_1,x\rangle,\ldots,\langle y_m,x\rangle) \in B\}$$

is empty, hence

$$\mu\{\omega:(L_{x_1}(\omega),\ldots,L_{x_n}(\omega),L_{y_1}(\omega),\ldots,L_{y_m}(\omega)) \in B\}$$

$$= 0,$$

from which the assertion.

Let $f \in \underline{A}_X$, say

$$f(x) = \Phi(\langle x_1,x\rangle,\ldots,\langle x_n,x\rangle).$$

Then the lifting of f is that element L_f of $M(\Omega,A,\mu)$ which is represented by

$$\Phi(L_{x_1},\ldots,L_{x_n}).$$

Therefore the lifting operation provides a filler for the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;L\;} & M(\Omega, A, \mu) \\ \downarrow & & \\ \underline{A}_X & & \end{array} \quad .$$

[Note: Matters are consistent in that $L_x = L_{<x,\_\_>}$ $\forall$ $x \in X$.]

N.B. It is not difficult to show that

$$\left[\begin{array}{l} L_{af+bg} = aL_f + bL_g \\ \\ L_{fg} = L_f L_g . \end{array}\right.$$

Therefore the arrow

$$\underline{A}_X \xrightarrow{\;L\;} M(\Omega, A, \mu)$$

is a homomorphism of algebras.

38.7 EXAMPLE Fix $P \in P_X$, let $B \in \text{Bor}(PX)$, and put $C = P^{-1}(B)$, thus $\chi_C \in \underline{A}_X$. Choose an orthonormal basis $e_1, \ldots, e_n$ for PX and define $\Phi : \underline{R}^n \to \underline{R}$ by

$$\Phi(t_1, \ldots, t_n) = \chi_B\left(\sum_{k=1}^{n} t_k e_k\right).$$

Then

$$\Phi(<e_1, x>, \ldots, <e_n, x>)$$

$$= \chi_B\left(\sum_{k=1}^{n} <e_k, x> e_k\right)$$

$$= \chi_B(Px)$$

$$= \chi_C(x),$$

so

$$L_{\chi_C} = \Phi(L_{e_1},\ldots,L_{e_n}),$$

or still,

$$L_{\chi_C} = \chi_B \circ \zeta = \chi_{\zeta^{-1}(B)},$$

where $\zeta:\Omega \to PX$ is the map

$$\zeta(\omega) = \sum_{k=1}^{n} L_{e_k}(\omega)e_k.$$

38.8 <u>LEMMA</u> $\forall\ f \in \underline{A}_X$, we have

$$\Pi \circ f^{-1} = \mu \circ L_f^{-1}$$

or still,

$$\Pi_f = \mu_{L_f}.$$

<u>PROOF</u> Define $\Theta:X \to \underline{R}^n$ by

$$\Theta(x) = (\langle x_1,x\rangle,\ldots,\langle x_n,x\rangle).$$

Then

$$\Pi \circ \Theta^{-1} = \mu \circ [L_{x_1},\ldots,L_{x_n}]^{-1}.$$

But $f = \Phi \circ \Theta$, thus

$$\Pi_f = \Pi \circ f^{-1}$$

$$= (\Pi \circ \Theta^{-1}) \circ \Phi^{-1}$$

$$= (\mu \circ [L_{x_1}, \ldots, L_{x_n}]^{-1}) \circ \Phi^{-1}$$

$$= \mu \circ L_f^{-1} = \mu_{L_f}.$$

38.9 <u>LEMMA</u> $\forall\ f \in b\underline{A}_X$, we have

$$\int_X f d\Pi = \int_\Omega L_f d\mu.$$

<u>PROOF</u> In fact, the LHS equals

$$\int_{\underline{R}} t d\Pi_f(t)$$

and the RHS equals

$$\int_{\underline{R}} t d\mu_{L_f}(t).$$

But $\Pi_f = \mu_{L_f}$ (cf. 38.8).

To force uniqueness of the model up to isomorphism, consider the reduction of L, i.e., the probability space (Ω, A_L, μ_L) -- then it is clear that

$$L(\underline{A}_X) \subset M(\Omega, A_L, \mu_L).$$

Moreover,

$$L(b\underline{A}_X) \subset bM(\Omega, A_L, \mu_L)$$

and the σ-algebra generated by $L(b\underline{A}_X)$ is A_L.

38.10 <u>LEMMA</u> $L(b\underline{A}_X)$ is dense in $L^2(\Omega,\mu_L)$.

<u>PROOF</u> If $I \subset \underline{R}$ is a finite interval, then the characteristic function of I is a uniformly bounded limit of polynomials, so $\forall\ \phi \in L(b\underline{A}_X)$, the characteristic function of $\{\omega:\phi(\omega) \in I\}$ is a uniformly bounded limit of a sequence of elements in $L(b\underline{A}_X)$. This said, let S denote the collection of all finite unions of sets of the form $\{\omega:\phi_i(\omega) \in I_i\ (i = 1,\ldots,n)\}$, where the $\phi_i \in L(b\underline{A}_X)$ and $I_i \subset \underline{R}$ is an interval (finite or infinite) -- then S is an algebra and the σ-algebra generated by S is A_L. Suppose that $\psi \in L^2(\Omega,\mu_L)$ is orthogonal to the elements of $L(b\underline{A}_X)$ -- then ψ is orthogonal to all uniformly bounded limits of sequences of elements in $L(b\underline{A}_X)$, hence, in view of what has been said above and the countable additivity of the indefinite integral, $\int_S \psi = 0\ \forall\ S \in S$. Since the collection of all measurable sets $A \in A_L$ such that $\int_A \psi = 0$ is closed under unions of monotone sequences and contains the algebra S, it follows that this collection contains the σ-algebra generated by S, i.e., A_L, thus $\psi = 0$ almost everywhere.

We shall now extend L to all of $M(X,C_X,\Pi)$.

38.11 LEMMA Let $f \in M(X, \mathcal{C}_X, \Pi)$ -- then there exists a random variable $\mathcal{L}_f$ on Ω such that the net $\{\mathcal{L}_{f \circ P} : P \in \mathcal{P}_X\}$ converges to $\mathcal{L}_f$ in measure:

$$\forall \ \varepsilon > 0, \ \exists \ P_\varepsilon \in \mathcal{P}_X : P \geq P_\varepsilon \Rightarrow$$

$$\mu(|\mathcal{L}_{f \circ P} - \mathcal{L}_f| > \varepsilon) < \varepsilon.$$

PROOF For each $k \geq 1$, choose $P_k \in \mathcal{P}_X : P_1, P_2 \in \mathcal{P}_X \ \& \ P_1 \geq P_k, \ P_2 \geq P_k$

$$\Rightarrow$$

$$\Pi(|f \circ P_1 - f \circ P_2| > \frac{1}{2^k}) < \frac{1}{2^k}$$

or still,

$$\mu(|\mathcal{L}_{f \circ P_1} - \mathcal{L}_{f \circ P_2}| > \frac{1}{2^k}) < \frac{1}{2^k}.$$

Without loss of generality, we can assume that $P_k \leq P_{k+1}$, hence

$$\mu(|\mathcal{L}_{f \circ P_k} - \mathcal{L}_{f \circ P_{k+1}}| > \frac{1}{2^k}) < \frac{1}{2^k}.$$

So, thanks to the Borel-Cantelli lemma,

$$\mu(\limsup |\mathcal{L}_{f \circ P_k} - \mathcal{L}_{f \circ P_{k+1}}| > \frac{1}{2^k}) = 0,$$

which implies that the sequence $\{\mathcal{L}_{f \circ P_k}\}$ converges almost everywhere to a random variable $\mathcal{L}_f$ on Ω. But

$$\mu(|\mathcal{L}_{f \circ P_{k+1}} - \mathcal{L}_f| > \frac{1}{2^k})$$

$$\leq \sum_{j=k+1}^{\infty} \mu(|L_{f \circ P_j} - L_{f \circ P_{j+1}}| > \frac{1}{2^j}) < \sum_{j=k+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^k} .$$

Accordingly, if $P \geq P_k$, then

$$\mu(|L_{f \circ P} - L_f| > \frac{1}{2^{k-1}})$$

$$\leq \mu(|L_{f \circ P} - L_{f \circ P_{k+1}}| > \frac{1}{2^k})$$

$$+ \mu(|L_{f \circ P_{k+1}} - L_f| > \frac{1}{2^k})$$

$$< \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}} .$$

Therefore the net $\{L_{f \circ P}:P \in \mathcal{P}_X\}$ converges to L_f in measure.

The <u>lifting</u> of L to $M(X,\mathcal{C}_X,\Pi)$ is the assignment $f \to L_f$.

[Note: Suppose that f is a cylinder function based at P_0 — then $\forall$ $P \geq P_0$, $f \circ P = f \Rightarrow L_{f \circ P} = L_f$, thus this definition is an extension of the earlier one for cylinder functions.]

Taking into account 37.13 and 38.11, $\forall$ $f \in M(X,\mathcal{C}_X,\Pi)$,

$$\Pi \circ f^{-1} = \mu \circ L_f^{-1}$$

or still,

$$\Pi_f = \mu_{L_f} \text{ (cf. 38.8)}.$$

Moreover,

$$f \in L^1(X,\Pi) \iff L_f \in L^1(\Omega,\mu)$$

and then

$$\int_X f d\Pi = \int_\Omega L_f d\mu.$$

38.12 LEMMA The arrow

$$M(X,C_X,\Pi) \xrightarrow{L} M(\Omega,A,\mu)$$

is a homomorphism of algebras.

[Note: If $f > 0$ and $L_f > 0$, then $\frac{1}{f} \in M(X,C_X,\Pi)$ and $L_{\frac{1}{f}} = \frac{1}{L_f}$.]

38.13 LEMMA Let $f \in M(X,C_X,\Pi)$ -- then $f \equiv 0 \bmod \Pi$ iff $L_f = 0$.

PROOF Recall that $f \equiv 0 \bmod \Pi$ iff $\Pi_f = \delta_0$ (cf. 37.14). But $\Pi_f = \mu_{L_f}$ and $\mu_{L_f} = \delta_0$ iff $L_f = 0$.

Write $M(X,C_X,\Pi)$ for the quotient $M(X,C_X,\Pi)/\sim$, where $\sim$ stands for $f \equiv g \bmod \Pi$ -- then 38.13 implies that the homomorphism

$$M(X,C_X,\Pi) \rightarrow M(\Omega,A,\mu)$$

of algebras is one-to-one.

38.14 LEMMA Let $f \in M(X,C_X,\Pi)$ — then $\exists$ an increasing sequence $P_n \in P_X$ which converges strongly to the identity I_X such that $L_{f \circ P_n} \rightarrow L_f$ a.e. $[\mu]$.

38.15 <u>LEMMA</u> Let $f \in L^1(X,\Pi)$ and suppose that $\int_C f d\Pi = 0 \ \forall \ C \in \mathcal{C}_X$ -- then $f \equiv 0 \bmod \Pi$.

[Here is a sketch of the proof, modulo measure theoretic technicalities (which can be handled by 38.14). Let $\mathcal{C}_\Omega$ be the set of subsets $A \in \mathcal{A}$ such that $\chi_A = L_{\chi_C}$ for some $C \in \mathcal{C}_X$ (cf. 38.7) -- then $\mathcal{C}_\Omega$ is an algebra. Write $\sigma(\mathcal{C}_\Omega)$ for the generated σ-algebra and consider the implications

$$\int_C f d\Pi = 0 \ \forall \ C \in \mathcal{C}_X$$

=>

$$\int_\Omega L_{\chi_C f} \, d\mu = 0 \ \forall \ C \in \mathcal{C}_X$$

=>

$$\int_\Omega L_{\chi_C} L_f \, d\mu = 0 \ \forall \ C \in \mathcal{C}_X$$

=>

$$\int_\Omega \chi_A L_f \, d\mu = 0 \ \forall \ A \in \mathcal{C}_\Omega$$

=>

$$\int_\Omega \chi_A L_f \, d\mu = 0 \ \forall \ A \in \sigma(\mathcal{C}_\Omega)$$

=>

$$L_f = 0$$

=>

$$f \equiv 0 \bmod \Pi \ \text{(cf. 38.13).]}$$

For later use, it is necessary to realize that the theory admits an obvious extension to function spaces over $\underline{C}$.

Let L be a model of Π -- then by $L^2(\Omega_\Pi,\mu_\Pi)$ we shall understand the space of complex valued square integrable functions per (Ω,A_L,μ_L), the reduction of L.

[Note: The rationale for the notation is that $L^2(\Omega_\Pi,\mu_\Pi)$ is a unitary invariant of $[L]$.]

Given $x \in X$, let

$$M_x:L^2(\Omega_\Pi,\mu_\Pi) \to L^2(\Omega_\Pi,\mu_\Pi)$$

be multiplication by $e^{\sqrt{-1}\, L_x}$ -- then the assignment $x \to M_x$ defines a homomorphism $X \to U(L^2(\Omega_\Pi,\mu_\Pi))$ which is continuous in the finite topology.

38.16 <u>LEMMA</u> The functions $e^{\sqrt{-1}\, L_x}$ $(x \in X)$ are total in $L^2(\Omega_\Pi,\mu_\Pi)$, hence 1 is a cyclic unit vector for M.

Therefore

$$M = U_{\hat{\Pi}} \quad \text{(cf. 14.10)}.$$

In fact,

$$\begin{aligned}\hat{\Pi}(x) &= \int_X \exp(\sqrt{-1}\, \langle x,y\rangle)\, d\Pi(y) \\ &= \int_\Omega e^{\sqrt{-1}\, L_x}\, d\mu \quad \text{(cf. 38.4)} \\ &= \langle 1, M_x 1\rangle.\end{aligned}$$

38.17 <u>REMARK</u> The completion of the pre-Hilbert space $L^2(X,\Pi)$ can be identified with $L^2(\Omega_\Pi,\mu_\Pi)$.

Suppose that $\mathcal{L}'$ and $\mathcal{L}''$ are LSPs on X -- then $\mathcal{L}'$ is said to be <u>weakly equivalent</u> to $\mathcal{L}''$ if $\exists$ a unitary map

$$U:L^2(\Omega',\mu_{\mathcal{L}'}) \to L^2(\Omega'',\mu_{\mathcal{L}''})$$

such that $\forall\ x \in X$,

$$UM_{\mathcal{L}'_x}U^{-1} = M_{\mathcal{L}''_x}.$$

[Note: $M_{\mathcal{L}'_x}$ and $M_{\mathcal{L}''_x}$ are the multiplication operators corresponding to $\mathcal{L}'_x$ and $\mathcal{L}''_x$.]

38.18 <u>REMARK</u> If $\mathcal{L}'$ and $\mathcal{L}''$ are equivalent, then $\mathcal{L}'$ and $\mathcal{L}''$ are weakly equivalent (but not conversely).

38.19 <u>LEMMA</u> Suppose that $\mathcal{L}'$ and $\mathcal{L}''$ are LSPs on X -- then $\mathcal{L}'$ and $\mathcal{L}''$ are weakly equivalent iff there exist nonnegative functions

$$\begin{cases} D' \in L^1(\Omega',\mu_{\mathcal{L}'}) \\ D'' \in L^1(\Omega'',\mu_{\mathcal{L}''}) \end{cases}$$

such that $\forall\ f \in b\underline{A}_X$,

$$\begin{cases} \int_{\Omega'} L'_f \, d\mu_{L'} = \int_{\Omega''} L''_f \, D'' \, d\mu_{L''} \\ \int_{\Omega''} L''_f \, d\mu_{L''} = \int_{\Omega'} L'_f \, D' \, d\mu_{L'} . \end{cases}$$

[Note: D' and D" are necessarily unique.]

38.20 <u>EXAMPLE</u> If $\Omega' = \Omega'' = \Omega$ and $A_{L'} = A_{L''} = A$, then L' and L'' are weakly equivalent iff $\mu_{L'}$ and $\mu_{L''}$ are mutually absolutely continuous.

§39. GROSS'S THEOREM

Recall the definition: A triple (X,Y,ι) is said to be an abstract Wiener space if

$$\begin{cases} X \text{ is a separable real Hilbert space } (\dim X = \infty) \\ Y \text{ is a separable real Banach space } (\dim Y = \infty) \end{cases}$$

and $\iota:X \to Y$ is a continuous linear injection with a dense range such that $||\cdot||_Y \circ \iota$ is tight, where $||\cdot||_Y$ is the norm on Y.

[Note: It will be convenient to assume outright that X is contained in Y.]

Let (X,Y,ι) be an abstract Wiener space. Consider the arrow of restriction $Y^* \to X^*$ and identify X^* with X -- then $\forall\ \lambda \in Y^*$, there is a unique vector $x_\lambda \in X$:

$$\lambda(x) = \langle x_\lambda, x\rangle \quad (x \in X).$$

It is clear that the map $\lambda \to x_\lambda$ is one-to-one. Moreover, the set $\{x_\lambda\}$ is total in X.

The following result was stated without proof in §36 (cf. 36.20).

39.1 THEOREM (Gross) Let (X,Y,ι) be an abstract Wiener space -- then the set function $\gamma_X \circ \iota^{-1}$ is countably additive on C_Y, hence can be extended to a centered gaussian measure γ_Y on Bor(Y).

PROOF Fix a model L of γ_X. Choose an increasing sequence $P_n \in P_X$ which converges strongly to the identity I_X such that

$$\gamma_X\{x:||Px||_Y > \frac{1}{2^n}\} < \frac{1}{2^n} \ \forall \ P \in \mathcal{P}_X : P \perp P_n.$$

Let $Q_n = P_{n+1} - P_n$ -- then $Q_n \perp P_n$, hence

$$\gamma_X\{x:||Q_n x||_Y > \frac{1}{2^n}\} < \frac{1}{2^n} .$$

Put

$$f(x) = ||x||_Y \quad (x \in X).$$

Thus $f \in M(X, \mathcal{C}_X, \gamma_X)$ (cf. 37.10) and

$$\mu\{\omega: L_{f \circ Q_n}(\omega) > \frac{1}{2^n}\} < \frac{1}{2^n} .$$

Let $d(n) = \dim P_n X$ ($\Rightarrow \dim Q_n X = d(n+1) - d(n)$). Fix an orthonormal basis $\{e_k : k = d(n)+1, \ldots, d(n+1)\}$ for $Q_n X$ -- then the collection $\{e_k : 1 \le k \le d(n)\}$ is an orthonormal basis for $P_n X$ and since $P_n \uparrow I_X$, the collection $\{e_k : k \ge 1\}$ is an orthonormal basis for X. Define $\Xi_n : \Omega \to Y$ by the prescription

$$\Xi_n(\omega) = \sum_{k=1}^{d(n)} L_{e_k}(\omega) e_k \quad (\omega \in \Omega).$$

On the basis of the definitions,

$$L_{f \circ Q_n} = L_{||\sum_{k=d(n)+1}^{d(n+1} <e_k, \_\_> e_k||_Y}$$

$$= ||\sum_{k=d(n)+1}^{d(n+1)} L_{e_k} e_k||_Y$$

$$= ||\Xi_{n+1} - \Xi_n||_Y$$

=>

$$\mu\{\omega: ||\Xi_{n+1}(\omega) - \Xi_n(\omega)||_Y > \frac{1}{2^n}\} < \frac{1}{2^n} .$$

Consequently, the sequence $\{\Xi_n\}$ is fundamental in measure. So: (1) $\exists$ a Borel measurable function $\Xi:\Omega \to Y$ such that $\Xi_n \to \Xi$ in measure and (2) $\exists$ a subsequence $\{\Xi_{n_j}\}$ of $\{\Xi_n\}$ which converges to Ξ a.e. $[\mu]$. Take now $\gamma_Y = \mu \circ \Xi^{-1}$ and consider its Fourier transform:

$$\hat{\gamma}_Y(\lambda) = \int_Y e^{\sqrt{-1}\ \lambda(y)}\ d\gamma_Y(y)$$

$$= \int_\Omega \exp(\sqrt{-1}\ \lambda(\Xi(\omega)))d\mu(\omega)$$

$$= \lim_{j \to \infty} \int_\Omega \exp(\sqrt{-1}\ \lambda(\Xi_{n_j}(\omega)))d\mu(\omega)$$

$$= \lim_{j \to \infty} \int_\Omega \exp(\sqrt{-1}\ \lambda(\sum_{k=1}^{d(n_j)} L_{e_k}(\omega)e_k))d\mu(\omega)$$

$$= \lim_{j \to \infty} \int_\Omega \exp(\sqrt{-1}\ \sum_{k=1}^{d(n_j)} L_{e_k}(\omega)\langle x_\lambda, e_k\rangle)d\mu(\omega)$$

$$= \lim_{j \to \infty} \int_X \exp(\sqrt{-1}\ \sum_{k=1}^{d(n_j)} \langle x_\lambda, e_k\rangle\langle e_k, x\rangle)d\gamma_X(x)$$

$$= \lim_{j \to \infty} \exp(-\frac{1}{2}\ ||\sum_{k=1}^{d(n_j)} \langle x_\lambda, e_k\rangle e_k||_X^2)$$

$$= \lim_{j \to \infty} \exp(-\frac{1}{2} \sum_{k=1}^{d(n_j)} |\langle x_\lambda, e_k\rangle|^2)$$

$$= \exp(-\frac{1}{2} \sum_{k=1}^{\infty} |\langle x_\lambda, e_k\rangle|^2)$$

$$= \exp(-\frac{1}{2} ||x_\lambda||_X^2)$$

$$= \hat{\gamma_X \circ \iota^{-1}}(\lambda).$$

Therefore

$$\gamma_Y | \mathcal{C}_Y = \gamma_X \circ \iota^{-1}.$$

39.2 <u>REMARK</u> The Cameron-Martin space $H(\gamma_Y)$ of γ_Y coincides with X (or, more precisely, $\iota(X)$).

Let (X,Y,ι) be an abstract Wiener space. On general grounds,

$$R_{\gamma_Y}: Y^*_{\gamma_Y} \to H(\gamma_Y)$$

and, by the above, $H(\gamma_Y) = X$, with

$$R_{\gamma_Y}(\lambda) = x_\lambda \quad (\lambda \in Y^*).$$

Given an arbitrary $x \in X$, let Φ_x be the element of $Y^*_{\gamma_Y}$ $(\subset L^2(Y,\gamma_Y))$ for which

$$R_{\gamma_Y}(\Phi_x) = x.$$

Then

$$\int_Y \exp(\sqrt{-1}\ \Phi_x)\, d\gamma_Y$$

$$= \exp(-\tfrac{1}{2}\ ||\Phi_x||^2_{L^2(\gamma_Y)}) \qquad (\text{cf. } 26.9)$$

$$= \exp(-\tfrac{1}{2}\ ||x||^2_X)$$

$$= \hat{\gamma}_X(x) \qquad (\text{cf. } 37.3).$$

And

$$\frac{d\gamma_{Y,x}}{d\gamma_Y} = \exp(\Phi_x - \tfrac{1}{2}\ ||x||^2_X).$$

So, $\forall\ f \in L^1(Y,\gamma_Y)$,

$$\int_Y f(x+y)\, d\gamma_Y(y)$$

$$= \int_Y f(y) \exp(\Phi_x(y) - \tfrac{1}{2}\ ||x||^2_X)\, d\gamma_Y(y).$$

39.3 <u>REMARK</u> We have (cf. §28)

$$\begin{array}{ccc} BO(Y^*_{\gamma_Y}) & \xrightarrow{T} & L^2(Y,\gamma_Y) \\ \updownarrow & & \\ BO(X), & & \end{array}$$

where T is the isometric isomorphism characterized by the relation

$$T\ \underline{\exp}(\Phi) = \Lambda_\Phi \quad (\Phi \in Y^*_{\gamma_Y}).$$

Let (X,Y,ι) be an abstract Wiener space -- then the assignment $x \to \Phi_x$ is a LSP on X and the completion of the pre-Hilbert space

$$\bigcup_{P \in \mathcal{P}_X} L^2(X,\mathcal{C}_P,\gamma_X)$$

can be identified with $L^2(Y,\gamma_Y)$ which in turn represents $L^2(\Omega_{\gamma_X},\mu_{\gamma_X})$.

39.4 <u>REMARK</u> Suppose that

$$\begin{cases} (X,Y',\iota') \\ (X,Y'',\iota'') \end{cases}$$

are abstract Wiener spaces -- then $\forall\ x \in X$,

$$\begin{cases} \int_{Y'} \exp(\sqrt{-1}\ \Phi_x')d\gamma_{Y'} \\ \int_{Y''} \exp(\sqrt{-1}\ \Phi_x'')d\gamma_{Y''} \end{cases} = \exp(-\frac{1}{2}\ ||x||_X^2),$$

thus Φ' and Φ'' are equivalent.

39.5 <u>LEMMA</u> Let $\phi: Y \to \underline{R}$ be continuous. Put $f = \phi \circ \iota$ -- then $f \in M(X,\mathcal{C}_X,\gamma_X)$ and $L_f = \phi$.

§40. *THE HEAT SEMIGROUP*

Let X be an infinite dimensional separable real Hilbert space -- then the canonical measure on X with variance $t > 0$ is the set function

$$\gamma_{X,t}:\mathcal{C}_X \to [0,1]$$

defined by the rule

$$\gamma_{X,t}(C) = \frac{1}{(2\pi t)^{n/2}} \int_B \exp(-\frac{1}{2t} ||x||^2) dx,$$

where $n = \dim PX$.

[Note: $\gamma_{X,t}$ is, of course, a cylinder measure on X with

$$\hat{\gamma}_{X,t}(x) = \exp(-\frac{t}{2} ||x||^2).]$$

Suppose now that (X,Y,ι) is an abstract Wiener space -- then $\forall\ t > 0$, the set function $\gamma_{X,t} \circ \iota^{-1}$ is countably additive on $\mathcal{C}_Y$, hence can be extended to a centered gaussian measure $\gamma_{Y,t}$ on Bor(Y) (argue as in 39.1).

Write p_t for the extension of $\gamma_{Y,t}$ to Bor(Y), thus

$$p_t(B) = p_1(\frac{1}{\sqrt{t}} B) \quad (B \in \text{Bor}(Y)).$$

In addition, abbreviate $\gamma_{X,t}$ to γ_t.

40.1 <u>LEMMA</u> $\forall\ f \in L^1(Y,p_t)$,

$$\int_Y f(y)dp_t(y) = \int_Y f(\sqrt{t}\, y)dp_1(y).$$

Set theoretically, $\forall$ t,

$$\begin{cases} Y^*_{p_t} = Y^*_{p_1} \\ H(p_t) = H(p_1) \quad (= X) \end{cases}$$

but the inner products are different.

To clarify the matter, observe first that

$$\int_Y \exp(\sqrt{-1}\ \Phi_x)dp_t = \exp(-\tfrac{t}{2}\ ||x||^2_X) \quad (x \in X).$$

[Note: Recall that $\Phi_x \in Y^*_{p_1}$ ($\subset L^2(Y,p_1)$) and $R_{p_1}(\Phi_x) = x$.]

Therefore

$$||\Phi_x||_{L^2(p_t)} = \sqrt{t}\ ||x||_X$$

$$= \sqrt{t}\ ||\Phi_x||_{L^2(p_1)}.$$

Let $H(p_t)$ be $H(p_1)$ (= X) equipped with the inner product derived from the norm

$$||x||_t = \frac{||x||_X}{\sqrt{t}}.$$

Put

$$\Phi_{x/t} = \frac{1}{t}\Phi_x \in Y^*_{p_t}.$$

Then

$$R_{p_t}(\Phi_{x/t}) = x.$$

In fact,

$$||\frac{1}{t}\Phi_x||_{L^2(p_t)} = \frac{1}{t}||\Phi_x||_{L^2(p_t)}$$

$$= \frac{1}{t}\sqrt{t}\;||x||_X$$

$$= \frac{1}{\sqrt{t}}\;||x||_X$$

$$= ||x||_t.$$

40.2 <u>REMARK</u> $\forall\, t > 0$, the assignment $x \to \Phi_x$ is a LSP on X (per the probability space $L^2(Y,p_t)$), call it L_t. Since

$$\hat{\gamma}_t(x) = \exp(-\frac{t}{2}||x||^2)$$

$$= \int_Y \exp(\sqrt{-1}\,\Phi_x)dp_t,$$

it follows that if $t_1 \neq t_2$, then $[L_{t_1}] \neq [L_{t_2}]$.

Given $h \in Y$, let $p_{t,h}$ be the image of p_t under the map $y \to y + h$ -- then $p_{t,h}$ is gaussian and, on general grounds (cf. 26.19),

$$H(p_t) = \{h \in Y:p_{t,h} \sim p_t\}.$$

40.3 <u>LEMMA</u> Suppose that $t_1 \neq t_2$ -- then $p_{t_1} \perp p_{t_2}$.

PROOF This is an application of 27.17. Indeed,

$$\left[\begin{array}{l} p_{t_1}(B) = p_1 (\frac{1}{\sqrt{t_1}} B) \\ \\ p_{t_2}(B) = p_1 (\frac{1}{\sqrt{t_2}} B) \end{array}\right. \qquad (B \in \mathrm{Bor}(Y)).]$$

40.4 LEMMA p_{t_1,h_1} and p_{t_2,h_2} are equivalent iff $t_1 = t_2$ and $h_1 - h_2 \in X$. Otherwise, they are mutually singular.

PROOF If $t_1 = t_2$, then $\forall\ h_1,h_2 \in X$,

$$\left[\begin{array}{l} p_{t_1,h_1} \sim p_{t_2,h_2} \text{ if } h_1 - h_2 \in X \\ \\ p_{t_1,h_1} \perp p_{t_2,h_2} \text{ if } h_1 - h_2 \notin X \end{array}\right. \qquad (\text{cf. } 27.2).$$

If $t_1 \neq t_2$, then $p_{t_1} \perp p_{t_2}$ (cf. 40.3), hence $p_{t_1,h_1} \perp p_{t_2,h_2}$ (cf. 27.3).

40.5 REMARK Let $x \in X$ — then

$$\begin{aligned} \frac{dp_{t,x}}{dp_t} &= \exp(\Phi_{x/t} - \frac{1}{2} ||x||_t^2) \\ &= \exp(\frac{1}{t} \Phi_x - \frac{1}{2} \frac{||x||_X^2}{t}) \\ &= \exp(\frac{1}{2t} (2\Phi_x - ||x||_X^2)). \end{aligned}$$

The generalities developed near the end of §32 can be specialized to the present situation:

$$\left[\begin{array}{l} X \rightarrow Y \\ \gamma \rightarrow p_1 \\ H(\gamma) \rightarrow X. \end{array}\right.$$

So, if $\phi:Y \rightarrow \underline{R}$ is bounded and Borel, then

$$\begin{aligned} P_t\phi(y) &= \int_Y \phi(y + \sqrt{t}\, y')dp_1(y') \\ &= \int_Y \phi(y + y')dp_t(y') \\ &= \phi * p_t(y). \end{aligned}$$

So, as in the finite dimensional case, the heat semigroup $\{P_t\}$ is generated by the one parameter family $\{p_t\}$ of gaussians.

[Note: The operator $-\Delta$ is essentially selfadjoint on $S(\underline{R}^n)$ and nonnegative, so its closure (denoted still by $-\Delta$) generates a semigroup on $L^2(\underline{R}^n)$. Put

$$u(x,t) = (e^{t\Delta}\phi)(x).$$

Then

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\underline{R}^n} e^{-(x-y)^2/4t}\, \phi(y)dy$$

and u(x,t) is a weak solution to the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t).$$

In addition,

$$e^{t\Delta/2} = p_t * .]$$

40.5 <u>LEMMA</u> $P_t\phi$ is infinitely H-differentiable (cf. 32.10).

§41. THE REAL WAVE REPRESENTATION

Let γ be a centered gaussian measure on X, where X is a separable LF-space. Given $h \in H(\gamma)$, determine $\hat{h} \in X^*_\gamma$ $(\subset L^2(X,\gamma))$ by $R_\gamma(\hat{h}) = h$.

Working over $\underline{C}$, we shall define two unitary representations of the additive group of $H(\gamma)$ on $L^2(X,\gamma)$.

[Note: Bear in mind that $H(\gamma)$ is a separable real Hilbert space.]

U: Given $h \in H(\gamma)$, let

$$U(h):L^2(X,\gamma) \to L^2(X,\gamma)$$

be the operator defined by the rule

$$U(h)\psi(x) = \psi(x+h) \left[\frac{d\gamma_{-h}}{d\gamma}(x) \right]^{1/2}.$$

V: Given $h \in H(\gamma)$, let

$$V(h):L^2(X,\gamma) \to L^2(X,\gamma)$$

be the operator defined by the rule

$$V(h)\psi(x) = e^{\sqrt{-1}\,\hat{h}(x)}\psi(x).$$

Ad U: We have

$$||U(h)\psi||^2_{L^2(\gamma)}$$

$$= \int_X |\psi(x+h)|^2 \frac{d\gamma_{-h}}{d\gamma}(x)d\gamma(x)$$

$$= \int_X |\psi(x+h)|^2 d\gamma_{-h}(x)$$

$$= \int_X |\psi(x+h-h)|^2 d\gamma(x)$$

$$= ||\psi||^2_{L^2(\gamma)}.$$

And

$$U(h_1+h_2)\psi(x)$$

$$= \psi(x+h_1+h_2)\left[\frac{d\gamma_{-h_1-h_2}}{d\gamma}(x)\right]^{1/2}$$

$$= \psi(x+h_1+h_2)\left[\frac{d\gamma_{-h_1}}{d\gamma}(x)\right]^{1/2}\left[\frac{d\gamma_{-h_2}}{d\gamma}(x+h_1)\right]^{1/2}$$

$$= U(h_1)(U(h_2)\psi)(x).$$

Ad V: We have

$$||V(h)\psi||^2_{L^2(\gamma)} = ||\psi||^2_{L^2(\gamma)}$$

and

$$V(h_1+h_2) = V(h_1)V(h_2).$$

41.1 LEMMA U and V satisfy the canonical commutation relations, i.e.,

$$U(h)V(h') = e^{\sqrt{-1}\,\langle h,h'\rangle_{H(\gamma)}}\,V(h')U(h).$$

PROOF Consider the LHS:

$$U(h)V(h')\psi|_x$$

$$= U(h)\,(e^{\sqrt{-1}\,\hat{h}'}\psi)\Big|_x$$

$$= e^{\sqrt{-1}\,\hat{h}'(x+h)}\psi(x+h)\left[\frac{d\gamma_{-h}}{d\gamma}(x)\right]^{1/2}.$$

But the RHS equals

$$e^{\sqrt{-1}\,\langle h,h'\rangle_{H(\gamma)}}\;V(h')\,(U(h)\psi)\Big|_x$$

$$= e^{\sqrt{-1}\,\langle h,h'\rangle_{H(\gamma)}}\;e^{\sqrt{-1}\,\hat{h}'(x)}\psi(x+h)\left[\frac{d\gamma_{-h}}{d\gamma}(x)\right]^{1/2}.$$

And

$$\hat{h}'(x+h) = \hat{h}'(x) + \hat{h}'(h)$$

$$= \hat{h}'(x) + \langle h,h'\rangle_{H(\gamma)}.$$

Applying now the standard procedure, put

$$W_{re}(h \oplus h') = \exp\left(\frac{\sqrt{-1}}{2}\langle h,h'\rangle_{H(\gamma)}\right)U(-h)V(h').$$

Then W_{re} defines a Weyl system over $H(\gamma) \oplus H(\gamma)$, the so-called <u>real wave representation</u>.

Explicitly,

$$W_{re}(h \oplus h')\psi(x)$$

$$= \exp\left(\frac{\sqrt{-1}}{2}\langle h,h'\rangle_{H(\gamma)}\right)U(-h)V(h')\psi)\Big|_x$$

$$= \exp(\frac{\sqrt{-1}}{2} \langle h,h'\rangle_{H(\gamma)}) \, e^{\sqrt{-1}\,\hat{h}'(x-h)} \psi(x-h) \left[\frac{d\gamma_h}{d\gamma}(x) \right]^{1/2}.$$

Since

$$\frac{d\gamma_h}{d\gamma}(x) = \exp(\hat{h}(x) - \frac{1}{2} ||h||^2_{H(\gamma)}),$$

it follows that

$$W_{re}(h \oplus h')\psi(x)$$

$$= \exp(\sqrt{-1}\,(\hat{h}'(x) - \langle h,h'\rangle_{H(\gamma)}/2))$$

$$\cdot [\exp(\hat{h}(x) - \frac{1}{2} ||h||^2_{H(\gamma)})]^{1/2} \psi(x-h).$$

41.2 <u>EXAMPLE</u> Take $X = \underline{R}^n$, $\gamma = \gamma_n$ -- then, as has been seen earlier (cf. 22.8), the prescription

$$W(a,b)\psi(x) = \exp(\sqrt{-1}\,(\langle x,b\rangle - \langle a,b\rangle/2))$$

$$\cdot [\exp(\langle x,a\rangle - a^2/2)]^{1/2} \psi(x-a)$$

defines a Weyl system over $\underline{R}^{2n} = \underline{R}^n \oplus \underline{R}^n$ which is unitarily equivalent to the Schrödinger system (cf. 10.4).

Working over $\underline{R}$, there is an isometric isomorphism

$$T:BO(X^*_\gamma) \rightarrow L^2(X,\gamma) \quad (\text{cf. §28}).$$

On the other hand, there is an isometric isomorphism

$$R_\gamma : X^*_\gamma \to H(\gamma)$$

with inverse

$$\wedge : H(\gamma) \to X^*_\gamma .$$

So

$$\begin{array}{ccc} BO(X^*_\gamma) & \xrightarrow{\quad T \quad} & L^2(X,\gamma) \\ {\scriptstyle \Gamma(\wedge)} \Big\uparrow & & \\ BO(H(\gamma)) & & \end{array} .$$

Here

$$T \circ \Gamma(\wedge)\underline{\exp}(h) = \Lambda_{\hat{h}} \quad (h \in H(\gamma)).$$

Now pass to the complexification $H(\gamma)_{\underline{C}}$ of $H(\gamma)$ and work over $\underline{C}$ to get an isometric isomorphism

$$\hat{T} : BO(H(\gamma)_{\underline{C}}) \to L^2(X,\gamma)$$

which sends

$$\underline{\exp}(h + \sqrt{-1}\, h')$$

to

$$\Lambda_{\hat{h} + \sqrt{-1}\, \hat{h}'} ,$$

where

$$\Lambda_{\hat{h} + \sqrt{-1}\, \hat{h}'}(x)$$

$$= \exp(\hat{h}(x) + \sqrt{-1}\, \hat{h}'(x) - \frac{1}{2} (h + \sqrt{-1}\, h')^2).$$

[Note: The symbol

$$(h + \sqrt{-1}\, h')^2$$

stands for the combination

$$\langle h - \sqrt{-1}\, h', h + \sqrt{-1}\, h'\rangle.]$$

Let

$$W_F : H(\gamma)_{\underline{C}} \to U(BO(H(\gamma)_{\underline{C}}))$$

be the Fock system (cf. 10.3).

41.3 LEMMA We have

$$\hat{T} W_F(-\frac{\sqrt{-1}}{\sqrt{2}} h)\hat{T}^{-1} \psi\Big|_x$$

$$= [\exp(\hat{h}(x) - \frac{1}{2} ||h||^2_{H(\gamma)})]^{1/2} \psi(x-h).$$

PROOF Take $\psi = \Lambda_f$, where $f \in X^*_\gamma$ (cf. 28.8) -- then $\hat{T}^{-1}\Lambda_f = \underline{\exp}(g)$ $(g = R_\gamma(f))$ and

$$W_F(-\frac{\sqrt{-1}}{\sqrt{2}} h)\underline{\exp}(g)$$

$$= \exp(-\frac{1}{4} ||\frac{h}{\sqrt{2}}||^2 - \frac{1}{\sqrt{2}} \langle\frac{h}{\sqrt{2}}, g\rangle)\underline{\exp}(\frac{1}{\sqrt{2}} \frac{h}{\sqrt{2}} + g) \qquad \text{(cf. 9.4)}$$

$$= \exp(-\frac{1}{8} ||h||^2 - \frac{1}{2} \langle h, g\rangle)\underline{\exp}(\frac{h}{2} + g).$$

Apply $\hat{T}$:

$$\hat{T}\underline{\exp}(\frac{h}{2} + g)\Big|_x$$

$$= \exp(\frac{\hat{h}}{2}(x) + \hat{g}(x) - \frac{1}{2} \langle \frac{h}{2} + g, \frac{h}{2} + g\rangle).$$

Then

$$- \frac{1}{2} \langle \frac{h}{2} + g, \frac{h}{2} + g\rangle$$

$$= - \frac{1}{2} (||\frac{h}{2}||^2 + 2\langle\frac{h}{2},g\rangle + ||g||^2)$$

$$= - \frac{1}{2} (\frac{1}{4} ||h||^2 + \langle h,g\rangle + ||g||^2).$$

Combining the exponential of this with

$$\exp(- \frac{1}{8} ||h||^2 - \frac{1}{2} \langle h,g\rangle)$$

gives

$$\exp(- \frac{1}{4} ||h||^2 - \langle h,g\rangle - \frac{1}{2} ||g||^2).$$

To complete the unraveling, consider

$$[\exp(\hat{h}(x) - \frac{1}{2} ||h||^2)]^{1/2} \Lambda_f(x-h)$$

or still,

$$\exp(\frac{\hat{h}}{2}(x) - \frac{1}{4} ||h||^2)\Lambda_{\hat{g}} (x-h),$$

thus reducing matters to the equality

$$\exp(\hat{g}(x) - \langle g,h\rangle - \frac{1}{2} ||g||^2)$$

$$= \Lambda_{\hat{g}} (x-h).$$

But, by definition,

$$\Lambda_{\hat{g}}(x-h) = \exp(\hat{g}(x-h) - \frac{1}{2}||g||^2)$$

$$= \exp(\hat{g}(x) - \langle g,h\rangle - \frac{1}{2}||g||^2),$$

thereby completing the proof.

41.4 LEMMA We have

$$\hat{T}W_F(\sqrt{2}\,h')\hat{T}^{-1}\psi\Big|_x$$

$$= e^{\sqrt{-1}\,\hat{h}'(x)}\psi(x).$$

PROOF Take $\psi = \Lambda_f$, where $f \in X_\gamma^*$ (cf. 28.8) -- then $\hat{T}^{-1}\Lambda_f = \underline{\exp}(g)$ $(g = R_\gamma(f))$ and

$$W_F(\sqrt{2}\,h')\underline{\exp}(g)$$

$$= \exp(-\frac{1}{4}||\sqrt{2}\,h'||^2 + \frac{\sqrt{-1}}{\sqrt{2}}\langle\sqrt{2}\,h',g\rangle)\underline{\exp}(\frac{\sqrt{-1}}{\sqrt{2}}\sqrt{2}\,h' + g) \quad \text{(cf. 9.4)}$$

$$= \exp(-\frac{1}{2}||h'||^2 + \sqrt{-1}\langle h',g\rangle)\underline{\exp}(\sqrt{-1}\,h' + g).$$

Apply $\hat{T}$:

$$\hat{T}\underline{\exp}(\sqrt{-1}\,h' + g)\Big|_x$$

$$= \exp(\sqrt{-1}\,\hat{h}'(x) + \hat{g}(x) - \frac{1}{2}(\sqrt{-1}\,h' + g)^2)$$

$$= e^{\sqrt{-1}\,\hat{h}'(x)} \exp(\hat{g}(x) - \frac{1}{2}\,||g||^2)$$

$$\cdot \exp(\frac{1}{2}\,||h'||^2 - \sqrt{-1}\,\langle h',g\rangle)$$

$$= e^{\sqrt{-1}\,\hat{h}'(x)} \Lambda_{\hat{g}}(x) \exp(\frac{1}{2}\,||h'||^2 - \sqrt{-1}\,\langle h',g\rangle).$$

Now cancel the exponentials to finish the verification.

The <u>canonical state</u> is, by definition, the function

$$\begin{cases} H(\gamma) \oplus H(\gamma) \rightarrow \underline{C} \\ (h,h') \rightarrow \langle 1, W_{re}(h \oplus h')1\rangle_{L^2(\gamma)} \end{cases}.$$

To calculate it, write

$$\langle 1, W_{re}(h \oplus h')1\rangle_{L^2(\gamma)}$$

$$= \langle 1, \exp(\frac{\sqrt{-1}}{2}\,\langle h,h'\rangle_{H(\gamma)})U(-h)V(h')1\rangle_{L^2(\gamma)}$$

$$= \exp(\frac{\sqrt{-1}}{2}\,\langle h,h'\rangle_{H(\gamma)})\langle 1,U(-h)V(h')1\rangle_{L^2(\gamma)}$$

$$= \exp(\frac{\sqrt{-1}}{2}\,\langle h,h'\rangle_{H(\gamma)})\langle 1,\hat{T}W_F(-\frac{\sqrt{-1}}{\sqrt{2}}\,h)\hat{T}^{-1}$$

$$\cdot \hat{T}W_F(\sqrt{2}\,h')\hat{T}^{-1}\,1\rangle_{L^2(\gamma)} \qquad \text{(cf. 41.3 \& 41.4)}$$

$$= \exp(\frac{\sqrt{-1}}{2}\,\langle h,h'\rangle_{H(\gamma)})\langle \Omega,W_F(-\frac{\sqrt{-1}}{\sqrt{2}}\,h)$$

$$\cdot W_F(\sqrt{2}\,h')\Omega>_{BO(H(\gamma)_{\underline{C}})}$$

$$= \exp(\frac{\sqrt{-1}}{2} <h,h'>_{H(\gamma)})$$

$$\times <\Omega,\ \exp(-\ \frac{\sqrt{-1}}{2}\ \mathrm{Im}<\ -\ \frac{\sqrt{-1}}{\sqrt{2}}\ h,\sqrt{2}\ h'>_{H(\gamma)_{\underline{C}}})$$

$$\cdot W_F(-\ \frac{\sqrt{-1}}{\sqrt{2}}\ h\ +\ \sqrt{2}\ h')\Omega>_{BO(H(\gamma)_{\underline{C}})}$$

$$= <\Omega,\ W_F(-\ \frac{\sqrt{-1}}{\sqrt{2}}\ (h\ +\ \sqrt{2}\ h')\Omega>_{BO(H(\gamma)_{\underline{C}})}$$

$$= \exp(\ -\ \frac{1}{4}\ ||-\ \frac{\sqrt{-1}}{\sqrt{2}}\ h\ +\ \sqrt{2}\ h'||^2_{H(\gamma)_{\underline{C}}})\qquad (\text{cf. } 9.5)$$

$$= \exp(\ -\ \frac{1}{4}\ (||\ \frac{1}{\sqrt{2}}\ h||^2_{H(\gamma)}\ +\ ||\sqrt{2}\ h'||^2_{H(\gamma)}))$$

$$= \exp(\ -\ \frac{1}{8}\ ||h||^2_{H(\gamma)}\ -\ \frac{1}{2}\ ||h'||^2_{H(\gamma)}).$$

In summary:

$$<1,W_{re}(h \oplus h')1>_{L^2(\gamma)}$$

$$= \exp(\ -\ \frac{1}{8}\ ||h||^2_{H(\gamma)}\ -\ \frac{1}{2}\ ||h'||^2_{H(\gamma)}).$$

[Note: This result leads to a simple proof of the continuity of W_{re}. Thus, from the explicit formula, it is clear that

$$<1, W_{re}(h \oplus h')1>$$

is a continuous function of (h,h'). But then, thanks to the Weyl relations, for fixed (h_1, h_1'), (h_2, h_2'),

$$<W_{re}(h_1 \oplus h_1')1, W_{re}(h \oplus h')W_{re}(h_2 \oplus h_2')1>$$

is a continuous function of (h,h'). Therefore W_{re} is continuous, 1 being a cyclic vector for W_{re}.]

41.5 <u>EXAMPLE</u> To run a reality check, take $X = \underline{R}$ and let

$$d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Consider

$$W(a,b)1(x) = \exp(\sqrt{-1}(xb - ab/2))\exp(\tfrac{1}{2}(xa - a^2/2)).$$

Then

$$<1, W(a,b)1>_{L^2(\gamma)}$$

$$= \exp(-\frac{\sqrt{-1}}{2} ab)\exp(-a^2/4)$$

$$\cdot \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \exp(\frac{ax}{2} + \sqrt{-1}\, bx)\ e^{-x^2/2} dx.$$

But $\forall z \in \underline{C}$,

$$\frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \exp(zx) e^{-x^2/2} dx = \exp(z^2/2) \quad \text{(cf. 24.6)}.$$

Therefore

$$\langle 1,W(a,b)1\rangle_{L^2(\gamma)}$$

$$= \exp(-\frac{\sqrt{-1}}{2}ab)\exp(-a^2/4)$$

$$\cdot \exp(\frac{1}{2}(\frac{a}{2}+\sqrt{-1}\,b)^2)$$

$$= \exp(-\frac{1}{8}a^2-\frac{1}{2}b^2).$$

Change of Variable:

$h \to \sqrt{2}\,h$: We have

$$\hat{T}W_F(-\sqrt{-1}\,h)\hat{T}^{-1}\psi\Big|_x$$

$$= [\exp(\sqrt{2}\,\hat{h}(x) - ||h||^2_{H(\gamma)})]^{1/2}\,\psi(x-\sqrt{2}\,h).$$

$h' \to \dfrac{h'}{\sqrt{2}}$: We have

$$\hat{T}W_F(h')\hat{T}^{-1}\psi\Big|_x = e^{\sqrt{-1}\,\frac{\hat{h}'(x)}{\sqrt{2}}}\,\psi(x).$$

[Note: The transformation

$$h+\sqrt{-1}\,h' \to \sqrt{2}\,h+\sqrt{-1}\,\frac{h'}{\sqrt{2}}$$

is a symplectic automorphism of $H(\gamma)_{\underline{C}}$ (per $\sigma = \mathrm{Im}\langle,\rangle_{H(\gamma)_{\underline{C}}}$).]

In view of these relations, modify the definition of the real wave representation:

$$W_{mod}(h \oplus h')\psi(x)$$

$$= \exp(\sqrt{-1}\ (\frac{\hat{h}'(x)}{\sqrt{2}} - \langle h,h' \rangle_{H(\gamma)}/2))$$

$$\cdot\ [\exp(\sqrt{2}\ \hat{h}(x) - ||h||^2_{H(\gamma)})]^{1/2}\ \psi(x - \sqrt{2}\ h).$$

Now go back to the Fock system:

$$W_F(h + \sqrt{-1}\ h').$$

Let $U:H(\gamma)_{\underline{C}} \rightarrow H(\gamma)_{\underline{C}}$ be multiplication by $-\sqrt{-1}$ -- then

$$\Gamma(U)W_F(h + \sqrt{-1}\ h')\Gamma(U)^{-1}$$

$$= W_F(-\sqrt{-1}\ (h + \sqrt{-1}\ h')) \qquad \text{(cf. 9.7)}$$

$$= W_F(-\sqrt{-1}\ h + h').$$

Therefore the Fock system is unitarily equivalent to the Weyl system

$$h + \sqrt{-1}\ h' \rightarrow W_F(-\sqrt{-1}\ h + h').$$

And, by the above,

$$\hat{T}W_F(-\sqrt{-1}\ h + h')\hat{T}^{-1}$$

$$= W_{mod}(h \oplus h').$$

Consequently, the Fock system is unitarily equivalent to the modified real wave representation.

41.6 <u>REMARK</u> Take $X = \underline{R}^n$, $\gamma = \gamma_n$ -- then the modified and unmodified real wave representations are unitarily equivalent. To see this, consider the map

$$S:\underline{C}^n \to \underline{C}^n$$

defined by

$$S(h + \sqrt{-1}\, h') = -\frac{\sqrt{-1}}{\sqrt{2}}\, h + \sqrt{2}\, h'.$$

Then S is a symplectic automorphism of $\underline{C}^n$ (viewed as a real vector space), hence by Shale's theorem is implementable (cf. 12.19): $\exists\ \Gamma_S \in \mathcal{U}(BO(\underline{C}^n))$ such that

$$\Gamma_S W_F(h + \sqrt{-1}\, h')\Gamma_S^{-1} = W_{F,S}(h + \sqrt{-1}\, h').$$

Therefore the Fock system is unitarily equivalent to the Weyl system

$$h + \sqrt{-1}\, h' \to W_F(-\frac{\sqrt{-1}}{\sqrt{2}}\, h + \sqrt{2}\, h'),$$

the latter being unitarily equivalent to the real wave representation.

[Note: In the finite dimensional case, the Hilbert-Schmidt condition on S is automatic. This, of course, is false in the infinite dimensional case: S is a symplectic automorphism of $H(\gamma)_{\underline{C}}$ but S is not implementable if $\dim H(\gamma)_{\underline{C}} = \infty$.]

§42. THE SCHRÖDINGER SYSTEM

Let (X,Y,ι) be an abstract Wiener space. Consider the real wave representation attached to p_t. Officially, this is a Weyl system over $H(p_t) \oplus H(p_t)$ which is realized on $L^2(Y,p_t)$:

$$W_{re}(x \oplus x')\psi(y)$$

$$= \exp(\sqrt{-1}\ (\hat{x}'(y) - \langle x,x'\rangle_t/2))$$

$$\cdot\ [\exp(\hat{x}(y) - \frac{1}{2}\ ||x||_t^2)]^{1/2}\ \psi(y-x).$$

Here (cf. §40)

$$\begin{cases} \hat{x} = \Phi_{x/t} = \dfrac{1}{t}\,\Phi_x \\ \hat{x}' = \Phi_{x'/t} = \dfrac{1}{t}\,\Phi_{x'} \end{cases}$$

and

$$\langle x,x'\rangle_t = \langle x,x'\rangle_X/t.$$

For later applications, it will be best to partially eliminate the parameter t. To this end, put

$$W_t(x \oplus x')\psi(y)$$

$$= \exp(\sqrt{-1}\ (\Phi_{x'}(y) - \langle x,x'\rangle_X/2))$$

$$\cdot\ [\exp(\Phi_{x/t}(y) - \frac{1}{2}\ ||x||_t^2)]^{1/2}\ \psi(y-x).$$

Then W_t is a Weyl system over $X \oplus X$ which is realized on $L^2(Y,p_t)$.

<u>N.B.</u> We have

$$W_t(x \oplus x') = \exp\left(\frac{\sqrt{-1}}{2} \langle x,x'\rangle_X\right) U(-x)V(x'),$$

where

$$\begin{cases} U(-x)\psi(y) = [\exp(\frac{1}{2t}(2\Phi_x - ||x||_X^2))]^{1/2}\,\psi(y-x) \\ V(x')\psi(y) = e^{\sqrt{-1}\,\Phi_{x'}(y)}\,\psi(y). \end{cases}$$

[Note: It is clear that the pair (U,V) satisfies the canonical commutation relations per $\langle,\rangle_X$.]

42.1 <u>REMARK</u> The W_t are irreducible. In addition, if $t' \neq t''$, then $W_{t'}$ is not unitarily equivalent to $W_{t''}$. To see this, let L' and L'' be the underlying LSPs:

$$\begin{cases} L'_x = \Phi_x \text{ per } L^2(Y,p_{t'}) \\ L''_x = \Phi_x \text{ per } L^2(Y,p_{t''}). \end{cases}$$

If $W_{t'}$ and $W_{t''}$ were unitarily equivalent, then L' and L'' would be weakly equivalent, hence $p_{t'}$ and $p_{t''}$ would be mutually absolutely continuous, a contradiction (cf. 40.3).

Let $\iota_t : X \to H(p_t)$ be the isometric isomorphism which sends x to $\sqrt{t}\,x$:

$$\iota_t x = \sqrt{t}\,x \quad (x \in X).$$

Therefore

$$X \xrightarrow{\iota_t} H(p_t) \xrightarrow{\hat{}_t} Y^*_{p_t}$$

=>

$$\begin{array}{ccccc} BO(X) & \xrightarrow{\Gamma(\iota_t)} & H(p_t) & \xrightarrow{\Gamma(\hat{}_t)} & BO(Y^*_t) \\ & & & & \downarrow T \\ & & & & L^2(Y,p_t). \end{array}$$

Passing to complexifications, put

$$T_t = T \circ \Gamma(\hat{}_t) \circ \Gamma(\iota_t)$$

and let

$$W_F : X_{\underline{C}} \to U(BO(X_{\underline{C}}))$$

be the Fock system (cf. 10.3).

Then we have

$$T_t W_F\left(- \frac{\sqrt{-1}}{\sqrt{2t}}\, x\right) T_t^{-1}\, \psi \Big|_Y$$

$$= \hat{T} W_F\left(\sqrt{t}\left(- \frac{\sqrt{-1}}{\sqrt{2t}}\right) x\right) \hat{T}^{-1}\, \psi \Big|_Y$$

$$= \hat{T} W_F\left(- \frac{\sqrt{-1}}{\sqrt{2}}\, x\right) \hat{T}^{-1}\, \psi \Big|_Y$$

$$= [\exp(\hat{x}(y) - \tfrac{1}{2}\, ||x||_t^2)]^{1/2}\, \psi(y-x) \qquad \text{(cf. 41.3)}$$

$$= [\exp(\Phi_{x/t}(y) - \frac{1}{2}||x||_t^2)]^{1/2}\ \psi(y-x)$$

$$= [\exp(\frac{1}{2t}\ (2\Phi_x - ||x||_X^2))]^{1/2}\ \psi(y-x)$$

and

$$T_t W_F(\sqrt{2t}\ x')T_t^{-1}\ \psi\Big|_y$$

$$= \hat{T}W_F(\sqrt{t}\ (\sqrt{2t})x')\hat{T}^{-1}\ \psi\Big|_y$$

$$= \hat{T}W_F(\sqrt{2}\ tx')\hat{T}^{-1}\ \psi\Big|_y$$

$$= e^{\sqrt{-1}\ t\hat{x}'(y)}\ \psi(y) \qquad \text{(cf. 41.4)}$$

$$= e^{\sqrt{-1}\ t\Phi_{x'/t}(y)}\ \psi(y)$$

$$= e^{\sqrt{-1}\ \Phi_{x'}(y)}\ \psi(y).$$

The canonical state at time t is, by definition, the function

$$\begin{cases} X \oplus X \to \underline{C} \\ (x,x') \to \langle 1, W_t(x \oplus x')1\rangle_{L^2(p_t)} \end{cases}.$$

To calculate it, write

$$\langle 1, W_t(x \oplus x')1\rangle_{L^2(p_t)}$$

$$= \langle 1, \exp(\frac{\sqrt{-1}}{2} \langle x,x'\rangle_X) U(-x) V(x') 1\rangle_{L^2(p_t)}$$

$$= \exp(\frac{\sqrt{-1}}{2} \langle x,x'\rangle_X) \langle 1, U(-x) V(x') 1\rangle_{L^2(p_t)}$$

$$= \exp(\frac{\sqrt{-1}}{2} \langle x,x'\rangle_X) \langle 1, T_t W_F(- \frac{\sqrt{-1}}{\sqrt{2t}} x) T_t^{-1}$$

$$\cdot T_t W_F(\sqrt{2t}\, x') T_t^{-1}\rangle_{L^2(p_t)}$$

$$= \exp(\frac{\sqrt{-1}}{2} \langle x,x'\rangle_X) \langle \Omega, W_F(- \frac{\sqrt{-1}}{\sqrt{2t}} x)$$

$$\cdot W_F(\sqrt{2t}\, x') \Omega\rangle_{BO(X_{\underline{C}})}$$

$$= \exp(\frac{\sqrt{-1}}{2} \langle x,x'\rangle_X)$$

$$\times \langle \Omega, \exp(- \frac{\sqrt{-1}}{2} \operatorname{Im} \langle - \frac{\sqrt{-1}}{\sqrt{2t}} x,\ \sqrt{2t}\, x'\rangle)$$

$$\cdot W_F(- \frac{\sqrt{-1}}{\sqrt{2t}} x + \sqrt{2t}\, x') \Omega\rangle_{BO(X_{\underline{C}})}$$

$$= \langle \Omega, W_F(- \frac{\sqrt{-1}}{\sqrt{2t}} x + \sqrt{2t}\, x') \Omega\rangle_{BO(X_{\underline{C}})}$$

$$= \exp(- \frac{1}{4} || - \frac{\sqrt{-1}}{\sqrt{2t}} x + \sqrt{2t}\, x' ||^2_{X_{\underline{C}}}) \qquad \text{(cf. 9.5)}$$

$$= \exp(-\frac{1}{4}(\frac{||x||_X^2}{2t} + 2t||x'||_X^2))$$

$$= \exp(-\frac{||x||_X^2}{8t} - \frac{t}{2}||x'||_X^2).$$

In particular: The canonical state at time $\frac{1}{2}$ is the function

$$\exp(-\frac{1}{4}(||x||_X^2 + ||x'||_X^2)).$$

We shall now compare

$$W_{mod} \text{ per } L^2(Y,p_1)$$

with

$$W_{1/2} \text{ per } L^2(Y,p_{1/2}).$$

These are Weyl systems over $X_{\underline{C}}$ and we claim that they are unitarily equivalent.

42.2 <u>REMARK</u> Recall that the Fock system over $X_{\underline{C}}$ is unitarily equivalent to the modified real wave representation realized on $L^2(Y,p_1)$. Granted the claim, it thus follows that the Fock system over $X_{\underline{C}}$ is unitarily equivalent to the Weyl system $W_{1/2}$.

By definition,

$$W_{mod}(x \oplus x')\psi\Big|_y$$

$$= \exp(\sqrt{-1}\,(\frac{\Phi_{x'}(y)}{\sqrt{2}} - \langle x,x'\rangle_X/2))$$

$$\cdot\ [\exp(\sqrt{2}\ \Phi_x(y) - ||x||_X^2)]^{1/2}\ \psi(y-\sqrt{2}\ x).$$

Let

$$D{:}L^2(Y,p_1) \to L^2(Y,p_{1/2})$$

be the isometric isomorphism defined by the rule

$$(D\psi)(y) = \psi(\sqrt{2}\ y) \qquad \text{(cf. 40.1)}.$$

Then $DW_{mod}D^{-1}$ is a Weyl system over $X_{\underline{C}}$ which is realized on $L^2(Y,p_{1/2})$:

$$DW_{mod}(x \oplus x')D^{-1}\ \psi\Big|_y$$

$$= W_{mod}(x \oplus x')D^{-1}\ \psi\Big|_{\sqrt{2}\ y}$$

$$= \exp(\sqrt{-1}\ (\Phi_{x'}(y) - \langle x,x'\rangle_X/2))$$

$$\cdot\ [\exp(2\Phi_x(y) - ||x||_X^2)]^{1/2}\ D^{-1}\ \psi(\sqrt{2}\ y - \sqrt{2}\ x)$$

$$= \exp(\sqrt{-1}\ (\Phi_{x'}(y) - \langle x,x'\rangle_X/2))$$

$$\cdot\ [\exp(2\Phi_x(y) - ||x||_X^2)]^{1/2}\ \psi(y-x).$$

On the other hand, $W_{1/2}$ is a Weyl system over $X_{\underline{C}}$ which is also realized on $L^2(Y,p_{1/2})$:

$$W_{1/2}(x \oplus x')\psi(y)$$

$$= \exp(\sqrt{-1}\ (\Phi_{x'}(y) - \langle x,x'\rangle_X/2))$$

$$\cdot\ [\exp(\Phi_{x/(1/2)}(y) - \frac{1}{2}\ ||x||^2_{1/2})]^{1/2}\ \psi(y-x)$$

$$= \exp(\sqrt{-1}\ (\Phi_{x'}(y) - \langle x,x'\rangle_X/2))$$

$$\cdot\ [\exp(2\Phi_x(y) - ||x||^2_X)]^{1/2}\ \psi(y-x).$$

Therefore

$$DW_{mod}D^{-1} = W_{1/2}.$$

At this point, it will be convenient to revert back to the traditional notation of the bosonic theory.

So let H be an infinite dimensional separable complex Hilbert space -- then a real part of H is a set H_0 of the form

$$\{f \in H : Cf = f\},$$

where C is a conjugation of H.

Let H_0 be a real part of H -- then $\forall\ f,g \in H_0$,

$$\langle f,g\rangle \in \underline{R}.$$

Since H_0 is necessarily closed, it follows that H_0 is an infinite dimensional separable real Hilbert space. Moreover, the complexification of H_0 is isomorphic as a complex Hilbert space to H.

Let C_1 and C_2 be conjugations of H and let H_1 and H_2 be the corresponding real parts of H. Consider abstract Wiener spaces

$$\begin{cases} (H_1, Y_1, \iota_1) \\ (H_2, Y_2, \iota_2). \end{cases}$$

Then this data gives rise to two Weyl systems over H:

$$\begin{cases} W^1_{1/2} \text{ per } L^2(Y_1, p_{1/2}) \\ W^2_{1/2} \text{ per } L^2(Y_2, p_{1/2}). \end{cases}$$

42.3 LEMMA $W^1_{1/2}$ and $W^2_{1/2}$ are unitarily equivalent.

PROOF Both are unitarily equivalent to the Fock system over H.

The Schrödinger system over H is $W_{1/2}$ taken over any real part of H.

[Note: The lemma implies that the Schrödinger system over H is unique up to unitary equivalence.]

42.4 REMARK When these considerations are specialized to the finite dimensional case, the resulting Schrödinger system is not the Schrödinger system of 10.4 (but the two are unitarily equivalent).

§43. *THE WIENER TRANSFORM*

Let $U:\underline{C} \to \underline{C}$ be multiplication by $\sqrt{-1}$ -- then U extends to a unitary operator $\Gamma(U)$ on $BO(\underline{C})$ which, in the n^{th} slot, is multiplication by $(\sqrt{-1})^n$, thus

$$\Gamma(U)\underline{\exp}(z) = \underline{\exp}(\sqrt{-1}\, z).$$

Put $W = T\Gamma(U)T^{-1}$ -- then

$$W:L^2(\underline{R},\gamma) \to L^2(\underline{R},\gamma)$$

is a unitary operator, the <u>Wiener transform</u>.

[Note: Here, as usual (cf. 6.10),

$$T:BO(\underline{C}) \to L^2(\underline{R},\gamma)$$

is the isometric isomorphism characterized by the relation

$$(T\,\underline{\exp}(z))(x) = e^{zx - \frac{1}{2}z^2}.]$$

43.1 <u>EXAMPLE</u> We have

$$W(\frac{H_n}{\sqrt{n!}}) = T\Gamma(U)T^{-1}(\frac{H_n}{\sqrt{n!}})$$

$$= T\Gamma(U)(1^{\otimes n})$$

$$= T((\sqrt{-1})^n\, 1^{\otimes n})$$

$$= (\sqrt{-1})^n\, T(1^{\otimes n})$$

$$= (\sqrt{-1})^n \frac{H_n}{\sqrt{n!}}$$

=>

$$W(H_n) = (\sqrt{-1})^n H_n.$$

43.2 <u>LEMMA</u> $\forall\ z \in \underline{C}$,

$$W(e^{zx}) = e^{\sqrt{-1}\ zx + z^2}.$$

<u>PROOF</u> Write

$$T\Gamma(U)T^{-1}(e^{zx})$$

$$= T\Gamma(U)T^{-1}(e^{\frac{1}{2} z^2} \underline{\exp}(z)(x))$$

$$= e^{\frac{1}{2} z^2} T \underline{\exp}(\sqrt{-1}\ z)$$

$$= e^{\frac{1}{2} z^2} e^{\sqrt{-1}\ zx} e^{-\frac{1}{2}(\sqrt{-1}\ z)^2}$$

$$= e^{\sqrt{-1}\ zx + z^2}.$$

43.3 <u>EXAMPLE</u> We have

$$W(x^n) = (\sqrt{-1}\ \sqrt{2})^n H_n(\frac{x}{\sqrt{2}}).$$

[In fact,

$$\sum_{n=0}^{\infty} z^n 2^{-n/2} \frac{W(x^n)}{n!}$$

$$= W\left(\sum_{n=0}^{\infty} z^n 2^{-n/2} \frac{x^n}{n!}\right)$$

$$= W\left(e^{\frac{zx}{\sqrt{2}}}\right)$$

$$= \exp\left(\sqrt{-1}\,\frac{zx}{\sqrt{2}} + \frac{z^2}{2}\right)$$

$$= \exp\left((\sqrt{-1}\,z)\,\frac{x}{\sqrt{2}} - \frac{1}{2}(\sqrt{-1}\,z)^2\right)$$

$$= \sum_{n=0}^{\infty} \frac{(\sqrt{-1}\,z)^n}{n!} H_n\left(\frac{x}{\sqrt{2}}\right)$$

=>

$$2^{-n/2}\, W(x^n) = (\sqrt{-1})^n H_n\left(\frac{x}{\sqrt{2}}\right)$$

=>

$$W(x^n) = (\sqrt{-1}\,\sqrt{2})^n H_n\left(\frac{x}{\sqrt{2}}\right).]$$

43.4 LEMMA Let $f = x^n$ — then

$$Wf\Big|_x = \int_{\underline{R}} f(\sqrt{-1}\,x + \sqrt{2}\,y)\,d\gamma(y).$$

<u>PROOF</u> From the above,

$$Wf\Big|_x = (\sqrt{-1}\ \sqrt{2})^n\ H_n\left(\frac{x}{\sqrt{2}}\right)$$

or still,

$$Wf\Big|_x = (\sqrt{-1}\ \sqrt{2})^n \int_{\underline{R}} \left(\frac{x}{\sqrt{2}} - \sqrt{-1}\ y\right)^n d\gamma(y).$$

But

$$\left(\frac{x}{\sqrt{2}} - \sqrt{-1}\ y\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{\sqrt{2}}\right)^k (-\sqrt{-1}\ y)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{\sqrt{2}}\right)^k (-\sqrt{-1})^{n-k}\ y^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{\sqrt{2}}\right)^k \left(\frac{1}{\sqrt{-1}}\right)^{n-k} y^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{\sqrt{2}}\right)^k (\sqrt{-1})^{k-n}\ y^{n-k}.$$

On the other hand,

$$(\sqrt{-1}\ x + \sqrt{2}\ y)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\sqrt{-1})^k x^k (\sqrt{2})^{n-k}\ y^{n-k}$$

$$= (\sqrt{2})^n \sum_{k=0}^{n} \binom{n}{k} (\sqrt{-1})^k \left(\frac{x}{\sqrt{2}}\right)^k y^{n-k}$$

$$= (\sqrt{-1}\ \sqrt{2})^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{\sqrt{2}}\right)^k \frac{(\sqrt{-1})^k}{(\sqrt{-1})^n}\ y^{n-k}$$

$$= (\sqrt{-1}\ \sqrt{2})^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{\sqrt{2}}\right)^k (\sqrt{-1})^{k-n}\ y^{n-k}.$$

43.5 REMARK The Ornstein-Uhlenbeck semigroup is defined on $L^2(\underline{R},\gamma)$ by

$$T_t f(x) = \int_{\underline{R}} f(e^{-t} x + \sqrt{1-e^{-2t}}\ y) d\gamma(y).$$

Here, of course, t is positive. However, if t is allowed to be complex, say $t = -\sqrt{-1}\ \frac{\pi}{2}$, then formally

$$T_{-\sqrt{-1}\ \frac{\pi}{2}} f(x) = \int_{\underline{R}} f(\sqrt{-1}\ x + \sqrt{2}\ y) d\gamma(y),$$

which is precisely the Wiener transform of f at x.

43.6 LEMMA $\forall\ f \in L^2(\underline{R},\gamma)$,

$$W^{-1} f(x) = Wf(-x).$$

PROOF It suffices to show that

$$W^2 f(x) = f(-x)$$

on a total set of functions, e.g., the exponentials $x \to e^{zx}$ $(z \in \underline{C})$. But

$$W(e^{zx}) = e^{\sqrt{-1}\ zx + z^2} \quad \text{(cf. 43.2)}$$

=>

$$W^2(e^{zx}) = e^{z^2} e^{(\sqrt{-1}\ z)^2} e^{-zx} = e^{-zx}.$$

Define

$$T_G : L^2(\underline{R},\gamma) \to L^2(\underline{R})$$

by

$$T_G f = f \cdot G,$$

where

$$G(x) = \frac{1}{(2\pi)^{1/4}} \exp\left(- \frac{x^2}{4}\right).$$

Define

$$U_F : L^2(\underline{R}) \to L^2(\underline{R})$$

by

$$U_F f = U_{1/2} \hat{f},$$

where

$$U_r \psi(x) = \sqrt{r}\ \psi(rx) \qquad (r > 0)$$

and

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} e^{\sqrt{-1}\ xy} f(y)\,dy.$$

43.7 <u>LEMMA</u> We have

$$W = T_G^{-1} U_F T_G.$$

PROOF If

$$\Lambda_z(x) = e^{zx - \frac{1}{2} z^2},$$

then

$$W\Lambda_z \Big|_x = e^{\sqrt{-1}\, zx + \frac{1}{2} z^2} \qquad \text{(cf. 43.2)}.$$

With this in mind, consider

$$T_G^{-1} U_F T_G \Lambda_z \Big|_x$$

or still,

$$\frac{1}{(2\pi)^{1/4}} e^{-\frac{1}{2} z^2} T_G^{-1} U_F[y \to \exp(-\frac{y^2}{4}) e^{zy}] \Big|_x .$$

But

$$\frac{1}{\sqrt{2\pi}} \int_{\underline{R}} e^{\sqrt{-1}\, xy} \exp(-\frac{y^2}{4}) e^{zy} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} e^{(\sqrt{-1}\, x+z)y} \exp(-\frac{y^2}{4}) dy$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2\sqrt{\pi} \exp((\sqrt{-1}\, x+z)^2)$$

$$= \sqrt{2} \exp((\sqrt{-1}\, x+z)^2).$$

Now apply $U_{1/2}$ -- then the resulting function of x is

$$\exp((\sqrt{-1}\, \frac{x}{2} + z)^2)$$

$$= \exp\left(- \frac{x^2}{4} + \sqrt{-1}\, zx + z^2\right).$$

We are thus left with

$$\frac{1}{(2\pi)^{1/4}} e^{-\frac{1}{2} z^2} \cdot (2\pi)^{1/4} \exp\left(\frac{x^2}{4}\right)$$

$$\cdot \exp\left(- \frac{x^2}{4} + \sqrt{-1}\, zx + z^2\right)$$

$$= e^{\sqrt{-1}\, zx + \frac{1}{2} z^2}.$$

Suppose now that X is a separable LF-space. Let γ be a centered gaussian measure on X -- then in view of what has been said in §28 (and passing from $\underline{R}$ to $\underline{C}$), there is an isometric isomorphism

$$T:BO((X_\gamma^*)_{\underline{C}}) \to L^2(X,\gamma)$$

characterized by the relation

$$T\, \underline{\exp}(f + \sqrt{-1}\, f') = \Lambda_{f + \sqrt{-1}\, f'},$$

where

$$\Lambda_{f + \sqrt{-1}\, f'}(x) = \exp(f(x) + \sqrt{-1}\, f'(x) - \frac{1}{2} (f + \sqrt{-1}\, f')^2).$$

[Note: The symbol

$$(f + \sqrt{-1}\, f')^2$$

stands for the combination

$$<f - \sqrt{-1}\, f',\ f + \sqrt{-1}\, f'>.]$$

Put $W = T\Gamma(U)T^{-1}$ — then

$$W: L^2(X,\gamma) \to L^2(X,\gamma)$$

is a unitary operator, the Wiener transform.

[Note: As at the beginning, $\Gamma(U)$ is the unitary operator on $BO((X^*_\gamma)_{\underline{C}})$ which, in the n^{th} slot, is multiplication by $(\sqrt{-1})^n$.]

Since W is unitary, it follows that

$$\int_X |W\psi|^2\, d\gamma = \int_X |\psi|^2\, d\gamma \quad (\psi \in L^2(X,\gamma)).$$

I.e.: The Plancherel formula is automatic.

There is also a version of the Parseval formula, viz.: $\forall\ \psi,\phi \in L^2(X,\gamma)$,

$$\int_X (W\psi)\phi\, d\gamma = \int_X \psi(W\phi)\, d\gamma.$$

Proof: It suffices to check this relation on functions of the form

$$\begin{cases} \psi = \Lambda_{f + \sqrt{-1}\, f'} \\ \phi = \Lambda_{g + \sqrt{-1}\, g'} \end{cases}.$$

LHS: We have

$$\int_X (W\psi)\phi\, d\gamma = \int_X \overline{\overline{(W\psi)}}\, \phi\, d\gamma$$

$$= \int_X \overline{\Lambda_{-\sqrt{-1}\, f-f'}}\ \Lambda_{g + \sqrt{-1}\, g'}\ d\gamma$$

$$= \exp(< - \sqrt{-1}\, f-f',\ g + \sqrt{-1}\, g'>)$$

$$= \exp(\sqrt{-1}\, <f,g> - <f,g'> - <f',g> - \sqrt{-1}\, <f',g'>).$$

<u>RHS</u>: We have

$$\int_X \psi(W\phi)\,d\gamma = \int_X \overline{\overline{\psi}}(W\phi)\,d\gamma$$

$$= \int_X \overline{\Lambda_{f - \sqrt{-1}\, f'}}\ \Lambda_{\sqrt{-1}\, g-g'}\ d\gamma$$

$$= \exp(<f - \sqrt{-1}\, f',\ \sqrt{-1}\, g-g'>)$$

$$= \exp(\sqrt{-1}\, <f,g> - <f,g'> - <f',g> - \sqrt{-1}\, <f',g'>).$$

Therefore

$$\text{LHS} = \text{RHS},$$

from which the result.

43.8 <u>REMARK</u> Suppose that $\psi = \Lambda_{f + \sqrt{-1}\, f'}$ -- then

$$W\psi = \sum_{n=0}^{\infty} (\sqrt{-1})^n\, I_n(\psi).$$

On the other hand, $\forall\ t > 0$,

$$T_t\psi = \sum_{n=0}^{\infty} e^{-nt}\, I_n(\psi).$$

So, passing into the complex domain, and taking $t = -\sqrt{-1}\,\frac{\pi}{2}$, we conclude that

$$W\psi = T_{-\sqrt{-1}\,\frac{\pi}{2}}\,\psi.$$

A polynomial on X is, by definition, any (complex valued) polynomial in a finite number of linear functionals on X.

[Note: Any polynomial on X admits a unique extension to the complexification $X_{\underline{C}}$ of X.]

43.9 LEMMA Let p be a polynomial on X — then

$$Wp\Big|_x = \int_X p(\sqrt{-1}\,x + \sqrt{2}\,y)\,d\gamma(y).$$

43.10 EXAMPLE Take $X = \underline{R}$ and let

$$d\gamma_t(x) = \frac{1}{\sqrt{2\pi t}}\,e^{-x^2/2t}\,dx.$$

Then in the notation introduced at the end of §8,

$$(X^*_{\gamma_t})_{\underline{C}} = \underline{C}_t.$$

This said, let

$$W_t : L^2(\underline{R},\gamma_t) \rightarrow L^2(\underline{R},\gamma_t)$$

be the Wiener transform at time t. Since

$$T = U_t \circ T_t \circ \Gamma(\iota_t),$$

it follows that

$$W_t = T_t \Gamma_t(U) T_t^{-1}$$

$$= U_t^{-1} \, T\Gamma(\iota_t)^{-1} \, \Gamma_t(U)\Gamma(\iota_t)T^{-1}U_t.$$

Here $\Gamma_t(U)$ refers to $BO_t(\underline{C})$. But

$$\Gamma(\iota_t)^{-1} \, \Gamma_t(U)\Gamma(\iota_t) = \Gamma(U),$$

where $\Gamma(U)$ refers to $BO(\underline{C})$. Indeed,

$$\Gamma(\iota_t)^{-1} \, \Gamma_t(U)\Gamma(\iota_t)\underline{\exp}(z)$$

$$= \Gamma(\iota_t)^{-1} \, \Gamma_t(U)\underline{\exp}(\sqrt{t}\; z)$$

$$= \Gamma(\iota_t)^{-1} \, \underline{\exp}(\sqrt{t}\,\sqrt{-1}\; z)$$

$$= \underline{\exp}(\frac{1}{\sqrt{t}}\,\sqrt{t}\,\sqrt{-1}\; z)$$

$$= \underline{\exp}(\sqrt{-1}\; z) = \Gamma(U)\underline{\exp}(z).$$

Therefore

$$W_t = U_t^{-1} \, T\Gamma(U)T^{-1}U_t$$

$$= U_t^{-1} \, WU_t.$$

Let p be a polynomial -- then the claim is that

$$W_t p\Big|_x = \int_{\underline{R}} p(\sqrt{-1}\, x + \sqrt{2}\, y)\, d\gamma_t(y)$$

or still,

$$W_t p\Big|_x = \int_{\underline{R}} p(\sqrt{-1}\, x + \sqrt{2}\, \sqrt{t}\, y)\, d\gamma(y).$$

To see this, put

$$p_t = U_t p \quad (\Rightarrow p_t(x) = p(\sqrt{t}\, x)).$$

Then

$$W_t p\Big|_x = U_t^{-1}\, W p_t\Big|_x$$

$$= W p_t\left(\frac{x}{\sqrt{t}}\right)$$

$$= \int_{\underline{R}} p_t\left(\frac{\sqrt{-1}\, x}{\sqrt{t}} + \sqrt{2}\, y\right) d\gamma(y) \qquad \text{(cf. 43.4)}$$

$$= \int_{\underline{R}} p(\sqrt{-1}\, x + \sqrt{2}\, \sqrt{t}\, y)\, d\gamma(y),$$

as claimed.

[Note: W_t can also be represented as an integral transform:

$$W_t f\Big|_x = \frac{1}{\sqrt{4\pi t}} \int_{\underline{R}} e^{-(\sqrt{-1}\, x - y)^2/4t}\, f(y)\, dy$$

or still,

$$W_t f\Big|_x = e^{\frac{x^2}{4t}} \frac{1}{\sqrt{4\pi t}} \int_{\underline{R}} e^{\frac{\sqrt{-1}\, xy}{2t}} f(y)\ e^{-y^2/4t}\, dy.$$

Thus let $f = \Lambda_z$, where

$$\Lambda_z(x) = e^{zx - \frac{1}{2} z^2}.$$

Then

$$W(e^{zx}) = e^{\sqrt{-1}\, zx + z^2} \qquad \text{(cf. 43.2)}$$

=>

$$W(e^{\sqrt{t}\, zx}) = e^{\sqrt{-1}\, \sqrt{t}\, zx + tz^2}$$

=>

$$W_t \Lambda_z\Big|_x = U_t^{-1}\, W U_t \Lambda_z\Big|_x$$

$$= e^{-\frac{1}{2} z^2}\ U_t^{-1}[e^{\sqrt{-1}\, \sqrt{t}\, zx + tz^2}]$$

$$= e^{-\frac{1}{2} z^2}\ e^{\sqrt{-1}\, zx}\ e^{tz^2}$$

$$= e^{(t - \frac{1}{2}) z^2}\ e^{\sqrt{-1}\, zx}.$$

Turning to the integral, we have

$$e^{\frac{x^2}{4t}} \frac{1}{\sqrt{4\pi t}} \int_{\underline{R}} e^{\frac{\sqrt{-1}\, xy}{2t}} \Lambda_z(y)\ e^{-y^2/4t}\, dy$$

$$= e^{\frac{x^2}{4t}}\, e^{-\frac{1}{2} z^2}\ \frac{1}{\sqrt{4\pi t}} \int_{\underline{R}} \exp\left(\left(\frac{\sqrt{-1}\, x}{2t} + z\right) y\right)\ e^{-y^2/4t}\, dy$$

$$= e^{\frac{x^2}{4t}} e^{-\frac{1}{2}z^2} \exp(t(\frac{\sqrt{-1}\,x}{2t} + z)^2)$$

$$= e^{-\frac{1}{2}z^2} e^{\frac{x^2}{4t}} \exp(t(-\frac{x^2}{4t^2} + \frac{\sqrt{-1}\,xz}{t} + z^2))$$

$$= e^{(t-\frac{1}{2})z^2} e^{\sqrt{-1}\,zx}.]$$

In the finite dimensional case, the Wiener transform is the gaussian version of the Fourier transform. But in the infinite dimensional case, the Wiener transform "is" the Fourier transform. Here is some additional evidence for this conclusion.

Let (X,Y,ι) be an abstract Wiener space, where X is an infinite dimensional separable real Hilbert space. Suppose that $f \in L^2(Y,p_1)$ is X-differentiable and

$$\partial_x f \in L^2(Y,p_1) \;\; \forall\, x \in X.$$

Then it can be shown that $\forall\, x_1,x_2 \in X$,

$$[\int_Y |\hat{x}_1(y)f(y)|^2\, dp_1(y)] \cdot [\int_Y |\hat{x}_2(y)Wf(y)|^2\, dp_1(y)]$$

$$\geq \langle x_1,x_2\rangle^2\, ||f||^4_{L^2(p_1)}.$$

Therefore this result is an infinite dimensional version of the inequality:

$$[\int_{\underline{R}^n} ||x||^2\, |f(x)|^2 dx] \cdot [\int_{\underline{R}^n} ||x||^2\, |\hat{f}(x)|^2 dx] \geq \frac{n^2}{4}\, ||f||^4$$

valid for any $f \in L^2(\underline{R}^n)$.

§44. *BARGMANN SPACE*

This is the set $A^2(\underline{C}^n)$ of all holomorphic functions F on $\underline{C}^n$ such that

$$||F||^2 = \frac{1}{\pi^n} \int_{\underline{C}^n} |F(z)|^2 e^{-|z|^2} dz < \infty.$$

It is a complex Hilbert space with inner product

$$\langle F,G\rangle = \frac{1}{\pi^n} \int_{\underline{C}^n} \overline{F(z)}\, G(z) e^{-|z|^2} dz.$$

44.1 REMARK $A^2(\underline{C}^n)$ is a closed subspace of $L^2(\underline{C}^n,\mu)$, where

$$d\mu(z) = \frac{1}{\pi^n} e^{-|z|^2} dz.$$

[Note: To be completely precise, $L^2(\underline{C}^n,\mu) = L^2(\underline{R}^{2n},p_{1/2})$.]

44.2 LEMMA The functions

$$\zeta_I(z) = \frac{z^I}{\sqrt{I!}}$$

are an orthonormal basis for $A^2(\underline{C}^n)$.

[Note: Here I is an arbitrary multiindex.]

The series

$$\sum_I \langle\zeta_I(w),\zeta_I(z)\rangle$$

is absolutely convergent $\forall\ w,z \in \underline{C}^n$. Call its sum $K(w,z)$ -- then

$$K(w,z) = e^{\langle w,z \rangle}.$$

And, $\forall\, F \in A^2(\underline{C}^n)$,

$$\begin{cases} F(z) = \frac{1}{\pi^n} \int_{\underline{C}^n} K(w,z) F(w) e^{-|w|^2} \, dw \\ |F(z)|^2 \le K(z,z) \; ||F||^2. \end{cases}$$

[Note: Let

$$E_w(z) = e^{\langle w,z \rangle}.$$

Then

$$||E_w||^2 = e^{|w|^2}$$

and the set $\{E_w : w \in \underline{C}^n\}$ is total in $A^2(\underline{C}^n)$. Its elements are called <u>coherent states</u>.]

Put

$$B(z,x) = \exp(-\tfrac{1}{2}(z^2+x^2) + \sqrt{2}\, z \cdot x),$$

where

$$\begin{cases} z^2 = z_1^2 + \cdots + z_n^2 \\ x^2 = x_1^2 + \cdots + x_n^2 \end{cases}$$

and

$$z \cdot x = z_1 x_1 + \cdots + z_n x_n.$$

Then the Bargmann transform is the map

$$B:L^2(\underline{R}^n) \to A^2(\underline{C}^n)$$

defined by the rule

$$Bf(z) = \frac{1}{\pi^{n/4}} \int_{\underline{R}^n} B(z,x)f(x)dx.$$

44.3 LEMMA B is an isometric isomorphism.

[Note: B^{-1} is the map

$$A^2(\underline{C}^n) \to L^2(\underline{R}^n)$$

defined by the rule

$$B^{-1}F(x) = \frac{1}{\pi^n} \int_{\underline{C}^n} B(\overline{z},x)F(z)e^{-|z|^2} dz$$

provided the integral is absolutely convergent, e.g., if F is a polynomial. In general, one can compute $B^{-1}F$ by applying it to the partial sums of the Taylor series of F (which converge to F in the topology of $A^2(\underline{C}^n)$) and taking the limit of the resulting functions in the L^2 norm.]

44.4 REMARK We have

$$\left[\begin{array}{l} B\left(\dfrac{Q_j + \sqrt{-1}\, P_j}{\sqrt{2}}\right) B^{-1} = \dfrac{\partial}{\partial z_j} \\ \\ B\left(\dfrac{Q_j - \sqrt{-1}\, P_j}{2}\right) B^{-1} = z_j \end{array}\right. \qquad (j=1,\ldots,n).$$

[Take n = 1 and ignore all issues of domain.

• $\frac{d}{dz} Bf(z)$

$$= \frac{1}{\pi^{1/4}} \int_{\underline{R}} \frac{d}{dz} \exp(- \frac{1}{2} (z^2+x^2) + \sqrt{2}\, zx) f(x) dx$$

$$= \frac{1}{\pi^{1/4}} \int_{\underline{R}} (- z + \sqrt{2}\, x) \exp(- \frac{1}{2} (z^2+x^2) + \sqrt{2}\, zx) f(x) dx$$

$$= - zBf(z) + \sqrt{2}\, B(xf(x))(z)$$

=>

$$\frac{d}{dz} B = - zB + \sqrt{2}\, BQ.$$

• $B[\frac{df}{dx}](z)$

$$= \frac{1}{\pi^{1/4}} \int_{\underline{R}} \exp(- \frac{1}{2} (z^2+x^2) + \sqrt{2}\, zx) \frac{df}{dx} dx$$

$$= - \frac{1}{\pi^{1/4}} \int_{\underline{R}} (\sqrt{2}\, z - x) \exp(- \frac{1}{2} (z^2+x^2) + \sqrt{2}\, zx) f(x) dx$$

$$= - \sqrt{2}\, zBf(z) + B(xf(x))(z)$$

=>

$$B \frac{d}{dx} = - \sqrt{2}\, zB + BQ.$$

The rest is elementary algebra.]

If these considerations are transferred to $L^2(\underline{R}^n, p_1)$, then the Bargmann transform is the map

$$L^2(\underline{R}^n,p_1) \to A^2(\underline{C}^n)$$

which sends f to the function

$$z \to \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} e^{-(z-x)^2/2} f(x)dx$$

or still, to the function

$$z \to e^{-\frac{1}{2}z^2} \int_{\underline{R}^n} e^{z\cdot x} f(x)dp_1(x),$$

where now

$$\left[\begin{array}{l} \frac{\partial}{\partial x_j} \longleftrightarrow \frac{\partial}{\partial z_j} \\ \\ x_j - \frac{\partial}{\partial x_j} \longleftrightarrow z_j \end{array}\right. \qquad (j=1,\ldots,n).$$

[Note: To convince ourselves of this, take n = 1 -- then, in the notation of §8,

$$L^2(\underline{R},p_1) \xrightarrow{T_G} L^2(\underline{R}) \xrightarrow{U_{\sqrt{2}}} L^2(\underline{R}) \xrightarrow{B} A^2(\underline{C}),$$

the claim being that

$$BU_{\sqrt{2}} T_G f\Big|z$$

$$= e^{-\frac{1}{2}z^2} \int_{\underline{R}} e^{zx} f(x) dp_1(x).$$

First,

$$T_G f\Big|_x = \frac{1}{(2\pi)^{1/4}} \exp(-\frac{x^2}{4}) f(x).$$

Second,

$$U_{\sqrt{2}} T_G f\Big|_x = \frac{1}{(2\pi)^{1/4}} (\sqrt{2})^{1/2} \exp(-\frac{(\sqrt{2}\ x)^2}{4}) f(\sqrt{2}\ x)$$

$$= \frac{1}{\pi^{1/4}} \exp(-\frac{x^2}{2}) f(\sqrt{2}\ x).$$

Third,

$$BU_{\sqrt{2}} T_G f\Big|_z$$

$$= \frac{1}{\pi^{1/4}} \int_{\underline{R}} \exp(-\frac{1}{2}(z^2+x^2) + \sqrt{2}\ zx) \frac{1}{\pi^{1/4}} \exp(-\frac{x^2}{2}) f(\sqrt{2}\ x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \exp(-\frac{1}{2}(z^2 + \frac{u^2}{2}) + \sqrt{2}\ z \frac{u}{\sqrt{2}}) \exp(-\frac{u^2}{4}) f(u) du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \exp(-\frac{1}{2} z^2 + zu - \frac{1}{2} u^2) f(u) du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \exp(-\frac{1}{2}(z-u)^2) f(u) du.$$

I.e.:

$$BU_{\sqrt{2}} T_G f\Big|_z$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} e^{-(z-x)^2/2} f(x)dx.$$

Finally (cf. §8),

$$\left[\begin{array}{l} U_{\sqrt{2}} T_G(\frac{d}{dx}) T_G^{-1} U_{\sqrt{2}}^{-1} = U_{\sqrt{2}} (\frac{x}{2} + \frac{d}{dx}) U_{\sqrt{2}}^{-1} = \frac{1}{\sqrt{2}} (Q + \sqrt{-1}\ P) \\ \\ U_{\sqrt{2}} T_G(x - \frac{d}{dx}) T_G^{-1} U_{\sqrt{2}}^{-1} = U_{\sqrt{2}} (\frac{x}{2} - \frac{d}{dx}) U_{\sqrt{2}}^{-1} = \frac{1}{\sqrt{2}} (Q - \sqrt{-1}\ P).\end{array}\right]$$

44.5 <u>EXAMPLE</u> By definition,

$$\underline{H}_{k_1,\ldots,k_n}(x_1,\ldots,x_n)$$

$$= \frac{H_{k_1}(x_1)}{\sqrt{k_1!}} \cdots \frac{H_{k_n}(x_n)}{\sqrt{k_n!}}$$

$$= \frac{(x_1 - \frac{\partial}{\partial x_1})^{k_1}}{\sqrt{k_1!}} \cdots \frac{(x_n - \frac{\partial}{\partial x_n})^{k_n}}{\sqrt{k_n!}} 1.$$

Therefore

$$\underline{H}_{k_1,\ldots,k_n} \longleftrightarrow \zeta_{k_1,\ldots,k_n} \quad \text{(cf. 44.2)}.$$

Strictly speaking, B maps $L^2(\underline{R}^n)$ to $A^2(\underline{C}^n)$ but when the context is clear, the same symbol is used to denote its transfer to $L^2(\underline{R}^n, p_1)$.

44.6 REMARK Since

$$BO(\underline{C}^n) \longleftrightarrow L^2(\underline{R}^n, p_1),$$

it follows that

$$BO(\underline{C}^n) \longleftrightarrow A^2(\underline{C}^n).$$

[Note: Recall that the arrow

$$T: BO(\underline{C}^n) \rightarrow L^2(\underline{R}^n, p_1)$$

is characterized by the relation

$$(T\ \underline{\exp}(z))(x) = e^{z \cdot x - \frac{1}{2} z^2}.$$

If z is fixed, then the Bargmann transform of $e^{z \cdot x}$, as a function of $w \in \underline{C}^n$, is

$$e^{\langle \bar{w}, z \rangle + \frac{1}{2} z^2}.$$

Therefore the composition

$$BO(\underline{C}^n) \rightarrow L^2(\underline{R}^n, p_1) \rightarrow A^2(\underline{C}^n)$$

sends $\underline{\exp}(z)$ to the coherent state $E_{\bar{z}}$:

$$E_{\bar{z}}(w) = e^{\langle \bar{z}, w \rangle} = e^{\overline{\langle w, \bar{z} \rangle}} = e^{\langle \bar{w}, z \rangle}.]$$

Before proceeding further, we shall define two unitary representations of the additive group of $\underline{R}^n$ on $L^2(\underline{R}^{2n}, p_{1/2})$, which will play a fundamental role in the sequel.

$\underline{U}$: Given $a \in \underline{R}^n$, define

$$U(a): L^2(\underline{R}^{2n}, p_{1/2}) \to L^2(\underline{R}^{2n}, p_{1/2})$$

by

$$U(a)\psi(z) = e^{-||a||^2/4} e^{-\langle z, a/\sqrt{2}\rangle} \psi(z + \frac{a}{\sqrt{2}}).$$

$\underline{V}$: Given $b \in \underline{R}^n$, define

$$V(b): L^2(\underline{R}^{2n}, p_{1/2}) \to L^2(\underline{R}^{2n}, p_{1/2})$$

by

$$V(b)\psi(z) = e^{-||b||^2/4} e^{\langle z, \sqrt{-1}\, b/\sqrt{2}\rangle} \psi(z - \frac{\sqrt{-1}\, b}{\sqrt{2}}).$$

[Note: Here

$$\left[\begin{array}{l} \underline{C}^n \longleftrightarrow \underline{R}^{2n} \\ \\ z \longleftrightarrow (x,y) \end{array}\right.$$

and the inner products are complex.]

That U(a) and V(b) are really unitary requires a verification.

<u>Ad U</u>: We have

$$\frac{dp_{1/2,-a\sqrt{2}}}{dp_{1/2}}(z) = \exp(-\sqrt{2}\langle x,a\rangle - \frac{1}{2}||a||^2).$$

Therefore

$$||U(a)\psi||^2_{L^2(p_{1/2})}$$

$$= \int_{\underline{R}^{2n}} |U(a)\psi(z)|^2 \, dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |e^{-||a||^2/4} \; e^{-<z,a/\sqrt{2}>} \; |^2 |\psi(z + \frac{a}{\sqrt{2}})|^2 \, dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |e^{-||a||^2/4} \; e^{-<x+ \sqrt{-1}\, y,a/\sqrt{2}>} \; |^2 |\psi(z + \frac{a}{\sqrt{2}})|^2 \, dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |\psi(z + \frac{a}{\sqrt{2}})|^2 \exp(- \sqrt{2} <x,a> - \frac{1}{2} ||a||^2) dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |\psi(z + \frac{a}{\sqrt{2}})|^2 \frac{dp_{1/2,-a/\sqrt{2}}}{dp_{1/2}} (z) dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |\psi(z + \frac{a}{\sqrt{2}})|^2 \, dp_{1/2,-a/\sqrt{2}}(z)$$

$$= \int_{\underline{R}^{2n}} |\psi(z + \frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}})|^2 \, dp_{1/2}(z)$$

$$= ||\psi||^2_{L^2(p_{1/2})} .$$

<u>Ad V:</u> We have

$$\frac{dp_{1/2,b/\sqrt{2}}}{dp_{1/2}} (z) = \exp(\sqrt{2} <y,b> - \frac{1}{2} ||b||^2).$$

Therefore

$$||V(b)\psi||^2_{L^2(p_{1/2})}$$

$$= \int_{\underline{R}^{2n}} |V(b)\psi(z)|^2 \, dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |e^{-||b||^2/4} e^{<z,\sqrt{-1}\, b/\sqrt{2}>}|^2 |\psi(z - \frac{\sqrt{-1}\, b}{\sqrt{2}})|^2 \, dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |e^{-||b||^2/4} e^{<x+\sqrt{-1}\, y,\sqrt{-1}\, b/\sqrt{2}>}|^2 |\psi(z - \frac{\sqrt{-1}\, b}{\sqrt{2}})|^2 \, dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |\psi(z - \frac{\sqrt{-1}\, b}{\sqrt{2}})|^2 \exp(\sqrt{2} <y,b> - \frac{1}{2} ||b||^2) dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |\psi(z - \frac{\sqrt{-1}\, b}{\sqrt{2}})|^2 \frac{dp_{1/2,b/\sqrt{2}}}{dp_{1/2}} (z) dp_{1/2}(z)$$

$$= \int_{\underline{R}^{2n}} |\psi(z - \frac{\sqrt{-1}\, b}{\sqrt{2}})|^2 \, dp_{1/2,b/\sqrt{2}}(z)$$

$$= \int_{\underline{R}^{2n}} |\psi(z - \frac{\sqrt{-1}\, b}{\sqrt{2}} + \frac{\sqrt{-1}\, b}{\sqrt{2}})|^2 \, dp_{1/2}(z)$$

$$= ||\psi||^2_{L^2(p_{1/2})}.$$

[Note: Needless to say, the convention is that

$$\left[\begin{array}{l} \dfrac{a}{\sqrt{2}} \longleftrightarrow (\dfrac{a}{\sqrt{2}},0) \\ \\ \dfrac{\sqrt{-1}\, b}{\sqrt{2}} \longleftrightarrow (0, \dfrac{b}{\sqrt{2}}) \end{array}\right. .]$$

44.7 LEMMA U and V satisfy the canonical commutation relations, i.e.,

$$U(a)V(b) = e^{\sqrt{-1}\,<a,b>}\,V(b)U(a).$$

PROOF Consider the LHS:

$$U(a)V(b)\psi\Big|_z$$

$$= U(a)\ \ [e^{-||b||^2/4}\, e^{<z,\sqrt{-1}\,b/\sqrt{2}>}\, \psi(z - \frac{\sqrt{-1}\,b}{\sqrt{2}})]$$

$$= e^{-||a||^2/4}\, e^{-||b||^2/4}\, e^{-<z,a/\sqrt{2}>}\, e^{<z + a/\sqrt{2},\sqrt{-1}\,b/\sqrt{2}>}$$

$$\cdot\ \psi(z + \frac{a}{\sqrt{2}} - \frac{\sqrt{-1}\,b}{2})$$

$$= e^{\frac{\sqrt{-1}}{2}<a,b>}\, e^{-(||a||^2 + ||b||^2)/4}\, e^{<z,- a/\sqrt{2} + \sqrt{-1}\,b/\sqrt{2}>}$$

$$\cdot\ \psi(z + \frac{a}{\sqrt{2}} - \frac{\sqrt{-1}\,b}{\sqrt{2}}).$$

But the RHS equals:

$$e^{\sqrt{-1}\,<a,b>}\, V(b)U(a)\psi\Big|_z$$

$$= e^{\sqrt{-1}\,<a,b>}\, V(b)\ \ [e^{-||a||^2/4}\, e^{-<z,a/\sqrt{2}>}\psi(z + \frac{a}{\sqrt{2}})]$$

$$= e^{\sqrt{-1}\,<a,b>}\, e^{-||b||^2/4}\, e^{-||a||^2/4}\, e^{<z,\sqrt{-1}\,b/\sqrt{2}>}\, e^{-<z- \sqrt{-1}\,b/\sqrt{2},a/\sqrt{2}>}$$

$$\cdot\ \psi(z + \frac{a}{\sqrt{2}} - \frac{\sqrt{-1}\,b}{\sqrt{2}})$$

$$= e^{\frac{\sqrt{-1}}{2}<a,b>} e^{-(||a||^2 + ||b||^2)/4} e^{<z,-a/\sqrt{2} + \sqrt{1}\, b/\sqrt{2}>}$$

$$\cdot\ \psi(z + \frac{a}{\sqrt{2}} - \frac{\sqrt{-1}\, b}{\sqrt{2}}).$$

Consequently, the prescription

$$W(a \oplus b) = \exp(\frac{\sqrt{-1}}{2} <a,b>)U(-a)V(b)$$

defines a Weyl system over $\underline{R}^n \oplus \underline{R}^n$ (or still, over $\underline{C}^n$).

Explicitly:

$$W(a \oplus b)\psi(z)$$

$$= e^{-(||a||^2 + ||b||^2)/4} e^{<z,a/\sqrt{2} + \sqrt{-1}\, b/\sqrt{2}>}$$

$$\cdot\ \psi(z - \frac{a}{\sqrt{2}} - \frac{\sqrt{-1}\, b}{\sqrt{2}}).$$

To simplify this, put

$$c = a + \sqrt{-1}\, b.$$

Then

$$W(c)\psi(z) = \exp(<z,c>/\sqrt{2} - <c,c>/4)\psi(z - \frac{c}{\sqrt{2}}).$$

In what follows, it will be convenient to work with $\bar{A}^2(\underline{C}^n)$, the antiholomorphic counterpart of $A^2(\underline{C}^n)$, writing $\bar{B}$ for the map

$$L^2(\underline{R}^n, p_1) \to \bar{A}^2(\underline{C}^n)$$

which sends f to the function

$$z \to \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} e^{-(\bar{z}-x)^2/2} f(x)dx$$

or still, to the function

$$z \to e^{-\frac{1}{2}\bar{z}^2} \int_{\underline{R}^n} e^{\bar{z}\cdot x} f(x)dp_1(x).$$

44.8 REMARK W is not irreducible. In fact, $\bar{A}^2(\underline{C}^n)$ is a closed invariant subspace of $L^2(\underline{C}^n,\mu)$ ($= L^2(\underline{R}^{2n},p_{1/2})$).

It was shown in §41 that the Fock system over $\underline{C}^n$ is unitarily equivalent to the modified real wave representation realized on $L^2(\underline{R}^n,p_1)$:

$$W_{mod}(a + \sqrt{-1}\, b)\psi(x)$$

$$= \exp(\sqrt{-1}\ (\frac{<x,b>}{\sqrt{2}} - <a,b>/2))$$

$$\cdot [\exp(\sqrt{2}\ <x,a> - ||a||^2)]^{1/2}\ \psi(x- \sqrt{2}\ a)$$

or, more succinctly,

$$W_{mod}(c)\psi(x) = \exp(<x,c>/\sqrt{2} - \frac{1}{2} <a,c>)\psi(x- \sqrt{2}\ a).$$

Put

$$W_{cx} = W|\bar{A}^2(\underline{C}^n).$$

44.9 <u>LEMMA</u> We have

$$\bar{B}W_{mod} = W_{cx}\bar{B}.$$

[Note: Therefore W_{mod} and W_{cx} are unitarily equivalent.]

It suffices to check the lemma on functions of the form $x \to e^{w\cdot x}$ and for this, one can take $w = 0$ and compare

$$\bar{B}[x \to \exp(\langle x,c\rangle/\sqrt{2} - \frac{1}{2}\langle a,c\rangle)]$$

with $W_{cx}1$, i.e., with

$$z \to \exp(\langle z,c\rangle/\sqrt{2} - \langle c,c\rangle/4).$$

By definition,

$$\bar{B}[x \to \exp(\langle x,c\rangle/\sqrt{2} - \frac{1}{2}\langle a,c\rangle)]\Big|_z$$

$$= e^{-\frac{1}{2}\bar{z}^2} \int_{\underline{R}^n} e^{\bar{z}\cdot x}\, e^{\frac{c}{\sqrt{2}}\cdot x}\, e^{-\frac{1}{2}\langle a,c\rangle}\, dp_1(x).$$

But

$$\int_{\underline{R}^n} e^{(\bar{z} + \frac{c}{\sqrt{2}})\cdot x}\, dp_1(x) = e^{\frac{1}{2}(\bar{z} + \frac{c}{\sqrt{2}})^2}.$$

Matters thus reduce to

$$\exp(-\frac{1}{2}\bar{z}^2 + \frac{1}{2}\bar{z}^2 + \langle z,c\rangle/\sqrt{2} + \frac{c^2}{4} - \frac{1}{2}\langle a,c\rangle)$$

$$= \exp(\langle z,c\rangle/\sqrt{2} + \frac{c^2}{4} - \frac{1}{2}\langle a,c\rangle).$$

However

$$\frac{c^2}{4} - \frac{1}{2} \langle a,c\rangle$$

$$= \frac{a^2 + 2\sqrt{-1}\,\langle a,b\rangle - b^2}{4} - \frac{1}{2} a^2 - \frac{\sqrt{-1}}{2} \langle a,b\rangle$$

$$= -\frac{1}{4} (a^2 + b^2)$$

$$= -\frac{1}{4} (||a||^2 + ||b||^2)$$

$$= - \langle c,c\rangle/4.$$

And this completes the proof.

N.B. W_{cx} is called the complex wave representation.

So, the Fock system is unitarily equivalent to the modified real wave representation which in turn is unitarily equivalent to the complex wave representation.

§45. *HOLOMORPHIC FUNCTIONS*

Let (X,Y,ι) be an abstract Wiener space -- then a complex structure on (X,Y,ι) is a complex structure J on Y such that $JX \subset X$.

Suppose that J is a complex structure on (X,Y,ι) -- then J is said to be isometric if

$$\begin{cases} ||Jy||_Y = ||y||_Y \ \forall \ y \in Y \\ ||Jx||_X = ||x||_X \ \forall \ x \in X. \end{cases}$$

45.1 EXAMPLE Take for (X,Y,ι) the triple $(W_0^{2,1}([0,1];\underline{R}^2), C_0([0,1];\underline{R}^2),\iota)$ and define

$$J:C_0([0,1];\underline{R}^2) \to C_0([0,1];\underline{R}^2)$$

by

$$Jf = J(f_1,f_2) = (-f_2,f_1).$$

Then J leaves $W_0^{2,1}([0,1];\underline{R}^2)$ invariant. Since the norm on $C_0([0,1];\underline{R}^2)$ is

$$||f||_\infty = \sup_{0\leq t\leq 1} ||f(t)||_{\underline{R}^2}$$

and since the norm on $W_0^{2,1}([0,1];\underline{R}^2)$ is

$$||h||_2 = \left(\int_0^1 ||h'(t)||^2_{\underline{R}^2}\, dt\right)^{1/2},$$

it follows that J is isometric.

Let J be an isometric complex structure on (X,Y,ι) -- then the norm $||\cdot||_Y$ is said to be rotation invariant if $\forall\ y \in Y$,

$$||(a + bJ)y||_Y = |a + \sqrt{-1}\, b|\ ||y||_Y \quad (a,b \in \underline{R}).$$

[Note: This condition implies that $Y^\sim$ is a Banach space over $\underline{C}$.]

45.2 REMARK If $||\cdot||_Y$ is not rotationally invariant, then $||\cdot||_Y$ can always be replaced by an equivalent norm $||\cdot||_{Y,J}$ that is rotationally invariant, viz.

$$||y||_{Y,J} = \sup_{0\leq\theta\leq 2\pi} ||(\cos\theta + J\sin\theta)y||_Y.$$

[Note: Because $||\cdot||_{Y,J}$ is equivalent to $||\cdot||_Y$, the restriction $||\cdot||_{Y,J} \circ \iota$ is tight.]

Suppose that J is an isometric complex structure on (X,Y,ι) under which $||\cdot||_Y$ is rotationally invariant. Let $Y^*_{\underline{C}} = Y^* \oplus \sqrt{-1}\, Y^*$ -- then the elements of $Y^*_{\underline{C}}$ are the continuous $\underline{R}$-linear complex valued functions on Y. Put

$$\begin{cases} Y^{*(1,0)} = \{\lambda \in Y^*_{\underline{C}}: J^*\lambda = \sqrt{-1}\,\lambda\} \\ Y^{*(0,1)} = \{\lambda \in Y^*_{\underline{C}}: J^*\lambda = -\sqrt{-1}\,\lambda\}. \end{cases}$$

Then $Y^{*(1,0)}$ and $Y^{*(0,1)}$ are complex subspaces of $Y^*_{\underline{C}}$ and

$$Y^*_{\underline{C}} = Y^{*(1,0)} \oplus Y^{*(0,1)}.$$

Moreover, the elements of $Y^{*(1,0)}$ are the continuous $\underline{C}$-linear complex valued functions on Y, i.e., $Y^{*(1,0)}$ is the dual of $Y^{\sim}$:

$$\begin{aligned}\lambda(\sqrt{-1}\,y) &= \langle\sqrt{-1}\,y,\lambda\rangle \\ &= \langle Jy,\lambda\rangle \\ &= \langle y,J^*\lambda\rangle \\ &= \langle y,\sqrt{-1}\,\lambda\rangle = \sqrt{-1}\,\lambda(y).\end{aligned}$$

[Note: The definitions of $X^*_{\underline{C}}$, $X^{*(1,0)}$, and $X^{*(0,1)}$ are analogous.]

A function $F:Y \to \underline{C}$ is a <u>holomorphic polynomial</u> if it has the form

$$F = f(\lambda_1,\ldots,\lambda_n),$$

where $\lambda_i \in Y^{*(1,0)}$ $(i = 1,\ldots,n)$ and $f:\underline{C}^n \to \underline{C}$ is a polynomial.

[Note: Antiholomorphic polynomials are defined by replacing $Y^{*(1,0)}$ with $Y^{*(0,1)}$.]

Write $P_H(Y)$ for the set of holomorphic polynomials on Y.

45.3 <u>LEMMA</u> Let $F \in P_H(Y)$ — then

$$F(y) = \int_Y F(y+y')dp_{1/2}(y').$$

Let $F \in L^2(Y,p_{1/2})$ — then F is said to be an L^2-<u>holomorphic function</u> if

$$F \in \overline{P_H(Y)} \ (\subset L^2(Y,p_{1/2})).$$

Denote by $A^2(Y)$ the set of L^2-holomorphic functions on Y.

45.4 REMARK In general, an L^2-holomorphic function F is neither continuous nor X-differentiable (but it is true that $\forall\, x \in X$,

$$\frac{d}{dt} F(y+tx) \Big|_{t=0}$$

exists a.e. $[p_{1/2}]$). Furthermore, there are elements of $A^2(Y)$ which are not in the Sobolev space $W^{2,1}(Y,p_{1/2})$ (cf. 45.9).

45.5 SPLITTING PRINCIPLE Fix $\lambda \in Y^*: ||x_\lambda||_X = 1$. Let $X(\lambda)$ be the linear span of x_λ and Jx_λ; let $X' = X(\lambda)^\perp$ and let

$$P':X \to X'$$

be the associated orthogonal projection. Assuming that X is contained in Y, call Y' the closure of X' in Y and extend P' continuously to Y':

$$Q':Y \to Y'.$$

Define a bijection

$$\underline{R}^2 \times Y' \to Y$$

by

$$\begin{cases} (a,0) \to ax_\lambda \\ \\ (0,b) \to bJx_\lambda \end{cases}, \quad y' \to y'.$$

Then

$$\mu_{\underline{C}} \times \mu' \longleftrightarrow p_{1/2}.$$

Here

$$d\mu_{\underline{C}}(z) = \frac{1}{\pi} e^{-|z|^2} dz \text{ and } \mu' = p_{1/2} \circ (Q')^{-1}.$$

Suppose now that F is an L^2-holomorphic function. View F as a function of (z,y'). Fix a sequence $\{F_n\}$ of holomorphic polynomials: $F_n \xrightarrow[L^2]{} F$, arranging matters so that

$$\lim_{n \to \infty} \int_{\underline{C}} |F_n(z,y') - F(z,y')|^2 d\mu_{\underline{C}}(z)$$

$$= 0$$

for μ' - a.e. y'. For such a y', the sequence $\{F_n(z,y')\}$ converges uniformly on compacta. Therefore $F(z,y')$ is holomorphic in z (change values on a $\mu_{\underline{C}}$-null set if necessary).

45.6 LEMMA Let $F_1,F_2 \in A^2(Y)$. Assume:

$$p_{1/2}\{y:F_1(y) = F_2(y)\} > 0.$$

Then

$$F_1 = F_2 \text{ a.e. } [p_{1/2}].$$

PROOF Take $F_2 = 0$, put $F = F_1$, and let

$$B = \{y:F(y) = 0\}.$$

Then for μ' - a.e. y',

$$\mu_{\underline{C}}\{z:F(z,y') = 0\} = 0 \text{ or } 1 \quad (\text{cf. } 45.5),$$

thus

$$p_{1/2}(B\Delta(B + x_\lambda)) = 0$$

or still,

$$p_{1/2}(B + x_\lambda) = p_{1/2}(B).$$

Since λ is arbitrary subject to $||x_\lambda||_X = 1$ and since by assumption $p_{1/2}(B) > 0$, the conclusion is that $p_{1/2}(B) = 1$ (see the proof of 26.33).

Fix a sequence $\{\lambda_n\} \subset Y^{*(1,0)}$ with the property that $\{\lambda_n\}$ is an orthonormal basis for $X^{*(1,0)}$ (hence that $\{\bar{\lambda}_n\}$ is an orthonormal basis for $X^{*(0,1)}$).

45.7 <u>LEMMA</u> The functions

$$\prod_{j=1}^{\infty} \frac{H_{a_j,b_j}(\lambda_j,\bar{\lambda}_j)}{\sqrt{a_j!\,b_j!}}$$

constitute an orthonormal basis for $L^2(Y,p_{1/2})$ (cf. 28.6).

[Note: Here $\{a_j\}$ and $\{b_j\}$ are sequences of nonnegative integers, almost all of whose terms are zero.]

Let $W_{a,b}$ denote the closed linear subspace of $L^2(Y,p_{1/2})$ generated by the

$$\prod_{j=1}^{\infty} \frac{H_{a_j,b_j}(\lambda_j,\bar{\lambda}_j)}{\sqrt{a_j!b_j!}},$$

where $\sum_j a_j = a$, $\sum_j b_j = b$, and let $I_{a,b}$ denote the orthogonal projection of $L^2(Y,p_{1/2})$ onto $W_{a,b}$ -- then

$$(a,b) \neq (c,d) \Rightarrow W_{a,b} \perp W_{c,d}$$

and

$$W_n = \bigoplus_{a+b=n} W_{a,b}.$$

45.8 <u>LEMMA</u> Let $F \in L^2(Y,p_{1/2})$ -- then $F \in A^2(Y)$ iff $\forall\ b \geq 1$,

$$I_{a,b}(F) = 0.$$

[Note: So, if $F \in A^2(Y)$, then

$$F = \sum_{a=0}^{\infty} I_{a,0}(F).]$$

Given $\underline{a} = (a_1,a_2,\ldots)$ $(|\underline{a}| \equiv \sum_j a_j,\ a_j = 0\ (j >> 0))$, put

$$F_{\underline{a}} = \frac{1}{\sqrt{a_1!a_2!\cdots}} \prod_{j=1}^{\infty} \lambda_j^{a_j}$$

$$\left(= \prod_{j=1}^{\infty} \frac{H_{a_j,0}(\lambda_j,\bar{\lambda}_j)}{\sqrt{a_j!0!}}\right).$$

Then the $F_{\underline{a}}$ form an orthonormal basis for $A^2(Y)$, thus $\forall\ F \in A^2(Y)$,

$$F = \sum_{\underline{a}} c_{\underline{a}} F_{\underline{a}},$$

where

$$c_{\underline{a}} = \int_Y \bar{F}_{\underline{a}} F dp_{1/2}.$$

[Note: This expansion is called the L^2-Taylor series of F.]

45.9 EXAMPLE Let

$$F = \sum_{n=1}^{\infty} \frac{1}{(n+1)} \frac{\lambda_n^n}{\sqrt{n!}} .$$

Then $F \in A^2(Y)$, but $F \notin W^{2,1}(Y,p_{1/2})$. In fact,

$$(I-L)^{1/2}F = \sum_{n=1}^{\infty} \frac{1}{(n+1)} \sqrt{n+1}\, \frac{\lambda_n^n}{\sqrt{n!}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \frac{\lambda_n^n}{\sqrt{n!}} .$$

Therefore

$$||F||_{2,1}^2 = ||(I-L)^{1/2}F||_2^2$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)} \left|\left| \frac{H_{n,0}(\lambda_n,\bar{\lambda}_n)}{\sqrt{n!}} \right|\right|_2^2$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)} = \infty$$

=>

$$F \notin W^{2,1}(Y,p_{1/2}).$$

45.10 REMARK Let

$$\lambda_{\underline{a}} = \left[\frac{|\underline{a}|!}{a_1!a_2!\cdots}\right]^{1/2} P_{|\underline{a}|}(\lambda_1^{a_1} \otimes \lambda_2^{a_2} \otimes \cdots).$$

Then the $\lambda_{\underline{a}}$ form an orthonormal basis for $BO(X^{*(1,0)})$ and the arrow

$$\begin{cases} BO(X^{*}(1,0)) \rightarrow A^2(Y) \\ \lambda_{\underline{a}} \rightarrow F_{\underline{a}} \end{cases}$$

is an isometric isomorphism.

[Note: $X^{*(1,0)}$ is the dual of $X^{\sim}$.]

45.11 LEMMA Let $F \in P_H(Y)$ -- then

$$F(e^{-t}y) = \int_Y F(e^{-t}y + (1-e^{-2t})^{1/2} y')d_{p_{1/2}}(y'),$$

i.e.,

$$F(e^{-t}y) = T_tF(y).$$

PROOF This is obvious if $F = F_{\underline{a}}$, which suffices.

[Note: Therefore

$$F(ty) = T_{-\log t} F(y) \quad (0 < t < 1)$$

=>

$$\int_Y |F(ty)|^2 dp_{1/2}(y) \leq \int_Y |F(y)|^2 dp_{1/2}(y).]$$

§46. *SKELETONS*

Fix an abstract Wiener space (X,Y,ι) and keep to the assumptions and notation of §45.

Given $\theta \in \underline{R}$, define

$$U_\theta : L^2(Y,p_{1/2}) \to L^2(Y,p_{1/2})$$

by

$$U_\theta F\Big|_y = F((\cos\theta + J\sin\theta)y).$$

46.1 <u>LEMMA</u> Let $F \in A^2(Y)$ -- then

$$I_n(F) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}\, n\theta}\, U_\theta F d\theta.$$

<u>PROOF</u> There is no loss of generality in supposing that F is a holomorphic polynomial. Since $U_\theta I_m(F) = e^{\sqrt{-1}\, m\theta}\, I_m(F)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}\, n\theta}\, U_\theta F d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}\, n\theta}\, U_\theta \sum_m I_m(F) d\theta$$

$$= \sum_m \frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}\, n\theta}\, e^{\sqrt{-1}\, m\theta}\, I_m(F) d\theta$$

$$= \sum_m \frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}\, n\theta}\, e^{\sqrt{-1}\, m\theta}\, d\theta \times I_m(F)$$

$$= I_n(F),$$

from which the lemma.

46.2 LEMMA Let $B_r = \{y \in Y: ||y||_Y < r\}$ — then $\forall\ F \in A^2(Y)$,

$$\frac{1}{p_{1/2}(B_r)} \int_{B_r} F\, dp_{1/2} = \int_Y F\, dp_{1/2}.$$

PROOF One has only to note that

$$p_{1/2}(B_r) \times \int_Y F\, dp_{1/2}$$

$$= \int_{B_r} 1\, dp_{1/2}(y) \cdot \frac{1}{2\pi} \int_0^{2\pi} U_\theta F\Big|_Y\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{B_r} U_\theta F\Big|_Y\, dp_{1/2}(y)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{B_r} F(y)\, dp_{1/2}(y)$$

$$= \int_{B_r} F\, dp_{1/2}.$$

46.3 REMARK If $F \in A^2(Y)$ is continuous, then

$$F(0) = \lim_{r \to 0} \frac{1}{p_{1/2}(B_r)} \int_{B_r} F\, dp_{1/2} = \int_Y F\, dp_{1/2}.$$

Let $F \in A^2(Y)$ -- then the skeleton of F is the function

$$S_F : X \to \underline{C}$$

defined by

$$S_F(x) = \int_Y F(x+y)\,dp_{1/2}(y) \quad \text{(cf. 26.16)}.$$

46.4 <u>REMARK</u> We have

$$S_F(x) = \int_Y F(y) \frac{dp_{1/2,x}}{dp_{1/2}}(y)\,dp_{1/2}(y)$$

$$= \int_Y F(y)\exp(2\Phi_x(y) - ||x||_X^2)\,dp_{1/2}(y).$$

[Note: The functions

$$y \to \exp(2\Phi_x(y) - ||x||_X^2) \quad (x \in X)$$

are total in $L^2(Y,p_{1/2})$ (cf. 28.8). Consequently, $S_{F_1} = S_{F_2}$ iff $F_1 = F_2$ a.e. $[p_{1/2}]$.]

46.5 <u>LEMMA</u> Fix $x \in X$ — then $\forall$ $F \in A^2(Y)$, $S_F(x)$ is the Lebesgue density of F at x:

$$\lim_{r \to 0} \frac{1}{p_{1/2}(B_r)} \int_{B_r} F(x+y)\,dp_{1/2}(y).$$

If $F \in A^2(Y)$ is continuous, then $\forall$ $x \in X$, $S_F(x) = F(x)$. I.e.:

$$S_F = F|X.$$

In general, F always admits a version for which this is true, a fact which is not obvious and requires some preliminaries.

Given a sequence $\{F_n\}$ of holomorphic polynomials such that

$$\sum_n ||F_n||_{L^2(p_{1/2})} < \infty,$$

put

$$N_2(\{F_n\}) = \{y: \sum_n |F_n(y)| = \infty\}.$$

46.6 <u>LEMMA</u> Under the above assumptions,

$$p_{1/2}(N_2(\{F_n\})) = 0.$$

<u>PROOF</u> In fact,

$$\int_Y \sum_n |F_n(y)| dp_{1/2}(y)$$

$$= \sum_n \int_Y |F_n(y)| dp_{1/2}(y)$$

$$\leq \sum_n (\int_Y |F_n(y)|^2 dp_{1/2}(y))^{1/2}$$

$$< \infty.$$

Therefore

$$y \to \sum_n |F_n(y)| \in L^1(Y, p_{1/2}).$$

46.7 <u>LEMMA</u> $\forall\, x \in X$,

$$\sum_n |F_n(x)| < \infty.$$

PROOF Write

$$\sum_n |F_n(x)| = \sum_n |\int_Y F_n(x+y)\,dp_{1/2}(y)| \quad \text{(cf. 45.3)}$$

$$= \sum_n |\int_Y F_n(y)\exp(2\Phi_x(y) - ||x||_X^2)\,dp_{1/2}(y)|$$

$$\leq \sum_n ||F_n||_{L^2(p_{1/2})} \cdot ||\exp(2\Phi_x(\cdot) - ||x||_X^2)||_{L^2(p_{1/2})}$$

$$< \infty.$$

Let $F \in A^2(Y)$. Choose a sequence $\{F_n\} \subset P_H(Y)$ subject to the following conditions:

(1) $||F_n - F||_{L^2(p_{1/2})} \to 0;$

(2) $\sum_n ||F_{n+1} - F_n||_{L^2(p_{1/2})} < \infty.$

Let

$$\tilde{F}(y) = \begin{cases} \lim F_n(y) & (y \notin N_2(\{F_{n+1} - F_n\})) \\ 0 & (y \in N_2(\{F_{n+1} - F_n\})). \end{cases}$$

Then $\tilde{F} = \tilde{F}$ a.e. $[p_{1/2}]$, thus $\tilde{F} \in A^2(Y)$.

46.8 LEMMA $\forall\, x \in X,$

$$S_{\tilde{F}}(x) = \tilde{F}(x).$$

[Since $x \notin N_2(\{F_{n+1} - F_n\})$ (cf. 46.7),

$$F_n(x) \rightarrow \tilde{F}(x).$$

On the other hand,

$$F_n(x) = \int_Y F_n(x+y)\,dp_{1/2}(y) \qquad \text{(cf. 45.3)}$$

$$\rightarrow \int_Y \tilde{F}(x+y)\,dp_{1/2}(y) = S_{\tilde{F}}(x).$$

Given $F \in A^2(Y)$, it will be assumed henceforth that

$$S_F = F|X.$$

On general grounds, S_F is locally bounded and differentiable (cf. §32).

[Note: We have

$$|S_F(x)| \leq e^{||x||^2}||F||_{L^2(p_{1/2})}.$$

In this connection, observe that

$$\left(\int_Y \exp(2\Phi_x(y))^2\,dp_{1/2}(y)\right)^{1/2}$$

$$= \left(\int_Y \exp(2\cdot 2\Phi_x(y))\,dp_{1/2}(y)\right)^{1/2}$$

$$= \left(\exp\left(\frac{4}{2}\cdot ||2\Phi_x||^2_{L^2(p_{1/2})}\right)\right)^{1/2} \qquad \text{(cf. 26.17)}$$

$$= (\exp(\frac{4}{2} \cdot ||x||^2_{1/2}))^{1/2} \quad \text{(cf. §40)}$$

$$= \exp(\frac{||x||^2_X}{1/2})$$

$$= \exp(2||x||^2).]$$

One can also view S_F as a function on $X^{\sim}$. As such, for any choice of x_0 and $x_i\,(i=1,\ldots,n)$ in $X^{\sim}$, the function $\underline{C}^n \to \underline{C}$ defined by

$$(z_1,\ldots,z_n) \to S_F(x_0 + z_1x_1 + \cdots + z_nx_n)$$

is holomorphic.

46.9 RAPPEL Let $\mathcal{H}$ be a separable complex Hilbert space -- then a function $F:\mathcal{H} \to \underline{C}$ is said to be holomorphic if F is locally bounded and holomorphic on each finite dimensional subspace of $\mathcal{H}$.

Accordingly, $\forall\ F \in A^2(Y)$,

$$S_F:X^{\sim} \to \underline{C}$$

is holomorphic.

46.10 LEMMA Suppose that $\exists\ M > 0$:

$$|S_F(x)| \le M\ \forall\ x \in X.$$

Then $\exists$ a constant $C:F = C$ a.e. $[p_{1/2}]$.

PROOF $\forall x \in X$, the function $z \to S_F(zx)$ is holomorphic, hence is constant. Therefore

$$S_F(x) = S_F(0) \quad (x \in X).$$

Let $C = S_F(0)$ -- then the function $y \to C$ is in $A^2(Y)$ and $S_F = S_C$, thus $F = C$ a.e. $[p_{1/2}]$ (cf. 46.4).

46.11 LEMMA Suppose that $\exists$ an open subset $O \subset X$:

$$S_F(x) = 0 \ \forall \ x \in O.$$

Then $F = 0$ a.e. $[p_{1/2}]$.

PROOF Fix $x_0 \in O$ and consider the holomorphic function $z \to S_F(x_0 + zx)$ $(x \in X)$. If $|z|$ is sufficiently small, say $|z| < \varepsilon$, then $x_0 + zx \in O$, hence $S_F(x_0 + zx) = 0$ $(|z| < \varepsilon)$. But this implies that

$$S_F(x_0 + zx) = 0$$

for all z, in particular

$$S_F(x_0 + x) = 0.$$

Therefore

$$S_F(x) = S_F(x_0 + (x-x_0))$$

$$= 0$$

$$\Rightarrow$$

$$F = 0 \text{ a.e. } [p_{1/2}] \quad (\text{cf. } 46.4).$$

Denote by $A^2(X)$ the set of all functions F on X of the form

$$F = \sum_{\underline{a}} c_{\underline{a}} S_{F_{\underline{a}}} \ ,$$

where

$$\sum_{\underline{a}} |c_{\underline{a}}|^2 < \infty.$$

Then $A^2(X)$ is a complex Hilbert space with inner product

$$\langle F,F'\rangle = \sum_{\underline{a}} \bar{c}_{\underline{a}} c'_{\underline{a}} \ .$$

46.12 <u>$A^2(Y)$ vs. $A^2(X)$</u> The connection between the two is simply this: The arrow

$$\begin{cases} A^2(Y) \to A^2(X) \\ F \to S_F \end{cases}$$

is an isometric isomorphism.

<u>N.B.</u> It follows that the elements of $A^2(X)$ are holomorphic (in the sense of 46.9).

Let $\langle , \rangle^{\sim}$ $(= \langle , \rangle_J)$ be the inner product on $X^{\sim}$:

$$\langle x,x'\rangle^{\sim} = \langle x,x'\rangle - \sqrt{-1}\ \langle x,Jx'\rangle \quad \text{(cf. 19.2)}.$$

46.13 <u>LEMMA</u> Let $F \in A^2(X)$ — then $\forall\ x \in X$,

$$|F(x)| \le ||F||e^{\langle x,x\rangle^{\sim}/2}.$$

Consequently, the evaluation

$$\begin{cases} A^2(X) \rightarrow \underline{C} \\ x \rightarrow F(x) \end{cases}$$

is continuous, hence there exists a unique element $E_x \in A^2(X)$ such that $\forall\ F \in A^2(X)$,

$$F(x) = \langle E_x, F\rangle.$$

The set $\{E_x : x \in X\}$ is total in $A^2(X)$. Its elements are called <u>coherent states</u>. One has

$$\begin{cases} E_x(x') = e^{\langle x,x'\rangle^{\sim}} \\ \langle E_x, E_{x'}\rangle = e^{\langle x',x\rangle^{\sim}}. \end{cases}$$

[Note: Recall that $X^{*(1,0)}$ is the dual of $X^{\sim}$. Given $\lambda, \eta \in X^{*(1,0)}$, determine $e_\lambda, e_\eta \in X^{\sim}$ by

$$\begin{cases} \lambda(x) = \langle e_\lambda, x\rangle^{\sim} \\ \eta(x) = \langle e_\eta, x\rangle^{\sim} \end{cases} \quad (x \in X).$$

Then the inner product $\langle\lambda,\eta\rangle$ per $X^{*(1,0)}$ is $\langle e_\eta, e_\lambda\rangle^{\sim}$. And the arrow

$$\begin{cases} \underline{BO}(X^{*(1,0)}) \rightarrow A^2(X) \\ \underline{\exp}(\lambda) \rightarrow E_{e_\lambda} \end{cases}$$

is an isometric isomorphism:

$$\begin{aligned}
<E_{e_\lambda}, E_{e_\eta}> &= e^{<e_\eta, e_\lambda>^\sim} \\
&= e^{<\lambda,\eta>} \\
&= <\underline{\exp}(\lambda), \underline{\exp}(\eta)>.]
\end{aligned}$$

46.14 <u>LEMMA</u> Let V_n be the span of $\{e_{\lambda_1}, \ldots, e_{\lambda_n}\}$ and put $d_n = \dim V_n$ — then $\forall\ F \in A^2(X)$,

$$||F||^2 = \lim_{n \to \infty} \frac{1}{\pi^{d_n}} \int_{V_n} |F(v)|^2\, e^{-<v,v>^\sim}\, dv.$$

46.15 <u>REMARK</u> Let $\bar{A}^2(Y)$ and $\bar{A}^2(X)$ be the antiholomorphic versions of $A^2(Y)$ and $A^2(X)$ — then $\bar{A}^2(Y) \approx \bar{A}^2(X)$ and $\bar{A}^2(X) \approx \underline{BO}(X^{*(0,1)})$ or still, $\bar{A}^2(X) \approx \underline{BO}(X^\sim)$, the point being that $X^{*(0,1)}$ is the antidual of $X^\sim$, hence is isometrically isomorphic to $X^\sim$.

§47. *THE COMPLEX WAVE REPRESENTATION*

Let X be an infinite dimensional separable complex Hilbert space. Fix a real part X_0 of X and let (X_0,Y_0,ι) be an abstract Wiener space -- then $(X_0 \times X_0, Y_0 \times Y_0, \iota \times \iota)$ is an abstract Wiener space.

[Note: The exchange

$$(y_0,y_0') \to (-y_0',y_0)$$

is an isometric complex structure on $(X_0 \times X_0, Y_0 \times Y_0, \iota \times \iota)$.]

47.1 <u>REMARK</u> The finite dimensional model is

$$X = \underset{\sim}{C}^n,\ X_0 = \underset{\sim}{R}^n (= Y_0),\ X_0 \times X_0 = \underset{\sim}{R}^{2n}.$$

It was shown in §41 that the Fock system over X $(= X_0 + \sqrt{-1}\, X_0)$ is unitarily equivalent to the modified real wave representation realized on $L^2(Y_0,p_1)$:

$$W_{mod}(a + \sqrt{-1}\, b)\psi\Big|_{y_0}$$

$$= \exp(\sqrt{-1}\ (\frac{\Phi_b(y_0)}{\sqrt{2}} - \langle a,b\rangle/2))$$

$$\cdot [\exp(\sqrt{2}\ \Phi_a(y_0) - ||a||^2)]^{1/2}\ \psi(y_0 - \sqrt{2}\ a).$$

In the finite dimensional model, the modified real wave representation is also unitarily equivalent to the complex wave representation (cf. §44). Objective:

Extend these considerations to the infinite dimensional situation.

To begin with, let us recall that $L^2(Y_0,p_1)$ is the completion of the pre-Hilbert space

$$\bigcup_{P\in \mathcal{P}_{X_0}} L^2(X_0,\mathcal{C}_P;\gamma_{X_0}).$$

This said, the infinite dimensional version of the Bargmann transform is the isometric isomorphism

$$B:L^2(Y_0,p_1) \to A^2(Y) \quad (Y = Y_0 \times Y_0)$$

characterized by the following property: For all $f \in L^2(X_0,\mathcal{C}_P;\gamma_{X_0})$,

$$S_{Bf}(c) = e^{-\langle\bar{c},c\rangle/2} \int_{X_0} e^{\langle x,c\rangle} f(x)d\gamma_{X_0}(x).$$

[Note:

$$\begin{cases} c = a + \sqrt{-1}\, b \\ \\ \bar{c} = a - \sqrt{-1}\, b \end{cases} \qquad (a,b \in X_0)$$

and S_{Bf} is the skeleton of Bf.]

N.B. There is, of course, an antiholomorphic version of B, call it $\bar{B}$.

Define now a Weyl system over X, realized on $L^2(Y,p_{1/2})$, by the following prescription:

$$W(c)\psi\Big|_{(y_0,y_0')}$$

$$= \exp(\Phi_{a/\sqrt{2}}(y_0) + \Phi_{b/\sqrt{2}}(y_0') + \sqrt{-1}\,(\Phi_{b/\sqrt{2}}(y_0) - \Phi_{a/\sqrt{2}}(y_0')) - \langle c,c\rangle/4)$$

$$\cdot\ \psi((y_0,y_0') - (\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})).$$

[Note: Obviously,

$$|\exp(\Phi_{a/\sqrt{2}}(y_0) + \Phi_{b/\sqrt{2}}(y_0') + \sqrt{-1}(\Phi_{b/\sqrt{2}}(y_0) - \Phi_{a/\sqrt{2}}(y_0')) - \langle c,c\rangle/4)|^2$$

$$= \exp(\sqrt{2}\ \Phi_a(y_0) + \sqrt{2}\ \Phi_b(y_0') - (||a||^2 + ||b||^2)/2).$$

On the other hand,

$$\frac{dp_{1/2,(a/\sqrt{2},b/\sqrt{2})}}{dp_{1/2}}\ (y_0,y_0')$$

$$= \exp(2\Phi_{(a/\sqrt{2},b/\sqrt{2})}(y_0,y_0') - ||\ (\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})\ ||^2)$$

$$= \exp(\sqrt{2}\Phi_a(y_0) + \sqrt{2}\Phi_b(y_0') - (||a||^2 + ||b||^2)/2).]$$

Let $(x_0,x_0') \in X_0 \times X_0$ — then

$$\langle x_0 + \sqrt{-1}\ x_0',c\rangle/\sqrt{2}$$

$$= \langle x_0 + \sqrt{-1}\ x_0',\ a + \sqrt{-1}\ b\rangle/\sqrt{2}$$

$$= \langle x_0,a\rangle/\sqrt{2} + \langle x_0',b\rangle/\sqrt{2} + \sqrt{-1}\ (\langle x_0,b\rangle/\sqrt{2} - \langle x_0',a\rangle/\sqrt{2})$$

$$= \Phi_{a/\sqrt{2}}(x_0) + \Phi_{b/\sqrt{2}}(x_0') + \sqrt{-1}\,(\Phi_{b/\sqrt{2}}(x_0) - \Phi_{a/\sqrt{2}}(x_0')).$$

Since $< \text{—},c>/\sqrt{2}$ belongs to $\bar{A}^2(X)$, it follows that the multiplier

$$\exp(\Phi_{a/\sqrt{2}}(y_0) + \Phi_{b/\sqrt{2}}(y_0') + \sqrt{-1}\,(\Phi_{b/\sqrt{2}}(y_0) - \Phi_{a/\sqrt{2}}(y_0')) - <c,c>/4)$$

belongs to $\bar{A}^2(Y)$. Therefore $\bar{A}^2(Y)$ is W-invariant.

What was said in the finite dimensional case then goes through in the infinite dimensional case: Put

$$W_{cx} = W|\bar{A}^2(Y).$$

47.2 LEMMA We have

$$\bar{B}W_{mod} = W_{cx}\bar{B}.$$

[Note: Therefore W_{mod} and W_{cx} are unitarily equivalent.]

N.B. W_{cx} is called the complex wave representation.

So, the Fock system is unitarily equivalent to the modified real wave representation which in turn is unitarily equivalent to the complex wave representation.

§48. *REVIEW OF DEFINITIONS*

Working first in $\underline{R}^n$, consider the laplacian Δ -- then (cf. 1.15)

1. Δ is selfadjoint.

2. $\Delta | C_c^\infty(\underline{R}^n)$ is essentially selfadjoint.

48.1 <u>REMARK</u> The spectrum of $-\Delta$ is $[0,\infty[$, thus $-m^2$ $(m > 0)$ is in the resolvent of $-\Delta$. Therefore

$$(-\Delta + m^2)^{-1}$$

is a bounded linear operator on $L^2(\underline{R}^n)$.

Equip $C_c^\infty(\underline{R}^n)$ with the norm

$$||f||_{2,r} = ||(1 - \Delta)^{r/2} f||_{L^2} \qquad (r \in \underline{R}).$$

Then its completion is the Sobolev space $W^{2,r}(\underline{R}^n)$. In particular:

$$\mathrm{Dom}(\Delta) = W^{2,2}(\underline{R}^n).$$

Suppose now that M is an n-dimensional connected C^∞ manifold.

I. Assume that M is compact. Fix a finite covering of M by coordinate charts $\{(U_i,\phi_i)\}$ and let $\{\kappa_i\}$ be a subordinate partition of unity. Given a distribution T on M, write $T \in W^{2,r}(M)$ if for each i, the pushforward $(\phi_i)_*(\kappa_i T)$

is an element of $W^{2,r}(\underline{R}^n)$. This definition is intrinsic, i.e., independent of the choices made for U_i, ϕ_i, and κ_i. And $W^{2,r}(M)$ is a Hilbert space with norm

$$||T||_{2,r} = (\sum_i ||(\phi_i)_*(\kappa_i T)||_{2,r}^2)^{1/2}.$$

II. Assume that M admits a complete riemannian structure g — then the laplacian Δ_g is the divergence of the gradient, thus locally

$$\Delta_g f = \frac{1}{|g|^{1/2}} \partial_i (g^{ij}|g|^{1/2}\partial_j f),$$

and a theorem due to Gaffney says that $\Delta_g | C_c^\infty(M)$ is essentially selfadjoint. One then defines $W_g^{2,r}(M)$ as the completion of $C_c^\infty(M)$ w.r.t. the norm

$$||f||_{2,r} = ||(1 - \Delta_g)^{r/2} f||_{L^2} \quad (r \in \underline{R}).$$

[Note: The space $W_g^{2,r}(M)$ depends on g but if M is compact, then $W_g^{2,r}(M) = W^{2,r}(M)$.]

48.2 <u>LEMMA</u> Let (M,g), (M',g') be two complete riemannian manifolds. Suppose that $\Psi: M \to M'$ is a diffeomorphism — then for any open, relatively compact set $O \subset M$, $\exists\ C_1 > 0$, $C_2 > 0$ such that $\forall\ f \in C_c^\infty(O)$,

$$C_1||f||_{2,r} \leq ||f \circ \Psi^{-1}||_{2,r} \leq C_2||f||_{2,r} \quad (r \in \underline{R}).$$

[Note: Take M = M', Ψ = id — then the topology on $C_c^\infty(O)$ induced by $W_g^{2,r}(M)$ is equivalent to the topology on $C_c^\infty(O)$ induced by $W_{g'}^{2,r}(M)$.]

§49. A CLASSICAL EXAMPLE

Suppose that (M,g) is a complete riemannian manifold. Let

$$E = C_c^\infty(M) \oplus C_c^\infty(M)$$

and put

$$\sigma((f_1,f_2),(f_1',f_2')) = \int_M (f_1 f_2' - f_1' f_2) d\mu_g.$$

Then the pair (E,σ) is a symplectic vector space.

[Note: μ_g is the riemannian measure attached to g.]

49.1 <u>LEMMA</u> Define $J:E \to E$ by

$$J(f_1,f_2) = (-f_2,f_1).$$

Then J is a Kähler structure on (E,σ).

<u>PROOF</u> There are two points.

- $\sigma(J(f_1,f_2),J(f_1',f_2'))$

$$= \sigma((-f_2,f_1),(-f_2',f_1'))$$

$$= \langle -f_2,f_1'\rangle - \langle -f_2',f_1\rangle$$

$$= \langle f_1,f_2'\rangle - \langle f_1',f_2\rangle$$

$$= \sigma((f_1,f_2),(f_1',f_2')).$$

• $\sigma((f_1,f_2),J(f_1,f_2))$

$$= \sigma((f_1,f_2),(-f_2,f_1))$$

$$= \langle f_1,f_1\rangle - \langle -f_2,f_2\rangle$$

$$= \langle f_1,f_1\rangle + \langle f_2,f_2\rangle$$

$$> 0 \quad (f_1 \neq 0 \ \& \ f_2 \neq 0).$$

The <u>energy inner product</u> μ_E on

$$E = C_c^\infty(M) \oplus C_c^\infty(M)$$

is defined by

$$\mu_E((f_1,f_2),(f_1',f_2'))$$

$$= \int_M (f_1(1-\Delta_g)f_1' + f_2f_2')d\mu_g.$$

49.2 <u>LEMMA</u> We have

$$\mu_E \in IP(E,\sigma).$$

<u>PROOF</u> View the pairs

$$\begin{bmatrix} (f_1,f_2) \\ (f_1',f_2') \end{bmatrix}$$

as elements of $L^2(M,\mu_g)$:

$$\left|\begin{array}{l} f_1 + \sqrt{-1}\, f_2 \\ \\ f_1' + \sqrt{-1}\, f_2'. \end{array}\right.$$

Then

$$\begin{aligned} &\langle f_1 + \sqrt{-1}\, f_2, f_1' + \sqrt{-1}\, f_2'\rangle \\ &= \langle f_1, f_1'\rangle + \langle f_2, f_2'\rangle \\ &\qquad + \sqrt{-1}\, [\langle f_1, f_2'\rangle - \langle f_1', f_2\rangle] \\ &= \langle f_1, f_1'\rangle + \langle f_2, f_2'\rangle \\ &\qquad + \sqrt{-1}\, \sigma((f_1,f_2),(f_1',f_2')). \end{aligned}$$

Therefore

$$\begin{aligned} &|\sigma((f_1,f_2),(f_1',f_2'))|^2 \\ &\le ||f_1 + \sqrt{-1}\, f_2||^2 \cdot ||f_1' + \sqrt{-1}\, f_2'||^2 \\ &= (\langle f_1,f_1\rangle + \langle f_2,f_2\rangle)\cdot(\langle f_1',f_1'\rangle + \langle f_2',f_2'\rangle) \\ &\le (\langle f_1,(1 - \Delta_g)f_1\rangle + \langle f_2,f_2\rangle)\cdot(\langle f_1',(1 - \Delta_g)f_1'\rangle + \langle f_2',f_2'\rangle) \\ &= \mu_E\,((f_1,f_2),(f_1,f_2)) \cdot \mu_E\,((f_1',f_2'),(f_1',f_2')). \end{aligned}$$

The energy inner product μ_E is not pure. To compute its purification, observe first that

$$H_{\mu_E} = W_g^{2,1}(M) \oplus L^2(M,\mu_g),$$

where, of course, the spaces are taken over $\underline{R}$. Recall now that

$$A_{\mu_E}: H_{\mu_E} \to H_{\mu_E}$$

is characterized by the condition

$$\sigma_{\mu_E}(x,y) = \mu_E(x,A_{\mu_E}y) \quad (x,y \in H_{\mu_E}).$$

Agreeing to regard the elements of E as column vectors, we then claim that

$$A_{\mu_E} = \begin{bmatrix} 0 & (1-\Delta_g)^{-1} \\ \\ -I & 0 \end{bmatrix}.$$

In fact,

$$\mu_E((f_1,f_2),A_{\mu_E}(f_1',f_2'))$$

$$= \mu_E((f_1,f_2),((1-\Delta_g)^{-1}f_2', -f_1'))$$

$$= \langle f_1,(1-\Delta_g)(1-\Delta_g)^{-1}f_2'\rangle + \langle f_2,-f_1'\rangle$$

$$= \langle f_1,f_2'\rangle - \langle f_1',f_2\rangle$$

$$= \sigma((f_1,f_2),(f_1',f_2')).$$

[Note: It follows that A_{μ_E} is injective, hence σ_{μ_E} is symplectic (cf. 20.12).]

49.3 <u>REMARK</u> The operator $(1 - \Delta_g)^{-1}$ is a bounded linear transformation from $L^2(M,\mu_g)$ to $W_g^{2,2}(M) \subset W_g^{2,1}(M)$.

49.4 <u>LEMMA</u> Let

$$A_{\mu_E} = J_{\mu_E}|A_{\mu_E}|$$

be the polar decomposition of A_{μ_E} -- then

$$J_{\mu_E} = \begin{bmatrix} 0 & (1 - \Delta_g)^{-1/2} \\ -(1 - \Delta_g)^{1/2} & 0 \end{bmatrix}$$

and

$$|A_{\mu_E}| = \begin{bmatrix} (1 - \Delta_g)^{-1/2} & 0 \\ 0 & (1 - \Delta_g)^{-1/2} \end{bmatrix}.$$

<u>PROOF</u> It is clear that

$$A_{\mu_E} = J_{\mu_E}|A_{\mu_E}|.$$

J_{μ_E} is orthogonal: We have

$$< J_{\mu_E} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, J_{\mu_E} \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} >$$

$$= < \begin{bmatrix} (1 - \Delta_g)^{-1/2} f_2 \\ -(1 - \Delta_g)^{1/2} f_1 \end{bmatrix}, \begin{bmatrix} (1 - \Delta_g)^{-1/2} f_2' \\ -(1 - \Delta_g)^{1/2} f_1' \end{bmatrix} >$$

$$= < (1 - \Delta_g)^{-1/2} f_2, (1 - \Delta_g)^{-1/2} f_2' >_{W^{2,1}}$$

$$+ < (1 - \Delta_g)^{1/2} f_1, (1 - \Delta_g)^{1/2} f_1' >_{L^2}$$

$$= < (1 - \Delta_g)^{-1/2} f_2, (1 - \Delta_g)(1 - \Delta_g)^{-1/2} f_2' >_{L^2}$$

$$+ < f_1, (1 - \Delta_g) f_1' >_{L^2}$$

$$= < f_2, f_2' >_{L^2} + < f_1, f_1' >_{W^{2,1}}$$

$$= < \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} >.$$

$|A_{\mu_E}|$ is nonnegative: We have

$$< \begin{bmatrix} f_1 \\ \\ f_2 \end{bmatrix}, |A_{\mu_E}| \begin{bmatrix} f_1 \\ \\ f_2 \end{bmatrix} >$$

$$= < \begin{bmatrix} f_1 \\ \\ f_2 \end{bmatrix}, \begin{bmatrix} (1 - \Delta_g)^{-1/2} f_1 \\ \\ (1 - \Delta_g)^{-1/2} f_2 \end{bmatrix} >$$

$$= < f_1, (1 - \Delta_g)^{-1/2} f_1 >_{W^{2,1}} + < f_2, (1 - \Delta_g)^{-1/2} f_2 >_{L^2}$$

$$= < f_1, (1 - \Delta_g)^{1/2} f_1 >_{L^2} + < f_2, (1 - \Delta_g)^{-1/2} f_2 >_{L^2}$$

$$\geq 0.$$

Write $\mu_{E,p}$ for the purification of μ_E:

$$\mu_{E,p}((f_1,f_2), (f_1',f_2'))$$

$$= \mu_E((f_1,f_2), |A_{\mu_E}|(f_1',f_2'))$$

$$= \mu_E((f_1,f_2), ((1 - \Delta_g)^{-1/2} f_1', (1 - \Delta_g)^{-1/2} f_2'))$$

$$= \langle f_1, (1 - \Delta_g)(1 - \Delta_g)^{-1/2} f_1' \rangle + \langle f_2, (1 - \Delta_g)^{-1/2} f_2' \rangle$$

$$= \langle f_1, (1 - \Delta_g)^{1/2} f_1' \rangle + \langle f_2, (1 - \Delta_g)^{-1/2} f_2' \rangle$$

$$= \langle f_1, f_1' \rangle_{2,1/2} + \langle f_2, f_2' \rangle_{2,-1/2},$$

the Sobolev inner product per

$$\mathcal{H}_{\mu_{E,p}} = W_g^{2,1/2}(M) \oplus W_g^{2,-1/2}(M).$$

Here $|A_{\mu_{E,p}}| = I$ (cf. 20.25) and

$$J_{\mu_{E,p}} = \begin{bmatrix} 0 & (1 - \Delta_g)^{-1/2} \\ -(1 - \Delta_g)^{1/2} & 0 \end{bmatrix}.$$

Proof:

$$\mu_{E,p}((f_1, f_2), J_{\mu_{E,p}}(f_1', f_2'))$$

$$= \mu_{E,p}((f_1, f_2), ((1 - \Delta_g)^{-1/2} f_2', -(1 - \Delta_g)^{1/2} f_1'))$$

$$= \langle f_1, (1 - \Delta_g)^{1/2} (1 - \Delta_g)^{-1/2} f_2' \rangle$$

$$+ \langle f_2, (1 - \Delta_g)^{-1/2} -(1 - \Delta_g)^{1/2} f_1' \rangle$$

$$= \langle f_1, f_2' \rangle - \langle f_1', f_2 \rangle$$

$$= \sigma((f_1, f_2), (f_1', f_2')).$$

49.5 REMARK The operators

$$\begin{cases} (1 - \Delta_g)^{-1/2} : W_g^{2,-1/2}(M) \to W_g^{2,1/2}(M) \\ (1 - \Delta_g)^{1/2} : W_g^{2,1/2}(M) \to W_g^{2,-1/2}(M) \end{cases}$$

are bounded linear transformations, so everything makes sense.

Since $\mu_{E,p}$ is pure, one can realize 20.19 directly (see the discussion after 20.27): Use the isometric complex structure

$$- J_{\mu_{E,p}} : H_{\mu_{E,p}} \to H_{\mu_{E,p}}$$

to convert $H_{\mu_{E,p}}$ into a complex Hilbert space $H^{\sim}_{\mu_{E,p}}$ with inner product

$$\langle x,y \rangle = \mu_{E,p}(x,y) - \sqrt{-1}\, \mu_{E,p}(x, -J_{\mu_{E,p}} y)$$

or still,

$$\langle x,y \rangle = \mu_{E,p}(x,y) + \sqrt{-1}\, \sigma_{\mu_{E,p}}(x,y).$$

Now take

$$\begin{cases} x = (f_1, f_2) \\ y = (f_1', f_2') \end{cases}$$

and let $k_\mu : E \rightarrow \mathcal{H}^\sim_{\mu_{E,p}}$ be the inclusion -- then

$$< k_\mu(f_1,f_2),\ k_\mu(f_1',f_2') >$$

$$= \mu_{E,p}((f_1,f_2),(f_1',f_2')) + \sqrt{-1}\ \sigma((f_1,f_2),(f_1',f_2')),$$

as desired.

[Note: According to the theory, the assignment

$$\delta_{(f_1,f_2)} \rightarrow W(k_\mu(f_1,f_2))$$

defines an irreducible representation of $\mathcal{W}(E,\sigma)$ on $BO(\mathcal{H}^\sim_{\mu_{E,p}})$ which is the GNS representation associated with the state

$$\omega_{\mu_{E,p}}(\delta_{(f_1,f_2)})$$

$$= \exp(-\frac{1}{4}\mu_{E,p}((f_1,f_2),(f_1,f_2))).]$$

Specialize and take $M = \underline{R}^n$ (g = euclidean metric) -- then

$$\mathcal{H}_{\mu_{E,p}} = W^{2,1/2}(\underline{R}^n) \oplus W^{2,-1/2}(\underline{R}^n).$$

Let

$$Q(f) = <f,\ (1 - \Delta)^{1/2}f>_{L^2(\underline{R}^n)} \quad (f \in \mathcal{S}(\underline{R}^n)).$$

Since $\mathcal{S}(\underline{R}^n)$ is nuclear, $e^{-Q/2}$ is the Fourier transform of a unique gaussian measure γ on $\mathcal{S}(\underline{R}^n)^*$ (cf. §34 (e.g. 34.15)). Here

$$\begin{array}{ccc} W^{2,1/2}(\underline{R}^n) \approx S(\underline{R}^n)_\gamma & \longrightarrow & L^2(S(\underline{R}^n)^*,\gamma) \\ R_\gamma \downarrow & & \\ H(\gamma) \approx W^{2,1/2}(\underline{R}^n). & & \end{array}$$

[Note: On general grounds (cf. 34.14), there is an isometric isomorphism

$$BO(W^{2,1/2}(\underline{R}^n)) \xrightarrow{T} L^2(S(\underline{R}^n)^*,\gamma).]$$

Denote by

$$[\ ,\ ]:W^{2,1/2}(\underline{R}^n) \times W^{2,-1/2}(\underline{R}^n) \to \underline{R}$$

the canonical pairing -- then $\forall\ \phi \in S(\underline{R}^n)_\gamma$, $\exists$ a unique $\lambda_\phi \in W^{2,-1/2}(\underline{R}^n)$ such that

$$\phi(h) = [h,\lambda_\phi] \quad (h \in W^{2,1}(\underline{R}^n)).$$

49.6 LEMMA The arrow

$$\begin{cases} S(\underline{R}^n)_\gamma \to W^{2,-1/2}(\underline{R}^n) \\ \phi \to \lambda_\phi \end{cases}$$

is bijective with inverse

$$\begin{cases} W^{2,-1/2}(\underline{R}^n) \to S(\underline{R}^n) \\ \lambda \to \phi_\lambda. \end{cases}$$

Passing from $\underline{R}$ to $\underline{C}$ and imitating what was done in the formulation of the real wave representation, we shall now construct a Weyl system over

$$W^{2,1/2}(\underline{R}^n) \oplus W^{2,-1/2}(\underline{R}^n).$$

$\underline{U}$: Given $h \in W^{2,1/2}(\underline{R}^n)$, let

$$U(h):L^2(S(\underline{R}^n)^*,\gamma) \to L^2(S(\underline{R}^n)^*,\gamma)$$

be the operator defined by the rule

$$U(h)\psi(x) = \psi(x+h)\left[\frac{d\gamma_{-h}}{d\gamma}(x)\right]^{1/2}.$$

$\underline{V}$: Given $\lambda \in W^{2,-1/2}(\underline{R}^n)$, let

$$V(\lambda):L^2(S(\underline{R}^n)^*,\gamma) \to L^2(S(\underline{R}^n)^*,\gamma)$$

be the operator defined by the rule

$$V(\lambda)\psi(x) = e^{\sqrt{-1}\,\phi_\lambda(x)}\psi(x).$$

The definitions then imply that

$$U(h)V(\lambda) = \exp(\sqrt{-1}\,[h,\lambda])V(\lambda)U(h).$$

[Note: Observe that

$$\phi_\lambda(x+h) = \phi_\lambda(x) + \phi_\lambda(h)$$

$$= \phi_\lambda(x) + [h,\lambda_{\phi_\lambda}]$$

$$= \phi_\lambda(x) + [h,\lambda].]$$

Following the standard procedure, put

$$W(h \oplus \lambda) = \exp(\frac{\sqrt{-1}}{2}[h,\lambda])U(-h)V(\lambda).$$

Then W defines a Weyl system over

$$W^{2,1/2}(\underline{R}^n) \oplus W^{2,-1/2}(\underline{R}^n).$$

[Note: The underlying symplectic structure σ is induced from [,] in the usual way:

$$\sigma((h,\lambda),(h',\lambda')) = [h,\lambda'] - [h',\lambda].$$

Since

$$f_1,f_2 \in C_c^\infty(\underline{R}^n) \Rightarrow [f_1,f_2] = \langle f_1,f_2\rangle_{L^2(\underline{R}^n)},$$

it follows that W restricts to a Weyl system over (E,σ).]

Next

$$\langle 1,W(h \oplus \lambda)1\rangle_{L^2(\gamma)}$$

$$= \langle 1, \exp(\frac{\sqrt{-1}}{2}[h,\lambda])U(-h)V(\lambda)1\rangle_{L^2(\gamma)}$$

$$= \exp(\frac{\sqrt{-1}}{2}[h,\lambda])\langle 1,U(-h)V(\lambda)1\rangle_{L^2(\gamma)}$$

$$= \exp(-\frac{\sqrt{-1}}{2}[h,\lambda])\exp(-\frac{1}{4}||h||^2_{2,1/2})$$

$$\times \int_{S(\underline{R}^n)^*} \exp(\sqrt{-1}\,\phi_\lambda(x) + \hat{h}(x)/2)d\gamma(x)$$

$$= \exp(-\frac{\sqrt{-1}}{2}[h,\lambda])\exp(-\frac{1}{4}||h||^2_{2,1/2})$$

$$\times \exp\left(\frac{1}{2}\left(\frac{1}{4}\ ||h||^2_{2,1/2} + \sqrt{-1}\ [h,\lambda] - ||\lambda||^2_{2,-1/2}\right)\right)$$

$$= \exp\left(- \frac{1}{8}\ ||h||^2_{2,1/2} - \frac{1}{2}\ ||\lambda||^2_{2,-1/2}\right).$$

This makes it plain that it is best to work with W_{mod}, since

$$<1,W_{mod}(h \oplus \lambda)1>_{L^2(\gamma)}$$

$$= \exp\left(- \frac{1}{4}\ (||h||^2_{2,1/2} + ||\lambda||^2_{2,-1/2})\right).$$

In particular: $\forall\ f_1, f_2 \in C^\infty_c(\underline{R}^n)$,

$$<1,W_{mod}(f_1 \oplus f_2)1>_{L^2(\gamma)}$$

$$= \exp\left(- \frac{1}{4}\ (<f_1,f_1>_{2,1/2} + <f_2,f_2>_{2,-1/2})\right)$$

$$= \exp\left(- \frac{1}{4}\ \mu_{E,p}((f_1,f_2),(f_1,f_2))\right).$$

Consequently, the assignment

$$\delta_{(f_1,f_2)} \rightarrow W_{mod}(f_1 \oplus f_2)$$

defines a representation of $\mathcal{W}(E,\sigma)$ on $L^2(\mathcal{S}(\underline{R}^n)^*,\gamma)$ which is the GNS representation associated with the state $\omega_{\mu_{E,p}}$ corresponding to $\mu_{E,p}$.

[Note: The functions $e^{\sqrt{-1}\ <f,\text{—}>}$ $(f \in C^\infty_c(\underline{R}^n))$ are dense in $L^2(\mathcal{S}(R^n)^*,\gamma)$, thus 1 is cyclic.]

49.7 <u>REMARK</u> Define

$$U: \mathcal{H}_{\mu_{E,p}}^{\sim} \rightarrow W^{2,1/2}(\underline{R}^n)_{\underline{C}}$$

by

$$U(f_1,f_2) = (f_1, (1 - \Delta)^{-1/2} f_2).$$

Then it is clear that U is bijective and

$$||U(f_1,f_2)|| = ||(f_1,f_2)||.$$

In addition, U is complex linear:

$$U(- J_{\mu_{E,p}})(f_1,f_2)$$

$$= U \begin{bmatrix} 0 & - (1 - \Delta)^{-1/2} \\ (1 - \Delta)^{1/2} & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$= U(-(1 - \Delta)^{-1/2} f_2, (1 - \Delta)^{1/2} f_1)$$

$$= (-(1 - \Delta)^{-1/2} f_2, f_1),$$

while

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ (1 - \Delta)^{-1/2} f_2 \end{bmatrix}$$

$$= (-(1 - \Delta)^{-1/2} f_2, f_1).$$

§50. *EQUATIONS OF MOTION*

Suppose that $\mathcal{H}$ is a finite dimensional complex Hilbert space, $A:\mathcal{H} \to \mathcal{H}$ a selfadjoint operator -- then the <u>quantization</u> of the pair $(\mathcal{H},A)$ is the pair

$$(BO(\mathcal{H}),\ d\Gamma(A) + \frac{1}{2}\operatorname{tr}(A)).$$

50.1 <u>EXAMPLE</u> (The Harmonic Oscillator) In the (q,p)-plane, let

$$H(q,p) = \frac{1}{2}(q^2 + p^2).$$

Then H is the hamiltonian for the harmonic oscillator, viewed as a classical mechanical system. To quantize it, we shall first convert to an equivalent quantum mechanical system. To this end, take $\mathcal{H} = \underline{C}$ and $A = I$ -- then the Schrödinger equation per $(\underline{C},I)$ is equivalent to the equations of motion

$$\begin{cases} \dot{q} = p \\ \dot{p} = -q \end{cases}$$

per H. Thus fix (q_0,p_0) -- then the classical trajectory through (q_0,p_0) is

$$\begin{cases} q(t) = q_0 \cos t + p_0 \sin t \\ p(t) = -q_0 \sin t + p_0 \cos t. \end{cases}$$

On the other hand, put

$$(Q(t),P(t)) = e^{-\sqrt{-1}\,tI}(q_0,p_0).$$

Then the Schrödinger equation is

$$\sqrt{-1}\,\frac{d}{dt}\,(e^{-\sqrt{-1}\,tI}\,(q_0,p_0)) = (Q(t),P(t)).$$

But

$$e^{-\sqrt{-1}\,tI}\,(q_0,p_0) = e^{-\sqrt{-1}\,t}\,(q_0,p_0)$$

$$= (\cos t - \sqrt{-1}\sin t)(q_0 + \sqrt{-1}\,p_0)$$

$$= q_0 \cos t + p_0 \sin t + \sqrt{-1}\,(- q_0 \sin t + p_0 \cos t)$$

=>

$$\sqrt{-1}\,e^{-\sqrt{-1}\,tI}\,(q_0,p_0)$$

$$= q_0 \sin t - p_0 \cos t + \sqrt{-1}\,(q_0 \cos t + p_0 \sin t)$$

=>

$$\frac{d}{dt}\,(q_0 \sin t - p_0 \cos t, q_0 \cos t + p_0 \sin t)$$

$$= (q_0 \cos t + p_0 \sin t, - q_0 \cos t + p_0 \cos t)$$

=>

$$\left[\begin{array}{l} Q(t) = q(t) \\ P(t) = p(t). \end{array}\right.$$

Applying now the quantization procedure to the pair $(\underline{C},I)$ gives the pair $(BO(\underline{C}),N + \frac{1}{2})$ and when transferred to $L^2(\underline{R})$, we have (cf. 8.7)

$$U_{\sqrt{2}} T_G T(N + \frac{1}{2}) T^{-1} T_G^{-1} U_{\sqrt{2}}^{-1}$$

$$= \frac{1}{2} [- \frac{d^2}{dx^2} + x^2],$$

the hamiltonian for the harmonic oscillator, viewed as a quantum mechanical system.

It is a standard observation that a quantum mechanical system can always be viewed as a classical mechanical system in the sense that the Schrödinger equations are an instance of Hamilton's equations.

Thus suppose that H is a complex Hilbert space. Let $A:\text{Dom}(A) \to H$ be selfadjoint. Put $X_A = - \sqrt{-1}\, A$ and define

$$< A > :\text{Dom}(A) \to \underline{R}$$

by

$$< A >(x) = \frac{1}{2} <x,Ax>.$$

50.2 <u>LEMMA</u> On Dom(A),

$$\iota_{X_A} \text{Im} < , > = d< A >.$$

I.e.: $\forall\ x,y \in \text{Dom}(A)$,

$$\text{Im}<X_A x,y> = d< A >\Big|_x (y).$$

PROOF We have

$$d< A >\Big|_x (y)$$

$$= \frac{d}{d\varepsilon} < A >(x + \varepsilon y) \Big|_{\varepsilon=0}$$

$$= \frac{1}{2} \frac{d}{d\varepsilon} <x + \varepsilon y, A(x + \varepsilon y)> \Big|_{\varepsilon=0}$$

$$= \frac{1}{2} (<y,Ax> + <x,Ay>)$$

$$= \frac{1}{2} (<y,Ax> + <Ax,y>)$$

$$= \frac{1}{2} (<y,Ax> + \overline{<y,Ax>})$$

$$= \mathrm{Re}<y,Ax>$$

$$= \mathrm{Re}\overline{<y,Ax>}$$

$$= \mathrm{Re}<Ax,y>$$

$$= \mathrm{Re}<\sqrt{-1}\, X_A x,y>$$

$$= \mathrm{Im}<X_A x,y>.$$

Therefore X_A is a hamiltonian vector field with energy $< A >$. This said, the flow of X_A is the function

$$\phi_A : \underline{R} \times \mathrm{Dom}(A) \to \mathrm{Dom}(A)$$

defined by

$$\phi_A(t,x) = (e^{tX_A})x$$

$$\equiv x_t,$$

the curve $t \to x_t$ being the trajectory of X_A through x:

$$\dot{x}_t = X_A x_t,$$

which are Hamilton's equations for $< A >$.

N.B.

$$X_A x_t = - \sqrt{-1}\, A x_t$$

$$\Rightarrow$$

$$\sqrt{-1}\, \dot{x}_t = A x_t,$$

the Schrödinger equation.

Suppose now that H_0 is a real Hilbert space.

50.3 LEMMA Let $T:\text{Dom}(T) \to H_0$ be densely defined and closed -- then on $\text{Dom}(T)$, the prescription

$$<\psi,\psi'>_T = <\psi,\psi'> + <T\psi,T\psi'>$$

equips $\text{Dom}(T)$ with the structure of a real Hilbert space.

[Note: Assume that T is selfadjoint and $\geq I$ -- then $\text{Dom}(T^{1/2})$ is a real Hilbert space with inner product

$$<\psi,\psi'>_{T^{1/2}} = <T^{1/2}\psi,T^{1/2}\psi'>.$$

In fact, $\forall\ \psi \in \text{Dom}(T^{1/2})$,

$$||T^{1/2}\psi||^2 \le ||\psi||^2 + ||T^{1/2}\psi||^2 \le 2||T^{1/2}\psi||^2.]$$

50.4 <u>EXAMPLE</u> (The Abstract Wave Equation) Assume that $T:\text{Dom}(T) \to \mathcal{H}_0$ is selfadjoint and $\ge I$ -- then

$$\mathcal{H}_T = \text{Dom}(T^{1/2}) \oplus \mathcal{H}_0$$

is a real Hilbert space with norm

$$||(\psi,x)||_{\mathcal{H}_T} = [\langle T^{1/2}\psi, T^{1/2}\psi\rangle + \langle x,x\rangle]^{1/2}.$$

Define $\sigma:\mathcal{H}_T \times \mathcal{H}_T \to \underline{R}$ by

$$\sigma((\psi,x),(\psi',x')) = \langle\psi,x'\rangle - \langle\psi',x\rangle.$$

Then the pair $(\mathcal{H}_T,\sigma)$ is a symplectic vector space. Put

$$E(\psi,x) = \frac{1}{2}[\langle\psi,T\psi\rangle + \langle x,x\rangle]$$

and let

$$X = \begin{bmatrix} 0 & I \\ -T & 0 \end{bmatrix},$$

where

$$\text{Dom}(X) = \text{Dom}(T) \oplus \text{Dom}(T^{1/2}).$$

The definitions then imply that on Dom(X),

$$\iota_X\sigma = dE \quad (\text{cf. } 50.2),$$

so X is a hamiltonian vector field, thus the equations of motion are

$$\dot{\gamma}(t) = X\gamma(t).$$

Written out, if $\gamma(t) = (\psi(t), x(t))$, then

$$\begin{bmatrix} \dot{\psi}(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -T & 0 \end{bmatrix} \cdot \begin{bmatrix} \psi(t) \\ x(t) \end{bmatrix}$$

$$= \begin{bmatrix} x(t) \\ -T\psi(t) \end{bmatrix}$$

=>

$$\left[\begin{array}{l} \dot{\psi}(t) = x(t) \\ \dot{x}(t) = -T\psi(t) \end{array}\right.$$

or still,

$$\ddot{\psi}(t) + T\psi(t) = 0.$$

Now let

$$J = \begin{bmatrix} 0 & T^{-1/2} \\ -T^{1/2} & 0 \end{bmatrix}.$$

Then

$$J: \mathcal{H}_T \to \mathcal{H}_T$$

is an isometric complex structure, hence $\mathcal{H}_T^{\sim}$ is a complex Hilbert space, the inner product being

$$< \ , \ >_{\mathcal{H}_T} - \sqrt{-1} < \ ,J \ >_{\mathcal{H}_T}.$$

It is straightforward to check that X is skewadjoint (note that X commutes with J), thus

$$H = \sqrt{-1}\, X$$

is selfadjoint. Here

$$\exp(-\sqrt{-1}\, tH) = \begin{bmatrix} \cos(tT^{1/2}) & T^{-1/2}\sin(tT^{1/2}) \\ -T^{1/2}\sin(tT^{1/2}) & \cos(tT^{1/2}) \end{bmatrix}.$$

Given $(\psi,x) \in \mathrm{Dom}(H)$ $(= \mathrm{Dom}(X))$, let

$$\gamma(t) = \exp(-\sqrt{-1}\, tH)(\psi,x).$$

Then

$$\sqrt{-1}\,\dot{\gamma}(t) = H\gamma(t).$$

I.e.:

$$\dot{\gamma}(t) = X\gamma(t).$$

Therefore the Schrödinger equation per H and the Hamilton equation per X are one and the same.

[Note: The pair $(\mathcal{H}_T, X)$ is a classical mechanical system.]

50.5 REMARK X stays skewadjoint if J is replaced by -J and

$$-\sqrt{-1}\, tH = -JtJX = -(-J)t(-J)X.$$

To realize this set up, let (M,g) be a complete riemannian manifold and take

$$\left[\begin{array}{l} \mathcal{H}_0 = L^2(M,\mu_g) \\ \\ T = 1 - \Delta_g. \end{array}\right.$$

Then

$$\text{Dom}(T^{1/2}) = W_g^{2,1}(M)$$

=>

$$\mathcal{H}_T = W_g^{2,1}(M) \oplus L^2(M,\mu_g).$$

The hamiltonian vector field X is defined on the dense subspace

$$W_g^{2,2}(M) \oplus W_g^{2,1}(M)$$

and the equations of motion become

$$\partial_t^2\psi + (1 - \Delta_g)\psi = 0.$$

50.6 REMARK Return to 50.4 and consider the pair $(\text{BO}(\mathcal{H}_T^{\sim}),d\Gamma(H))$ — then one may attach to each $(\psi,x) \in \mathcal{H}_T^{\sim}$ the Weyl operator

$$W(\psi,x) = \exp(\sqrt{-1}\ \overline{Q(\psi,x)}) \qquad (\text{cf. } 10.3)$$

and 9.7 implies that

$$\Gamma(\exp(-\sqrt{-1}\ tH))W(\psi,x)\Gamma(\exp(\sqrt{-1}\ tH))$$

$$= W(\exp(-\sqrt{-1}\ tH)(\psi,x))$$

$$= W(\gamma(t)).$$

Formally, therefore,

$$\frac{d}{dt} W(\gamma(t)) = W(\dot{\gamma}(t))$$

$$= W(X\gamma(t)).$$

[Note: It is not difficult to make this rigorous.]

§51. *EXTENSION OF THE THEORY*

Suppose that (M,g) is a complete riemannian manifold -- then the restriction to $C_c^\infty(M)$ of the laplacian Δ_g is essentially selfadjoint and the energy inner product μ_E on

$$E = C_c^\infty(M) \oplus C_c^\infty(M)$$

is defined by

$$\mu_E((f_1,f_2),(f_1',f_2'))$$

$$= \int_M (f_1(1 - \Delta_g)f_1' + f_2f_2')d\mu_g.$$

These considerations will now be generalized. Thus fix $\alpha \in C^\infty(M):1 \leq \alpha \leq C$ and put

$$\mu_\alpha((f_1,f_2),(f_1',f_2'))$$

$$= \int_M (f_1\alpha(1 - \Delta_g)f_1' + \alpha f_2f_2')d\mu_g$$

$$- \int_M f_1g(d\alpha,df_1')d\mu_g.$$

[Note: Take $\alpha \equiv 1$ -- then $\mu_1 = \mu_E$.]

51.1 <u>LEMMA</u> We have

$$\mu_\alpha \in IP(E,\sigma) \quad \text{(cf. 49.2)}.$$

The proof of this hinges on an integral formula.

51.2 <u>LEMMA</u> Let $f,f' \in C_c^\infty(M)$; let $\alpha \in C^\infty(M)$ — then

$$\int_M f(\alpha(-\Delta_g f') - g(d\alpha, df'))d\mu_g$$

$$= \int_M \alpha g(df, df')d\mu_g.$$

<u>PROOF</u>

1. We have

$$\text{grad}(f'\alpha) = (\text{grad } f')\alpha + f'(\text{grad } \alpha).$$

Therefore

$$\int_M \alpha g(df, df')d\mu_g$$

$$= \int_M \alpha g(\text{grad } f, \text{grad } f')d\mu_g$$

$$= \int_M g(\text{grad } f, (\text{grad } f')\alpha)d\mu_g$$

$$= \int_M g(\text{grad } f, \text{grad}(f'\alpha))d\mu_g$$

$$- \int_M g(\text{grad } f, f'(\text{grad } \alpha))d\mu_g$$

$$= - \int_M f\Delta_g(f'\alpha)d\mu_g$$

$$- \int_M g(\text{grad } f, f'(\text{grad } \alpha))d\mu_g.$$

2. We have

$$\text{grad}(f'f) = (\text{grad } f')f + f'(\text{grad } f).$$

Therefore

$$\int_M g(\text{grad } f, f'(\text{grad } \alpha))d\mu_g$$

$$= \int_M g(f'(\text{grad } f), \text{grad } \alpha)d\mu_g$$

$$= \int_M g(\text{grad}(f'f), \text{grad } \alpha)d\mu_g$$

$$- \int_M g((\text{grad } f')f, \text{grad } \alpha)d\mu_g$$

$$= - \int_M f(f'\Delta_g\alpha)d\mu_g$$

$$- \int_M fg(\text{grad } f', \text{grad } \alpha)d\mu_g.$$

Combine terms to get

$$\int_M \alpha g(df, df')d\mu_g$$

$$= - \int_M f\Delta_g(f'\alpha)d\mu_g + \int_M f(f'(\Delta_g\alpha))d\mu_g$$

$$+ \int_M fg(\text{grad } f', \text{grad } \alpha)d\mu_g.$$

But

$$\Delta_g(f'\alpha)$$

$$= f'(\Delta_g\alpha) + \alpha(\Delta_g f') + 2g(\text{grad } f', \text{grad } \alpha).$$

Inserting this then leads to the stated formula.

Thanks to 51.2, μ_α is symmetric. Next

$$\mu_\alpha((f_1,f_2),(f_1,f_2))$$

$$= \int_M \alpha[(f_1)^2 + g(df_1,df_1) + (f_2)^2]d\mu_g$$

$$\geq \int_M [(f_1)^2 + g(df_1,df_1) + (f_2)^2]d\mu_g$$

$$= \int_M [(f_1)^2 - f_1(\Delta_g f_1) + (f_2)^2]d\mu_g$$

$$= \int_M (f_1(1 - \Delta_g)f_1 + f_2^2)d\mu_g$$

$$= \mu_E((f_1,f_2),(f_1,f_2)).$$

Ditto if (f_1,f_2) is replaced by (f_1',f_2'). But then

$$|\sigma((f_1,f_2),(f_1',f_2'))|^2$$

$$\leq \mu_E((f_1,f_2),(f_1,f_2)) \cdot \mu_E((f_1',f_2'),(f_1',f_2'))$$

$$\leq \mu_\alpha((f_1,f_2),(f_1,f_2)) \cdot \mu_\alpha((f_1',f_2'),(f_1',f_2'))$$

$\Rightarrow$

$$\mu_\alpha \in \text{IP}(E,\sigma).$$

51.3 <u>LEMMA</u> Let

$$A:C_c^\infty(M) \to C_c^\infty(M)$$

be defined by

$$Af = \alpha(1 - \Delta_g)f - g(d\alpha, df).$$

Then A is essentially selfadjoint.

[Note: The closure $\bar{A}$ is selfadjoint, $\geq I$, and has a bounded inverse.]

51.4 <u>REMARK</u> Due to our assumption on α, the multiplication operator M_α is bounded and selfadjoint with inverse $M_{1/\alpha}$.

In what follows, we shall omit the overbar that signifies closure and identify a multiplication operator with its underlying function.

Like μ_E, μ_α is not pure. Here

$$H_{\mu_\alpha} = \text{Dom}(A^{1/2}) \oplus \text{Dom}(\alpha^{1/2})$$

and

$$A_{\mu_\alpha} : H_{\mu_\alpha} \to H_{\mu_\alpha}$$

is characterized by the condition

$$\sigma_{\mu_\alpha}(x,y) = \mu_\alpha(x, A_{\mu_\alpha} y) \quad (x,y \in H_{\mu_\alpha}).$$

One can be explicit:

$$A_{\mu_\alpha} = \begin{bmatrix} 0 & A^{-1} \\ & \\ -\alpha^{-1} & 0 \end{bmatrix}.$$

For

$$\mu_\alpha((f_1,f_2),A_{\mu_\alpha}(f_1',f_2'))$$

$$= \mu_\alpha((f_1,f_2),(A^{-1}f_2', -\alpha^{-1}f_1'))$$

$$= \langle f_1,AA^{-1}f_2'\rangle + \langle \alpha f_2, -\alpha^{-1}f_1'\rangle$$

$$= \langle f_1,f_2'\rangle - \langle f_1',f_2\rangle$$

$$= \sigma((f_1,f_2),(f_1',f_2')).$$

[Note: It follows that A_{μ_α} is injective, hence σ_{μ_α} is symplectic (cf. 20.12).]

51.5 LEMMA Let

$$A_{\mu_\alpha} = J_{\mu_\alpha}|A_{\mu_\alpha}|$$

be the polar decomposition of A_{μ_α}. Put

$$A_\alpha = \alpha^{1/2}A\alpha^{1/2}.$$

Then

$$J_{\mu_\alpha} = \begin{bmatrix} 0 & \alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2} \\ -\alpha^{-1/2}A_\alpha^{1/2}\alpha^{-1/2} & 0 \end{bmatrix}$$

and

$$|A_{\mu_\alpha}| = \begin{bmatrix} \alpha^{1/2}A_\alpha^{-1/2}\alpha^{-1/2} & 0 \\ 0 & \alpha^{-1/2}A_\alpha^{-1/2}\alpha^{1/2} \end{bmatrix}.$$

PROOF It is clear that

$$A_{\mu_\alpha} = J_{\mu_\alpha}|A_{\mu_\alpha}|.$$

J_{μ_α} is orthogonal: We have

$$< J_{\mu_\alpha}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, J_{\mu_\alpha}\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} >$$

$$= < \begin{bmatrix} \alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2 \\ -\alpha^{-1/2}A_\alpha^{1/2}\alpha^{-1/2}f_1 \end{bmatrix}, \begin{bmatrix} \alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2' \\ -\alpha^{-1/2}A_\alpha^{1/2}\alpha^{-1/2}f_1' \end{bmatrix} >$$

$$= <\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2,\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2'>_{A^{1/2}}$$

$$+ <\alpha^{-1/2}A_\alpha^{1/2}\alpha^{-1/2}f_1,\alpha^{-1/2}A_\alpha^{1/2}\alpha^{-1/2}f_1'>_{\alpha^{1/2}}$$

$$= <A^{1/2}\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2,A^{1/2}\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2'>_{L^2}$$

$$+ \langle \alpha^{1/2}\alpha^{-1/2}A_\alpha^{1/2}\alpha^{-1/2}f_1, \alpha^{1/2}\alpha^{-1/2}A_\alpha^{1/2}\alpha^{-1/2}f_1' \rangle_{L^2}$$

$$= \langle A^{1/2}\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2, A^{1/2}\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2' \rangle_{L^2}$$

$$+ \langle A_\alpha^{1/2}\alpha^{-1/2}f_1, A_\alpha^{1/2}\alpha^{-1/2}f_1' \rangle_{L^2} .$$

And

- $\langle A^{1/2}\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2, A^{1/2}\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2' \rangle_{L^2}$

$$= \langle \alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2, A\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2' \rangle_{L^2}$$

$$= \langle \alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2, \alpha^{-1/2}A_\alpha\alpha^{-1/2}\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2' \rangle_{L^2}$$

$$= \langle A_\alpha^{-1/2}\alpha^{1/2}f_2, A_\alpha^{1/2}\alpha^{1/2}f_2' \rangle_{L^2}$$

$$= \langle \alpha^{1/2}f_2, \alpha^{1/2}f_2' \rangle_{L^2}$$

$$= \langle f_2, f_2' \rangle_{\alpha^{1/2}} ;$$

- $\langle A_\alpha^{1/2}\alpha^{-1/2}f_1, A_\alpha^{1/2}\alpha^{-1/2}f_1' \rangle_{L^2}$

$$= \langle \alpha^{-1/2}f_1, A_\alpha\alpha^{-1/2}f_1' \rangle_{L^2}$$

$$= \langle \alpha^{-1/2} f_1, \alpha^{1/2} A \alpha^{1/2} \alpha^{-1/2} f_1' \rangle_{L^2}$$

$$= \langle f_1, A f_1' \rangle_{L^2}$$

$$= \langle A^{1/2} f_1, A^{1/2} f_1' \rangle_{L^2}$$

$$= \langle f_1, f_1' \rangle_{A^{1/2}} .$$

<u>$|A_{\mu_\alpha}|$ is nonnegative:</u> We have

$$\left\langle \begin{bmatrix} f_1 \\ \\ f_2 \end{bmatrix}, \quad |A_{\mu_\alpha}| \begin{bmatrix} f_1 \\ \\ f_2 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} f_1 \\ \\ f_2 \end{bmatrix}, \begin{bmatrix} \alpha^{1/2} A_\alpha^{-1/2} \alpha^{-1/2} f_1 \\ \\ \alpha^{-1/2} A_\alpha^{-1/2} \alpha^{1/2} f_2 \end{bmatrix} \right\rangle$$

$$= \langle f_1, \alpha^{1/2} A_\alpha^{-1/2} \alpha^{-1/2} f_1 \rangle_{A^{1/2}}$$

$$+ \langle f_2, \alpha^{-1/2} A_\alpha^{-1/2} \alpha^{1/2} f_2 \rangle_{\alpha^{1/2}}$$

$$= \langle A^{1/2} f_1, A^{1/2} \alpha^{1/2} A_\alpha^{-1/2} \alpha^{-1/2} f_1 \rangle_{L^2}$$

$$+ <\alpha^{1/2}f_2,\alpha^{1/2}\alpha^{-1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2>_{L^2}$$

$$= <f_1,\alpha^{-1/2}A_\alpha\alpha^{-1/2}\alpha^{1/2}A_\alpha^{-1/2}\alpha^{-1/2}f_1>_{L^2}$$

$$+ <\alpha^{1/2}f_2,A_\alpha^{-1/2}\alpha^{1/2}f_2>_{L^2}$$

$$= <\alpha^{-1/2}f_1,A_\alpha^{1/2}\alpha^{-1/2}f_1>_{L^2}$$

$$+ <\alpha^{1/2}f_2,A_\alpha^{-1/2}\alpha^{1/2}f_2>_{L^2}$$

$$\geq 0.$$

Let $\mu_{\alpha,p}$ be the purification of μ_α -- then

$$\mu_{\alpha,p}((f_1,f_2),(f_1',f_2'))$$

$$= \mu_\alpha((f_1,f_2),|A_{\mu_\alpha}|(f_1',f_2'))$$

$$= <f_1,\alpha^{-1/2}A_\alpha^{1/2}\alpha^{-1/2}f_1'>_{L^2}$$

$$+ <f_2,\alpha^{1/2}A_\alpha^{-1/2}\alpha^{1/2}f_2'>_{L^2}$$

and

$$\mu_{\alpha,p}((f_1,f_2),J_{\mu_\alpha}(f_1',f_2'))$$

$$= \sigma((f_1,f_2),(f_1',f_2')).$$

Bearing in mind that

$$H_{\mu_\alpha} = \mathrm{Dom}(A^{1/2}) \oplus \mathrm{Dom}(\alpha^{1/2}),$$

put

$$E_\alpha(\psi,x) = \frac{1}{2}\,[\langle\psi,A\psi\rangle + \langle x,\alpha x\rangle]$$

and let

$$X_\alpha = \begin{bmatrix} 0 & \alpha \\ -A & 0 \end{bmatrix},$$

where

$$\mathrm{Dom}(X^\alpha) = \mathrm{Dom}(A) \oplus \mathrm{Dom}(A^{1/2}\alpha).$$

Proceeding now as in the discussion of the abstract wave equation, one finds that X_α is a hamiltonian vector field with energy E_α. So, if $\gamma(t) = (\psi(t),x(t))$ is an integral curve for X_α, i.e., if

$$\dot{\gamma}(t) = X_\alpha \gamma(t),$$

then

$$\begin{bmatrix} \dot{\psi}(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ -A & 0 \end{bmatrix} \begin{bmatrix} \psi(t) \\ x(t) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x(t) \\ -A\psi(t) \end{bmatrix}$$

=>

$$\begin{cases} \dot{\psi}(t) = \alpha x(t) \\ \dot{x}(t) = -A\psi(t) \end{cases}$$

or still,

$$\ddot{\psi}(t) + \alpha A\psi(t) = 0.$$

51.6 <u>REMARK</u> J_{μ_α} is an isometric complex structure on H_{μ_α}. Observing that $X_\alpha J_{\mu_\alpha} = J_{\mu_\alpha} X_\alpha$, hence that on $\tilde{H}_{\mu_\alpha}$, X_α is complex linear, one can then show that X_α is skewadjoint. Therefore $\sqrt{-1}\, X_\alpha$ is selfadjoint and

$$\sqrt{-1}\, \dot{\gamma}(t) = \sqrt{-1}\, X_\alpha \gamma(t) \quad \text{(Schrödinger)}$$

<=>

$$\dot{\gamma}(t) = X_\alpha \gamma(t) \quad \text{(Hamilton)}.$$

The final step in the analysis is the introduction of a vector field $\vec{\beta} \in \mathcal{D}^1(M)$.

<u>Assumption</u>

$$\alpha - \frac{g(\vec{\beta},\vec{\beta})}{\alpha} \geq 1.$$

With this understanding, the <u>hamiltonian</u> of the theory is the function

$$H:E \rightarrow \underline{R}$$

defined by

$$H(f_1,f_2) = E_\alpha(f_1,f_2) + \langle L_{\vec{\beta}} f_1, f_2\rangle.$$

[Note: As above

$$E_\alpha(f_1,f_2) = \frac{1}{2} [\langle f_1, Af_1\rangle + \langle f_2, \alpha f_2\rangle].]$$

51.7 REMARK We have

$$\int_M (L_{\vec{\beta}} f_1) f_2 d\mu_g + \int_M f_1 (L_{\vec{\beta}} f_2) d\mu_g$$

$$= \int_M L_{\vec{\beta}} (f_1 f_2) d\mu_g$$

$$= - \int_M f_1 f_2 \cdot \text{div } \vec{\beta} d\mu_g.$$

Let

$$X = \begin{bmatrix} L_{\vec{\beta}} & \alpha \\ -A & L_{\vec{\beta}} + \text{div } \vec{\beta} \end{bmatrix}.$$

Then

$$H(f_1,f_2) = \frac{1}{2} \langle (f_1,f_2), JX(f_1,f_2)\rangle_{L^2 + L^2} \qquad \text{(cf. 49.1)}$$

or still,

$$H(f_1,f_2) = \frac{1}{2} \sigma((f_1,f_2), -X(f_1,f_2))$$

$$= \frac{1}{2} \sigma(X(f_1,f_2), (f_1,f_2)).$$

51.8 <u>LEMMA</u> We have

$$\iota_X \sigma = dH.$$

[Write

$$X = X_\alpha + X_{\vec{\beta}},$$

where

$$X_\alpha = \begin{bmatrix} 0 & \alpha \\ -A & 0 \end{bmatrix}$$

and

$$X_{\vec{\beta}} = \begin{bmatrix} L_{\vec{\beta}} & 0 \\ 0 & L_{\vec{\beta}} + \operatorname{div} \vec{\beta} \end{bmatrix}.$$

Then

$$\iota_X \sigma = \iota_{X_\alpha} \sigma + \iota_{X_{\vec{\beta}}} \sigma.$$

Here

$$\iota_{X_\alpha} \sigma = E_\alpha$$

and from the definitions

$$\sigma(X_{\vec{\beta}} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} f_1' \\ f_2' \end{bmatrix})$$

$$= \sigma\left(\begin{bmatrix} L_{\vec{\beta}} & 0 \\ 0 & L_{\vec{\beta}} + \operatorname{div} \vec{\beta} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} , \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} \right)$$

$$= \sigma\left(\begin{bmatrix} L_{\vec{\beta}} f_1 \\ L_{\vec{\beta}} f_2 + (\operatorname{div} \vec{\beta}) f_2 \end{bmatrix} , \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} \right)$$

$$= \langle L_{\vec{\beta}} f_1, f_2' \rangle - \langle L_{\vec{\beta}} f_2 + (\operatorname{div} \vec{\beta}) f_2, f_1' \rangle$$

$$= \langle L_{\vec{\beta}} f_1, f_2' \rangle + \langle L_{\vec{\beta}} f_1', f_2 \rangle$$

$$= \frac{d}{d\varepsilon} \langle L_{\vec{\beta}} (f_1 + \varepsilon f_1'), f_2 + \varepsilon f_2' \rangle \Big|_{\varepsilon=0}$$

$$= d\langle L_{\vec{\beta}} \text{---},\text{---} \rangle_{(f_1,f_2)} (f_1', f_2').$$

Put

$$\mu_{\alpha,\vec{\beta}} ((f_1,f_2),(f_1',f_2'))$$

$$= \langle (f_1,f_2), JX(f_1',f_2') \rangle_{L^2+L^2} .$$

Then we claim that

$$\mu_{\alpha,\vec{\beta}} \in \mathbb{P}(E,\sigma).$$

<u>$\mu_{\alpha,\vec{\beta}}$ is symmetric:</u> In fact,

$$<(f_1,f_2),JX(f_1',f_2')>_{L^2+L^2}$$

$$= \sigma((f_1,f_2),-X(f_1',f_2'))$$

$$= \sigma(X(f_1,f_2),(f_1',f_2'))$$

$$= -\ \sigma((f_1',f_2'),X(f_1,f_2))$$

$$= \sigma((f_1',f_2'),-X(f_1,f_2))$$

$$= <(f_1',f_2'),JX(f_1,f_2)>_{L^2+L^2}\ .$$

<u>$\mu_{\alpha,\vec{\beta}}$ is positive definite:</u> In fact,

$$\mu_{\alpha,\vec{\beta}}((f_1,f_2),(f_1,f_2))$$

$$= \int_M \alpha[(f_1)^2 + g(df_1,df_1) + (f_2)^2]d\mu_g$$

$$+ 2\int_M (L_{\vec{\beta}} f_1)f_2 d\mu_g$$

$$= \int_M \alpha[(f_1)^2$$

$$+ g(\text{grad } f_1 + \frac{f_2}{\alpha}\vec{\beta}, \text{ grad } f_1 + \frac{f_2}{\alpha}\vec{\beta})$$

$$+ (f_2)^2 (1 - \frac{g(\vec{\beta},\vec{\beta})}{\alpha^2}]d\mu_g$$

$$\geq \int_M \alpha[(f_1)^2 + (f_2)^2 (1 - \frac{g(\vec{\beta},\vec{\beta})}{\alpha^2})]d\mu_g$$

$$= \int_M [\alpha(f_1)^2 + (f_2)^2 (\alpha - \frac{g(\vec{\beta},\vec{\beta})}{\alpha})]d\mu_g$$

$$\geq \int_M [(f_1)^2 + (f_2)^2]d\mu_g$$

$$= \langle f_1,f_1\rangle + \langle f_2,f_2\rangle.$$

[Note: We have

$$\alpha g(\text{grad } f_1 + \frac{f_2}{\alpha}\vec{\beta}, \text{ grad } f_1 + \frac{f_2}{\alpha}\vec{\beta})$$

$$= \alpha g(\text{grad } f_1, \text{ grad } f_1)$$

$$+ 2f_2\, g(\text{grad } f_1,\vec{\beta}) + \frac{(f_2)^2}{\alpha} g(\vec{\beta},\vec{\beta})$$

$$= \alpha g(df_1,df_1) + 2f_2(L_{\vec{\beta}} f_1) + \frac{(f_2)^2}{\alpha} g(\vec{\beta},\vec{\beta}).]$$

To conclude that

$$\mu_{\alpha,\vec{\beta}} \in \mathrm{IP}(E,\sigma),$$

it remains only to recall that

$$|\sigma((f_1,f_2),(f_1',f_2'))|^2$$

$$\leq (\langle f_1,f_1\rangle + \langle f_2,f_2\rangle)\cdot(\langle f_1',f_1'\rangle + \langle f_2',f_2'\rangle).$$

One can then pass to $\mathcal{H}_{\mu_{\alpha,\vec{\beta}}}$, where

$$A_{\mu_{\alpha,\vec{\beta}}} = -X^{-1}.$$

Now form $\mathcal{H}^{\sim}_{\mu_{\alpha,\vec{\beta}}}$ (taken per $J_{\mu_{\alpha,\vec{\beta}}}$) -- then X is skewadjoint, hence $\sqrt{-1}\,X$ is self-adjoint and once again "Schrödinger = Hamilton".

<u>Definition</u> The <u>Ashtekar-Magnon state</u> is the pure state on $\mathcal{W}(E,\sigma)$ determined by $\mu_{\alpha,\vec{\beta},p}$.

In particular: If $\alpha = 1$ and $\vec{\beta} = 0$, then the Ashtekar-Magnon state is the pure state on $\mathcal{W}(E,\sigma)$ determined by $\mu_{E,p}$.

§52. *KLEIN-GORDON*

Let M be a connected C^∞ manifold of dimension n. Denote by $\underline{M}$ the set of semiriemannian structures on M, thus

$$\underline{M} = \coprod_{0 \leq k \leq n} \underline{M}_{k,n-k},$$

where $\underline{M}_{k,n-k}$ is the set of semiriemannian structures on M of signature (k,n-k).

[Note: Our convention is

$$\begin{bmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \quad (0 \leq k \leq n).]$$

It will not be unduly restrictive to assume that M is orientable with orientation μ, vol_g then standing for the unique n-form on M such that $\forall\ x \in M$ and every oriented orthonormal basis $\{E_1,\ldots,E_n\} \subset T_xM$,

$$\text{vol}_g\big|_x (E_1,\ldots,E_n) = 1.$$

[Note: In a connected open set $U \subset M$ equipped with coordinates $x^1,\ldots,x^n$ consistent with μ, i.e., such that

$$[\frac{\partial}{\partial x^1}\Big|_x,\ldots,\frac{\partial}{\partial x^n}\Big|_x] \in \mu_x \ \forall\ x \in U,$$

$$\text{vol}_g = |g|^{1/2}\, dx^1 \wedge \cdots \wedge dx^n.]$$

Given $g \in \underline{M}$, the laplacian Δ_g is, by definition, div $\circ$ grad.

<u>N.B.</u> If $g \in \underline{M}_{1,n-1}$, then it is customary to write $\Box_g$ in place of Δ_g.

E.g.: In Minkowski space (a.k.a. $\underline{R}^{1,3}$),

$$\Box_g = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2.$$

Fix $m > 0$ -- then an element $f \in C_c^\infty(M)$ is said to be a solution to the <u>Klein-Gordon equation</u> provided

$$(\Delta_g - m^2) f = 0.$$

<u>Functional Derivatives</u> There is a pairing

$$< , > : \begin{cases} C_c^\infty(M) \times C_c^\infty(M) \to \underline{R} \\ (f_1, f_2) \to \int_M f_1 f_2 \, \text{vol}_g. \end{cases}$$

So, if

$$L : C_c^\infty(M) \to \underline{R},$$

then $\frac{\delta L}{\delta f}$ is the element of $C_c^\infty(M)$ such that

$$\frac{d}{d\varepsilon} L(f + \varepsilon \delta f)\Big|_{\varepsilon=0} = <\delta f, \frac{\delta L}{\delta f}>$$

for all $\delta f \in C_c^\infty(M)$.

Fix $m > 0$ -- then the <u>Klein-Gordon lagrangian</u> is the functional

$$L_{KG}:C_c^\infty(M) \to \underline{R}$$

defined by the prescription

$$L_{KG}(f) = -\frac{1}{2}\int_M (g(\text{grad } f,\text{grad } f) + m^2f^2)\text{vol}_g.$$

52.1 <u>LEMMA</u> We have

$$\frac{\delta L_{KG}}{\delta f} = (\Delta_g - m^2)f.$$

<u>PROOF</u> In fact,

$$-\frac{1}{2}\int_M \frac{d}{d\varepsilon}\, g(\text{grad}(f + \varepsilon\delta f),\text{grad}(f + \varepsilon\delta f))\Big|_{\varepsilon=0}\text{vol}_g$$

$$= -\frac{1}{2}\int_M (g(\text{grad } f,\text{grad } \delta f) + g(\text{grad } \delta f,\text{grad } f))\text{vol}_g$$

$$= -\int_M g(\text{grad } \delta f,\text{grad } f)\text{vol}_g$$

$$= \int_M \delta f(\Delta_g f)\text{vol}_g$$

and

$$-\frac{1}{2}\int_M m^2 \frac{d}{d\varepsilon}(f + \varepsilon\delta f)^2\Big|_{\varepsilon=0}\text{vol}_g$$

$$= \int_M \delta f(-m^2f)\text{vol}_g.$$

Therefore

$$\frac{\delta L_{KG}}{\delta f} = (\Delta_g - m^2)f.$$

A <u>critical point</u> for L_{KG} is an element $f \in C_c^\infty(M)$ such that

$$\frac{\delta L_{KG}}{\delta f} = 0.$$

Accordingly, f is a critical point for L_{KG} iff f is a solution to the Klein-Gordon equation:

$$(\Delta_g - m^2)f = 0.$$

§53. *HAMILTONIAN ANALYSIS*

Let M be a connected C^∞ manifold of dimension n. Suppose that

$$M = \underline{R} \times \Sigma,$$

where Σ is a connected orientable C^∞ manifold of dimension n-1.

- A lapse is a strictly positive time dependent C^∞ function N on Σ:

$$N_t(x) = N(t,x) \quad (x \in \Sigma).$$

- A shift is a time dependent vector field $\vec{N}$ on Σ:

$$\vec{N}_t(x) = \vec{N}(t,x) \quad (x \in \Sigma).$$

Fix a lapse N, a shift $\vec{N}$, and let $t \rightarrow q_t (= q(t))$ be a path in Q (the set of riemannian structures on Σ) -- then the prescription

$$g_{(t,x)}((r,X),(s,Y))$$

$$= - rs(N_t^2(x) - q_x(t)(\vec{N}_t|_x,\vec{N}_t|_x))$$

$$+ sq_x(t)(X,\vec{N}_t|_x) + rq_x(t)(Y,\vec{N}_t|_x)$$

$$+ q_x(t)(X,Y) \quad (r,s \in \underline{R} \ \& \ X,Y \in T_x\Sigma)$$

defines an element g of $\underline{M}_{1,n-1}$.

[Note: In adapted coordinates (with $\vec{N} = N^a\partial_a$),

$$[g_{ij}] = \begin{bmatrix} -N^2 + N^a N_a & N_b \\ & \\ N_a & q_{ab} \end{bmatrix}$$

and

$$[g^{ij}] = \frac{1}{N^2} \begin{bmatrix} -1 & N^b \\ & \\ N^a & N^2 q^{ab} - N^a N^b \end{bmatrix} .]$$

Put

$$\underline{n} = \frac{1}{N} (\partial/\partial t - \vec{N}).$$

53.1 <u>LEMMA</u> We have

$$g(\underline{n}, \partial_a) = 0.$$

<u>PROOF</u> For

$$g(\underline{n}, \partial_a) = \frac{1}{N} g(\partial/\partial t - \vec{N}, \partial_a)$$

$$= \frac{1}{N} (g(\partial_0, \partial_a) - N^b g(\partial_b, \partial_a))$$

$$= \frac{1}{N} (N_a - N^b q_{ab})$$

$$= \frac{1}{N} (N_a - N_a) = 0.$$

53.2 LEMMA We have

$$g(\underline{n},\underline{n}) = -1.$$

PROOF For

$$g(\underline{n},\underline{n}) = \frac{1}{N^2} g(\partial/\partial t - \vec{N}, \partial/\partial t - \vec{N})$$

$$= \frac{1}{N^2} (g(\partial_0,\partial_0) - 2g(\partial_0,\vec{N}) + g(\vec{N},\vec{N}))$$

$$= \frac{1}{N^2} (g_{00} - 2N^a g_{0a} + N^a N^b q_{ab})$$

$$= \frac{1}{N^2} (- N^2 + N^a N_a - 2N^a N_a + N^a N_a)$$

$$= - \frac{N^2}{N^2} = -1.$$

Let $\Sigma_t = \{t\} \times \Sigma$ and call $i_t : \Sigma \approx \Sigma_t \rightarrow M$ the embedding -- then $\forall f \in C_c^\infty(M)$,

$$L_{KG}(f)$$

$$= - \frac{1}{2} \int_{\underline{R}} dt \int_\Sigma (g(\text{grad } f, \text{grad } f) \circ i_t + m^2 (f \circ i_t)^2) i_t^* (\iota_{\partial/\partial t} \text{vol}_g)$$

$$= - \frac{1}{2} \int_{\underline{R}} dt \int_\Sigma (g(\text{grad } f, \text{grad } f) \circ i_t + m^2 (f \circ i_t)^2) N_t \text{ vol}_{q_t}.$$

Put

$$f_t = f \circ i_t$$

and

$$\dot{f}_t = (L_{\partial/\partial t} f) \circ i_t.$$

Then

$$(L_{\underline{n}} f) \circ i_t = \frac{\dot{f}_t - L_{\vec{N}_t} f_t}{N_t}.$$

53.3 <u>LEMMA</u> $\forall\ f \in C_c^\infty(M)$,

$$g(\text{grad } f, \text{grad } f) \circ i_t$$

$$= - \left[\frac{\dot{f}_t - L_{\vec{N}_t} f_t}{N_t} \right]^2 + q_t(\text{grad } f_t, \text{grad } f_t).$$

Let $C = C_c^\infty(\Sigma)$ -- then

$$TC = C \times C_c^\infty(\Sigma)$$

is the <u>velocity phase space</u> of the theory.

[Note: Elements of TC are pairs $(u, \dot{u})$.]

53.4 <u>REMARK</u> Each $f \in C_c^\infty(M)$ determines a path $t \rightarrow (f_t, \dot{f}_t)$ in TC.

The <u>lagrangian</u> of the theory at time t is the function

$$L_t : TC \rightarrow \underline{R}$$

defined by the rule

$$L_t(u,\dot{u}) = -\frac{1}{2}\int_\Sigma\left(-\left[\frac{\dot{u} - L_{\vec{N}_t} u}{N_t}\right]^2 + q_t(du,du) + m^2u^2\right)N_t \text{vol}_{q_t}.$$

53.5 EXAMPLE Suppose that $\forall$ t, $N_t = 1$ and $\vec{N}_t = \vec{0}$ -- then

$$L_t(u,\dot{u}) = -\frac{1}{2}\int_\Sigma (-\dot{u}^2 + q_t(du,du) + m^2u^2)\text{vol}_{q_t}$$

$$= \frac{1}{2}\int_\Sigma \dot{u}^2\text{vol}_{q_t} + L_{KG}(u)\Big|_t.$$

N.B. From the above,

$$L_{KG}(f) = \int_{\underline{R}} L_t(f_t,\dot{f}_t)dt.$$

Thinking of TC as the tangent bundle of C, put

$$T^*C = C \times C_d^\infty(\Sigma)$$

and call it the momentum phase space of the theory.

[Note: Elements of T^*C are pairs (u,π).]

In terms of the pairing

$$< , > : \begin{cases} C_c^\infty(\Sigma) \times C_d^\infty(\Sigma) \to \underline{R} \\ \\ (u,\pi) \to \int_\Sigma u\pi, \end{cases}$$

the functional derivative $\frac{\delta L_t}{\delta \dot{u}}$ is the element of $C_d^\infty(\Sigma)$ such that

$$\frac{d}{d\varepsilon} L_t(u,\dot{u} + \varepsilon\delta\dot{u})\Big|_{\varepsilon=0} = \langle \delta\dot{u}, \frac{\delta L_t}{\delta \dot{u}} \rangle$$

for all $\delta\dot{u} \in C_c^\infty(\Sigma)$. Explicated:

$$-\frac{1}{2}\int_\Sigma \frac{d}{d\varepsilon}\left[\frac{\dot{u} + \varepsilon\delta\dot{u} - L_{\vec{N}_t} u}{N_t}\right]^2\Bigg|_{\varepsilon=0} N_t \mathrm{vol}_{q_t}$$

$$= \int_\Sigma \delta\dot{u}\left[\frac{\dot{u} - L_{\vec{N}_t} u}{N_t}\right]\mathrm{vol}_{q_t}$$

=>

$$\frac{\delta L_t}{\delta \dot{u}} = \left[\frac{\dot{u} - L_{\vec{N}_t} u}{N_t}\right]|q_t|^{1/2}.$$

On general grounds, the <u>hamiltonian</u> of the theory at time t is the function

$$H_t : FL_t(TC) \to \underline{R}$$

given by

$$H_t \circ FL_t(u,\dot{u}) = \langle \dot{u}, \frac{\delta L_t}{\delta \dot{u}} \rangle - L_t(u,\dot{u}),$$

where

$$FL_t : TC \to T^*C$$

is the fiber derivative.

To simplify the RHS, let

$$\begin{cases} \kappa_t = \left[\dfrac{\dot{u} - L_{\vec{N}_t} u}{N_t} \right] \\ \\ \pi_t = \kappa_t |q_t|^{1/2} \end{cases}$$

and note that

$$\langle \dot{u}, \frac{\delta L_t}{\delta \dot{u}} \rangle$$

$$= \langle L_{\vec{N}_t} u + N_t \kappa_t, \pi_t \rangle$$

$$= \langle L_{\vec{N}_t} u, \pi_t \rangle + \langle N_t \kappa_t, \pi_t \rangle .$$

But

$$\langle N_t \kappa_t, \pi_t \rangle$$

$$= \int_\Sigma N_t \kappa_t \pi_t$$

$$= \int_\Sigma \kappa_t^2 N_t \mathrm{vol}_{q_t} .$$

In addition,

$$-\frac{1}{2} \int_\Sigma \left[\frac{\dot{u} - L_{\vec{N}_t} u}{N_t} \right]^2 N_t \mathrm{vol}_{q_t}$$

$$= -\frac{1}{2} \int_\Sigma \kappa_t^2 N_t \mathrm{vol}_{q_t} .$$

Therefore

$$H_t(u,\pi_t) = \langle L_{\vec{N}_t} u, \pi_t\rangle + \frac{1}{2} \langle\langle N_t \kappa_t, \pi_t\rangle\rangle$$

$$+ \frac{1}{2} \int_\Sigma (q_t(du,du) + m^2u^2) N_t \mathrm{vol}_{q_t}.$$

This conclusion provides the means to canonically extend H_t to all of T^*C. Thus take $\pi \in C_d^\infty(\Sigma)$ and write

$$\pi = \left(\frac{\pi}{|q_t|^{1/2}}\right) |q_t|^{1/2}.$$

Then

$$\kappa_t = \frac{\pi}{|q_t|^{1/2}}$$

is a density of weight 0, hence is an element of $C^\infty(\Sigma)$. And we put

$$H_t(u,\pi) = \langle L_{\vec{N}_t} u, \pi\rangle + \frac{1}{2} \langle\langle N_t \kappa_t, \pi\rangle\rangle$$

$$+ \frac{1}{2} \int_\Sigma (q_t(du,du) + m^2u^2) N_t \mathrm{vol}_{q_t}.$$

Now define

$$E_t : T^*C \to C_d^\infty(\Sigma)$$

by

$$E_t(u,\pi) = \frac{1}{2} (\kappa_t^2 + q_t(du,du) + m^2u^2) |q_t|^{1/2},$$

so

$$H_t(u,\pi) = \langle L_{\vec{N}_t} u, \pi\rangle + \int_\Sigma N_t E_t(u,\pi).$$

53.6 <u>LEMMA</u> The hamiltonian vector field

$$X_t : T^*C \to T^*C$$

attached to H_t is given by

$$X_t(u,\pi) = \left(\frac{\delta H_t}{\delta \pi}, -\frac{\delta H_t}{\delta u}\right).$$

[Note: The symplectic structure on T^*C is

$$\Omega((u_1,\pi_1),(u_2,\pi_2)) = \int_\Sigma (u_1\pi_2 - u_2\pi_2).]$$

• $\frac{\delta H_t}{\delta \pi}$: We have

$$\left\langle \frac{\delta H_t}{\delta \pi}, \delta\pi \right\rangle = \frac{d}{d\varepsilon} H_t(u, \pi + \varepsilon\delta\pi)\Big|_{\varepsilon=0}$$

$$= \langle L_{\vec{N}_t} u, \delta\pi\rangle + \langle N_t \kappa_t, \delta\pi\rangle$$

=>

$$\frac{\delta H_t}{\delta \pi} = L_{\vec{N}_t} u + N_t \kappa_t.$$

• $\frac{\delta H_t}{\delta u}$: We have

$$\left\langle \delta u, \frac{\delta H_t}{\delta u} \right\rangle = \frac{d}{d\varepsilon} H_t(u + \varepsilon\delta u, \pi)\Big|_{\varepsilon=0}$$

$$= \langle L_{\vec{N}_t} \delta u, \pi\rangle$$

$$+ \frac{1}{2} \int_\Sigma \frac{d}{d\varepsilon} (q_t(d(u + \varepsilon\delta u), d(u + \varepsilon\delta u)) + m^2(u + \varepsilon\delta u)^2)\Big|_{\varepsilon=0} N_t \mathrm{vol}_{q_t}$$

$$= - \langle\delta u, L_{\vec{N}_t} \pi\rangle$$

$$+ \int_\Sigma (q_t(d\delta u, du) + m^2(\delta u)u) N_t \mathrm{vol}_{q_t}$$

$$= - \langle\delta u, L_{\vec{N}_t} \pi\rangle$$

$$+ \int_\Sigma \delta u(N_t(- \Delta_{q_t} u) - q_t(du, dN_t)) \mathrm{vol}_{q_t} \qquad \text{(cf. 51.2)}$$

$$+ \int_\Sigma m^2(\delta u) u N_t \mathrm{vol}_{q_t}$$

$\Rightarrow$

$$\frac{\delta H_t}{\delta u} = - L_{\vec{N}_t} \pi$$

$$+ ((- \Delta_{q_t} u + m^2 u)N_t - q_t(du, dN_t)) |q_t|^{1/2}.$$

53.7 <u>REMARK</u> Since $\pi = \kappa_t |q_t|^{1/2}$, it follows that

$$L_{\vec{N}_t} \pi = (L_{\vec{N}_t} \kappa_t + \kappa_t (\mathrm{div}\, \vec{N}_t)) |q_t|^{1/2}.$$

Define $H: \underline{R} \times T^*C \to \underline{R}$ by

$$H(t, (u, \pi)) = H_t(u, \pi).$$

Then the time dependent hamiltonian vector field

$$X_H : \underline{R} \times T^*C \to T^*C$$

of the theory is

$$X_H(t,(u,\pi)) = X_t(u,\pi),$$

a curve $\gamma : \underline{R} \to T^*C$ being by definition an integral curve for X_H provided

$$\dot{\gamma}(t) = X_H(t,\gamma(t)).$$

So, if $\gamma(t) = (u_t,\pi_t)$, then

$$\dot{\gamma}(t) = X_t(u_t,\pi_t)$$

=>

$$\dot{u}_t = \frac{\delta H_t}{\delta \pi} = L_{\vec{N}_t} u_t + N_t \kappa_t$$

and

$$\dot{\pi}_t = -\frac{\delta H_t}{\delta u} = L_{\vec{N}_t} \pi_t$$

$$+ ((\Delta_{q_t} u_t - m^2 u_t) N_t + q_t(du_t, dN_t)) |q_t|^{1/2}.$$

Given $f \in C_c^\infty(M)$, write

$$\pi_t(f) = \kappa_t(f) |q_t|^{1/2},$$

where

$$\kappa_t(f) = \frac{\dot{f}_t - L_{\vec{N}_t} f_t}{N_t}.$$

53.8 THEOREM Let $f \in C_c^\infty(M)$ -- then f satisfies the Klein-Gordon equation, i.e.,

$$(\Delta_g - m^2)f = 0,$$

iff

$$\gamma_f(t) = (f_t, \pi_t(f))$$

is an integral curve for X_H.

[Note: While this can be checked by direct computation, it is simpler to use a variational argument.]

53.9 REMARK The relation

$$\dot{f}_t = \frac{\delta H_t}{\delta \pi} = L_{\vec{N}_t} f_t + N_t \kappa_t(f)$$

is automatic. In fact,

$$L_{\vec{N}_t} f_t + N_t \kappa_t(f)$$

$$= L_{\vec{N}_t} f_t + N_t \left[\frac{\dot{f}_t - L_{\vec{N}_t} f_t}{N_t} \right]$$

$$= L_{\vec{N}_t} f + \dot{f}_t - L_{\vec{N}_t} f$$

$$= \dot{f}_t.$$

53.10 EXAMPLE Take $M = \underline{R}^{1,3}$ (i.e., Minkowski space) -- then $N_t = 1$,

$\vec{N}_t = \vec{0}$, q_t = euclidean metric, and

$$\square_g = -\partial_t^2 + \partial^2_{x^1} + \partial^2_{x^2} + \partial^2_{x^3}.$$

Now explicate the momentum relation, thus

$$\ddot{f}_t = \Delta f_t - m^2 f_t$$

$\iff$

$$(\square_g - m^2)f = 0.$$

Take m = 1 — then the theory assigns to each instant of time a hamiltonian

$$H_t : T^*C \to \underline{R},$$

viz.

$$H_t(u,\pi) = \langle L_{\vec{N}_t} u, \pi\rangle + \int_\Sigma N_t E_t(u,\pi),$$

where

$$E_t(u,\pi) = \frac{1}{2}(\kappa_t^2 + q_t(du,du) + u^2)|q_t|^{1/2}.$$

To connect these facts with those of §51, assume that

$$1 \leq N_t \leq C_t$$

and

$$N_t - \frac{q_t(\vec{N}_t,\vec{N}_t)}{N_t} \geq 1.$$

One then has the following correspondences:

$$\left[\begin{array}{l} \Sigma \longleftrightarrow M \\ q_t \longleftrightarrow g \\ N_t \longleftrightarrow \alpha \\ \vec{N}_t \longleftrightarrow \vec{\beta}. \end{array}\right.$$

[Note: To be in agreement with the earlier considerations, assume that (Σ, q_t) is complete (which is automatic if Σ is compact).]

Nothing of substance is lost if $C_d^\infty(\Sigma)$ is replaced by $C_c^\infty(\Sigma)$, so H_t can be regarded as the function on

$$C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$$

that sends (u_1, u_2) to

$$E_{N_t}(u_1, u_2) + \langle L_{\vec{N}_t} u_1, u_2 \rangle_t .$$

Here

$$\langle \ , \ \rangle_t = \int_\Sigma \text{---} \, d\mu_{q(t)} .$$

With this understanding, H_t is precisely the "H" of §51.

53.11 <u>REMARK</u> Thanks to 53.6 and the accompanying calculations of the functional derivatives, the hamiltonian vector field X_t can be identified with the "X" of §51.

[Note: It is also necessary to utilize 53.7.]

§54. THE COVARIANT POINT OF VIEW

Let M be a connected C^∞ manifold of dimension 4. Fix $g \in \underline{M}_{1,3}$ -- then the pair (M,g) is said to be a spacetime if M is oriented and time oriented.

54.1 RAPPEL A spacetime (M,g) is globally hyperbolic if it is causal and $\forall\ p,q \in M$, $J^+(p) \cap J^-(q)$ is compact (hence $\forall$ compact $K,L \subset M$, $J^+(K) \cap J^-(L)$ is compact).

[Note: "Causal" means that no closed causal curve exists. The usual definition of globally hyperbolic imposes the condition "strongly causal". This, however, is overkill since "causal" + compactness of the diamonds $J^+(p) \cap J^-(q)$ implies "strongly causal".]

Suppose that (M,g) is globally hyperbolic -- then by the term Cauchy hypersurface we shall understand an embedded spacelike hypersurface Σ in M which is met exactly once by every inextendible timelike curve in M.

[Note: Cauchy hypersurfaces always exist (M being globally hyperbolic) and any such is necessarily closed and connected.]

54.2 LEMMA If Σ_1 and Σ_2 are Cauchy hypersurfaces in M, then Σ_1 and Σ_2 are diffeomorphic.

54.3 THEOREM (Bernal-Sánchez) Suppose that (M,g) is globally hyperbolic.

Let Σ be a Cauchy hypersurface in M -- then $\exists$ a foliation $\{\Sigma_t : t \in \underline{R}\}$ of M by Cauchy hypersurfaces Σ_t such that $\Sigma_0 = \Sigma$, hence

$$M = \coprod_t \Sigma_t.$$

[Note: One can construct a time function $\tau : M \to \underline{R}$ whose level sets $\tau^{-1}(\{t\})$ are the Σ_t.]

54.4 <u>REMARK</u> Let $q_t (= q(t))$ be the riemannian structure on Σ determined by pulling back g via the arrow

$$\Sigma \approx \{t\} \times \Sigma \xrightarrow{\Psi_t} \Sigma_t \xrightarrow{i_t} M.$$

Put

$$N_t(x) = \frac{1}{|g_{\Psi(t,x)}(\text{grad }\tau, \text{grad }\tau)|^{1/2}} \quad (x \in \Sigma).$$

Define $g_\tau \in \underline{M}_{1,n-1}$ (per $\underline{R} \times \Sigma$) by the prescription

$$(g_\tau)_{(t,x)}((r,X),(s,Y))$$

$$= - rsN_t^2(x) + q_x(t)(X,Y) \quad (r,s \in \underline{R} \ \& \ X,Y \in T_x\Sigma).$$

Then

$$g_\tau = \Psi^* g.$$

Assume still that (M,g) is globally hyperbolic. Let Ω be a connected open subset of M -- then Ω is <u>causally compatible</u> provided

$$J^+(p) \cap J^-(q)$$

is contained in Ω for all $p,q \in \Omega$.

54.5 <u>EXAMPLE</u> Given $x \in M$, put $J(x) = J^+(x) \cup J^-(x)$ -- then $\Omega = M - J(x)$ is causally compatible.

54.6 <u>LEMMA</u> If Ω is causally compatible, then Ω is globally hyperbolic.

<u>PROOF</u> To keep things straight, append subscripts and note first that $\forall\ x \in \Omega$,

$$J_\Omega^\pm(x) = J_M^\pm(x) \cap \Omega.$$

E.g.: Let $y \in J_M^+(x) \cap \Omega$ and let $\gamma:[0,1] \to M$ be a future directed causal curve with $\gamma(0) = x$, $\gamma(1) = y$ -- then $\gamma([0,1]) \subset J_M^+(x) \cap J_M^-(y) \subset \Omega$

$$\Rightarrow y \in J_\Omega^+(x) \Rightarrow J_M^+(x) \cap \Omega \subset J_\Omega^+(x).$$

So, $\forall\ p,q \in \Omega$,

$$J_\Omega^+(p) \cap J_\Omega^-(q) = J_M^+(p) \cap J_M^-(q) \cap \Omega$$

$$= J_M^+(p) \cap J_M^-(q)$$

is compact. Since Ω is obviously causal, it follows that Ω is globally hyperbolic (cf. 54.1).

GLOBHYP is the category whose objects are the globally hyperbolic space-times (M,g) and whose morphisms

$$\zeta:(M_1,g_1) \to (M_2,g_2)$$

are isometric embeddings that preserve the orientation and the time orientation and have the property that $\zeta(M_1)$ is a causally compatible subset of M_2.

N.B. $\zeta(M_1)$ is a globally hyperbolic sub-spacetime of M_2 (cf. 54.6).

C*-ALG is the category whose objects are the unital C*-algebras and whose morphisms

$$\phi:A_1 \to A_2$$

are injective and unit preserving.

54.7 DEFINITION A quantum field theory (QFT) is a functor

$$F:\underline{\text{GLOBHYP}} \to \underline{\text{C*-ALG}}.$$

To illustrate the definition, consider the Klein-Gordon operator $\square_g - m^2$, which is second order hyperbolic.

54.8 THEOREM (Dimock) Suppose that (M,g) is globally hyperbolic -- then $\exists$ continuous linear maps

$$E^{\pm}:C_c^\infty(M) \to C^\infty(M)$$

such that

$$\begin{cases} E^{\pm}(\Box_g - m^2)f = f \\ (\Box_g - m^2)E^{\pm}f = f. \end{cases}$$

Furthermore,

$$\text{spt } E^{\pm}f \subset J^{\pm}(\text{spt } f).$$

[Note: For sake of clarity, it is sometimes best to incorporate M into the notation: $E_M^{\pm}$.]

N.B. The stated properties characterize $E^{\pm}$ uniquely.

54.9 LEMMA Let $f_1, f_2 \in C_c^{\infty}(M)$ — then

$$\int_M (E^{\pm}f_1)f_2 \text{vol}_g = \int_M f_1(E^{\mp}f_2)\text{vol}_g.$$

PROOF We have

$$\int_M (E^{\pm}f_1)f_2 \text{vol}_g$$

$$= \int_M (E^{\pm}f_1)(\Box_g - m^2)E^{\mp}f_2 \text{vol}_g$$

$$= \int_M ((\Box_g - m^2)E^{\pm}f_1)(E^{\mp}f_2)\text{vol}_g$$

$$= \int_M f_1(E^{\mp}f_2)\text{vol}_g.$$

[Note: To justify the passage from the second line to the third, observe that

$$\mathrm{spt}\, E^{\pm} f_1 \cap \mathrm{spt}\, E^{\mp} f_2$$

$$\subset J^{\pm}(\mathrm{spt}\, f_1) \cap J^{\mp}(\mathrm{spt}\, f_2),$$

which is compact.]

Let

$$E = E^{+} - E^{-}.$$

Then $\forall\, f_1, f_2 \in C_c^{\infty}(M)$,

$$\int_M f_1 (E f_2) \mathrm{vol}_g$$

$$= \int_M f_1 (E^{+} f_2 - E^{-} f_2) \mathrm{vol}_g$$

$$= \int_M (E^{-} f_1 - E^{+} f_1) f_2 \mathrm{vol}_g$$

$$= \int_M (- E f_1) f_2 \mathrm{vol}_g$$

$$= - \int_M f_2 (E f_1) \mathrm{vol}_g.$$

Therefore the prescription

$$\sigma_g(f_1, f_2) = \int_M f_1 (E f_2) \mathrm{vol}_g$$

induces a symplectic structure on the quotient $C_c^\infty(M)/\text{Ker } E$. Denoting the latter by $E_m(M,g)$, it follows that the pair $(E_m(M,g),\sigma_g)$ is a symplectic vector space, from which the C*-algebra

$$\mathcal{W}(E_m(M,g),\sigma_g).$$

54.10 <u>THEOREM</u> (Brunetti-Fredenhagen-Verch) Fix $m > 0$ -- then the assignment

$$(M,g) \to \mathcal{W}(E_m(M,g),\sigma_g)$$

is a quantum field theory.

To prove this, we shall need a few more facts.

54.11 <u>LEMMA</u> $(\Box_g - m^2)|C_c^\infty(M)$ is injective and

$$\begin{cases} E \circ (\Box_g - m^2) = 0 \\ (\Box_g - m^2) \circ E = 0. \end{cases}$$

[This is clear.]

54.12 <u>LEMMA</u> Suppose that $f \in \text{Ker } E$ -- then $\exists\ f' \in C_c^\infty(M)$:

$$f = (\Box_g - m^2)f'.$$

<u>PROOF</u> $Ef = 0 \Rightarrow E^+f = E^-f$, call if f', thus

$$(\Box_g - m^2)f' = (\Box_g - m^2)E^{\pm}f = f.$$

On the other hand,

$$\text{spt } f' = \text{spt } E^+f \cap \text{spt } E^-f$$

$$\subset J^+(\text{spt } f) \cap J^-(\text{spt } f),$$

so spt f' is compact.

Let Ω be a connected open subset of M -- then there is an arrow

$$\begin{cases} C_c^\infty(\Omega) \to C_c^\infty(M) \\ f \to \text{ext } f, \end{cases}$$

viz. extension by zero.

54.13 LEMMA If Ω is causally compatible (cf. 54.6), then

$$E_\Omega^\pm f = (E_M^\pm \text{ ext } f)|\Omega.$$

PROOF Let

$$\begin{cases} D_M = \Box_g - m^2 \\ D_\Omega = \Box_{g|\Omega} - m^2. \end{cases}$$

Then $\forall f \in C_c^\infty(\Omega)$,

$$\begin{cases} E_M^\pm \text{ ext}(D_\Omega f)|\Omega = f \\ D_\Omega((E_M^\pm \text{ ext } f)|\Omega) = f. \end{cases}$$

E.g.:

$$E_M^{\pm}\ \mathrm{ext}(D_\Omega f)|\Omega$$

$$= E_M^{\pm}(D_M\ \mathrm{ext}\ f)|\Omega$$

$$= \mathrm{ext}\ f|\Omega = f.$$

In addition,

$$\mathrm{spt}((E_M^{\pm}\ \mathrm{ext}\ f)|\Omega)$$

$$= \mathrm{spt}(E_M^{\pm}\ \mathrm{ext}\ f) \cap \Omega$$

$$\subset J_M^{\pm}(\mathrm{spt\ ext}\ f) \cap \Omega$$

$$= J_M^{\pm}(\mathrm{spt}\ f) \cap \Omega$$

$$= J_\Omega^{\pm}(\mathrm{spt}\ f).$$

Now quote uniqueness.

Maintaining the assumption that Ω is causally compatible, we claim that

$$\mathrm{ext}(\mathrm{Ker}\ E_\Omega) \subset \mathrm{Ker}\ E_M.$$

For suppose that $E_\Omega f = 0$. Using the notation of 54.13, write $f = D_\Omega f'$ $(f' \in C_c^\infty(\Omega))$ (cf. 54.12) -- then

$$E_M\ \mathrm{ext}\ f = E_M\ \mathrm{ext}\ D_\Omega f'$$

$$= E_M D_M \text{ ext } f'$$

$$= 0 \text{ (cf. 54.11)}.$$

Accordingly,

$$\text{ext}: C_c^\infty(\Omega) \to C_c^\infty(M)$$

induces an $\underline{R}$-linear map

$$E_m(\Omega, g|\Omega) \to E_m(M,g)$$

on equivalence classes: $[f] \to [\text{ext } f]$. But

$$\sigma_{g|\Omega}(f_1, f_2)$$

$$= \int_\Omega f_1(E_\Omega f_2) \text{vol}_{g|\Omega}$$

$$= \int_\Omega f_1(E_M \text{ ext } f_2)|\Omega \text{ vol}_{g|\Omega} \quad \text{(cf. 54.13)}$$

$$= \int_M \text{ext } f_1(E_M \text{ ext } f_2)\text{vol}_g$$

$$= \sigma_g(\text{ext } f_1, \text{ext } f_2).$$

Applying 16.27 (the role of T being played by ext) thus leads to an injective morphism

$$W(E_m(\Omega, g|\Omega), \sigma_{g|\Omega}) \to W(E_m(M,g), \sigma_g)..$$

That 54.10 holds is then manifest.

The arrow

$$E_m(\Omega, g|\Omega) \to E_m(M,g)$$

is automatically injective and there are situations when it is surjective as well.

54.14 LEMMA Suppose that Ω is causally compatible (cf. 54.6). Assume: There is a Cauchy hypersurface Σ for Ω which is also a Cauchy hypersurface for M. Let $f \in C_c^\infty(M)$ — then $\exists\ \phi \in C_c^\infty(\Omega), \psi \in C_c^\infty(M)$:

$$f = \text{ext } \phi + (\Box_g - m^2)\psi.$$

[Note: Thanks to 54.11,

$$E \circ (\Box_g - m^2)\psi = 0.$$

Therefore

$$[f] = [\text{ext } \phi].]$$

Given a globally hyperbolic pair (M,g), let $K(M,g)$ be the collection of all subsets $O \subset M$, where O is open, connected, relatively compact, and causally compatible. Order the elements of $K(M,g)$ by inclusion and write

$$O \perp O' \iff J_M^{\pm}(\bar{O}) \cap \bar{O}' = \emptyset.$$

[Note: The symbol $O \perp O'$ signifies that there are no causal curves connecting a point in $\bar{O}$ with a point in $\bar{O}'$, a symmetric relation. I.e.:

$$O \perp O' \iff O' \perp O.]$$

N.B. The pair $(O,g|O)$ is globally hyperbolic (cf. 54.6).

54.15 LEMMA If $K \subset M$ is compact, then $\exists\ O \in K(M,g): K \subset O$.

This implies that $\mathcal{K}(M,g)$ is directed by inclusion: $\forall\ O_1,O_2 \in \mathcal{K}(M,g)$, $\exists\ O_3 \in \mathcal{K}(M,g):\bar{O}_1 \cup \bar{O}_2 \subset O_3$.

Given $O \in \mathcal{K}(M,g)$, put

$$\mathcal{A}_O = \mathcal{W}(E_m(O,g|O),\sigma_{g|O}).$$

View $\mathcal{A}_O$ as a C*-subalgebra of $\mathcal{W}(E_m(M,g),\sigma_g)$ and let $\mathcal{A}_M$ be the C*-subalgebra of $\mathcal{W}(E_m(M,g),\sigma_g)$ generated by the $\mathcal{A}_O$:

$$\mathcal{A}_M = C^*(\bigcup_O \mathcal{A}_O).$$

[Note: Trivially,

$$O_1 \subset O_2 \Rightarrow \mathcal{A}_{O_1} \subset \mathcal{A}_{O_2}.]$$

54.16 <u>LEMMA</u> We have

$$\mathcal{A}_M = \mathcal{W}(E_m(M,g),\sigma_g).$$

<u>PROOF</u> By definition,

$$\mathcal{A}_M \subset \mathcal{W}(E_m(M,g),\sigma_g).$$

To go the other way, take an $f \in C_c^\infty(M)$ -- then $\exists\ O \in \mathcal{K}(M,g):\text{spt } f \subset O$ (cf. 54.15), hence $W([\text{ext } f|O]) \in \mathcal{A}_O$.

54.17 <u>LEMMA</u> Let $O_1,O_2 \in \mathcal{K}(M,g)$. Assume: $O_1 \perp O_2$ -- then

$$[A_{O_1}, A_{O_2}] = 0.$$

I.e.: The subalgebras A_{O_1}, A_{O_2} of A_M commute.

<u>PROOF</u> Let

$$\begin{cases} f_1 \in C_c^\infty(O_1) \\ f_2 \in C_c^\infty(O_2). \end{cases}$$

Then

$$O_1 \perp O_2$$

$$\Rightarrow$$

$$\text{spt ext } f_1 \cap \text{spt } E_M \text{ ext } f_2 = \emptyset$$

$$\Rightarrow$$

$$\sigma_g(\text{ext } f_1, \text{ext } f_2) = 0$$

$$\Rightarrow$$

$$W([\text{ext } f_1])W([\text{ext } f_2])$$

$$= W([\text{ext } f_1] + [\text{ext } f_2])$$

$$= W([\text{ext } f_2])W([\text{ext } f_1]).$$

Therefore the generators of A_{O_1} commute with the generators of A_{O_2}.

There are two other properties possessed by the assignment

$$O \to A_O$$

that lie somewhat deeper.

54.18 <u>LEMMA</u> Let $O_1 \subset O_2$ be elements of $\mathcal{K}(M,g)$ which admit a common Cauchy hypersurface -- then $\mathcal{A}_{O_1} = \mathcal{A}_{O_2}$.

<u>PROOF</u> Apply 54.14 to

$$\begin{bmatrix} M = O_2 \\ \\ \Omega = O_1 \end{bmatrix}$$

and conclude that the injection

$$E_m(O_1,g|O_1) \to E_m(O_2,g|O_2)$$

is a surjection, so the inclusion $\mathcal{A}_{O_1} \subset \mathcal{A}_{O_2}$ is, in the case at hand, an equality.

54.19 <u>LEMMA</u> Let $O_1,O_2 \in \mathcal{K}(M,g)$. Suppose that O_1 is contained in the domain of dependence $D(O_2)$ of O_2 -- then $\mathcal{A}_{O_1} \subset \mathcal{A}_{O_2}$ provided $D(O_2)$ is relatively compact.

<u>PROOF</u> Fix a Cauchy hypersurface Σ per O_2. While Σ is not necessarily a Cauchy hypersurface in M, it is at least acausal, hence its domain of dependence is causally compatible. On the other hand, from the definitions, $D(\Sigma) = D(O_2)$, thus, by assumption, is relatively compact. The conclusion, therefore, is that $D(O_2) \in \mathcal{K}(M,g)$, so

$$\mathcal{A}_{O_2} = \mathcal{A}_{D(O_2)} \quad \text{(cf. 54.18)}$$

=>

$$\mathcal{A}_{O_1} \subset \mathcal{A}_{D(O_2)} = \mathcal{A}_{O_2}.$$

[Note: The domain of dependence $D(O)$ of an element $O \in \mathcal{K}(M,g)$ is, in general, not relatively compact.]

Denote by $C_{sc}^{\infty}(M)$ the subset of $C^{\infty}(M)$ consisting of those ϕ with the property that $\exists$ a compact subset $K \subset M$:

$$\text{spt } \phi \subset J^{+}(K) \cup J^{-}(K).$$

54.20 <u>REMARK</u> If Σ is a Cauchy hypersurface in M and if $K \subset M$ is compact, then

$$\Sigma \cap J^{\pm}(K)$$

is compact. So, $\forall\ \phi \in C_{sc}^{\infty}(M)$, $\text{spt } \phi|\Sigma$ is compact.

54.21 <u>LEMMA</u> Let $\phi \in C_{sc}^{\infty}(M)$. Assume: $(\square_g - m^2)\phi = 0$ -- then $\exists\ f \in C_c^{\infty}(M)$ such that $\phi = Ef$.

<u>PROOF</u> Choose a compact set K:

$$\text{spt } \phi \subset I^{+}(K) \cup I^{-}(K).$$

Using a C^{∞} partition of unity, write $\phi = \phi^{+} + \phi^{-}$, where

$$\begin{cases} \text{spt } \phi^{+} \subset I^{+}(K) \subset J^{+}(K) \\ \text{spt } \phi^{-} \subset I^{-}(K) \subset J^{-}(K). \end{cases}$$

Put

$$f = (\Box_g - m^2)\phi^+ = -(\Box_g - m^2)\phi^-$$

=>

$$\text{spt } f \subset J^+(K) \cap J^-(K)$$

=>

$$f \in C_c^\infty(M).$$

Then $\forall\ \chi \in C_c^\infty(M)$ (cf. 54.9):

- $\int_M \chi(E^+f)\text{vol}_g$

$$= \int_M (E^-\chi)f\ \text{vol}_g$$

$$= \int_M (E^-\chi)(\Box_g - m^2)\phi^+\ \text{vol}_g$$

$$= \int_M ((\Box_g - m^2)E^-\chi)\phi^+\ \text{vol}_g$$

$$= \int_M \chi\phi^+\ \text{vol}_g$$

=>

$$E^+f = \phi^+.$$

- $\int_M \chi(E^-f)\text{vol}_g$

$$= \int_M (E^+\chi)f\ \text{vol}_g$$

$$= \int_M (E^+\chi)(-(\Box_g - m^2)\phi^-)\text{vol}_g$$

$$= \int_M ((\Box_g - m^2)E^+\chi)(-\phi^-)\mathrm{vol}_g$$

$$= \int_M \chi(-\phi^-)\mathrm{vol}_g$$

=>

$$E^-f = -\phi^-.$$

Therefore

$$Ef = E^+f - E^-f$$

$$= \phi^+ - (-\phi^-) = \phi^+ + \phi^- = \phi.$$

§55. *CAUCHY DATA*

Suppose that (M,g) is globally hyperbolic. Fix a Cauchy hypersurface $\Sigma \subset M$. Given $\phi \in C^\infty(M)$, let

$$\begin{cases} \rho_\Sigma \phi = \phi | \Sigma \\ \partial_\Sigma \phi = \dfrac{\partial \phi}{\partial \underline{n}}, \end{cases}$$

where $\dfrac{\partial}{\partial \underline{n}}$ is defined using the future directed unit normal $\underline{n}$ along Σ.

55.1 <u>THEOREM</u> (Dimock) Let $u,v \in C_c^\infty(\Sigma)$ -- then there is a unique $\phi \in C^\infty(M)$ such that $(\square_g - m^2)\phi = 0$ and

$$\rho_\Sigma \phi = u, \partial_\Sigma \phi = v.$$

[Note: If $\text{spt } u \cup \text{spt } v \subset K$, where K is compact, then $\text{spt } \phi \subset J^+(K) \cup J^-(K)$, thus $\phi \in C_{sc}^\infty(M)$.]

In particular:

$$(\square_g - m^2)\phi = 0 \ \& \ \begin{cases} \rho_\Sigma \phi = 0 \\ \partial_\Sigma \phi = 0 \end{cases} \Rightarrow \phi = 0.$$

Let

$$\Gamma = C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$$

and put

$$\sigma((u,v),(u',v')) = \int_\Sigma (uv' - u'v)d\mu_q.$$

[Note: μ_q is the riemannian measure attached to q (= $g|\Sigma$).]

55.2 THEOREM (Dimock) The arrow

$$\begin{cases} (E_m(M,g),\sigma_g) \xrightarrow{T} (\Gamma,\sigma) \\ [f] \longrightarrow (\rho_\Sigma(Ef),\partial_\Sigma(Ef)) \end{cases}$$

is a symplectic isomorphism.

The first point to check is that $\rho_\Sigma(Ef)$ and $\partial_\Sigma(Ef)$ are actually compactly supported. This depends on the fact that Σ is a Cauchy hypersurface: $\forall$ compact set $K \subset M$,

$$\Sigma \cap J^{\pm}(K)$$

is compact (cf. 54.20). So, e.g.,

$$\text{spt } \rho_\Sigma(E^+f) \subset \Sigma \cap J^+(\text{spt } f)$$

is compact.

Injectivity: Suppose that

$$\rho_\Sigma(Ef_1) = \rho_\Sigma(Ef_2) \text{ and } \partial_\Sigma(Ef_1) = \partial_\Sigma(Ef_2).$$

Since

$$(\Box_g - m^2)E(f_1 - f_2) = 0 \quad \text{(cf. 54.11)},$$

it follows by uniqueness that $E(f_1 - f_2) = 0$, hence $[f_1] = [f_2]$.

Surjectivity: Given $u,v \in C_c^\infty(\Sigma)$, determine ϕ per 55.1 -- then $\exists\, f \in C_c^\infty(M)$ such that $\phi = Ef$ (cf. 54.21). Therefore $[f]$ is sent by T to

$$(\rho_\Sigma(Ef), \partial_\Sigma(Ef)) = (\rho_\Sigma\phi, \partial_\Sigma\phi) = (u,v).$$

The verification that

$$\sigma_g([f_1],[f_2]) = \sigma(T[f_1],T[f_2])$$

hinges on a variant of Green's identity.

55.3 <u>LEMMA</u> If $(\Box_g - m^2)\phi = 0$, then for any $f \in C_c^\infty(M)$,

$$\int_M f\phi \ \mathrm{vol}_g$$

$$= \int_\Sigma (\rho_\Sigma(Ef)\partial_\Sigma\phi - (\rho_\Sigma\phi)\partial_\Sigma(Ef))d\mu_g.$$

<u>PROOF</u> To begin with, M is the disjoint union of $I^-(\Sigma), \Sigma, I^+(\Sigma)$ and Σ is the common boundary of the open sets $I^-(\Sigma)$ and $I^+(\Sigma)$. This said, put $D_M = \Box_g - m^2$ -- then

- $\int_{I^-(\Sigma)} f\phi \ \mathrm{vol}_g$

$$= \int_{I^-(\Sigma)} (D_M E^+ f)\phi \ \mathrm{vol}_g$$

$$= \int_{I^-(\Sigma)} ((D_M E^+ f)\phi - (E^+ f)D_M\phi)\mathrm{vol}_g$$

$$= \int_\Sigma (\rho_\Sigma(E^+f)\partial_\Sigma\phi - (\rho_\Sigma\phi)\partial_\Sigma(E^+f))d\mu_q.$$

- $\int_{I^+(\Sigma)} f\phi \, \mathrm{vol}_g$

$$= \int_{I^+(\Sigma)} (D_M E^- f)\phi \, \mathrm{vol}_g$$

$$= \int_{I^+(\Sigma)} ((D_M E^- f)\phi - (E^- f)D_M\phi)\mathrm{vol}_g$$

$$= - \int_\Sigma (\rho_\Sigma(E^-f)\partial_\Sigma\phi - (\rho_\Sigma\phi)\partial_\Sigma(E^-f))d\mu_q.$$

Adding these relations leads to the stated formula.

Therefore

$$\sigma_g([f_1],[f_2])$$

$$= \int_M f_1(Ef_2)\mathrm{vol}_g$$

$$= \int_\Sigma (\rho_\Sigma(Ef_1)\partial_\Sigma(Ef_2) - \rho_\Sigma(Ef_2)\partial_\Sigma(Ef_1))d\mu_q$$

$$= \sigma(T[f_1],T[f_2]).$$

55.4 <u>LEMMA</u> T induces an isomorphism

$$\mathcal{W}(E_m(M,g),\sigma_g) \rightarrow \mathcal{W}(\Gamma,\sigma)$$

of C*-algebras.

Every quasifree state ω_μ on $\mathcal{W}(\Gamma,\sigma)$ thus gives rise to a quasifree state on $\mathcal{W}(E_m(M,g),\sigma_g)$ (cf. 20.6).

If $\mu \in IP(\Gamma,\sigma)$ and if

$$\lambda_\mu : \Gamma \times \Gamma \to \underline{C}$$

is its 2-point function, i.e.,

$$\lambda_\mu((u,v),(u',v'))$$

$$= \frac{1}{2} (\mu((u,v),(u',v')) + \sqrt{-1}\, \sigma((u,v),(u',v'))) \quad \text{(cf. 20.8 \& 20.9)},$$

then we shall define

$$\Lambda_\mu : C_c^\infty(M) \times C_c^\infty(M) \to \underline{C}$$

by pulling back the composition

$$E_m(M,g) \times E_m(M,g) \xrightarrow{T \times T} \Gamma \times \Gamma \xrightarrow{\lambda_\mu} \underline{C}$$

and lifting it to $C_c^\infty(M) \times C_c^\infty(M)$. Explicated:

$$\Lambda_\mu(f_1,f_2)$$

$$= \lambda_\mu((\rho_\Sigma Ef_1,\partial_\Sigma Ef_1),(\rho_\Sigma Ef_2,\partial_\Sigma Ef_2)).$$

Therefore

$$\text{Im}\, \Lambda_\mu(f_1,f_2)$$

$$= \text{Im}\, \lambda_\mu(T[f_1],T[f_2])$$

$$= \frac{1}{2}\,\sigma(T[f_1],T[f_2])$$

$$= \frac{1}{2}\int_M f_1(Ef_2)\mathrm{vol}_g.$$

55.5 <u>REMARK</u> If Λ_μ is separately continuous, then it determines a distribution on $M \times M$ denoted still by Λ_μ.

[Note: Put

$$\kappa(f_2)(f_1) = \Lambda_\mu(f_1,f_2).$$

Then for fixed f_2, $\Lambda_\mu(f_1,f_2)$ is continuous in f_1, thus $\kappa(f_2)$ is a distribution. Since $\kappa:C_c^\infty(M) \to C_c^\infty(M)^*$ is weakly sequentially continuous, the Schwartz kernel theorem implies that there exists a unique distribution K_κ on $M \times M$ such that

$$K_\kappa(f_1 \times f_2) = \kappa(f_2)(f_1).]$$

In practice, E is frequently regarded as an integral operator with kernel $E(x,y)$:

$$Ef(x) = \int_M E(x,y)f(y)\mathrm{vol}_g$$

subject to $E(x,y) = -E(y,x)$.

[Note: Technically, $E(x,y)$ is the distribution kernel of the operator E. Of course, the integral on the RHS represents the duality bracket between test functions and distributions (both w.r.t. the variable y). One should also observe that matters have been arranged so as to be consistent with the Schwartz kernel theorem. Indeed,

$$E:C_c^\infty(M) \to C^\infty(M)$$

is a continuous linear map and $\forall\ f_1,f_2 \in C_c^\infty(M)$,

$$\begin{aligned}\sigma_g(f_1,f_2) &= \int_M f_1(Ef_2)\mathrm{vol}_g \\ &= (Ef_2)(f_1) \\ &= E(f_1 \times f_2).]\end{aligned}$$

55.6 EXAMPLE Take $M = \underline{R}^{1,3}$ (i.e., Minkowski space) -- then

$$E((t,x),(s,y))$$

$$= \frac{1}{(2\pi)^3} \int_{\underline{R}^3} \sin((t-s)\lambda(\xi) - (x-y)\cdot\xi) \frac{d\xi}{\lambda(\xi)},$$

or still,

$$E((t,x),(s,y))$$

$$= \frac{1}{(2\pi)^3} \int_{\underline{R}^3} \sin((t-s)\lambda(\xi))e^{\sqrt{-1}\ (x-y)\cdot\xi} \frac{d\xi}{\lambda(\xi)},$$

where $\lambda(\xi) = (|\xi|^2 + m^2)^{1/2}$.

[Note: Here, of course

$$\left[\begin{array}{l} x \in \underline{R}^3 \\ y \in \underline{R}^3. \end{array}\right.$$

This said, put

$$\begin{cases} \underline{x} = (t,x) \\ \underline{y} = (s,y). \end{cases}$$

Then by definition,

$$\Delta(\underline{x}-\underline{y}) = E((t,x),(s,y)).]$$

N.B. Similar conventions apply to Λ_μ (if Λ_μ is actually a distribution (cf. 55.5)).

§56. *THE DEUTSCH-NAJMI CONSTRUCTION*

Assuming that (M,g) is globally hyperbolic, fix a Cauchy hypersurface $\Sigma \subset M$ and let $\mu \in \mathrm{IP}(\Gamma,\sigma)$ be pure — then, as we have seen (cf. 20.19 and 20.22), there exists a complex Hilbert space K_μ and a real linear map $k_\mu:\Gamma \rightarrow K_\mu$ such that

(1) k_μ is one-to-one and $k_\mu E$ is dense in K_μ;

(2) $\forall$ $(u,v),(u',v') \in \Gamma$,

$$<k_\mu(u,v),k_\mu(u',v')>$$

$$= \mu((u,v),(u',v')) + \sqrt{-1}\ \sigma((u,v),(u',v')).$$

It is also possible to reverse the procedure by first defining the pair (k,K) and then deducing what μ must be.

Consider $L^2(\Sigma,\mu_q)$ (taken over $\underline{C}$). Let R,S be densely defined linear operators on $L^2(\Sigma,\mu_q)$ whose domains contain $C_c^\infty(\Sigma)$ and which commute with the complex conjugation, subject to the following conditions:

(R) R is bounded and selfadjoint;

(S) S is selfadjoint, positive, and has a bounded inverse.

Define now a real linear map

$$k:\Gamma \rightarrow L^2(\Sigma,\mu_q)$$

by

$$k(u,v) = S^{-1/2}[(R - \sqrt{-1}\ S)u + v]$$

and let

$$\mathcal{K} = \overline{k\Gamma + \sqrt{-1}\, k\Gamma}.$$

56.1 LEMMA $\forall$ $(u,v),(u',v') \in \Gamma$,

$$\text{Im} \langle k(u,v), k(u',v')\rangle = \sigma((u,v),(u',v')).$$

PROOF In fact,

$$\text{Im} \langle k(u,v), k(u',v')\rangle$$

$$= \text{Im} \langle S^{-1/2}[(R - \sqrt{-1}\, S)u + v],\ S^{-1/2}[(R - \sqrt{-1}\, S)u' + v']\rangle$$

$$= \text{Im} \langle (R - \sqrt{-1}\, S)u + v,\ S^{-1}[(R - \sqrt{-1}\, S)u' + v']\rangle$$

$$= \text{Im} \langle Ru + v - \sqrt{-1}\, Su,\ S^{-1}Ru' + S^{-1}v' - \sqrt{-1}\, u'\rangle$$

$$= \langle Ru + v, - u'\rangle + \langle Su,\ S^{-1}Ru' + S^{-1}v'\rangle$$

$$= - \langle Ru,u'\rangle - \langle v,u'\rangle + \langle u,Ru'\rangle + \langle u,v'\rangle$$

$$= \langle u,v'\rangle - \langle u',v\rangle$$

$$= \sigma((u,v),(u',v')).$$

Inspection of this computation then gives

$$\text{Re} \langle k(u,v), k(u',v')\rangle$$

$$= \langle u,Su'\rangle + \langle Ru + v,\ S^{-1}(Ru' + v')\rangle.$$

Denote the latter by

$$\mu((u,v),(u',v')).$$

Since S is positive, it is clear that μ is a real valued inner product on Γ with

$$|\sigma((u,v),(u',v'))|^2 \leq \mu((u,v),(u,v))\mu((u',v'),(u',v')).$$

I.e.: $\mu \in IP(\Gamma,\sigma)$. And, by construction,

$$<k(u,v),k(u',v')>$$

$$= \mu((u,v),(u',v')) + \sqrt{-1}\,\sigma((u,v),(u',v')).$$

56.2 REMARK k is one-to-one. For suppose that $k(u,v) = 0$ -- then

$$\sigma((u,v),(u',v')) = 0 \ \forall \ (u',v') \in \Gamma,$$

which implies that $u = 0$ & $v = 0$.

It remains to establish that μ is pure. To this end, recall the definition of A_μ:

$$\sigma_\mu(x,y) = \mu(x,A_\mu y) \quad (x,y \in H_\mu).$$

56.3 LEMMA We have

$$A_\mu = \begin{bmatrix} S^{-1}R & S^{-1} \\ -RS^{-1}R - S & -RS^{-1} \end{bmatrix}.$$

PROOF Regarding the elements of Γ as column vectors,

$$\mu((u,v),A_\mu(u',v'))$$

$$= \mu((u,v),(S^{-1}Ru' + S^{-1}v', - RS^{-1}Ru' - Su' - RS^{-1}v'))$$

$$= \langle u,S(S^{-1}Ru' + S^{-1}v')\rangle$$

$$+ \langle Ru + v, S^{-1}(RS^{-1}Ru' + RS^{-1}v' - RS^{-1}Ru' - Su' - RS^{-1}v')\rangle$$

$$= \langle u,v'\rangle + \langle u,Ru'\rangle + \langle Ru + v, - u'\rangle$$

$$= \langle u,v'\rangle - \langle u',v\rangle$$

$$= \sigma((u,v),(u',v')).$$

But then $A_\mu^2 = - I$, thus $|A_\mu| = I$, so μ is pure (cf. 20.25).

[Note: Consequently, $k\Gamma$ is dense in K (cf. 20.24).]

56.4 REMARK Take $R = 0$ -- then matters simplify:

$$k(u,v) = - \sqrt{-1}\, S^{1/2}u + S^{-1/2}v$$

and

$$\mu((u,v),(u',v')) = \langle u,Su'\rangle + \langle v,S^{-1}v'\rangle.$$

[Note: Let

$$\tilde{k}(u,v) = \sqrt{-1}\, k(u,v),$$

hence

$$\tilde{k}(u,v) = S^{1/2}u + \sqrt{-1}\, S^{-1/2}v.$$

Since

$$<\tilde{k}(u,v),\tilde{k}(u',v')> = <k(u,v),k(u',v')>,$$

nothing is lost if we work with $\tilde{k}$ rather than k.]

56.5 <u>EXAMPLE</u> Suppose that the induced riemannian structure q on Σ is complete. Take R = 0 and $S = (-\Delta_q + m^2)^{1/2}$ -- then

$$\mu((u,v),(u',v')) = <u,(-\Delta_q + m^2)^{1/2}u'> + <v,(-\Delta_q + m^2)^{-1/2}v'>$$

and the associated quasifree state ω_μ on $\mathcal{W}(\Gamma,\sigma)$ leads to a quasifree state on $\mathcal{W}(E_m(m,g),\sigma_g)$ (cf. 55.4).

[Note: Put

$$A = -\Delta_q + m^2.$$

Then

$$\lambda_\mu((u,v),(u',v'))$$

$$= \frac{1}{2} <k(u,v),k(u',v')>$$

$$= \frac{1}{2} <\tilde{k}(u,v),\tilde{k}(u',v')>$$

$$= \frac{1}{2} <A^{1/4}u + \sqrt{-1}\, A^{-1/4}v,\ A^{1/4}u' + \sqrt{-1}\, A^{-1/4}v'>$$

$$= \frac{1}{2} <A^{-1/4}(A^{1/2}u + \sqrt{-1}\, v),\ A^{-1/4}(A^{1/2}u' + \sqrt{-1}\, v')>$$

$$= \frac{1}{2} <A^{1/2}u + \sqrt{-1}\, v,\ A^{-1/2}(A^{1/2}u' + \sqrt{-1}\, v')>$$

=>

$$\Lambda_\mu(f_1,f_2)$$

$$= \frac{1}{2} \langle (A^{1/2}\rho_\Sigma + \sqrt{-1}\ \partial_\Sigma)Ef_1,\ A^{-1/2}(A^{1/2}\rho_\Sigma + \sqrt{-1}\ \partial_\Sigma)Ef_2\rangle.]$$

N.B. This setup is realized if we let $M = \underline{R}^{1,3}$, $\Sigma = \underline{R}^3$ -- then

$$\Lambda_\mu((t,x),(s,y))$$

$$= \frac{1}{(2\pi)^3} \int_{\underline{R}^3} \exp(\sqrt{-1}\ ((t-s)\lambda(\xi) - (x-y)\ \cdot\xi))\ \frac{d\xi}{2\lambda(\xi)},$$

where $\lambda(\xi) = (|\xi|^2 + m^2)^{1/2}$.

[Note: To run a formal reality check, observe that

$$\Lambda_\mu((t,x),(s,y)) - \Lambda_\mu((s,y),(t,x))$$

$$= \sqrt{-1}\ E((t,x),(s,y)) \quad \text{(cf. 55.6)}.$$

Replacing Λ_μ by the symbol Δ_+ (which is traditional in this context), we can thus write

$$\Delta_+(\underline{x} - \underline{y}) - \Delta_+(\underline{y} - \underline{x}) = \sqrt{-1}\ \Delta(\underline{x} - \underline{y})$$

or still,

$$\Delta_+(\underline{x} - \underline{y}) - \overline{\Delta_+(\underline{x} - \underline{y})} = \sqrt{-1}\ \Delta(\underline{x} - \underline{y}).]$$

§57. *ULTRASTATIC SPACETIMES*

In this § we shall consider those objects in GLOBHYP that have the simplest structure.

57.1 LEMMA Suppose that Σ is a connected orientable C^∞ manifold of dimension 3. Let q be a complete riemannian structure on Σ. Put $M = \underline{R} \times \Sigma$ and define $g \in \underline{M}_{1,3}$ by

$$g_{(t,x)}((r,X),(s,Y))$$

$$= - rs + q_x(X,Y) \quad (r,s \in \underline{R} \ \& \ X,Y \in T_x\Sigma).$$

Then the pair (M,g) is globally hyperbolic.

[Note: Such a pair is said to be ultrastatic. In the terminology of §53, the lapse N is $\equiv 1$ and the shift $\vec{N}$ is $\equiv \vec{0}$.]

Assume henceforth that (M,g) is ultrastatic and denote the points in M by $\underline{x} = (t,x)$ $(t \in \underline{R},\ x \in \Sigma)$.

Put

$$A = - \Delta_q + m^2 \quad \text{(cf. §56)}.$$

Then the collection

$$\{ \frac{\sin(t\sqrt{A})}{\sqrt{A}} : t \in \underline{R} \}$$

is a one parameter family of densely defined linear operators on $L^2(\Sigma,\mu_q)$ and it is customary to write

$$\frac{\sin((t-s)\ \sqrt{A}\ )}{\sqrt{A}}\ u(x)$$

$$= \int_{\Sigma} \frac{\sin((t-s)\ \sqrt{A}\ )}{\sqrt{A}}\ (x,y)u(y)d\mu_q(y).$$

57.2 EXAMPLE Take $\Sigma = \underline{R}^3$, q = usual metric -- then $M = \underline{R}^{1,3}$ is Minkowski space. Since

$$Ae^{\sqrt{-1}\ x\cdot\xi} = (|\xi|^2 + m^2)e^{\sqrt{-1}\ x\cdot\xi}\ ,$$

it follows that

$$\frac{\sin((t-s)\ \sqrt{A}\ )}{\sqrt{A}}\ (x,y)$$

$$= \frac{1}{(2\pi)^3} \int_{\underline{R}^3} \frac{\sin((t-s)(|\xi|^2 + m^2)^{1/2})}{(|\xi|^2 + m^2)^{1/2}}\ e^{\sqrt{-1}\ (x-y)\cdot\xi}\ d\xi$$

$$= E((t,x),(s,y)) \quad \text{(cf. 55.6).}$$

In the Cauchy problem per 55.1, let $u = 0$ but let v be arbitrary. Define $\phi \in C^\infty(M)$ by

$$\phi(t,x) = \frac{\sin(t\ \sqrt{A}\ )}{\sqrt{A}}\ v(x).$$

Then

$$(\Box_g - m^2)\phi$$

$$= (-\partial_t^2 + \Delta_q - m^2)\phi$$

$$= -(\partial_t^2 + A)\phi$$

$$= -(-\sin(t\sqrt{A})\sqrt{A}\,v + \sin(t\sqrt{A})\sqrt{A}\,v)$$

$$= 0.$$

And

$$\begin{cases} \phi(0,x) = 0 \\ \frac{\partial\phi}{\partial t}(0,x) = v. \end{cases}$$

On the other hand, the function

$$(t,x) \to \int_\Sigma E((t,x),(0,y))v(y)d\mu_q(y)$$

has exactly the same properties (observe that

$$\int_\Sigma \frac{\partial}{\partial t} E((0,x),(0,y))v(y)d\mu_q(y)$$

$$= \int_\Sigma \delta(x,y)v(y)d\mu_q(y)$$

$$= v(x)).$$

Therefore

$$\frac{\sin(t\sqrt{A})}{\sqrt{A}}(x,y) = E((t,x),(0,y)).$$

57.3 <u>LEMMA</u> We have

$$\frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}(x,y) = E((t,x),(s,y)).$$

PROOF Repeat the foregoing discussion, working instead with the Cauchy hypersurface $\{s\} \times \Sigma$.

57.4 EXAMPLE Take $\Sigma = [0,L]^3/\sim$, q = usual metric -- then the orthonormal eigenfunctions of A are the $L^{-3/2} e^{\sqrt{-1}\, k\cdot x}$ $(k = \frac{2\pi}{L} n,\ n \in \underline{Z}^3)$ with

$$\begin{aligned} AL^{-3/2} e^{\sqrt{-1}\, k\cdot x} &= \left|\frac{2\pi}{L} n\right|^2 + m^2 \\ &\equiv \Lambda(n). \end{aligned}$$

Consequently,

$$\begin{aligned} &\frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}(x,y) \\ &= \frac{1}{L^3} \sum_{n\in\underline{Z}^3} \frac{\sin((t-s)\sqrt{\Lambda(n)})}{\sqrt{\Lambda(n)}} e^{\sqrt{-1}\,\frac{2\pi}{L} n\cdot(x-y)} \end{aligned}$$

and we claim that

$$\begin{aligned} &\frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}(x,y) \\ &= \sum_{n\in\underline{Z}^3} \Delta(t-s,x-y+nL), \end{aligned}$$

Δ being as in 55.6. In fact,

$$\sum_{n\in Z^3} \Delta(t-s,x-y+nL)$$

§58. *PSEUDODIFFERENTIAL OPERATORS*

It is a question here of formulating those definitions and results from the theory that will be needed later on.

Notation

: $x = (x^1,\ldots,x^n) \in \underline{R}^n$

: $\xi = (\xi_1,\ldots,\xi_n) \in \underline{R}^n$

: $x\xi = x^1\xi_1 + \cdots + x^n\xi_n$

: $|\xi| = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$

: $\alpha = (\alpha_1,\ldots,\alpha_n) \in \underline{Z}^n_{\geq 0}$

: $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$

: $\beta = (\beta_1,\ldots,\beta_n) \in \underline{Z}^n_{\geq 0}$

: $|\beta| = |\beta_1| + \cdots + |\beta_n|$

: $D_x^\alpha = \left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x^n}\right)^{\alpha_n}$

: $D_\xi^\beta = \left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial \xi_1}\right)^{\beta_1} \cdots \left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial \xi_n}\right)^{\beta_n}$.

[Note: Conceptually, x is a vector and ξ is a covector, the arrow

$$\begin{cases} \underline{R}^n \times \underline{R}^n \to \underline{R} \\ (x,\xi) \to x\xi \end{cases}$$

being the duality.]

N.B. The sign convention on Fourier transforms is "minus", i.e.,

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} e^{-\sqrt{-1}\, x\xi} f(x)dx.$$

Let X be a nonempty open subset of $\underline{R}^n$. Let m be any real number -- then by

$$S^m(X \times \underline{R}^n)$$

we understand the set of C^∞ functions $a:X \times \underline{R}^n \to \underline{C}$ which have the property that for all compact sets $K \subset X$ and all multiindices α,β, $\exists$ a constant $C_{K,\alpha,\beta} > 0$: $\forall\ x \in K\ \&\ \forall\ \xi \in \underline{R}^n$,

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{K,\alpha,\beta}\ (1 + |\xi|)^{m - |\beta|}.$$

The elements of $S^m(X \times \underline{R}^n)$ are called the symbols of degree $\leq m$.

58.1 LEMMA $S^m(X \times \underline{R}^n)$ is a Fréchet space when equipped with the topology induced by the seminorms

$$p_{K,\alpha,\beta}(a) = \sup_{x\in K,\ \xi\in\underline{R}^n} (1 + |\xi|)^{-m + |\beta|} |D_x^\alpha D_\xi^\beta a(x,\xi)|,$$

where K ranges over the compact subsets of X and α,β ranges over the pairs of multiindices.

Obviously,

$$m' < m \Rightarrow S^{m'}(X \times \underline{R}^n) \subset S^m(X \times \underline{R}^n)$$

and the canonical injection

$$S^{m'}(X \times \underline{R}^n) \to S^m(X \times \underline{R}^n)$$

is continuous.

58.2 LEMMA The closure of $C_c^\infty(X \times \underline{R}^n)$ in $S^m(X \times \underline{R}^n)$ contains $S^{m'}(X \times \underline{R}^n)$ for all $m' < m$.

Put

$$\begin{cases} S^{-\infty}(X \times \underline{R}^n) = \bigcap_{m \in \underline{R}} S^m(X \times \underline{R}^n) \\ S^{\infty}(X \times \underline{R}^n) = \bigcup_{m \in \underline{R}} S^m(X \times \underline{R}^n). \end{cases}$$

Given $a, a' \in S^\infty(X \times \underline{R}^n)$, one writes

$$a \sim a'$$

if

$$a - a' \in S^{-\infty}(X \times \underline{R}^n).$$

Let $a \in S^m(X \times \underline{R}^n)$. Suppose $\exists\, a_j \in S^{m_j}(X \times \underline{R}^n)$, where

$$m = m_0 > m_1 > \cdots > m_j \to -\infty \ (j \to \infty),$$

such that

$$a - \sum_{0 \le j < k} a_j \in S^{m_k}(X \times \underline{R}^n)$$

for every positive integer k -- then the sequence $\{a_j : j \ge 0\}$ is called an <u>asymptotic expansion</u> of a.

58.3 <u>LEMMA</u> Let $\{m_j : j \ge 0\}$ be a strictly decreasing sequence of real numbers with $\lim_{j \to \infty} m_j = -\infty$. Suppose that $\forall$ j,

$$a_j \in S^{m_j}(X \times \underline{R}^n).$$

Then $\exists$

$$a \in S^{m_0}(X \times \underline{R}^n)$$

such that

$$a - \sum_{0 \le j < k} a_j \in S^{m_k}(X \times \underline{R}^n)$$

for every positive integer k.

[Note: The symbol a is unique modulo $S^{-\infty}(X \times \underline{R}^n)$. For if a' is another symbol with the stated property, then

$$a - a' = (a - \sum_{0 \le j < k} a_j) - (a' - \sum_{0 \le j < k} a_j) \in S^{m_k}(X \times \underline{R}^n)$$

=>

$$a - a' \in S^{-\infty}(X \times \underline{R}^n).]$$

Let $a \in S^m(X \times \underline{R}^n)$ -- then the pseudodifferential operator A_a attached to a is the continuous linear map

$$A_a : C_c^\infty(X) \to C^\infty(X)$$

defined by the rule

$$A_a f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} e^{\sqrt{-1}\, x\xi}\, a(x,\xi)\hat{f}(\xi)d\xi.$$

[Note: Since the Fourier transform $\hat{f}$ is rapidly decreasing and since $|a(x,\xi)| \leq C_x(1 + |\xi|)^m$, it follows that the function

$$\xi \to a(x,\xi)\hat{f}(\xi)$$

is integrable for all $x \in X$.]

58.4 REMARK If $A_a = A_{a'}$, then it is not necessarily true that $a = a'$ but at least $a \sim a'$.

Let

$$\Psi^m(X) = \{A_a : a \in S^m(X \times \underline{R}^n)\}$$

and put

$$\begin{cases} \Psi^{-\infty}(X) = \bigcap\limits_{m \in \underline{R}} \Psi^m(X) \\ \Psi^{\infty}(X) = \bigcup\limits_{m \in R} \Psi^m(X). \end{cases}$$

Given $A, A' \in \Psi^{\infty}(X)$, one writes

$$A \sim A'$$

if

$$A - A' \in \Psi^{-\infty}(X).$$

[Note: The elements of $\Psi^m(X)$ are said to have order $\leq m$ and the elements of

$$\Psi^m(X) - \bigcup_{m'<m} \Psi^{m'}(X)$$

are said to have order m.]

58.5 LEMMA The map

$$\begin{cases} S^m(X \times \underline{R}^n) \rightarrow \Psi^m(X) \\ \qquad a \rightarrow A_a \end{cases}$$

induces a linear bijection

$$S^m(X \times \underline{R}^n)/S^{-\infty}(X \times \underline{R}^n) \rightarrow \Psi^m(X)/\Psi^{-\infty}(X)$$

i.e., induces a linear bijection

$$S^m(X \times \underline{R}^n)/\sim \rightarrow \Psi^m(X)/\sim.$$

58.6 EXAMPLE Let

$$A = \sum_{|\alpha| \leq m} a_{\alpha}(x) D_x^{\alpha} \quad (a_{\alpha} \in C^{\infty}(X))$$

be a linear differential operator on X. Put

$$\xi^\alpha = (\xi_1)^{\alpha_1} \cdots (\xi_n)^{\alpha_n}.$$

Then

$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \in S^m(X \times \underline{R}^n).$$

But $\forall\, f \in C_c^\infty(X)$,

$$(Af)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} e^{\sqrt{-1}\, x\xi}\, a(x,\xi)\hat{f}(\xi)d\xi.$$

Therefore

$$A = A_a \Rightarrow A \in \Psi^m(X).$$

58.7 EXAMPLE Take $X = \underline{R}^n$ and let Δ be the laplacian -- then

$$(1 - \Delta)^{m/2} f(x)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} e^{\sqrt{-1}\, x\xi}\, (1 + |\xi|^2)^{m/2}\hat{f}(\xi)d\xi,$$

so

$$(1 - \Delta)^{m/2} \in \Psi^m(\underline{R}^n).$$

58.8 EXAMPLE Let $\phi \in C_c^\infty(X)$ and put

$$a_\phi(x,\xi) = \int_{\underline{R}^n} \phi(x-y)e^{\sqrt{-1}\,(y-x)\xi}\, dy.$$

Then a_ϕ is rapidly decreasing in ξ. And, $\forall\, f \in C_c^\infty(X)$,

$$\frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} e^{\sqrt{-1}\, x\xi}\, a_\phi(x,\xi)\hat{f}(\xi)d\xi$$

$$= \int_{\underline{R}^n} \phi(x-y)\ [\frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} e^{\sqrt{-1}\, y\xi}\hat{f}(\xi)d\xi]dy$$

$$= \int_{\underline{R}^n} \phi(x-y)f(y)dy$$

$$= \phi * f(x),$$

thus the convolution $\phi*$— is a pseudodifferential operator:

$$\phi * \text{—} \in \Psi^{-\infty}(X).$$

Given $a \in S^m(X \times \underline{R}^n)$, let K_a be the distribution on $X \times X$ corresponding to A_a via the Schwartz kernel theorem. Symbolically:

$$K_a(x,y) = \frac{1}{(2\pi)^n} \int_{\underline{R}^n} e^{\sqrt{-1}\,(x-y)\xi} a(x,\xi)d\xi.$$

In this connection, observe that $\forall\ f_1, f_2 \in C_c^\infty(X)$,

$$\langle f_1, A_a f_2\rangle$$

$$= \int_{\underline{R}^n} \int_{\underline{R}^n} K_a(x,y) f_1(x) f_2(y) dxdy.$$

58.9 <u>LEMMA</u> K_a is C^∞ off the diagonal $\Delta(X \times X)$ of $X \times X$.

58.10 <u>REMARK</u> The distribution kernel K_A of a pseudodifferential operator $A \in \Psi^{\infty}(X)$ is a C^{∞} function on $X \times X$ iff $A \in \Psi^{-\infty}(X)$.

[Note: The elements of $\Psi^{-\infty}(X)$ are called <u>smoothing operators</u>. They are regularizing in the sense that each such extends to a continuous linear map $C^{\infty}(X)^* \to C^{\infty}(X)$.]

58.11 <u>EXAMPLE</u> Take $X = \underline{R} - \{0\}$ and let

$$\tilde{f}(x) = f(-x) \quad (f \in C_c^{\infty}(X)).$$

Then the assignment $f \to \tilde{f}$ is not a pseudodifferential operator. Indeed,

$$\tilde{f}(x) = \int_{\underline{R}} \delta(x+y) f(y) dy$$

but $\delta(x+y)$ is not C^{∞} off the diagonal of $X \times X$ (cf. 58.9).

The support of K_a is a closed subset of $X \times X$. We shall then term A_a <u>properly supported</u> if both projections from spt $K_a \subset X \times X$ to X are proper maps.

58.12 <u>EXAMPLE</u> Let

$$A = \sum_{|\alpha| \leq m} a_{\alpha}(x) D_x^{\alpha} \quad (a_{\alpha} \in C^{\infty}(X))$$

be a linear differential operator on X (cf. 58.6) -- then

$$K_A(x,y) = \frac{1}{(2\pi)^n} \int_{\underline{R}^n} \sum_{|\alpha| \leq m} e^{\sqrt{-1}\,(x-y)\xi} a_{\alpha}(x) \xi^{\alpha} d\xi$$

$$= \sum_{|\alpha| \leq m} a_\alpha(x) \frac{1}{(2\pi)^n} \int_{\underline{R}^n} e^{\sqrt{-1}\,(x-y)\,\xi} \xi^\alpha d\xi$$

$$= \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha \delta(x-y)$$

=>

$$\text{spt } K_A \subset \Delta(X \times X).$$

Therefore A is properly supported.

58.13 <u>LEMMA</u> Let $A \in \Psi^m(X)$ -- then $A = A' + A''$, where $A' \in \Psi^m(X)$ is properly supported and $A'' \in \Psi^{-\infty}(X)$.

58.14 <u>REMARK</u> In general, a pseudodifferential operator sends $C_c^\infty(X)$ continuously to $C^\infty(X)$ but a properly supported pseudodifferential operator sends $C_c^\infty(X)$ continuously to itself (and, in addition, gives rise to a continuous map $C^\infty(X) \to C^\infty(X)$). Observe too that a properly supported smoothing operator sends $C_c^\infty(X)^*$ continuously to $C^\infty(X)$ (cf. 58.10).

58.15 <u>LEMMA</u> If $A \in \Psi^m(X)$ is properly supported, then for any $A' \in \Psi^{m'}(X)$, the compositions

$$\begin{cases} A \circ A' \\ A' \circ A \end{cases}$$

make sense and lie in $\Psi^{m+m'}(X)$.

[Note: We have

$$\begin{cases} C_c^\infty(X) \xrightarrow{A'} C^\infty(X) \xrightarrow{A} C^\infty(X) \\ C_c^\infty(X) \xrightarrow{A} C_c^\infty(X) \xrightarrow{A'} C^\infty(X).] \end{cases}$$

Let $\zeta:X \to X'$ ($\subset \underline{R}^n$) be a diffeomorphism. Suppose that $A \in \Psi^m(X)$. Define

$$A_\zeta:C_c^\infty(X') \to C^\infty(X')$$

by

$$A_\zeta f = A(f \circ \zeta) \circ \zeta^{-1} \quad (f \in C_c^\infty(X')).$$

Then $A_\zeta \in \Psi^m(X')$.

[Note: A_ζ is properly supported provided A is properly supported.]

58.16 <u>LEMMA</u> If $A = A_a$ and $A_\zeta = A_{a_\zeta}$, then

$$a_\zeta(x',\xi') - a(\zeta^{-1}x',({}^tD\zeta^{-1}(x'))^{-1}\xi') \in S^{m-1}(X' \times \underline{R}^n).$$

Suppose that M is a C^∞ manifold of dimension n. Let $A:C_c^\infty(M) \to C^\infty(M)$ be a continuous linear map. Given a chart (X,ζ) in M, define

$$A_\zeta:C_c^\infty(\zeta X) \to C^\infty(\zeta X)$$

by

$$A_\zeta f = A(f \circ \zeta) \circ \zeta^{-1} \quad (f \in C_c^\infty(\zeta X)).$$

Then A is a pseudodifferential operator (of order $\leq$ m) if $\forall$ pair (X,ζ),

$$A_\zeta \in \Psi^m(\zeta X).$$

[Note: Employ the obvious notation, viz.

$$\Psi^m(M), \Psi^{-\infty}(M), \Psi^{\infty}(M).]$$

58.17 <u>REMARK</u> There is a small matter of consistency. Thus let X be a nonempty open subset of $\underline{R}^n$. Viewing X as a C^∞ manifold, suppose that $A \in \Psi^m(X)$. Let $X' \subset X$ be open -- then $A|X' \in \Psi^m(X')$ and for any diffeomorphism $\zeta':X' \to \zeta'X'$, $(A|X')_{\zeta'} \in \Psi^m(\zeta'X')$ (cf. supra).

[Note: The other direction is, of course, trivial (take $\zeta = I_X$).]

Assume again that X is a nonempty open subset of $\underline{R}^n$. Let $A \in \Psi^m(X)$ be of order m -- then A is said to have a <u>principal symbol</u> if for some $a \in S^m(X \times \underline{R}^n)$ such that $A_a = A$, there is a decomposition

$$a = \sigma + a' \quad (|\xi| >> 0),$$

where a' is a symbol of degree < m and $\sigma(x,\xi)$ is of class C^∞ in $X \times (\underline{R}^n - \{0\})$, is positively homogeneous of degree m in ξ, and is not identically zero.

[Note: If $m < 0$, then σ is not a symbol.]

58.18 <u>LEMMA</u> If σ exists, then σ is unique.

[The point is that a positively homogeneous function of degree m in $|\xi|$ can be bounded above by $C(1 + |\xi|)^{m-\varepsilon}$ for $|\xi|$ large $(C > 0, \varepsilon > 0)$ only if it is identically zero.]

N.B. Any other symbol for A admits an analogous decomposition with the same function σ, denote it by σ_A.

[Note: σ_A is called the principal symbol for A.]

58.19 EXAMPLE The principal symbol of a linear differential operator

$$A = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha \quad (a_\alpha \in C^\infty(X))$$

is the function

$$\sigma_A(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \quad \text{(cf. 58.6)}.$$

Let $A \in \Psi^m(X)$ be of order m. Suppose that A has principal symbol σ_A -- then for any diffeomorphism $\zeta: X \to X' (\subset \underline{R}^n)$, $A_\zeta \in \Psi^m(X')$ is of order m and has principal symbol σ_{A_ζ}, where

$$\sigma_{A_\zeta}(x',\xi') = \sigma_A(\zeta^{-1}x', ({}^tD\zeta^{-1}(x'))^{-1}\xi').$$

58.20 REMARK In the manifold situation, the agreement is that $A \in \Psi^m(M)$ has a principal symbol if this is the case of the A_ζ, thus σ_A is a C^∞ function on $T^*M\backslash 0$ (the complement of the zero section in T^*M).

[Note: When X is a nonempty open subset of $\underline{R}^n$, we have

$$T^*X\backslash 0 = X \times (\underline{R}^n - \{0\})$$

but the definition of principal symbol in the manifold sense is more restrictive (e.g., a symbol $a \in S^m(X \times \underline{R}^n)$ might vanish identically in some nonempty open subset of X).]

A symbol $a \in S^m(X \times \underline{R}^n)$ is said to be <u>elliptic</u> of degree m if $\forall$ compact subset $K \subset X$, $\exists\ C_K > 0$ & $R > 0$:

$$|a(x,\xi)| \geq C_K|\xi|^m \quad (x \in K, |\xi| > R).$$

[Note: The pseudodifferential operator A_a determined by a is called <u>elliptic</u> of order m.]

58.21 <u>EXAMPLE</u> Consider a linear differential operator

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha \quad (a_\alpha \in C^\infty(X))$$

on X (cf. 58.6) -- then the usual terminology is that A is elliptic if

$$\sigma_A(x,\xi) \neq 0 \ \forall \ (x,\xi) \in X \times (\underline{R}^n - \{0\}),$$

in which case

$$\Big|\sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha\Big| \geq C_x|\xi|^m$$

for some positive C_x and, of course, C_x can be chosen independent of x so long as x varies in a compact subset of X.

Let $A \in \Psi^{\infty}(X)$ be properly supported -- then A induces arrows

$$\begin{cases} C_c^{\infty}(X) \to C_c^{\infty}(X) \\ \\ C^{\infty}(X) \to C^{\infty}(X) \end{cases} \qquad \text{(cf. 58.14)}$$

denoted still by A. This said, a parametrix for A is a continuous linear map

$$Q:C_c^{\infty}(X) \to C^{\infty}(X)$$

such that

$$\begin{cases} A \circ Q - I \in \Psi^{-\infty}(X) \\ \\ Q \circ A - I \in \Psi^{-\infty}(X). \end{cases}$$

[Note: We have

$$\begin{cases} C_c^{\infty}(X) \xrightarrow{Q} C^{\infty}(X) \xrightarrow{A} C^{\infty}(X) \\ \\ C_c^{\infty}(X) \xrightarrow{A} C_c^{\infty}(X) \xrightarrow{Q} C^{\infty}(X).] \end{cases}$$

58.22 LEMMA If $A \in \Psi^{m}(X)$ is properly supported, then A is elliptic iff A admits a parametrix $Q \in \Psi^{-m}(X)$.

58.23 <u>REMARK</u> Let $\zeta:X \to X'$ $(\subset \underline{R}^n)$ be a diffeomorphism. Suppose that $A \in \Psi^m(X)$ is elliptic — then $A_\zeta \in \Psi^m(X')$ is elliptic.

To extend the foregoing considerations to a C^∞ manifold M of dimension n, one simply stipulates that an element $A \in \Psi^m(M)$ is elliptic of order m provided that this is so of the

$$A_\zeta:C_c^\infty(\zeta X) \to C^\infty(\zeta X),$$

where (X,ζ) is any chart in M. The notion of parametrix is then defined in the obvious way and 58.22 remains valid.

58.24 <u>EXAMPLE</u> Suppose that (M,g) is riemannian — then the laplacian Δ_g is elliptic of order 2.

58.25 <u>EXAMPLE</u> Suppose that (M,g) is globally hyperbolic. Define

$$E^{\pm}:C_c^\infty(M) \to C^\infty(M)$$

as in 54.8 -- then $E^{\pm}$ are parametrices for $\square_g - m^2$ but $E^{\pm}$ are not pseudodifferential operators.

§59. *WAVE FRONT SETS*

Let X be a nonempty open subset of $\underline{R}^n$. Suppose that

$$T \in C_c^\infty(X)^*$$

is a distribution on X — then the singular support of T, written

$$\text{sing spt } T,$$

is the complement in X of the largest open subset of X on which T is a C^∞ function, thus

$$\text{sing spt } T \subset \text{spt } T.$$

So, e.g., $\forall\ x \in X$,

$$\text{sing spt } \delta_x = \{x\}.$$

59.1 EXAMPLE If $A \in \Psi^\infty(X)$ is a pseudodifferential operator and if K_A is its distribution kernel, then

$$\text{sing spt } K_A \subset \Delta(X \times X) \quad (\text{cf. } 58.9).$$

59.2 LEMMA Let $A \in \Psi^\infty(X)$ be a pseudodifferential operator -- then A can be extended to a continuous linear map

$$A:C^\infty(X)^* \to C_c^\infty(X)^*$$

and $\forall\ T \in C^\infty(X)^*$,

$$\text{sing spt } AT \subset \text{sing spt } T.$$

[Note: If, in addition, A is properly supported, then A can be extended to a continuous linear map

$$A:C_c^\infty(X)^* \to C_c^\infty(X)^*$$

and $\forall\ T \in C_c^\infty(X)^*$,

$$\text{sing spt } AT \subset \text{sing spt } T.]$$

To accommodate certain applications, it is necessary to slightly enlarge the symbol concept: For any real number m and for any positive integer N,

$$S^m(X \times \underline{R}^N)$$

stands for the set of C^∞ functions $a:X \times \underline{R}^N \to \underline{C}$ which have the property that for all compact sets $K \subset X$ and all multiindices α,β, $\exists$ a constant $C_{K,\alpha,\beta} > 0$: $\forall\ x \in K$ & $\forall\ \xi \in \underline{R}^N$,

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{K,\alpha,\beta}(1 + |\xi|)^{m-|\beta|}.$$

[Note: $S^m(X \times \underline{R}^N)$ is a Fréchet space (cf. 58.1).]

A real valued C^∞ function θ on $X \times (\underline{R}^N - \{0\})$ is called a <u>phase function</u> if $\theta(x,\rho\xi) = \rho\theta(x,\xi)$ $(\rho > 0)$ and $d_{(x,\xi)}\theta \neq 0$. E.g.: $\theta(x,\xi) = x\xi$ $(N = n)$ is a phase function.

[Note: Since

$$d_{(x,\xi)}\theta = \sum_{i=1}^n \frac{\partial\theta}{\partial x^i}\,dx^i + \sum_{j=1}^N \frac{\partial\theta}{\partial \xi_j}\,d\xi_j,$$

the condition $d_{(x,\xi)}\theta \neq 0$ means that at every point $(x,\xi) \in X \times (\underline{R}^N - \{0\})$, one or more of the partial derivatives $\frac{\partial\theta}{\partial x^i}, \frac{\partial\theta}{\partial\xi_j}$ does not vanish.]

59.3 THEOREM (Hörmander) Fix a phase function θ. Given $a \in S^m(X \times \underline{R}^N)$ and $\chi \in C_c^\infty(\underline{R}^N):\chi(0) = 1$, put

$$<f,I_\chi(\theta,a)> = \lim_{\varepsilon \to 0} \iint e^{\sqrt{-1}\,\theta(x,\xi)} \chi(\varepsilon\xi)a(x,\xi)f(x)dxd\xi,$$

where $f \in C_c^\infty(X)$ -- then $I_\chi(\theta,a)$ is a distribution on X, which is independent of χ.

N.B. Call this distribution $I(\theta,a)$ -- then the assignment

$$a \to I(\theta,a)$$

is a continuous linear map from $S^m(X \times \underline{R}^N)$ to $C_c^\infty(X)^*$.

[Note: If a has compact support in ξ, then $I(\theta,a)$ is the C^∞ function

$$\int_{\underline{R}^N} e^{\sqrt{-1}\,\theta(x,\xi)} a(x,\xi)d\xi.]$$

It is customary to abuse notation and denote $I(\theta,a)$ by

$$\int e^{\sqrt{-1}\,\theta(x,\xi)} a(x,\xi)d\xi,$$

referring to it as an oscillatory integral.

59.4 EXAMPLE Take $N = n$, $X = \underline{R}^n$, $\theta(x,\xi) = x\xi$, $a(x,\xi) = 1$ and consider

$$\int e^{\sqrt{-1}\, x\xi}\, d\xi.$$

Then $\forall\ f \in C_c^\infty(\underline{R}^n)$,

$$\lim_{\varepsilon \to 0} \iint e^{\sqrt{-1}\, x\xi}\, \chi(\varepsilon\xi) f(x)\, dx d\xi$$

$$= (2\pi)^{n/2} \lim_{\varepsilon \to 0} \int_{\underline{R}^n} \Big(\frac{1}{(2\pi)^{n/2}} \int_{\underline{R}^n} e^{\sqrt{-1}\, x\xi}\, \chi(\varepsilon\xi)\, d\xi\Big) f(x)\, dx$$

$$= (2\pi)^{n/2} \lim_{\varepsilon \to 0} \int_{\underline{R}^n} \frac{1}{\varepsilon^n} \hat{\chi}(-\, x/\varepsilon) f(x)\, dx$$

$$= (2\pi)^{n/2} \lim_{\varepsilon \to 0} \int_{\underline{R}^n} \hat{\chi}(-\, x) f(\varepsilon x)\, dx$$

$$= (2\pi)^{n/2} f(0) \int_{\underline{R}^n} \hat{\chi}(x)\, dx$$

$$= (2\pi)^n f(0)\chi(0) = (2\pi)^n f(0)$$

=>

$$\int e^{\sqrt{-1}\, x\xi}\, d\xi = (2\pi)^n\, \delta_0.$$

Given a phase function θ, let

$$C(\theta) = \{(x,\xi) \in X \times (\underline{R}^N - \{0\}) : d_\xi\theta(x,\xi) = 0\}.$$

Spelled out, $C(\theta)$ consists of those points $(x,\xi) \in X \times (\underline{R}^N - \{0\})$ such that

$$\left(\frac{\partial\theta}{\partial\xi_1}, \ldots, \frac{\partial\theta}{\partial\xi_N}\right)\Big|_{(x,\xi)} = 0.$$

[Note: If $(x,\xi) \in C(\theta)$, then $d_x\theta(x,\xi) \neq 0$.]

Let

$$\pi_X : X \times (\underline{R}^N - \{0\}) \to X$$

be the projection.

59.5 LEMMA We have

$$\text{sing spt } I(\theta,a) \subset \pi_X C(\theta).$$

59.6 EXAMPLE Take $X = \underline{R}^4$, $N = 3$, and consider

$$\Delta_+(\underline{x}) = \frac{1}{(2\pi)^3} \int_{\underline{R}^3} \exp(\sqrt{-1}\,(t\lambda(\xi) - x\cdot\xi)) \frac{d\xi}{2\lambda(\xi)} \quad (\text{cf. } \S 56),$$

where $\lambda(\xi) = (|\xi|^2 + m^2)^{1/2}$ $(\underline{x} = (t,x), x \in \underline{R}^3)$. Let

$$\theta(\underline{x},\xi) = t|\xi| - x\cdot\xi$$

and

$$a(\underline{x},\xi) = \frac{1}{(2\pi)^3} \exp(\sqrt{-1}\,(t\lambda(\xi) - t|\xi|)) \frac{1}{2\lambda(\xi)}.$$

Then it is clear that θ is a phase function (for $\frac{\partial\theta}{\partial t} = |\xi| > 0$). On the other hand, a is not C^∞ ($|\xi|$ is not smooth at the origin), but for $|\xi|$ large, it behaves like an element of $S^{-1}(\underline{R}^4 \times \underline{R}^3)$. So, strictly speaking, our integral is not actually oscillatory but it is a distribution whose singular support can be estimated by 59.5.

[Note: Since

$$d_\xi \theta(\underline{x},\xi) = \frac{t\xi}{|\xi|} - x,$$

it follows that

$$C(\theta) = \{(\underline{x},\xi):\underline{x} = \underline{0}\}$$

$$\cup \{(\underline{x},\xi):|t| = |x| \neq 0 \ \& \ \frac{\xi}{|\xi|} = \frac{x}{t}\}.]$$

59.7 <u>RAPPEL</u> Let T be a compactly supported distribution on $\underline{R}^n$ -- then T is a C^∞ function iff its Fourier transform $\hat{T}$ is rapidly decreasing, i.e., $\forall\ N \in \underline{N}$, $\exists\ C_N > 0$:

$$|\hat{T}(\xi)| \leq C_N(1 + |\xi|)^{-N}$$

for all $\xi \in \underline{R}^n$.

Suppose that

$$T \in C_c^\infty(X)^*$$

is a distribution on X. If $x \in X$ is not in sing spt T, then $\exists$ a neighborhood U of x such that the restriction of T to U is a C^∞ function. Accordingly, $\forall\ f \in C_c^\infty(U)$, $fT \in C_c^\infty(R^n)$ (extension by zero), so its Fourier transform is rapidly decreasing. Conversely, if $\exists$ a neighborhood U of x such that $\forall$ $f \in C_c^\infty(U)$, $\widehat{fT}$ is rapidly decreasing, then fT is a C^∞ function, hence $x \notin$ sing spt T.

59.8 <u>RAPPEL</u> Let T be a compactly supported distribution on $\underline{R}^n$ — then the <u>regularity set</u> reg T of T is the maximal open conic subset of $\underline{R}^n - \{0\}$ on which its Fourier transform $\hat{T}$ is rapidly decreasing.

Fact:

$\forall$ $f \in C_c^\infty(\underline{R}^n)$,

$$\text{reg } fT \supset \text{reg } T.$$

[Note: The <u>singularity set</u> sing T of T is the complement of reg T, thus sing T is a closed conic subset of $\underline{R}^n - \{0\}$ and is empty iff T is a C^∞ function.]

Suppose that

$$T \in C_c^\infty(X)^*$$

is a distribution on X. Put

$$\Sigma_x(T) = \bigcap_f \text{sing } fT \quad (f \in C_c^\infty(X), f(x) \neq 0).$$

59.9 <u>LEMMA</u> $\Sigma_x(T) = \emptyset$ iff $x \notin$ sing spt T.

The <u>wave front set</u> of T is the closed conic subset of $X \times (\underline{R}^n - \{0\})$ defined

by

$$WF(T) = \{(x,\xi) \in X \times (\underline{R}^n - \{0\}) : \xi \in \Sigma_x(T)\}.$$

So, e.g., $\forall\ x \in X$,

$$WF(\delta_x) = \{x\} \times (\underline{R}^n - \{0\}).$$

[Note: $WF(T) = \emptyset$ iff T is a C^∞ function.]

59.10 <u>EXAMPLE</u> Take $X = \underline{R}^n$ and fix $f \in C_c^\infty(\underline{R}^n) : \hat{f} \geq 0$ & $\hat{f}(0) = 1$. Given $\xi \in \underline{R}^n - \{0\}$, put

$$F(x) = \sum_{k=1}^{\infty} \frac{f(kx)}{k^2} e^{\sqrt{-1}\, k^2 x\xi}.$$

Then f is continuous, C^∞ on $\underline{R}^n - \{0\}$, and

$$WF(F) = \{(0,t\xi)\ \ (t > 0)\}.$$

[Note: It is an interesting point of detail that for any closed conic subset Γ of $X \times (\underline{R}^n - \{0\})$, $\exists\ T \in C_c^\infty(X)^* : WF(T) = \Gamma$.]

59.11 <u>LEMMA</u> The projection of WF(T) in the first variable is sing spt T.

59.12 <u>REMARK</u> If $X = \underline{R}^n$ and if T is compactly supported, then the projection of WF(T) in the second variable is sing T.

59.13 LEMMA $\forall\ T_1, T_2 \in C_c^\infty(X)^*$,

$$WF(T_1 + T_2) \subset WF(T_1) \cup WF(T_2).$$

[Note: If $f \in C^\infty(X)$, then

$$WF(T + f) = WF(T).$$

For

$$WF(T + f) \subset WF(T) \cup WF(f)$$

$$= WF(T).$$

But

$$WF(T) = WF(T + f - f)$$

$$\subset WF(T + f) \cup WF(- f)$$

$$= WF(T + f).]$$

59.14 LEMMA $\forall\ f \in C_c^\infty(X)$,

$$WF(fT) \subset WF(T).$$

59.15 LEMMA (cf. 59.2) Let $A \in \Psi^\infty(X)$ be a pseudodifferential operator -- then A can be extended to a continuous linear map

$$A:C^\infty(X)^* \to C_c^\infty(X)^*$$

and $\forall\ T \in C^\infty(X)^*$,

$$WF(AT) \subset WF(T).$$

[Note: If, in addition, A is properly supported, then A can be extended to a continuous linear map

$$A:C_c^\infty(X)^* \to C_c^\infty(X)^*$$

and $\forall$ $T \in C_c^\infty(X)^*$,

$$WF(AT) \subset WF(T).]$$

59.16 <u>EXAMPLE</u> If $A \in \Psi^m(X)$ is properly supported and elliptic, then $\forall$ $T \in C_c^\infty(X)^*$,

$$WF(AT) = WF(T).$$

Thus choose $Q \in \Psi^{-m}(X)$ per 58.22. In view of 58.13, there is no loss of generality in taking Q properly supported. This said, write

$$T = QAT + (I - QA)T.$$

Then $(I - QA)T \in C^\infty(X)$, hence

$$\begin{aligned} WF(T) &\subset WF(QAT + (I - QA)T) \\ &\subset WF(QAT) + WF((I - QA)T) \quad \text{(cf. 59.13)} \\ &= WF(QAT) \\ &\subset WF(AT) \quad \text{(cf. 59.15)} \\ &\subset WF(T) \quad \text{(cf. 59.15)}, \end{aligned}$$

from which the assertion.

[Note: For a case in point, let $A = \Delta$, the laplacian — then $\forall$ $T \in C_c^\infty(X)^*$,

$$WF(\Delta T) = WF(T).$$

Therefore

$$\Delta T = 0 \Rightarrow WF(T) = \emptyset$$

$$\Rightarrow T \in C^\infty(X).$$

I.e.: T is a harmonic function.]

59.17 <u>RAPPEL</u> If T is a distribution on X, then its <u>conjugate</u> is the distribution $\bar{T}$ on X defined by

$$\bar{T}(f) = \overline{T(\bar{f})} \quad (f \in C_c^\infty(X)).$$

59.18 <u>EXAMPLE</u> $\forall$ θ & $\forall$ a,

$$\overline{I(\theta,a)} = I(-\theta, \bar{a}).$$

59.19 <u>LEMMA</u> Let $T \in C_c^\infty(X)^*$ -- then

$$WF(\bar{T}) = \{(x,\xi) \in X \times (\underline{R}^n - \{0\}) : (x,-\xi) \in WF(T)\}.$$

Given a phase function θ, let

$$SP(\theta) = \{(x, d_x\theta(x,\xi)) : (x,\xi) \in C(\theta)\}.$$

Then $SP(\theta)$ is a closed conic subset of $X \times (\underline{R}^n - \{0\})$.

[Note: In this context, $\xi \in \underline{R}^N - \{0\}$, while

$$d_x\theta(x,\xi) = \left(\frac{\partial\theta}{\partial x^1}, \ldots, \frac{\partial\theta}{\partial x^n}\right)\Big|_{(x,\xi)}$$

is a nonzero element of $\underline{R}^n$.]

59.20 LEMMA (cf. 59.5) We have

$$WF(I(\theta,a)) \subset SP(\theta).$$

59.21 REMARK In general, 59.20 overestimates $WF(I(\theta,a))$, the point being that the growth of a has not been taken into account. E.g. (cf. 59.27):

$$a \in S^{-\infty}(X \times \underline{R}^N) \Rightarrow WF(I(\theta,a)) = \emptyset.$$

59.22 EXAMPLE Let $a \in S^m(X \times \underline{R}^n)$ and let K_a be the distribution kernel corresponding to A_a, thus

$$K_a(x,y) = \frac{1}{(2\pi)^n} \int_{\underline{R}^n} e^{\sqrt{-1}\,(x-y)\xi}\, a(x,\xi)\, d\xi.$$

Define a phase function

$$\theta: X \times X \times (\underline{R}^n - \{0\}) \to \underline{R}$$

by

$$\theta((x,y),\xi) = (x-y)\xi.$$

Then

$$\begin{cases} d_{(x,y)}\theta((x,y),\xi) = (\xi,-\xi) \in \underline{R}^{2n} \\ d_\xi\theta((x,y),\xi) = x-y \in \underline{R}^n. \end{cases}$$

Therefore 59.20 implies that

$$WF(K_a) \subset \{((x,x),(\xi,-\xi)):x,\xi \in \underline{R}^n,\xi \neq 0\}.$$

59.23 EXAMPLE Keeping to the assumptions and notation of 59.6, recall that

$$C(\theta) = \{(\underline{x},\xi):\underline{x} = 0\}$$

$$\cup \{(\underline{x},\xi):|t| = |x| \neq 0 \ \& \ \frac{\xi}{|\xi|} = \frac{x}{t}\}.$$

Since

$$d_x\theta(x,\xi) = (|\xi|,-\xi),$$

$SP(\theta)$ decomposes into three pieces:

$$SP(\theta) = SP_0(\theta) \cup SP_+(\theta) \cup SP_-(\theta),$$

where

$$SP_0(\theta) = \{(\underline{0},(|\xi|,-\xi)):\xi \in \underline{R}^3 - \{0\}\}$$

and

$$\begin{cases} SP_+(\theta) = \{((|x|,x),(\lambda|x|,-\lambda x)):\underline{x} \neq \underline{0},\lambda > 0\} \\ SP_-(\theta) = \{((-|x|,x),(\lambda|x|,\lambda x)):\underline{x} \neq \underline{0},\lambda > 0\}. \end{cases}$$

To confirm the description of $SP_{\pm}(\theta)$, take $\underline{x} \neq \underline{0}$ and $|t| = |x| \neq 0$ — then there are two possibilities:

$$(+) \ t > 0 \text{ or } (-) \ t < 0.$$

Consider the first of these, thus $t = |x| \Rightarrow \underline{x} = (|x|,x)$. The condition on ξ is: $\frac{\xi}{|\xi|} = \frac{x}{t}$, so the admissible ξ are precisely the λx $(\lambda > 0)$. Proof:

$$\frac{\lambda x}{|\lambda x|} = \frac{\lambda x}{\lambda |x|} = \frac{x}{|x|} = \frac{x}{t} .$$

In the second case, $t = -|x|$ and the signs change:

$$\frac{-\lambda x}{|-\lambda x|} = \frac{-\lambda x}{\lambda |x|} = -\frac{x}{|x|} = \frac{x}{-|x|} = \frac{x}{t} .$$

Therefore (cf. 59.20)

$$WF(\Delta_+) \subset SP_0(\theta) \cup SP_+(\theta) \cup SP_-(\theta).$$

[Note: The singular support of Δ_+ is

$$\{\underline{0}\} \cup \{\underline{x} \neq \underline{0}: |t| = |x|\}$$

as can be seen from the classical expansion of Δ_+ in terms of J_1, K_1, N_1 etc.]

A symbol $a \in S^{\infty}(X \times \underline{R}^n)$ is said to be smoothing at $(x_0, \xi_0) \in X \times (\underline{R}^n - \{0\})$ if $\exists$ a conic neighborhood Γ_0 of (x_0, y_0) such that $\forall\ M > 0\ \&\ \forall\ (\alpha, \beta) \in \underline{Z}^n_{\geq 0} \times \underline{Z}^n_{\geq 0}$, $\exists\ C_{M,\alpha,\beta} > 0$:

$$|D_x^{\alpha} D_{\xi}^{\beta} a(x,\xi)| \leq C_{M,\alpha,\beta}(1 + |\xi|)^{-M} \quad ((x,\xi) \in \Gamma_0).$$

The conic support $\Gamma(a)$ of a is the complement in $X \times (\underline{R}^n - \{0\})$ of the set on which a is smoothing.

[Note: $\Gamma(a)$ is a closed conic set.]

59.24 LEMMA Let $a \in S^{\infty}(X \times \underline{R}^n)$ -- then $a \in S^{-\infty}(X \times \underline{R}^n)$ iff its conic support

$\Gamma(a)$ is empty.

Suppose that $A \in \Psi^{\infty}(X)$, say $A = A_a$ -- then the microsupport of A, written

$$\mu\text{spt } A,$$

is the conic support $\Gamma(a)$ of a.

59.25 EXAMPLE Consider a linear differential operator

$$A = \sum_{|\alpha| \leq m} a_{\alpha}(x) D_x^{\alpha} \quad (a_{\alpha} \in C^{\infty}(X))$$

on X (cf. 58.6) -- then

$$\mu\text{spt } A = X \times (\underline{R}^n - \{0\})$$

unless

$$a(x,\xi) = \sum_{|\alpha| \leq m} a_{\alpha}(x) \xi^{\alpha}$$

vanishes identically in some nonempty open subset of X.

59.26 LEMMA (cf. 59.15) Let $A \in \Psi^{\infty}(X)$ be a pseudodifferential operator -- then A can be extended to a continuous linear map

$$A: C^{\infty}(X)^* \to C_c^{\infty}(X)^*$$

and $\forall\ T \in C^{\infty}(X)^*$,

$$WF(AT) \subset WF(T) \cap \mu\text{spt } A.$$

[Note: If, in addition, A is properly supported, then A can be extended to a continuous linear map

$$A:C_c^\infty(X)^* \to C_c^\infty(X)^*$$

and $\forall\ T \in C_c^\infty(X)^*$,

$$WF(AT) \subset WF(T) \cap \mu\text{spt}\ A.]$$

59.27 <u>REMARK</u> The estimate figuring in 59.20 can also be improved. Thus put

$$SP(\theta,a) = \{(x,d_x\theta(x,\xi)):(x,\xi) \in C(\theta) \cap \Gamma(a)\}.$$

Then

$$WF(I(\theta,a)) \subset SP(\theta,a).$$

In particular:

$$a \in S^{-\infty}(X \times \underline{R}^n)$$

$$\Rightarrow \Gamma(a) = \emptyset \quad (\text{cf. } 59.24)$$

$$\Rightarrow SP(\theta,a) = \emptyset$$

$$\Rightarrow WF(I(\theta,a)) = \emptyset$$

$$\Rightarrow I(\theta,a) \in C^\infty(X).$$

59.28 <u>EXAMPLE</u> (cf. 59.22) Let $a \in S^m(X \times \underline{R}^n)$ and let K_a be the distribution kernel corresponding to A_a, thus

$$K_a(x,y) = \frac{1}{(2\pi)^n} \int_{\underline{R}^n} e^{\sqrt{-1}\,(x-y)\xi}\, a(x,\xi)\,d\xi.$$

Then

$$WF(K_a) \subset \{((x,x),(\xi,-\xi)):(x,\xi) \in \mu\text{spt } A_a\}.$$

[Note: It is not difficult to show that the containment is actually an equality.]

Suppose now that

$$X_i \subset \underline{R}^{n_i} \quad (i = 1,2,3)$$

are open and nonempty. Let

$$\begin{cases} K_1 \in C_c^\infty(X_1 \times X_2)^* \\ K_2 \in C_c^\infty(X_2 \times X_3)^*. \end{cases}$$

Then

$$\begin{cases} WF(K_1) \subset X_1 \times X_2 \times (\underline{R}^{n_1+n_2} - \{(0,0)\}) \\ WF(K_2) \subset X_2 \times X_3 \times (\underline{R}^{n_2+n_3} - \{(0,0)\}) \end{cases}$$

and we put

$$\begin{cases} WF_{X_1}(K_1) = \{(x_1,\xi_1) \in X_1 \cap (\underline{R}^{n_1} - \{0\}):((x_1,x_2),(\xi_1,0)) \in WF(K_1)(\exists\ x_2 \in X_2)\} \\ WF_{X_2}(K_2) = \{(x_2,\xi_2) \in X_2 \cap (\underline{R}^{n_2} - \{0\}):((x_2,x_3),(\xi_2,0)) \in WF(K_2)(\exists\ x_3 \in X_3)\}. \end{cases}$$

It will also be convenient to introduce

$$\left[\begin{array}{l} WF'(K_1) = \{((x_1,x_2),(\xi_1,-\xi_2)):((x_1,x_2),(\xi_1,\xi_2)) \in WF(K_1)\} \\ \\ WF'(K_2) = \{((x_2,x_3),(\xi_2,-\xi_3)):((x_2,x_3),(\xi_2,\xi_3)) \in WF(K_2)\} \end{array}\right.$$

and

$$\left[\begin{array}{l} WF'_{X_2}(K_1) = \{(x_2,\xi_2) \in X_2 \cap (\underline{R}^{n_2} - \{0\}):((x_1,x_2),(0,\xi_2)) \in WF'(K_1)(\exists\ x_1 \in X_1)\} \\ \\ WF'_{X_3}(K_2) = \{(x_3,\xi_3) \in X_3 \cap (\underline{R}^{n_3} - \{0\}):((x_2,x_3),(0,\xi_3)) \in WF'(K_2)(\exists\ x_2 \in X_2)\}. \end{array}\right.$$

59.29 <u>LEMMA</u> Assume that K_1,K_2 are properly supported and

$$WF'_{X_2}(K_1) \cap WF_{X_2}(K_2) = \emptyset.$$

Then the composite

$$K_1 \circ K_2$$

exists as a distribution on $X_1 \times X_3$ and

$$WF'(K_1 \circ K_2) \subset WF'(K_1) \circ WF'(K_2)$$

$$\cup\ (WF_{X_1}(K_1) \times X_3 \times \{0\}) \cup (X_1 \times \{0\} \times WF'_{X_3}(K_2)).$$

[Note: Here

$$WF'(K_1 \circ K_2) = \{((x_1,x_3),(\xi_1,-\xi_3)):((x_1,x_3),(\xi_1,\xi_3) \in WF(K_1 \circ K_2)\}$$

and

$$WF'(K_1) \circ WF'(K_2)$$

is set theoretic composition.]

N.B. Formally,

$$\int_{X_2} K_1(x_1,x_2)K_2(x_2,x_3)\,dx_2$$

represents

$$(K_1 \circ K_2)(x_1,x_3).$$

59.30 EXAMPLE Let $M = \underline{R} \times \Sigma$ be ultrastatic, where Σ is a connected open subset of $\underline{R}^3$, and consider the vacuum state ω_μ on $\mathcal{W}(\Gamma,\sigma)$. Pass to $\Lambda_\mu \in C^\infty(M \times M)^*$, thus

$$\Lambda_\mu(f_1,f_2)$$

$$= \frac{1}{2} \langle (A^{1/2}\rho_\Sigma + \sqrt{-1}\ \partial_\Sigma)Ef_1, A^{-1/2}(A^{1/2}\rho_\Sigma + \sqrt{-1}\ \partial_\Sigma)Ef_2 \rangle$$

or, in kernel notation,

$$\Lambda_\mu(\underline{x}_1,\underline{x}_2)$$

$$= -\frac{1}{2}\int_\Sigma \overline{(A^{1/2} + \sqrt{-1}\ \frac{\partial}{\partial t})E(\underline{x}_1,(0,x))}A^{-1/2}(A^{1/2} + \sqrt{-1}\ \frac{\partial}{\partial t})E((0,x),\underline{x}_2)\,d\mu_q(x).$$

Define

$$\begin{cases} K_1 \in C_c^\infty(M \times \Sigma)^* \\ K_2 \in C_c^\infty(\Sigma \times M)^* \end{cases}$$

by

$$\begin{cases} K_1(\underline{x}_1,x) = \frac{1}{\sqrt{2}} \overline{(A^{1/2} + \sqrt{-1}\,\frac{\partial}{\partial t})E(\underline{x}_1,(0,x))} \\ \\ K_2(x,\underline{x}_2) = -\frac{1}{\sqrt{2}} A^{-1/2}(A^{1/2} + \sqrt{-1}\,\frac{\partial}{\partial t})E((0,x),\underline{x}_2). \end{cases}$$

Then it seems plausible that

$$\Lambda_\mu = K_1 \circ K_2$$

but this is not automatic due to the issue of whether K_1, K_2 are properly supported.

[Note: There is another subtlety. To appreciate the point, take $\Sigma = \underline{R}^3$, q = euclidean metric — then $A = (-\Delta_q + m^2)^{1/2}$ and the "symbol" of

$$(-\Delta_q + m^2)^{1/2} - \frac{\partial}{\partial t}$$

is

$$(|\xi|^2 + m^2)^{1/2} - \sqrt{-1}\,\xi_0$$

which is not an element of $S^1(\underline{R}^4 \times \underline{R}^4)$ (differentiation w.r.t. ξ_i does not lower the order w.r.t. ξ_0 below 0).

Let $A \in \Psi^m(X)$. Assume: A has principal symbol σ_A. Put

$$\text{char } A = \{(x,\xi) \in X \times (\underline{R}^n - \{0\}) : \sigma_A(x,\xi) = 0\}.$$

59.31 LEMMA (cf. 59.15) $\forall\, T \in C^\infty(X)^*$,

$$\mathrm{WF}(T) \subset \mathrm{WF}(AT) \cup \mathrm{char}\ A.$$

[Note: If, in addition, A is properly supported, then $\forall\ T \in C_c^\infty(X)^*$,

$$\mathrm{WF}(T) \subset \mathrm{WF}(AT) \cup \mathrm{char}\ A.]$$

Consequently,

$$AT \in C^\infty(X) \Rightarrow \mathrm{WF}(AT) = \emptyset$$

$$\Rightarrow \mathrm{WF}(T) \subset \mathrm{char}\ A.$$

59.32 <u>RAPPEL</u> Suppose that f is a real valued C^∞ function defined on some open subset of $X \times \underline{R}^n$. Put

$$H_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x^j} - \frac{\partial f}{\partial x^j} \frac{\partial}{\partial \xi_j}\right).$$

Then H_f is the hamiltonian vector field attached to f and along an integral curve $\gamma(\tau) = (x(\tau), \xi(\tau))$ of H_f, we have

$$\begin{cases} \dot{x}^j = \dfrac{\partial f}{\partial \xi_j} \\ \dot{\xi}_j = -\dfrac{\partial f}{\partial x^j}. \end{cases}$$

Moreover, f is constant on γ. Proof:

$$\frac{d}{d\tau} f(x(\tau), \xi(\tau))$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \dot{x}^j + \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j} \dot{\xi}_j$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \frac{\partial f}{\partial \xi_j} + \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j} \left(- \frac{\partial f}{\partial x^j}\right)$$

$$= 0.$$

Let $A \in \Psi^m(X)$. Assume: A has principal symbol σ_A, which is real valued. Because

$$\sigma_A \in C^\infty(X \times (\underline{R}^n - \{0\})),$$

it makes sense to form H_{σ_A}, the integral curves of H_{σ_A} being called the <u>bicharacteristics</u> of A.

[Note: A bicharacteristic of A is either entirely contained in char A or never intersects char A.]

59.33 <u>THEOREM</u> (Duistermaat-Hörmander) (Propagation of Singularities) Take A properly supported and let γ be a bicharacteristic of A. Fix $I = [a,b] \subset \text{Dom}\ \gamma$ and given $T \in C_c^\infty(X)^*$, suppose that

$$\gamma(I) \cap WF(AT) = \emptyset.$$

Then either

$$\gamma(I) \subset WF(T) \text{ or } \gamma(I) \cap WF(T) = \emptyset.$$

59.34 REMARK Assume that

$$\gamma(I) \subset WF(T).$$

Then, in view of 59.31,

$$\gamma(I) \subset WF(AT) \cup \text{char } A.$$

But

$$\gamma(I) \cap WF(AT) = \emptyset.$$

Therefore

$$\gamma(I) \subset \text{char } A \Rightarrow \sigma_A|\gamma(I) = 0.$$

Since σ_A is constant on γ (cf. 59.32), it follows that

$$\sigma_A|\gamma = 0.$$

59.35 EXAMPLE Maintain the setup of 59.23 and take $A = \Box_g - m^2$ -- then A is properly supported (cf. 58.12) and

$$- \partial_t^2 + \partial_{x^1}^2 + \partial_{x^2}^2 + \partial_{x^3}^2 - m^2$$

$$= D_t^2 - D_{x^1}^2 - D_{x^2}^2 - D_{x^3}^2 - m^2.$$

Therefore $\sigma_A(\underline{x},\underline{\xi})$ ($\underline{x} = (t,x)$, $\underline{\xi} = (\xi_0,\xi)$) equals

$$\xi_0^2 - |\xi|^2 \ (= - g^{k\ell}(\underline{x}) \ \xi_k\xi_\ell)$$

and the bicharacteristics of A are the integral curves of the system

$$\begin{cases} \dfrac{dt}{d\tau} = \dfrac{\partial \sigma_A}{\partial \xi_0} = 2\xi_0 \\ \\ \dfrac{dx^j}{d\tau} = \dfrac{\partial \sigma_A}{\partial \xi_j} = -\ 2\xi_j \end{cases} , \quad \begin{cases} \dfrac{d\xi_0}{d\tau} = -\ \dfrac{\partial \sigma_A}{\partial t} = 0 \\ \\ \dfrac{d\xi_j}{d\tau} = -\ \dfrac{\partial \sigma_A}{\partial x^j} = 0. \end{cases}$$

By inspection, the solutions are (ξ_0,ξ) a constant and

$$\begin{cases} t(\tau) = 2\xi_0\tau \\ \\ x^j(\tau) = C_j - 2\xi_j\tau, \end{cases}$$

the C_j being constants. We have seen earlier that

$$WF(\Delta_+) \subset SP_0(\theta) \cup SP_+(\theta) \cup SP_-(\theta) \qquad \text{(cf. 59.23)}$$

and we claim that equality prevails. To establish this, note first that

$$(\Box_g - m^2)\Delta_+ = 0,$$

so by 59.31,

$$WF(\Delta_+) \subset \text{char } \Box_g - m^2.$$

If $\underline{x} \neq \underline{0}$ is lightlike, then $\underline{x} \in$ sing spt Δ_+, thus $\exists\ (\xi_0,\xi) \in \Sigma_{\underline{x}}(\Delta_+)$ with $\xi_0^2 = |\xi|^2$. Consider the situation when $\underline{x} = (|x|,x)$, hence

$$\underline{x} \in SP_+(\theta) \Rightarrow (\xi_0,\xi) = (\lambda|x|,-\ \lambda x) \quad (\exists\ \lambda > 0).$$

Since $WF(\Delta_+)$ is conic, $\forall\ r > 0$,

$$(\underline{x}, r(\xi_0,\xi)) \in WF(\Delta_+).$$

It is thus clear that

$$SP_+(\theta) \cup SP_-(\theta) \subset WF(\Delta_+).$$

To deal with $SP_0(\theta)$, let $\xi \in \underline{R}^3 - \{0\}$:

$$(\underline{0}, (|\xi|, -\xi)) \in SP_0(\theta).$$

Form the bicharacteristic

$$((2|\xi|\tau, 2\xi\tau), (|\xi|, -\xi)) \qquad (\tau \in \underline{R}).$$

Fix $\tau > 0$ and put $x = 2\xi\tau$ ($\Rightarrow |x| = 2|\xi|\tau$) -- then

$$-\frac{x}{2\tau} = -\xi.$$

This means that

$$((2|\xi|\tau, 2\xi\tau), (|\xi|, -\xi))$$

has the form

$$((|x|, x), (\lambda|x|, -\lambda x))$$

if $\lambda = \frac{1}{2\tau}$. But (cf. supra)

$$((|x|, x), (\lambda|x|, -\lambda x)) \in WF(\Delta_+).$$

Accordingly (cf. 59.33)

$$(\underline{0}, (|\xi|, -\xi)) \in WF(\Delta_+).$$

To recapitulate:

$$WF(\Delta_+) = SP_0(\theta) \cup SP_+(\theta) \cup SP_-(\theta).$$

[Note: Take an element

$$((\pm |x|,x),(\lambda|x|,\mp \lambda x)) \in SP_{\pm}(\theta).$$

Then $(\lambda|x|,\mp \lambda x)$ is technically a covector. Since the signature of g is - + + +, the associated vector

$$g^{\#}(\lambda|x|,\mp \lambda x)$$

is

$$(-\lambda|x|,\mp \lambda x)$$

which is parallel to $(\pm |x|,x)$:

$$\begin{cases} (-\lambda|x|,-\lambda x) = -\lambda(|x|,x) \\ (-\lambda|x|,+\lambda x) = \lambda(-|x|,x). \end{cases}$$

59.36 REMARK As was pointed out in §56,

$$\sqrt{-1}\,\Delta = \Delta_+ - \bar{\Delta}_+.$$

Therefore

$$WF(\Delta) \subset WF(\Delta_+) \cup WF(\bar{\Delta}_+) \qquad \text{(cf. 59.13)}$$

and $WF(\bar{\Delta}_+)$ is computable in terms of $WF(\Delta_+)$ via 59.19. On the other hand, the singular support of Δ is

$$\{\underline{0}\} \cup \{\underline{x} \neq \underline{0}: |t| = |x|\}.$$

From these observations, it is then straightforward to show that

$$WF(\Delta) = WF(\Delta_+) \cup WF(\bar{\Delta}_+).$$

Working still in Minkowski space, given a nonzero vector $\underline{x}$ and a nonzero covector $\underline{\xi}$, let us agree to write $\underline{x}||\underline{\xi}$ provided $\underline{x}||g^{\#}\underline{\xi}$. We shall also signify that $\underline{x}$ or $\underline{\xi}$ is lightlike by writing $\underline{x}^2 = 0$ or $\underline{\xi}^2 = 0$.

These conventions then allow one to describe $WF(\Delta_+)$ in a compact fashion, viz.

$$WF(\Delta_+) = \{(\underline{0},\underline{\xi}):\underline{\xi}^2 = 0 \ \& \ \xi_0 > 0\}$$

$$\cup \{(\underline{x},\underline{\xi}):\underline{x}^2 = 0,\underline{\xi}^2 = 0,\underline{x}||\underline{\xi},\xi_0 > 0\}.$$

[Note: Analogously,

$$WF(\Delta) = \{(\underline{0},\underline{\xi}):\underline{\xi}^2 = 0\}$$

$$\cup \{(\underline{x},\underline{\xi}):\underline{x}^2 = 0,\underline{\xi}^2 = 0,\underline{x}||\underline{\xi}\}.]$$

59.37 <u>EXAMPLE</u> The methods employed in 59.35 can also be used to compute the wavefront set of Λ_μ, where

$$\Lambda_\mu(\underline{x},\underline{y}) = \Delta_+(\underline{x}-\underline{y}) \ (\underline{x},\underline{y} \in \underline{R}^{1,3}) \quad \text{(cf. §56)}.$$

Thus take $X = \underline{R}^4 \times \underline{R}^4$, $N = 3$, and let

$$\theta((\underline{x},\underline{y}),\xi) = (t-s)|\xi| - (x-y)\cdot\xi \quad \text{(cf. 59.6)}.$$

Then θ is a phase function and

$$\begin{cases} d_\xi\theta((\underline{x},\underline{y}),\xi) = \dfrac{(t-s)\xi}{|\xi|} - (x-y) \\ d_{(\underline{x},\underline{y})}\theta((\underline{x},\underline{y}),\xi) = ((|\xi|,-\xi),(-|\xi|,\xi)). \end{cases}$$

Explicating $C(\theta)$, one finds that there are two contributions to $WF(\Lambda_\mu)$.

Case 1 $(t = s)$ Here $x = y$ and we get

$$\{(\underline{x},\underline{\xi}_1),(\underline{x},\underline{\xi}_2) \in \underline{R}^4 \times (\underline{R}^4 - \{0\}):$$

$$\underline{\xi}_1^2 = 0, (\xi_1)_0 > 0, \underline{\xi}_1 + \underline{\xi}_2 = 0\}.$$

Case 2 $(t \neq s)$ Here $x \neq y$ but $(\underline{x}-\underline{y})^2 = 0$ and we get

$$\{(\underline{x},\underline{\xi}),(\underline{y},\underline{\eta}) \in \underline{R}^4 \times (\underline{R}^4 - \{0\}):$$

$$\underline{x} \neq \underline{y}, (\underline{x}-\underline{y})^2 = 0, \underline{\xi}^2 = 0, (\underline{x}-\underline{y}) \,||\, \underline{\xi}, \xi_0 > 0, \underline{\xi} + \underline{\eta} = \underline{0}\}.$$

Let $\zeta:X \to X'$ $(\subset \underline{R}^n)$ be a diffeomorphism -- then ζ induces isomorphisms

$$\begin{cases} \zeta^*:C_c^\infty(X') \to C_c^\infty(X) \\ \zeta_*:C_c^\infty(X)^* \to C_c^\infty(X')^*. \end{cases}$$

There is also an associated diffeomorphism

$$\zeta_*:X \times (\underline{R}^n - \{0\}) \to X' \times (\underline{R}^n - \{0\}),$$

namely

$$\zeta_*(x,\xi) = (\zeta(x), ({}^tD\zeta(x))^{-1}\xi).$$

59.38 LEMMA $\forall\, T \in C_c^\infty(X)^*$, we have

$$WF(\zeta_*T) = \zeta_*WF(T).$$

Suppose that M is a C^∞ manifold of dimension n -- then the transformation property encoded in 59.38 enables one to extend the notion of wave front set to M, hence $\forall\ T \in C_c^\infty(M)^*$, WF(T) is a closed conic subset of T*M and the earlier theory goes through essentially without change.

[Note: As regards notation, $(x,\xi) \in T^*M$ iff $\xi \in T_x^*M$.]

59.39 RAPPEL Let Σ be a closed submanifold of M -- then the conormal bundle $N^*\Sigma \to \Sigma$ has for its fiber $N_x\Sigma$ over $x \in \Sigma$ the kernel of the arrow $T_x^*M \to T_x^*\Sigma$.

[Note: If $\iota:\Sigma \to M$ is the inclusion, then

$$N_x^*\Sigma = \{(x,\xi):\xi(v) = 0\ \forall\ v \in T_{\iota(x)}\Sigma\}.$$

In particular: N*M is the zero section of T*M.]

59.40 EXAMPLE Let μ be a C^∞ density on Σ and assume that spt $\mu = \Sigma$. Define a distribution $\delta_\mu \in C_c^\infty(M)^*$ by the rule

$$f \to \int_\Sigma (f|\Sigma)\mu \quad (f \in C_c^\infty(M)).$$

Then

$$WF(\delta_\mu) = N^*\Sigma\backslash 0.$$

[Note: Take $M = \underline{R}^n$, $\Sigma = \underline{R}^n$, $\mu = dx$ -- then the wave front set of the distribution

$$f \to \int_{\underline{R}^n} f dx \quad (f \in C_c^\infty(\underline{R}^n))$$

is $N^*M\backslash 0$, i.e., is the empty set per prediction (1 <—> dx). At the other extreme, if $\Sigma = \{0\}$ and μ = unit point mass at 0, then

$$\int_{\{0\}} f d\mu = f(0) = \delta_0(f)$$

and $N^*\{0\} = \{0\} \times \underline{R}^n$, thus

$$WF(\delta_0) = \{0\} \times (\underline{R}^n - \{0\}),$$

thereby providing yet another reality check on the theory.]

Put

$$\mathcal{D}_\Sigma(M) = \{T \in C_c^\infty(M)^* : WF(T) \cap N^*\Sigma = \emptyset\}.$$

Then

$$T \in C^\infty(M) \Rightarrow WF(T) = \emptyset$$

$$\Rightarrow C^\infty(M) \subset \mathcal{D}_\Sigma(M).$$

59.41 <u>LEMMA</u> The pullback ι^*T can be defined for all $T \in \mathcal{D}_\Sigma(M)$ in such a way that it is equal to $\iota^*T (= T \circ \iota)$ when $T \in C^\infty(M)$. And

$$WF(\iota^*T) \subset \iota^*WF(T).$$

[Note: One writes $T|\Sigma$ in place of ι^*T and calls it the <u>restriction</u> of T to Σ.]

59.42 <u>EXAMPLE</u> (Products) Given $T_1, T_2 \in C_c^\infty(M)^*$, their <u>direct product</u> $T_1 \times T_2$ is that element of $C_c^\infty(M \times M)^*$ characterized by the property

$$(T_1 \times T_2)(f_1 \times f_2) = T_1(f_1)T_2(f_2)$$

and we have

$$\mathrm{WF}(T_1 \times T_2) \subset \mathrm{WF}(T_1) \times \mathrm{WF}(T_2)$$

$$\cup\ (\mathrm{WF}(T_1) \times (\mathrm{spt}\ T_2 \times \{0\})) \cup ((\mathrm{spt}\ T_1 \times \{0\}) \times \mathrm{WF}(T_2)).$$

In contrast to the direct product, the pointwise product can only be defined under certain conditions which, in the present setting, can be formulated in terms of wave front sets, the motivation being that $f_1(x)f_2(x)$ $(x \in M)$ is the restriction to the diagonal of $(f_1 \times f_2)(x_1,x_2) = f_1(x_1)f_2(x_2)$ $(x_1,x_2 \in M)$. With this in mind, let us impose the following condition on T_1,T_2:

- $(\mathrm{WF}(T_1) \times \mathrm{WF}(T_2)) \cap N^*\Delta(M \times M) = \emptyset.$

Taking into account the foregoing estimate for $\mathrm{WF}(T_1 \times T_2)$ in conjunction with the fact that $N^*\Delta(M \times M)$ is the subset of $T^*(M \times M)$ consisting of those points of the form $((x,x),(\xi,-\xi))$, we see that this condition implies that

$$T_1 \times T_2 \in \mathcal{D}_{\Delta(M \times M)}(M \times M).$$

Therefore $T_1 \times T_2|\Delta(M \times M)$ makes sense (cf. 59.41). When construed as an element of $C_c^\infty(M)^*$ via the identification

$$\begin{cases} M \to \Delta(M \times M) \\ x \longrightarrow (x,x) \end{cases},$$

one writes instead $T_1 \cdot T_2$ and calls it the <u>pointwise product</u> of T_1,T_2. If $T_1 \in C^\infty(M)$, then, of course, $\mathrm{WF}(T_1) = \emptyset$ and the condition is automatic (in this situation,

$T_1 \cdot T_2(f) = T_2(T_1 f)$ $(f \in C_c^\infty(M)))$.

[Note: To facilitate matters, put

$$WF(T_1) \oplus WF(T_2) = \{(x, \xi_1 + \xi_2) : (x, \xi_i) \in WF(T_i) \, (i = 1,2)\}.$$

Then

$$(WF(T_1) \times WF(T_2)) \cap N^*\Delta(M \times M) = \emptyset$$

iff $\forall \, x \in M$,

$$(x,0) \notin WF(T_1) \oplus WF(T_2)$$

and,

$$WF(T_1 \cdot T_2) \subset WF(T_1) \cup WF(T_2) \cup (WF(T_1) \oplus WF(T_2)).]$$

Let $A \in \Psi^m(X)$. Assume: A has principal symbol σ_A (cf. 58.20). Put

$$\text{char } A = \{(x,\xi) \in T^*M \backslash 0 : \sigma_A(x,\xi) = 0\}.$$

Then 59.31 remains in force. If further, σ_A is real valued, then 59.33 holds, hence the wave front set of a distribution T with AT = 0 is made up of integral curves of H_{σ_A} in char A and their projections onto M constitute the singular support of T.

<u>N.B.</u> Locally, the hamiltonian vector field H_{σ_A} attached to σ_A is given by

$$H_{\sigma_A} = \sum_{j=1}^{n} \left(\left(\frac{\partial}{\partial \xi_j} \sigma_A \right) \frac{\partial}{\partial x^j} - \left(\frac{\partial}{\partial x^j} \sigma_A \right) \frac{\partial}{\partial \xi_j} \right).$$

59.43 <u>EXAMPLE</u> Suppose that (M,g) is globally hyperbolic. Take $A = \Box_g - m^2$ --

then

$$\sigma_A(x,\xi) = - g^{k\ell}(x)\xi_k\xi_\ell.$$

Therefore

$$\sigma_A(x,\xi) = 0 \Rightarrow \xi \text{ lightlike.}$$

Here the equations of Hamilton are

$$\begin{cases} \dot{x}^j = -2g^{jk}\xi_k \\ \dot{\xi}_j = \partial_j g^{k\ell}\xi_k\xi_\ell \end{cases}$$

and if $\tau \to \gamma(\tau) = (x(\tau),\xi(\tau))$ is a bicharacteristic of A in char A, then $\tau \to x(\tau)$ is a lightlike geodesic.

[Note: Due to the assumption that (M,g) is globally hyperbolic, no complete lightlike geodesic remains within a compact subset of M.]

59.44 <u>REMARK</u> Suppose that $(\Box_g - m^2)T = 0$ $(T \in C_c^\infty(M)^*)$ -- then T can be restricted to any Cauchy hypersurface Σ (cf. 59.41) (for WF(T) contains only lightlike directions).

§60. BISOLUTIONS

Suppose that (M,g) is globally hyperbolic. Let $\Lambda \in C_c^\infty(M \times M)^*$ -- then Λ is said to be a bisolution mod C^∞ for $\square_g - m^2$ if $\exists$

$$\begin{cases} K_\ell \in C^\infty(M \times M) \\ K_r \in C^\infty(M \times M) \end{cases}$$

such that $\forall\ f_1, f_2 \in C_c^\infty(M)$,

$$\begin{cases} \Lambda((\square_g - m^2)f_1 \times f_2) = \iint_{M \times M} f_1(x_1)f_2(x_2)K_\ell(x_1,x_2)d\mu_g(x_1)d\mu_g(x_2) \\ \Lambda(f_1 \times (\square_g - m^2)f_2) = \iint_{M \times M} f_1(x_1)f_2(x_2)K_r(x_1,x_2)d\mu_g(x_1)d\mu_g(x_2). \end{cases}$$

[Note: If

$$\begin{cases} K_\ell = 0 \\ K_r = 0, \end{cases}$$

then one simply says that Λ is a bisolution for $\square_g - m^2$, thus, operationally,

$$\begin{cases} ((\square_g - m^2) \otimes 1)\Lambda = 0 \\ (1 \otimes (\square_g - m^2))\Lambda = 0. \end{cases}$$

N.B. Define distributions $\Lambda_\ell, \Lambda_r \in C_c^\infty(M \times M)^*$ by

$$\begin{cases} \Lambda_\ell(f_1 \times f_2) = \Lambda((\square_g - m^2)f_1 \times f_2) \\ \Lambda_r(f_1 \times f_2) = \Lambda(f_1 \times (\square_g - m^2)f_2). \end{cases}$$

Then

$$\left[\begin{array}{l} WF(\Lambda_\ell) \\ \\ WF(\Lambda_r) \end{array}\right. \subset WF(\Lambda) \quad \text{(cf. 59.15)}$$

and Λ is a bisolution mod C^∞ for $\square_g - m^2$ iff

$$\left[\begin{array}{l} WF(\Lambda_\ell) = \emptyset \\ \\ WF(\Lambda_r) = \emptyset. \end{array}\right.$$

60.1 <u>EXAMPLE</u> The quasifree states on $\mathcal{W}(E_m(M,g),\sigma_g)$ are in a one-to-one correspondence with the elements

$$\mu \in \mathbb{IP}(E_m(M,g),\sigma_g)$$

and the 2-point function Λ_μ attached to ω_μ is the bilinear functional

$$C_c^\infty(M)/\ker E \times C_c^\infty(M)/\ker E \to \underline{C}$$

which sends $([f_1],[f_2])$ to

$$\frac{1}{2}\,(\mu([f_1],[f_2]) + \sqrt{-1}\,\sigma_g\,([f_1],[f_2])).$$

Denote its lift to $C_c^\infty(M) \times C_c^\infty(M)$ by the same symbol -- then we shall term μ (or ω_μ) <u>physical</u> provided Λ_μ is separately continuous, hence determines a distribution on $M \times M$ that will also be called Λ_μ (cf. 55.5). We then claim that

Λ_μ is a bisolution for $\Box_g - m^2$. E.g.:

$$\Lambda_\mu((\Box_g - m^2)f_1 \times f_2)$$

$$= \Lambda_\mu((\Box_g - m^2)f_1, f_2)$$

$$= \Lambda_\mu([(\Box_g - m^2)f_1], [f_2]).$$

But $(\Box_g - m^2)f_1 \in \ker E$ (cf. 54.11). Therefore

$$[(\Box_g - m^2)f_1] = 0$$

$$\Rightarrow$$

$$\Lambda_\mu((\Box_g - m^2)f_1 \times f_2) = 0.$$

Put

$$N = \text{char } \Box_g - m^2 \subset T^*M\backslash 0.$$

Given $(x_1,\xi_1),(x_2,\xi_2)$ in N, write

$$(x_1,\xi_1) \sim (x_2,\xi_2)$$

if $x_1 = x_2$ & $\xi_1 = \xi_2$ or if there is a lightlike geodesic $\tau \to x(\tau)$ such that

$$\left[\begin{array}{l} x(\tau_1) = x_1 \\ \\ x(\tau_2) = x_2 \end{array}\right. \qquad (x_1 \neq x_2)$$

and

$$\begin{cases} \xi_{1k} = \dot{x}^j(\tau_1)g_{jk}(x_1) \\ \xi_{2k} = \dot{x}^j(\tau_2)g_{jk}(x_2). \end{cases}$$

Then it is clear that $\sim$ is an equivalence relation and we let $B(x,\xi) = [(x,\xi)]$ be the equivalence class of $(x,\xi) \in N$ per $\sim$.

Put

$$N_0 = N \cup M \times \{0\}.$$

60.2 THEOREM (Duistermaat-Hörmander) If Λ is a bisolution mod C^∞ for $\square_g - m^2$, then

$$WF(\Lambda) \subset N_0 \times N_0$$

and

$$((x_1,\xi_1),(x_2,\xi_2)) \in WF(\Lambda)$$

$$\Rightarrow$$

$$B(x_1,\xi_1) \times B(x_2,\xi_2) \subset WF(\Lambda).$$

[Note: This result is a variant on 59.33 but, strictly speaking, is not a corollary thereof. It is to be stressed that here both $\xi_1 \neq 0$ and $\xi_2 \neq 0$. However, a priori, $WF(\Lambda)$ might also contain elements of the form

$$\begin{cases} ((x_1,\xi_1),(x_2,0)) & (\xi_1 \neq 0) \\ ((x_1,0),(x_2,\xi_2)) & (\xi_2 \neq 0). \end{cases}$$

On the other hand, the points

$$((x_1,0),(x_2,0))$$

are automatically excluded (since $WF(\Lambda) \subset T^*(M \times M)\backslash 0$).

60.3 <u>REMARK</u> Let $\Sigma \subset M$ be a Cauchy hypersurface -- then any inextendible lightlike geodesic intersects Σ. Let

$$((x_1,\xi_1),(x_2,\xi_2)) \in WF(\Lambda)$$

and assume that $x_1 \neq x_2$ -- then

$$((x_1',\xi_1'),(x_2',\xi_2')) \in WF(\Lambda),$$

where $((x_1',\xi_1'),(x_2',\xi_2'))$ is the (unique) element of $B(x_1,\xi_1) \times B(x_2,\xi_2)$ with $x_1',x_2' \in \Sigma$.

Define a diffeomorphism

$$\tau:T^*(M \times M) \rightarrow T^*(M \times M)$$

by

$$\tau((x_1,x_2),(\xi_1,\xi_2)) = ((x_2,x_1),(\xi_2,\xi_1)).$$

60.4 <u>EXAMPLE</u> Let

$$N^{\pm} = \{(x,\xi) \in N: \pm\, \xi > 0\},$$

where $\xi > 0$ means that the vector $\xi^j = g^{jk}\xi_k$ is future pointing and nonzero -- then

$$\tau(N^+ \times N^-) = N^- \times N^+.$$

Given $\Lambda \in C_c^\infty(M \times M)^*$, define

$$\Lambda^{\pm} \in C_c^\infty(M \times M)^*$$

by

$$\begin{cases} \Lambda^+(f_1 \times f_2) = \frac{1}{2}\,(\Lambda(f_1 \times f_2) + \Lambda(f_2 \times f_1)) \\ \Lambda^-(f_1 \times f_2) = \frac{1}{2}\,(\Lambda(f_1 \times f_2) - \Lambda(f_2 \times f_1)). \end{cases}$$

Then Λ^+ is symmetric, i.e.,

$$\Lambda^+(f_1 \times f_2) = \Lambda^+(f_2 \times f_1),$$

and Λ^- is antisymmetric, i.e.,

$$\Lambda^-(f_1 \times f_2) = -\,\Lambda^-(f_2 \times f_1).$$

In addition,

$$\Lambda = \Lambda^+ + \Lambda^-.$$

60.5 <u>LEMMA</u> If Λ is symmetric, then

$$WF(\Lambda) = \tau WF(\Lambda).$$

60.6 <u>EXAMPLE</u> Suppose that Λ is symmetric and $WF(\Lambda) \subset N^+ \times N^-$ -- then $WF(\Lambda) = \emptyset$. In fact,

$$\begin{aligned} WF(\Lambda) &= \tau WF(\Lambda) \quad (\text{cf. } 60.5) \\ &\subset \tau(N^+ \times N^-) \end{aligned}$$

$$= N^- \times N^+ \quad \text{(cf. 60.4)}.$$

But

$$(N^+ \times N^-) \cap (N^- \times N^+) = \emptyset.$$

§61. DISTINGUISHED PARAMETRICES

Assuming still that (M,g) is globally hyperbolic, in the discussion prefacing 58.22 take $A = \square_g - m^2$ -- then a <u>parametrix</u> for $\square_g - m^2$ is a continuous linear map

$$Q: C_c^\infty(M) \to C^\infty(M)$$

such that

$$\begin{cases} (\square_g - m^2) \circ Q - I \in \Psi^{-\infty}(M) \\ Q \circ (\square_g - m^2) - I \in \Psi^{-\infty}(M). \end{cases}$$

[Note: Q has a distribution kernel $K_Q \in C_c^\infty(M \times M)^*$ which, abusively, will be denoted by Q. E.g.: $\forall \ f_1, f_2 \in C_c^\infty(M)$,

$$K_Q((\square_g - m^2)f_1 \times f_2)$$

$$\equiv Q((\square_g - m^2)f_1 \times f_2)$$

$$= Q(f_2)((\square_g - m^2)f_1)$$

$$= \int_M (\square_g - m^2)f_1 \ Qf_2 \ d\mu_g$$

$$= \int_M f_1((\square_g - m^2) \circ Q)f_2 \ d\mu_g$$

$$= \int_M f_1(f_2 + \cdots\cdots)d\mu_g.$$

Let us also remind ourselves that the distribution kernel associated with an element of $\Psi^{-\infty}(M)$ is necessarily a C^∞ function on $M \times M$ (cf. 58.10).]

61.1 <u>EXAMPLE</u> According to 54.8, $\exists$ continuous linear maps

$$E^{\pm}: C_c^\infty(M) \to C^\infty(M)$$

such that

$$\begin{cases} E^{\pm}(\Box_g - m^2)f = f \\ (\Box_g - m^2)E^{\pm}f = f. \end{cases}$$

Therefore $E^{\pm}$ are parametrices.

[Note: Recall that

$$\text{spt } E^{\pm}f \subset J^{\pm}(\text{spt } f)$$

and, by definition,

$$E = E^+ - E^-.]$$

Pass now to

$$N = \text{char } \Box_g - m^2 \subset T^*M\backslash 0 \quad (\text{cf. §60}).$$

Then the <u>characteristic relation</u> C of $\Box_g - m^2$ is the subset of $N \times N$ consisting of those pairs $(x_1,\xi_1),(x_2,\xi_2)$ in N such that $(x_1,\xi_1) \sim (x_2,\xi_2)$.

Let Δ_N be the diagonal of $N \times N$ -- then $\Delta_N \subset C$ and by an <u>orientation</u> of C we understand any decomposition

$$C\backslash\Delta_N = C^1 \coprod C^2$$

into disjoint open subsets that are inverse relations, i.e.,

$$((x_1,\xi_1),(x_2,\xi_2)) \in C^1 \iff ((x_2,\xi_2),(x_1,\xi_1)) \in C^2.$$

61.2 EXAMPLE Put

$$\begin{cases} C^+ = \{((x_1,\xi_1),(x_2,\xi_2)) \in C: x_1 \in J^+(x_2) \text{ if } \xi_1 > 0 \text{ or } x_1 \in J^-(x_2) \text{ if } \xi_1 < 0\} \\ C^- = \{((x_1,\xi_1),(x_2,\xi_2)) \in C: x_1 \in J^+(x_2) \text{ if } \xi_1 < 0 \text{ or } x_1 \in J^-(x_2) \text{ if } \xi_1 > 0\}. \end{cases}$$

Then

$$C \backslash \Delta_N = C^+ \coprod C^-$$

is an orientation of C.

It turns out that C admits precisely 4 orientations. To describe them, let

$$\begin{cases} N_1^1 = N, \ N_1^2 = \emptyset \\ N_2^1 = N^+, \ N_2^2 = N^- \\ N_3^1 = N^-, \ N_3^2 = N^+ \\ N_4^1 = \emptyset, \ N_4^2 = N. \end{cases}$$

Then

$$N = N_i^1 \coprod N_i^2 \quad (i = 1,2,3,4).$$

Set

$$C^{\pm}(x,\xi) = C^{\pm} \cap (B(x,\xi) \times B(x,\xi))$$

and put

$$\begin{cases} C_i^1 = (\bigcup_{N_i^1} C^+(x,\xi)) \cup (\bigcup_{N_i^2} C^-(x,\xi)) \\ C_i^2 = (\bigcup_{N_i^1} C^-(x,\xi)) \cup (\bigcup_{N_i^2} C^+(x,\xi)). \end{cases}$$

Then

$$C\backslash\Delta_N = C_i^1 \coprod C_i^2 \qquad (i = 1,2,3,4)$$

are the 4 orientations of C.

<u>N.B.</u> We have

$$\begin{cases} C_1^1 = C^+ = C_4^2 \\ C_4^1 = C^- = C_1^2 \end{cases} , \quad \begin{cases} C_2^1 = C_3^2 \\ C_3^1 = C_2^2. \end{cases}$$

Therefore the different possible orientations of C are the pairs

$$(C_1^1,C_4^1),\ (C_2^1,C_3^1),\ (C_3^1,C_2^1),\ (C_4^1,C_1^1).$$

To simplify the writing, given a distribution $T \in C_c^\infty(M \times M)^*$, let

$$WF'(T) = \{((x_1,x_2),(\xi_1,-\xi_2)):((x_1,x_2),(\xi_1,\xi_2)) \in WF(T)\}$$

and call Δ^* the diagonal of

$$(T^*M\backslash 0) \times (T^*M\backslash 0) \subset T^*(M \times M)\backslash 0,$$

thus

$$WF'(I) = \Delta^*.$$

61.3 <u>THEOREM</u> (Duistermaat-Hörmander) Associated with each orientation $C \backslash \Delta_N = C_i^1 \coprod C_i^2$ of C, there are parametrices Q_i^1 and Q_i^2 for $\square_g - m^2$ such that

$$\begin{cases} WF'(Q_i^1) = \Delta^* \cup C_i^1 \\ WF'(Q_i^2) = \Delta^* \cup C_i^2. \end{cases}$$

Furthermore

$$WF'(Q_i^1 - Q_i^2) = C.$$

61.4 <u>THEOREM</u> (Duistermaat-Hörmander) If Q is a parametrix for $\square_g - m^2$ and if

$$WF'(Q) \subset \Delta^* \cup C_i^1 \text{ or } WF'(Q) \subset \Delta^* \cup C_i^2,$$

then

$$Q = Q_i^1 \text{ or } Q = Q_i^2$$

modulo a smooth kernel.

<u>N.B.</u> The parametrices Q_i^1, Q_i^2 are said to be <u>distinguished</u>.

61.5 <u>LEMMA</u> We have

$$\begin{cases} Q_2^1 = E^+ \\ Q_3^1 = E^- \end{cases}$$

modulo smooth kernels.

<u>PROOF</u> C_2^1 (C_3^1) is nonempty only if $x_1 \in J^+(x_2)$ $(x_1 \in J^-(x_2))$.

Therefore

$$E = E^+ - E^-$$

$$= Q_2^1 - Q_3^1 + K \quad (K \in C^\infty(M \times M))$$

$\Rightarrow$

$$WF'(E) = WF'(Q_2^1 - Q_3^1)$$

$$= C \quad \text{(cf. 61.3)}.$$

61.6 <u>EXAMPLE</u> (cf. 59.37) Take for M Minkowski space $\underline{R}^{1,3}$ -- then WF(E) is the union

$$\{(\underline{x},\underline{\xi}_1),(\underline{x},\underline{\xi}_2) \in \underline{R}^4 \times (\underline{R}^4 - \{0\}):$$

$$\underline{\xi}_1^2 = 0,\ \underline{\xi}_1 + \underline{\xi}_2 = \underline{0}\}$$

$\cup$

$$\{(\underline{x},\underline{\xi}),(\underline{y},\underline{\eta}) \in \underline{R}^4 \times (\underline{R}^4 - \{0\}):$$

$$\underline{x} \neq \underline{y}, (\underline{x} - \underline{y})^2 = 0,\ \underline{\xi}^2 = 0,\ (\underline{x} - \underline{y}) || \underline{\xi},\ \underline{\xi} + \underline{\eta} = \underline{0}\}.$$

Put

$$\begin{cases} E_F^+ = Q_1^1 \\ E_F^- = Q_4^1, \end{cases}$$

the subscript standing for Feynmann.

61.7 <u>LEMMA</u> We have

$$E^+ + E^- = E_F^+ + E_F^-$$

modulo a smooth kernel.

§62. HADAMARD STATES

Let (M,g) be globally hyperbolic -- then a distribution $\Lambda \in C_c^\infty(M \times M)^*$ is said to satisfy the <u>microlocal spectrum condition</u> if

$$WF(\Lambda) = \{((x_1,\xi_1),(x_2,\xi_2)) \in N_+ \times N_- : (x_1,\xi_1) \sim (x_2,-\xi_2)\}.$$

Suppose that

$$\mu \in IP(E_m(M,g),\sigma_g)$$

is physical (cf. 60.1), hence that the 2-point function Λ_μ is a distribution on $M \times M$. Since Λ_μ is a bisolution for $\Box_g - m^2$, it follows that

$$WF(\Lambda_\mu) \subset N_0 \times N_0 \quad \text{(cf. 60.2)}.$$

We then call ω_μ an Hadamard state provided Λ_μ fulfills the microlocal spectrum condition.

62.1 <u>REMARK</u> The original definition of "Hadamard state" differs from that given above. That the two are equivalent is a fundamental result due to Radzikowski, our position on the matter being a reflection of the old adage "good theorems become definitions".

62.2 <u>EXAMPLE</u> Take $M = \underline{R} \times \Sigma$ ultrastatic. Define $\mu \in IP(\Gamma,\sigma)$ as in 56.5 -- then it can be shown that Λ_μ is Hadamard.

[Note: This was established in 59.37 for the special case of Minkowski space.]

N.B. The derivation of the fact that the vacuum state in an ultrastatic spacetime is Hadamard uses the "old" definition. An attempt to prove it using the "new" definition and microlocal techniques has been made by Junker. To simplify, he took Σ compact. Even so, his argument contained mistakes which were subsequently dealt with in an erratum. Unfortunately, this erratum is incomplete and gaps still remain, thus the issue is problematic.

62.3 REMARK The special nature of the setup in 62.2 is crucial. Indeed, it is clear that if (M,g) is globally hyperbolic and if $\Sigma \subset M$ is a Cauchy hypersurface, then the same construction can be carried out but, in general, the resulting quasi-free state is not Hadamard!

62.4 LEMMA Suppose that $\Omega \subset M$ is causally compatible -- then there is an injective morphism

$$\mathcal{W}(E_m(\Omega,g|\Omega),\sigma_{g|\Omega}) \rightarrow \mathcal{W}(E_m(M,g),\sigma_g)$$

and for any Hadamard state

$$\omega_\mu \in \mathcal{S}(\mathcal{W}(E_m(M,g),\sigma_g)),$$

the restriction

$$\omega_\mu|\mathcal{S}(\mathcal{W}(E_m(\Omega,g|\Omega),\sigma_{g|\Omega}))$$

is also Hadamard.

[Note: This is simply a reflection of the fact that the underlying singularity structure is local.]

62.5 THEOREM (Fulling-Narcowich-Wald) On any globally hyperbolic spacetime (M,g), $\exists$ infinitely many Hadamard states.

62.6 REMARK If $\omega_{\mu_1}, \omega_{\mu_2}$ are Hadamard, then

$$\Lambda_{\mu_1} - \Lambda_{\mu_2} \in C^\infty(M \times M).$$

Thus write

$$\begin{cases} \Lambda_{\mu_1} = \frac{1}{2}(\mu_1 + \sqrt{-1}\,\sigma_g) \\ \Lambda_{\mu_2} = \frac{1}{2}(\mu_2 + \sqrt{-1}\,\sigma_g). \end{cases}$$

Then

$$\Lambda_{\mu_1} - \Lambda_{\mu_2} = \frac{1}{2}(\mu_1 - \mu_2),$$

so $\Lambda_{\mu_1} - \Lambda_{\mu_2}$ is symmetric. But

$$WF(\Lambda_{\mu_1} - \Lambda_{\mu_2}) \subset WF(\Lambda_{\mu_1}) \cup WF(\Lambda_{\mu_2}) \qquad \text{(cf. 59.13)}$$

$$\subset N_+ \times N_-.$$

Therefore

$$WF(\Lambda_{\mu_1} - \Lambda_{\mu_2}) = \emptyset \qquad \text{(cf. 60.6)}$$

=>

$$\Lambda_{\mu_1} - \Lambda_{\mu_2} \in C^\infty(M \times M).$$

62.7 <u>THEOREM</u> (Verch) Let $\omega_{\mu_1}, \omega_{\mu_2}$ be quasifree states on $W(E_m(M,g), \sigma_g)$ and let π_1, π_2 be their associated GNS representations. Assume: $\omega_{\mu_1}, \omega_{\mu_2}$ are Hadamard -- then $\forall\, O \in K(M,g)$, the restrictions

$$\begin{cases} \pi_1 | A_O \\ \pi_2 | A_O \end{cases}$$

are geometrically equivalent.

There is one final point of interest. Suppose that ω_μ is Hadamard and consider the combinations

$$\Lambda_\mu^{\pm} = \sqrt{-1}\ \Lambda_\mu \pm E^{\pm}.$$

Then

$$\begin{aligned}\Lambda_\mu^{\pm} &= \sqrt{-1}\ (\tfrac{1}{2}(\mu + \sqrt{-1}\ E)) \pm E^{\pm} \\ &= \frac{\sqrt{-1}}{2}\mu - \frac{1}{2}(E^+ - E^-) \pm E^{\pm} \\ &= \frac{\sqrt{-1}}{2}\mu \pm \frac{1}{2}(E^+ + E^-).\end{aligned}$$

Thus $\Lambda_\mu^{\pm}$ is symmetric (cf. 54.9), so 60.5 is applicable.

62.8 <u>LEMMA</u> We have

$$\begin{cases} \Lambda_\mu^{+} = E_F^{-} \\ \Lambda_\mu^{-} = E_F^{+} \end{cases}$$

modulo smooth kernels.

PROOF It suffices to deal with Λ_μ^+. In view of 61.5 and 61.3,

$$\mathrm{WF}'(E^+) = \Delta^* \cup C_2^1.$$

Therefore

$$\mathrm{WF}'(\Lambda_\mu^+)\Big|_{x_1 \neq x_2} = C_4^1.$$

To determine $\mathrm{WF}'(\Lambda_\mu^+)$ on the diagonal, observe first that

$$\Lambda_\mu^+((\Box_g - m^2)f_1 \times f_2)$$

$$= \sqrt{-1}\ \Lambda_\mu((\Box_g - m^2)f_1 \times f_2) + E^+((\Box_g - m^2)f_1 \times f_2)$$

$$= E^+((\Box_g - m^2)f_1 \times f_2) \quad \text{(cf. 60.1)}$$

$$= \int_M ((\Box_g - m^2)f_1)(E^+f_2)d\mu_g$$

$$= \int_M (E^-(\Box_g - m^2)f_1)f_2\ d\mu_g \quad \text{(cf. 54.9)}$$

$$= \int_M f_1(x)f_2(x)d\mu_g(x).$$

I.e.:

$$((\Box_g - m^2) \otimes 1)\Lambda_\mu^+ = \delta(x_1 - x_2),$$

the kernel of the identity map I. Consequently (cf. 59.15),

$$\mathrm{WF}'(\Lambda_\mu^+) \supset \mathrm{WF}'(((\Box_g - m^2) \otimes 1)\Lambda_\mu^+)$$

$$= \mathrm{WF}'(I) = \Delta^*.$$

On the other hand,

$$\mathrm{WF}'(\Lambda_\mu^+) \subset \mathrm{WF}'(\Lambda_\mu) \cup \mathrm{WF}'(E^+) \qquad \text{(cf. 59.13)}$$

$$= \mathrm{WF}'(\Lambda_\mu) \cup \Delta^* \cup C_2^1.$$

But

$$\begin{cases} \mathrm{WF}'(\Lambda_\mu)\Big|_{x_1 = x_2} \subset \Delta^* \\ \\ C_2^1 \cap \Delta^* = \emptyset. \end{cases}$$

Hence

$$\mathrm{WF}'(\Lambda_\mu^+)\Big|_{x_1 = x_2} = \Delta^*,$$

so, altogether,

$$\mathrm{WF}'(\Lambda_\mu^+) = \Delta^* \cup C_4^1.$$

However (see above), Λ_μ^+ is a parametrix for $\Box_g - m^2$. Accordingly (cf. 61.4),

$$\Lambda_\mu^+ = Q_4^1 \equiv E_F^-$$

modulo a smooth kernel, which completes the proof.

[Note: It is also true that

$$\Lambda_\mu^+ = E^+ + E^- - E_F^+$$

modulo a smooth kernel (cf. 61.7).]

§63. *HODGE CONVENTIONS*

Let M be a connected C^∞ manifold of dimension n, which we take to be oriented. Fix a semiriemannian structure $g \in \underline{M}$ and consider the star operator

$$*:\Lambda^p(M) \rightarrow \Lambda^{n-p}(M).$$

Then

$$**\alpha = (-1)^\iota(-1)^{p(n-p)}\alpha$$

and

$$\begin{cases} *f = f\mathrm{vol}_g \\ *(f\mathrm{vol}_g) = (-1)^\iota f. \end{cases}$$

[Note: Here $\iota \in \{0,1\}$ is the index of g.]

63.1 <u>EXAMPLE</u> $\forall\ X \in \mathcal{D}^1(M)$,

$$*(\mathrm{div}\ X) = (\mathrm{div}\ X)\mathrm{vol}_g = L_X\mathrm{vol}_g.$$

Let $q \leq p$ -- then there is a bilinear map

$$\begin{cases} \iota:\Lambda^q(M) \times \Lambda^p(M) \rightarrow \Lambda^{p-q}(M) \\ (\beta,\alpha) \longrightarrow \iota_\beta\alpha \end{cases}$$

which is characterized by the following properties:

$$\forall\ \alpha,\beta \in \Lambda^1(M),\ \iota_\beta\alpha = g(\alpha,\beta),$$

$$\iota_\beta(\alpha_1 \wedge \alpha_2) = \iota_\beta\alpha_1 \wedge \alpha_2 + (-1)^{p_1}\alpha_1 \wedge \iota_\beta\alpha_2 \quad (\alpha_i \in \Lambda^{p_i}(M), \beta \in \Lambda^1(M)),$$

$$\iota_{\beta_1 \wedge \beta_2} = \iota_{\beta_2} \circ \iota_{\beta_1}.$$

[Note: One calls ι the <u>interior product</u> on $\Lambda^p(M)$. If $\beta \in \Lambda^0(M) = C^\infty(M)$, then ι_β is simply multiplication by β.]

63.2 <u>REMARK</u> $\forall\ X \in \mathcal{D}^1(M)$,

$$\iota_X = \iota_{g^\flat X}.$$

Take $q = p$ -- then $\iota_\beta\alpha \in C^\infty(M)$ and we set, by definition,

$$g(\alpha,\beta) = \iota_\beta\alpha = \iota_\alpha\beta.$$

If now $\alpha \in \Lambda^p(M)$, $\beta \in \Lambda^q(M)$ $(q < p)$, then $\forall\ \gamma \in \Lambda^{p-q}(M)$,

$$\begin{aligned} g(\iota_\beta\alpha,\gamma) &= \iota_\gamma\iota_\beta\alpha \\ &= \iota_{\beta \wedge \gamma}\alpha \\ &= g(\alpha,\beta \wedge \gamma). \end{aligned}$$

In other words, the operations

$$\begin{cases} \iota_\beta : \Lambda^p(M) \to \Lambda^{p-q}(M) \\ \beta \wedge \text{—} : \Lambda^{p-q}(M) \to \Lambda^p(M) \end{cases}$$

are mutually adjoint.

63.3 <u>LEMMA</u> $\forall\ \alpha \in \Lambda^p(M)$,

$$*\alpha = \iota_\alpha \text{vol}_g.$$

63.4 <u>EXAMPLE</u> Let $\alpha = 1$ -- then

$$*1 = \text{vol}_g$$

=>

$$*\text{vol}_g = **1 = (-1)^\iota$$

=>

$$\begin{aligned} g(\text{vol}_g, \text{vol}_g) &= \iota_{\text{vol}_g} \text{vol}_g \\ &= *\text{vol}_g \\ &= (-1)^\iota. \end{aligned}$$

63.5 <u>EXAMPLE</u> Let $\alpha \in \Lambda^p(M)$, $\beta \in \Lambda^{n-p}(M)$ -- then

$$\begin{aligned} g(\alpha \wedge \beta, \text{vol}_g) &= \iota_{\alpha \wedge \beta} \text{vol}_g \\ &= \iota_\beta \iota_\alpha \text{vol}_g \\ &= \iota_\beta *\alpha \\ &= g(*\alpha, \beta). \end{aligned}$$

63.6 RULES In what follows, $\alpha \in \Lambda^p(M)$ and $\beta \in \Lambda^q(M)$ (subject to the obvious restrictions).

- $\iota_\beta * \alpha = *(\alpha \wedge \beta)$.

- $* \iota_\beta \alpha = (-1)^{q(n-q)} * \alpha \wedge \beta$.

- $\alpha \wedge *\beta = g(\alpha,\beta)\mathrm{vol}_g = \beta \wedge *\alpha$

- $g(*\alpha, *\beta) = (-1)^\iota g(\alpha,\beta)$.

The interior derivative

$$\delta : \Lambda^p(M) \to \Lambda^{p-1}(M)$$

is

$$\delta = (-1)^\iota (-1)^{np + n+1} * \circ d \circ *.$$

[Note: Therefore $\delta f = 0$ $(f \in C^\infty(M))$.]

63.7 LEMMA We have

$$\delta \circ \delta = 0.$$

PROOF For $* \circ * = \pm 1$ and $d \circ d = 0$.

63.8 EXAMPLE Take $M = \underline{R}^{1,3}$ -- then

$$(-1)^\iota (-1)^{np + n+1} = (-1)^1 (-1)^{4p + 4+1} = 1,$$

so in this case,

$$\delta\alpha = *d*\alpha.$$

63.9 <u>REMARK</u> The exterior derivative d does not depend on g. By contrast, the interior derivative δ depends on g (and the underlying orientation).

Write $\Lambda_c^p(M)$ for the space of compactly supported p-forms on M and put

$$<\alpha,\beta>_g = \int_M g(\alpha,\beta)\mathrm{vol}_g \quad (\alpha,\beta \in \Lambda_c^p(M)).$$

63.10 <u>LEMMA</u> Let $\alpha \in \Lambda_c^p(M)$, $\beta \in \Lambda_c^{p+1}(M)$ -- then

$$<d\alpha,\beta>_g = <\alpha,\delta\beta>_g.$$

<u>PROOF</u> We have

$$g(\alpha,\delta\beta)\mathrm{vol}_g = \alpha \wedge *\delta\beta$$

$$= - (-1)^{\iota}(-1)^{n(p+2)}\alpha \wedge **d*\beta$$

$$= - (-1)^{\iota}(-1)^{np}\alpha \wedge (-1)^{\iota}(-1)^{(n-p)p}d*\beta$$

$$= - (-1)^{p^2}\alpha \wedge d*\beta$$

$$= - (-1)^{p}\alpha \wedge d*\beta.$$

Therefore

$$g(d\alpha,\beta)\mathrm{vol}_g - g(\alpha,\delta\beta)\mathrm{vol}_g$$

$$= d\alpha \wedge *\beta + (-1)^p \alpha \wedge d*\beta$$

$$= d(\alpha \wedge *\beta).$$

And, by Stokes' theorem,

$$\int_M d(\alpha \wedge *\beta) = 0,$$

from which the result.

63.11 RAPPEL Let $f \in C_c^\infty(M)$ -- then $\forall\ X \in \mathcal{D}^1(M)$,

$$\int_M (\text{div } fX)\text{vol}_g = \int_M L_{fX}\text{vol}_g$$

$$= \int_M (\iota_{fX} \circ d + d \circ \iota_{fX})\text{vol}_g$$

$$= \int_M d(\iota_{fX}\text{vol}_g) = 0.$$

Consequently,

$$0 = \int_M (Xf + f(\text{div } X))\text{vol}_g$$

or still,

$$\int_M Xf\ \text{vol}_g = - \int_M f(\text{div } X)\text{vol}_g.$$

63.12 LEMMA Let $X \in \mathcal{D}^1(M)$ -- then

$$\text{div } X = - \delta g^\flat X.$$

PROOF In fact, $\forall\ f \in C_c^\infty(M)$,

$$\langle f, \delta g^\flat X\rangle_g = \langle df, g^\flat X\rangle_g \quad \text{(cf. 63.10)}$$

$$= \int_M g(df, g^\flat X)\,\text{vol}_g$$

$$= \int_M g(g^\flat g^\# df, g^\flat X)\,\text{vol}_g$$

$$= \int_M g(g^\flat \text{grad } f, g^\flat X)\,\text{vol}_g$$

$$= \int_M g(\text{grad } f, X)\,\text{vol}_g$$

$$= \int_M Xf \,\text{vol}_g$$

$$= - \int_M f(\text{div } X)\,\text{vol}_g \quad (\text{cf. } 63.11)$$

$$= - \langle f, \text{div } X\rangle_g$$

=>

$$\text{div } X = - \delta g^\flat X.$$

Recall now that

$$\Delta_g = \text{div} \circ \text{grad}$$

$$= \text{div} \circ g^\# \circ d.$$

But

$$\text{div} = - \delta \circ g^\flat \quad (\text{cf. } 63.12).$$

Therefore

$$\Delta_g = - \delta \circ g^\flat \circ g^\# \circ d$$

$$= - \delta \circ d.$$

With this in mind, the laplacian

$$\Delta_g : \Lambda^p(M) \to \Lambda^p(M)$$

is then defined by

$$\Delta_g = -(d \circ \delta + \delta \circ d).$$

63.13 LEMMA We have

$$(1)\ d \circ \Delta_g = \Delta_g \circ d;\ (2)\ \delta \circ \Delta_g = \Delta_g \circ \delta;\ (3)\ * \circ \Delta_g = \Delta_g \circ *.$$

63.14 LEMMA Let $f \in C^\infty(M)$, $\alpha \in \Lambda^p M$ — then

$$\Delta_g(f\alpha) = (\Delta_g f)\alpha + f(\Delta_g \alpha) + 2\nabla_{\text{grad } f}\alpha.$$

[Note: On functions,

$$\Delta_g(f_1 f_2) = (\Delta_g f_1) f_2 + f_1(\Delta_g f_2) + 2g(\text{grad } f_1, \text{grad } f_2).]$$

Assume henceforth that (M,g) is riemannian with g complete and write $\Lambda_g^{2,p}(M)$ for the space of square integrable p-forms on M.

63.15 LEMMA $\Lambda_c^p(M)$ is dense in $\Lambda_g^{2,p}(M)$.

N.B. On $\Lambda_c^p(M)$, Δ_g is ≤ 0 and $\forall\ \alpha,\beta \in \Lambda_c^p(M)$,

$$<\Delta_g \alpha, \beta>_g = <\alpha, \Delta_g \beta>_g .$$

63.16 <u>LEMMA</u> The restriction $\Delta_g | \Lambda_c^p(M)$ is essentially selfadjoint.

[Note: Write

$$\bar{\Delta}_g = \overline{\Delta_g | \Lambda_c^p(M)} .]$$

<u>Domain Issues</u> Let

$$\text{Dom}(d) = \{\alpha \in \Lambda^p(M) \cap \Lambda_g^{2,p}(M) : d\alpha \in \Lambda_g^{2,p+1}(M)\}$$

and put

$$d_c = d | \Lambda_c^p(M) .$$

Then

$$\left[\begin{array}{l} d \\ \\ d_c \end{array}\right. \quad \text{admit closure:} \quad \left[\begin{array}{l} \bar{d} \\ \\ \bar{d}_c \end{array}\right. .$$

Analogous considerations apply to the interior derivative, thus

$$\left[\begin{array}{l} \delta \\ \\ \delta_c \end{array}\right. \quad \text{admit closure:} \quad \left[\begin{array}{l} \bar{\delta} \\ \\ \bar{\delta}_c \end{array}\right. .$$

So (cf. 1.6),

$$\left[\begin{array}{l} \bar{d} = d^{**} \\ \\ \bar{d}_c = d_c^{**} \end{array}\right. \quad \& \quad \left[\begin{array}{l} \bar{d}^* = d^* \\ \\ \bar{d}_c^* = d_c^* \end{array}\right.$$

and

$$\begin{cases} \bar{\delta} = \delta^{**} \\ \bar{\delta}_c = \delta_c^{**} \end{cases} \quad \& \quad \begin{cases} \bar{\delta}^* = \delta^* \\ \bar{\delta}_c^* = \delta_c^* \end{cases} .$$

63.17 <u>LEMMA</u> We have

$$\begin{cases} \bar{d} = \bar{d}_c = \delta_c^* \\ \bar{\delta} = \bar{\delta}_c = d_c^*. \end{cases}$$

Therefore

$$\begin{cases} \bar{d} = \bar{d}_c \Rightarrow \bar{d}^* = \bar{d}_c^* \Rightarrow d^* = d_c^* = \bar{\delta} \\ \bar{\delta} = \bar{\delta}_c \Rightarrow \bar{\delta}^* = \bar{\delta}_c^* \Rightarrow \delta^* = \delta_c^* = \bar{d}. \end{cases}$$

<u>N.B.</u> From the above

$$\begin{cases} \bar{d} \circ \bar{\delta} = \bar{d} \circ d^* = \bar{d} \circ \bar{d}^* \\ \bar{\delta} \circ \bar{d} = \bar{\delta} \circ \delta^* = \bar{\delta} \circ \bar{\delta}^*. \end{cases}$$

Accordingly,

$$\begin{cases} \bar{d} \circ \bar{\delta} \\ \bar{\delta} \circ \bar{d} \end{cases}$$

are selfadjoint (cf. 1.30).

63.18 THEOREM (Gaffney) Let $\alpha \in \text{Dom}(\bar{d})$ and $\beta \in \text{Dom}(\bar{\delta})$ -- then

$$<\bar{d}\alpha,\beta>_g = <\alpha,\bar{\delta}\beta>_g.$$

The domain of

$$\bar{d} \circ \bar{\delta} + \bar{\delta} \circ \bar{d}$$

is

$$\text{Dom}(\bar{d} \circ \bar{\delta}) \cap \text{Dom}(\bar{\delta} \circ \bar{d})$$

and

$$\begin{cases} \text{Dom}(\bar{d} \circ \bar{\delta}) = \{\alpha \in \text{Dom}(\bar{\delta}):\bar{\delta}\alpha \in \text{Dom}(\bar{d})\} \\ \text{Dom}(\bar{\delta} \circ \bar{d}) = \{\alpha \in \text{Dom}(\bar{d}):\bar{d}\alpha \in \text{Dom}(\bar{\delta})\}. \end{cases}$$

63.19 LEMMA The sum

$$\bar{d} \circ \bar{\delta} + \bar{\delta} \circ \bar{d}$$

is selfadjoint.

[Note: While individually, $\bar{d} \circ \bar{\delta}$ and $\bar{\delta} \circ \bar{d}$ are selfadjoint, this does not automatically guarantee that their sum is selfadjoint. However, since $\bar{d}$ and $\bar{\delta}$ are closed and densely defined, the operators

$$\begin{cases} (I + \bar{d} \circ \bar{\delta})^{-1} \\ (I + \bar{\delta} \circ \bar{d})^{-1} \end{cases}$$

are bounded and selfadjoint. In addition, it can be shown that here

$$(I + \bar{d} \circ \bar{\delta} + \bar{\delta} \circ \bar{d})^{-1}$$

$$= (I + \bar{d} \circ \bar{\delta})^{-1} + (I + \bar{\delta} \circ \bar{d})^{-1} - I,$$

hence

$$(I + \bar{d} \circ \bar{\delta} + \bar{\delta} \circ \bar{d})^{-1}$$

is selfadjoint. But this implies that

$$I + \bar{d} \circ \bar{\delta} + \bar{\delta} \circ \bar{d}$$

is selfadjoint, thus finally that

$$\bar{d} \circ \bar{\delta} + \bar{\delta} \circ \bar{d}$$

is selfadjoint.]

63.20 LEMMA We have

$$\bar{\Delta}_g = - (\bar{d} \circ \bar{\delta} + \bar{\delta} \circ \bar{d}).$$

PROOF By definition,

$$\Delta_g | \Lambda_c^p(M) = - (d \circ \delta + \delta \circ d) | \Lambda_c^p(M).$$

And, thanks to 63.19, $- (\bar{d} \circ \bar{\delta} + \bar{\delta} \circ \bar{d})$ is a selfadjoint extension of $\Delta_g | \Lambda_c^p(M)$. But $\Delta_g | \Lambda_c^p(M)$ is essentially selfadjoint (cf. 63.16). Therefore

$$\bar{\Delta}_g = - (\bar{d} \circ \bar{\delta} + \bar{\delta} \circ \bar{d}) \quad \text{(cf. 1.14)}.$$

Let $\alpha \in \text{Dom}(\bar{\Delta}_g)$ -- then (cf. 63.18)

$$- <\alpha, \bar{\Delta}_g \alpha>_g = <\bar{d}\alpha, \bar{d}\alpha>_g + <\bar{\delta}\alpha, \bar{\delta}\alpha>_g.$$

Therefore

$$\bar{\Delta}_g \alpha = 0 \iff \begin{cases} \bar{d}\alpha = 0 \\ \\ \bar{\delta}\alpha = 0. \end{cases}$$

Let $\alpha \in \Lambda_g^{2,p}(M)$ — then α is said to be <u>harmonic</u> if $\alpha \in \text{Dom}(\bar{\Delta}_g)$ and $\bar{\Delta}_g \alpha = 0$.

Denote the space of harmonic p-forms by $\underline{H}^p$ -- then the elements of $\underline{H}^p$ are necessarily C^∞.

63.21 <u>EXAMPLE</u> One has

$$\underline{H}^0 = \underline{H}^n = \begin{cases} 0 & \text{iff vol } M = \infty \\ \\ \underline{R} & \text{iff vol } M < \infty. \end{cases}$$

63.22 <u>EXAMPLE</u> Take $M = \underline{R}^n$ with g the usual metric -- then $\underline{H}^p = 0$ $(0 \leq p \leq n)$.

[Assume that $n > 1$, represent $\underline{R}^n$ as the product $\underline{R} \times \underline{R}^{n-1}$, and let ϕ_s be the flow attached to $\partial/\partial t$ — then $\forall$ s, $\phi_s : \underline{R}^n \to \underline{R}^n$ is an isometry, hence

$$\alpha \in \underline{H}^p \Rightarrow \phi_s^* \alpha \in \underline{H}^p.$$

Write

$$\frac{d}{ds} \phi_s^* \alpha = \phi_s^* L_{\partial/\partial t} \alpha$$

$$= \phi_s^* (d \circ \iota_{\partial/\partial t} + \iota_{\partial/\partial t} \circ d)\alpha$$

$$= \phi_s^* d\iota_{\partial/\partial t}\alpha$$

$$= d\phi_s^* \iota_{\partial/\partial t}\alpha$$

$$= d\iota_{\partial/\partial t}\phi_s^*\alpha$$

=>

$$\phi_t^*\alpha - \alpha = \phi_t^*\alpha - \phi_0^*\alpha$$

$$= \int_0^t \frac{d}{ds} \phi_s^*\alpha ds$$

$$= d\int_0^t \iota_{\partial/\partial t}\phi_s^*\alpha ds.$$

But

$$||\iota_{\partial/\partial t}\phi_s^*\alpha|| \leq ||\phi_s^*\alpha||.$$

Therefore $\phi_t^*\alpha$ is L^2-cohomologous to α, so $\phi_t^*\alpha = \alpha \ \forall\ t$, which is possible only if $\alpha = 0$.]

63.23 THEOREM (Kodaira) There is an orthogonal decomposition

$$\Lambda_g^{2,p}(M) = \overline{\delta\Lambda_c^{p+1}(M)} \oplus \overline{d\Lambda_c^{p-1}(M)} \oplus \underline{H}^p.$$

63.24 REMARK Let $\alpha \in \Lambda^p(M) \cap \Lambda_g^{2,p}(M)$ and write, in obvious notation,

$$\alpha = \alpha_\delta + \alpha_d + \alpha_{har}.$$

Then

$$\alpha_\delta, \alpha_d, \alpha_{har} \in \Lambda^p(M) \cap \Lambda_g^{2,p}(M).$$

63.25 LEMMA We have

$$\overline{d\Lambda_c^{p-1}(M)} = \overline{\text{Im } \bar{d}_{p-1}}.$$

[Note: In general, the range of $\bar{d}$ need not be closed.]

The L^2-cohomology groups of (M,g) are the

$$H_{(2)}^p(M) = \frac{\text{Ker } d_p}{\text{Im } d_{p-1}}.$$

63.26 LEMMA We have

$$H_{(2)}^p(M) \approx \frac{\text{Ker } \bar{d}_p}{\overline{\text{Im } \bar{d}_{p-1}}}.$$

63.27 LEMMA The canonical arrow

$$\underline{H}^p \to H_{(2)}^p(M)$$

is one-to-one.

PROOF Let $\alpha,\beta \in \underline{H}^p$ and suppose that $\alpha = \beta + \bar{d}\gamma$ -- then (cf. 63.18)

$$\langle\alpha-\beta,\alpha-\beta\rangle_g = \langle\bar{d}\gamma,\alpha-\beta\rangle_g$$

$$= \langle\gamma,\bar{\delta}(\alpha-\beta)\rangle_g$$

$$= \langle\gamma,0\rangle_g$$

$$\Rightarrow$$

$$\alpha = \beta.$$

Since

$$\text{Ker } \bar{d}_p = \underline{H}^p \oplus \overline{\text{Im } \bar{d}_{p-1}},$$

it follows that

$$H^p_{(2)}(M) = \underline{H}^p \oplus \frac{\overline{\text{Im } \bar{d}_{p-1}}}{\text{Im } \bar{d}_{p-1}} .$$

63.28 <u>EXAMPLE</u> Take $M = \underline{R}$ with g the usual metric -- then $\underline{H}^1$ is trivial but $H^1_{(2)}(\underline{R})$ is infinite dimensional.

The pair (M,g) satisfies the <u>closed range hypothesis</u> if $\forall$ p,

$$\overline{\text{Im } \bar{d}_{p-1}} = \text{Im } \bar{d}_{p-1}$$

or, equivalently, if $\forall$ p,

$$\overline{d\Lambda^{p-1}_c(M)} = \text{Im } \bar{d}_{p-1}.$$

[Note: If

$$\overline{\text{Im } \bar{d}_{p-1}} \neq \text{Im } \bar{d}_{p-1},$$

then

$$\frac{\overline{\text{Im } \bar{d}_{p-1}}}{\text{Im } \bar{d}_{p-1}}$$

is infinite dimensional.]

Thus, in the presence of the closed range hypothesis, L^2-cohomology is represented by harmonic forms.

63.29 LEMMA Suppose that the closed range hypothesis is in force -- then $\text{Im } \bar{\Delta}_g$ is closed and

$$\Lambda_g^{2,p}(M) = \text{Im } \bar{\Delta}_g \oplus \underline{H}^p.$$

63.30 REMARK If $\forall$ p, 0 is not in the essential spectrum of $\bar{\Delta}_g$, then the pair (M,g) satisfies the closed range hypothesis.

63.31 EXAMPLE Take $M = \underline{R}^n$ with g the usual metric -- then the closed range hypothesis is not satisfied. To see this, consider the situation when p = 0 and view the laplacian $\bar{\Delta}_g \equiv \Delta$ as a map

$$\Delta: W^{2,2}(\underline{R}^n) \to L^2(\underline{R}^n).$$

If the range of Δ were closed, then $\exists\ C > 0: \forall\ f \in W^{2,2}(\underline{R}^n)$,

$$||f||_{W^{2,2}} \leq C||\Delta f||_{L^2}.$$

But such a relation cannot be true. Thus let

$$(S_R f)(x) = f(Rx).$$

Then

$$\Delta S_R f = R^2 S_R \Delta f.$$

Therefore

$$\begin{aligned} ||f||_{L^2} &= R^{-n/2}||S_{1/R}f||_{L^2} \\ &\leq R^{-n/2}||S_{1/R}f||_{W^{2,2}} \\ &\leq CR^{-n/2}||\Delta S_{1/R}f||_{L^2} \\ &= CR^{-2}||\Delta f||_{L^2}, \end{aligned}$$

an impossibility.

Assume now that M is compact -- then the closed range hypothesis is automatic and $\forall$ p,

$$\underline{H}^p \approx H^p_{(2)}(M)$$

is finite dimensional.

63.32 <u>LEMMA</u> There is an orthogonal decomposition

$$\Lambda^p(M) = d(\Lambda^{p-1}(M)) \oplus \delta(\Lambda^{p+1}(M)) \oplus \underline{H}^p.$$

63.33 <u>EXAMPLE</u> Take M 3-dimensional and let $X \in \mathcal{D}^1(M)$ -- then $\exists\, f \in C^\infty(M)$ and $Y \in \mathcal{D}^1(M)$ such that

$$g^\flat X = g^\flat \text{grad } f + g^\flat \text{curl } Y + \gamma,$$

where $\gamma \in \underline{H}^1$. Here curl $Y \in \mathcal{D}^1(M)$ is determined by the equation

$$dg^\flat Y = *g^\flat \text{curl } Y.$$

[To see this, write

$$g^\flat X = df + \delta\alpha + \gamma \quad \text{(cf. 63.32)}$$

$$= g^\flat \text{grad } f + \delta\alpha + \gamma.$$

Define $Y \in \mathcal{D}^1(M)$ by the relation

$$*\alpha = g^\flat Y.$$

Then

$$\delta\alpha = *d*\alpha$$

$$= *dg^\flat Y$$

$$= **g^\flat \text{curl } Y$$

$$= g^\flat Y.]$$

Recalling 63.29, denote by $\underline{P}^p$ the orthogonal projection

$$\Lambda_g^{2,p}(M) \to \underline{H}^p$$

and given $\alpha \in \Lambda_g^{2,p}(M)$, let $G^p(\alpha)$ be the unique solution to

$$\bar{\Delta}_g(?) = \alpha - \underline{P}^p\alpha$$

in $(\underline{H}^p)^\perp$ -- then

$$G^p:\Lambda_g^{2,p}(M) \to (\underline{H}^p)^\perp$$

is a bounded linear operator.

<u>N.B.</u> Im $\bar{\Delta}_g$ is a Hilbert space and on Im $\bar{\Delta}_g$, $G^p = (\bar{\Delta}_g)^{-1}$. Furthermore, when viewed as a linear operator

$$\text{Im } \bar{\Delta}_g \to \text{Im } \bar{\Delta}_g,$$

G^p is compact and selfadjoint.

§64. ABSTRACT MAXWELL THEORY

Let (M,g) be a globally hyperbolic spacetime -- then its Cauchy hypersurfaces are either all compact or all noncompact (cf. 54.2) and it will be assumed in this section that we are in the compact situation.

[Note: In the literature, the respective terms are (M,g)

- <u>spatially compact</u>
- <u>spatially noncompact</u>.]

Suppose that $\Sigma \subset M$ is a Cauchy hypersurface and let $i:\Sigma \to M$ be the inclusion -- then $q = i^*(g)$ is a riemannian structure on Σ. To minimize the possibility of confusion, we shall append subscripts to distinguish $*$ and δ on M and Σ:

- $*_g, *_q$
- δ_g, δ_q.

Let $A \in \Lambda^1(M)$ -- then A is said to satisfy <u>Maxwell's equation</u> if

$$\delta_g dA = 0.$$

In terms of

$$\Box_g = -(d \circ \delta_g + \delta_g \circ d),$$

it is clear that A satisfies Maxwell's equation iff

$$\Box_g A + d\delta_g A = 0.$$

Given $A \in \Lambda^1(M)$, put

$$\begin{cases} A = i^*(A) \\ \Pi = *_q \circ i^* \circ *_g \circ dA. \end{cases}$$

64.1 LEMMA If $\delta_g dA = 0$, then $\delta_q \Pi = 0$.

PROOF In fact,

$$\begin{aligned} \delta_q \Pi &= - *_q \circ d_\Sigma \circ *_q \Pi \\ &= - *_q \circ d_\Sigma \circ *_q (*_q \circ i^* \circ *_g \circ d_M A) \\ &= - *_q \circ d_\Sigma \circ *_q^2 (i^* \circ *_g \circ d_M A) \\ &= - *_q \circ d_\Sigma \circ i^* \circ *_g \circ d_M A \\ &= - *_q \circ i^* \circ d_M \circ *_g \circ d_M A \\ &= - *_q \circ i^* \circ *_g \circ \delta_g \circ d_M A \\ &= 0. \end{aligned}$$

64.2 THEOREM (Dimock) Given $A, \Pi \in \Lambda^1(\Sigma)$ with $\delta_q \Pi = 0$, $\exists\ A \in \Lambda^1(M)$ with $\delta_g dA = 0$ such that

$$A = i^*(A) \ \& \ \Pi = *_q \circ i^* \circ *_g \circ dA.$$

Let $A, A' \in \Lambda^1(M)$ -- then A, A' are said to be gauge equivalent, written

$\mathcal{A} \sim \mathcal{A}'$, if $\exists\ f \in C^\infty(M)$ such that $\mathcal{A} = \mathcal{A}' + df$.

[Note: Obviously, if $\mathcal{A} \sim \mathcal{A}'$, then $\delta_g d\mathcal{A} = 0 \Longleftrightarrow \delta_g d\mathcal{A}' = 0$.]

64.3 <u>LEMMA</u> Fix $A, \Pi \in \Lambda^1(\Sigma)$ with $\delta_q \Pi = 0$ and let $\mathcal{A}, \mathcal{A}'$ be per 64.2 — then $\mathcal{A}, \mathcal{A}'$ are gauge equivalent.

The notion of gauge equivalence applies equally well to $\Lambda^1(\Sigma)$.

64.4 <u>LEMMA</u> Let $A, \Pi, A', \Pi' \in \Lambda^1(\Sigma)$ with $\delta_q \Pi = \delta_q \Pi' = 0$; let $\mathcal{A}, \mathcal{A}' \in \Lambda^1(M)$ with $\delta_g d\mathcal{A} = \delta_g d\mathcal{A}' = 0$. Assume:

$$\begin{cases} A = i^*(\mathcal{A}),\ \Pi = *_q \circ i^* \circ *_g \circ d\mathcal{A} \\ \\ A' = i^*(\mathcal{A}'),\ \Pi' = *_q \circ i^* \circ *_g \circ d\mathcal{A}'. \end{cases}$$

Then $A \sim A'$, $\Pi = \Pi'$ iff $\mathcal{A} \sim \mathcal{A}'$.

<u>PROOF</u> If $\mathcal{A} \sim \mathcal{A}'$, then it is clear that $A \sim A'$, $\Pi = \Pi'$. Turning to the converse, suppose that $A = A' + d\phi, \Pi = \Pi'$. Using standard extension theory, choose $f \in C^\infty(M): f|\Sigma = \phi$ and let $\mathcal{A}'' = \mathcal{A}' + df$ — then

$$\begin{aligned} i^*(\mathcal{A}'') &= i^*(\mathcal{A}') + i^*(df) \\ &= A' + d\phi \\ &= A \end{aligned}$$

and

$$dA'' = dA'$$

$$\Rightarrow$$

$$*_q \circ i^* \circ *_g \circ dA'' = \Pi' = \Pi.$$

Therefore $A'' \sim A$ (cf. 64.3). But $A'' \sim A'$, hence $A \sim A'$.

The preceding considerations can be summarized as follows: Given a gauge equivalence class $[A]$ in $\Lambda^1(\Sigma)$ and $\Pi \in \Lambda^1(\Sigma)$ with $\delta_q \Pi = 0$, there is a unique gauge equivalence class $[A]$ in $\Lambda^1(M)$ with $\delta_g d[A] = 0$ such that

$$[A] = i^*[A] \ \& \ \Pi = *_q \circ i^* \circ *_g \circ d[A].$$

64.5 <u>RAPPEL</u> The inner product on $\Lambda^1(\Sigma)$ is

$$\langle \alpha, \beta \rangle_q = \int_\Sigma \alpha \wedge *_q \beta = \int_\Sigma q(\alpha, \beta) \mathrm{vol}_q.$$

Let

$$E = \{([A], \Pi) : A, \Pi \in \Lambda^1(\Sigma), \delta_q \Pi = 0\}.$$

Put

$$\sigma(([A], \Pi), ([A'], \Pi'))$$

$$= \langle A, \Pi' \rangle_q - \langle A', \Pi \rangle_q.$$

<u>N.B.</u> σ is welldefined.

[For

$$<A + d\phi,\Pi'>_q - <A' + d\phi',\Pi>_q$$

$$= <A,\Pi'>_q - <A',\Pi>_q + <d\phi,\Pi'>_q - <d\phi',\Pi>_q$$

$$= <A,\Pi'>_q - <A',\Pi>_q + <\phi,\delta_q\Pi'>_q - <\phi',\delta_q\Pi>_q \quad \text{(cf. 63.10)}$$

$$= <A,\Pi'>_q - <A',\Pi>_q.]$$

64.6 LEMMA σ is nondegenerate.

PROOF Fix a pair $([A'],\Pi')$ and suppose that

$$\sigma(([A],\Pi),([A'],\Pi')) = 0$$

for all pairs $([A],\Pi)$ -- then the claim is that A' is exact and $\Pi' = 0$. Start by taking $A = \Pi'$, $\Pi = 0$ to get $<\Pi',\Pi'>_q = 0$, hence $\Pi' = 0$. We are thus left with

$$<A',\Pi>_q = 0$$

for all Π with $\delta_q\Pi = 0$. Bearing in mind that $\Lambda^1(\Sigma) = \text{Im } d \oplus \text{Ker } \delta_q$ (cf. 63.32), write $A' = d\phi' + B'$ $(\delta_q B' = 0)$ -- then

$$0 = <A',\Pi>_q$$

$$= <d\phi' + B',\Pi>_q$$

$$= <d\phi',\Pi>_q + <B',\Pi>_q$$

$$= <\phi',\delta_q\Pi>_q + <B',\Pi>_q \quad \text{(cf. 63.10)}$$

$$= \langle B',\Pi\rangle_q.$$

Now specialize and take $\Pi = B'$:

$$0 = \langle B',B'\rangle_q \Rightarrow B' = 0.$$

I.e.: A' is exact.

Therefore (E,σ) is a symplectic vector space.

64.7 REMARK If

$$\begin{cases} A \longleftrightarrow ([A],\Pi) \\ A' \longleftrightarrow ([A'],\Pi'), \end{cases}$$

then

$$\int_\Sigma i^*[A \wedge *_g \circ dA' - A' \wedge *_g \circ dA]$$

$$= \sigma(([A],\Pi),([A'],\Pi')).$$

Proof: We have

$$\int_\Sigma i^*[A \wedge *_g \circ dA' - A' \wedge *_g \circ dA]$$

$$= \int_\Sigma [A \wedge i^* \circ *_g \circ dA' - A' \wedge i^* \circ *_g \circ dA']$$

$$= \int_\Sigma [A \wedge *_q \circ *_q \circ i^* \circ *_g \circ dA' - A' \wedge *_q \circ *_q \circ i^* \circ *_g \circ dA']$$

$$= \int_\Sigma [A \wedge *_q\Pi' - A' \wedge *_q\Pi]$$

$$= \langle A,\Pi'\rangle_q - \langle A',\Pi\rangle_q$$

$$= \sigma(([A],\Pi),([A'],\Pi')).$$

[Note: Write

$$M = \coprod_t \Sigma_t \quad \text{(cf. 54.3)}$$

and work with Σ_t -- then the expression

$$\int_{\Sigma_t} i_t^*[A \wedge *_g \circ dA' - A' \wedge *_g \circ dA]$$

is independent of t.]

§65. *THE REDUCTION MECHANISM*

Let (M,g) be a globally hyperbolic spacetime which we shall assume is ultrastatic (cf. §57).

Given a p-form $\alpha \in \Lambda^p(M)$, write

$$\alpha = dt \wedge \alpha_0 + \alpha_\Sigma,$$

where

$$\alpha_0 = \iota_{\partial/\partial t}\alpha$$

and

$$\alpha_\Sigma = \alpha - dt \wedge \alpha_0.$$

[Note: Trivially,

$$\iota_{\partial/\partial t}\alpha_0 = 0.$$

On the other hand,

$$\iota_{\partial/\partial t}\alpha_\Sigma = \iota_{\partial/\partial t}\alpha - \iota_{\partial/\partial t}(dt \wedge \alpha_0)$$

$$= \alpha_0 - (\iota_{\partial/\partial t}dt \wedge \alpha_0 - dt \wedge \iota_{\partial/\partial t}\alpha_0)$$

$$= \alpha_0 - \alpha_0 = 0.]$$

Define an $\underline{R}$-linear map

$${}^3d:\Lambda^*(M) \rightarrow \Lambda^*(M)$$

by

$${}^3d = d - dt \wedge L_{\partial/\partial t}.$$

65.1 <u>LEMMA</u> We have

$$d\alpha = dt \wedge (L_{\partial/\partial t}\alpha_\Sigma - {}^3d\alpha_0) + {}^3d\alpha_\Sigma.$$

<u>PROOF</u> In fact,

$$\begin{aligned} d\alpha &= d(dt \wedge \alpha_0) + d\alpha_\Sigma \\ &= -\, dt \wedge d\alpha_0 + d\alpha_\Sigma \\ &= -\, dt \wedge ({}^3d\alpha_0 + dt \wedge L_{\partial/\partial t}\alpha_0) \\ &\quad + {}^3d\alpha_\Sigma + dt \wedge L_{\partial/\partial t}\alpha_\Sigma \\ &= dt \wedge (L_{\partial/\partial t}\alpha_\Sigma - {}^3d\alpha_0) + {}^3d\alpha_\Sigma. \end{aligned}$$

Let $\alpha,\beta \in \Lambda^p(M)$ -- then

$$\begin{aligned} g(\alpha,\beta) &= g(dt \wedge \alpha_0 + \alpha_\Sigma, dt \wedge \beta_0 + \beta_\Sigma) \\ &= g(dt \wedge \alpha_0, dt \wedge \beta_0) + g(\alpha_\Sigma, \beta_\Sigma). \end{aligned}$$

And

$$\begin{aligned} &g(dt \wedge \alpha_0, dt \wedge \beta_0) \\ &\quad = \iota_{dt \wedge \alpha_0}(dt \wedge \beta_0) \\ &\quad = \iota_{\alpha_0}\iota_{dt}(dt \wedge \beta_0) \\ &\quad = \iota_{\alpha_0}(\iota_{dt}dt \wedge \beta_0 - dt \wedge \iota_{dt}\beta_0) \end{aligned}$$

$$= \iota_{\alpha_0}(g(dt,dt)\beta_0 + dt \wedge \iota_{\partial/\partial t}\beta_0)$$

$$= - \iota_{\alpha_0}\beta_0$$

$$= - g(\alpha_0,\beta_0).$$

[Note: Tacitly,

$$\begin{cases} g(\alpha_\Sigma, dt \wedge \beta_0) = 0 \\ g(\beta_\Sigma, dt \wedge \alpha_0) = 0. \end{cases}$$

For example,

$$\begin{aligned} g(\alpha_\Sigma, dt \wedge \beta_0) &= \iota_{dt \wedge \beta_0}\alpha_\Sigma \\ &= \iota_{\beta_0}\iota_{dt}\alpha_\Sigma \\ &= - \iota_{\beta_0}\iota_{\partial/\partial t}\alpha_\Sigma \\ &= 0. \end{aligned}$$

In this connection, observe that

$$g^\flat(\partial/\partial t) = - dt$$

and keep in mind 63.2.]

Define t-dependent p-forms on Σ by

$$\begin{cases} \bar{\alpha}_0 = i_t^*\alpha_0 \qquad\qquad \bar{\alpha}_\Sigma = i_t^*\alpha_\Sigma \\ \qquad\qquad\qquad \& \\ \bar{\beta}_0 = i_t^*\beta_0 \qquad\qquad \bar{\beta}_\Sigma = i_t^*\beta_\Sigma. \end{cases}$$

Then

$$g(\alpha_0,\beta_0) \circ i_t = i_t^*(\iota_{\alpha_0}\beta_0)$$

$$= \iota_{i_t^*\alpha_0} i_t^*\beta_0$$

$$= q(\bar{\alpha}_0,\bar{\beta}_0)$$

and

$$g(\alpha_\Sigma,\beta_\Sigma) \circ i_t = i_t^*(\iota_{\alpha_\Sigma}\beta_\Sigma)$$

$$= \iota_{i_t^*\alpha_\Sigma} i_t^*\beta_\Sigma$$

$$= q(\bar{\alpha}_\Sigma,\bar{\beta}_\Sigma).$$

65.2 LEMMA Suppose that $\alpha,\beta \in \Lambda_c^p(M)$ -- then

$$\langle\alpha,\beta\rangle_g = \int_{\underline{R}} dt \int_\Sigma (q(\bar{\alpha}_\Sigma,\bar{\beta}_\Sigma) - q(\bar{\alpha}_0,\bar{\beta}_0))\text{vol}_q.$$

PROOF In view of the definitions and what has been said above,

$$\langle\alpha,\beta\rangle_g = \int_M g(\alpha,\beta)\text{vol}_g$$

$$= \int_{\underline{R}} dt \int_\Sigma i_t^*g(\alpha,\beta)\text{vol}_q$$

$$= \int_{\underline{R}} dt \int_\Sigma i_t^*(g(\alpha_\Sigma,\beta_\Sigma) - g(\alpha_0,\beta_0))\text{vol}_q$$

$$= \int_{\underline{R}} dt \int_\Sigma (g(\alpha_\Sigma,\beta_\Sigma) \circ i_t - g(\alpha_0,\beta_0) \circ i_t)\text{vol}_q$$

$$= \int_{\underline{R}} dt \int_{\Sigma} (q(\bar{\alpha}_{\Sigma},\bar{\beta}_{\Sigma}) - q(\bar{\alpha}_0,\bar{\beta}_0))\text{vol}_q.$$

65.3 RAPPEL Every connected orientable 3-manifold is parallelizable.

Therefore Σ is parallelizable, hence so is $M = \underline{R} \times \Sigma$.

Fix an orthonormal frame E_1,E_2,E_3 per q, put $E_0 = \partial/\partial t$, and let $\omega^0,\omega^1,\omega^2,\omega^3$ be the associated coframe (thus $\omega^i(E_j) = \delta^i_j$).

65.4 LEMMA

$$\begin{cases} *_g(\omega^0 \wedge \omega^1) = - \omega^2 \wedge \omega^3 \\ *_g(\omega^0 \wedge \omega^2) = \omega^1 \wedge \omega^3 \\ *_g(\omega^0 \wedge \omega^3) = - \omega^1 \wedge \omega^2 \end{cases}$$

and

$$\begin{cases} *_g(\omega^1 \wedge \omega^2) = \omega^0 \wedge \omega^3 \\ *_g(\omega^1 \wedge \omega^3) = - \omega^0 \wedge \omega^2 \\ *_g(\omega^2 \wedge \omega^3) = \omega^0 \wedge \omega^1. \end{cases}$$

Let

$$\bar{\omega}^a = i_t^*\omega^a \quad (a = 1,2,3).$$

65.5 LEMMA

$$\begin{cases} *_q(\bar{\omega}^1 \wedge \bar{\omega}^2) = \bar{\omega}^3 \\ *_q(\bar{\omega}^1 \wedge \bar{\omega}^3) = - \bar{\omega}^2 \\ *_q(\bar{\omega}^2 \wedge \bar{\omega}^3) = \bar{\omega}^1 \end{cases}$$

and

$$\begin{cases} *_q\bar{\omega}^1 = \bar{\omega}^2 \wedge \bar{\omega}^3 \\ *_q\bar{\omega}^2 = - \bar{\omega}^1 \wedge \bar{\omega}^3 \\ *_q\bar{\omega}^3 = \bar{\omega}^1 \wedge \bar{\omega}^2. \end{cases}$$

65.6 LEMMA Let $\alpha \in \Lambda^1(M)$ -- then

$$i_t^* *_g(dt \wedge \alpha) = - *_q\bar{\alpha} \quad (\bar{\alpha} = i_t^*\alpha).$$

PROOF Write

$$\alpha = C_0\omega^0 + C_1\omega^1 + C_2\omega^2 + C_3\omega^3.$$

Then

$$dt \wedge \alpha = C_1(\omega^0 \wedge \omega^1) + C_2(\omega^0 \wedge \omega^2) + C_3(\omega^0 \wedge \omega^3)$$

=>

$$*_g(dt \wedge \alpha) = - C_1(\omega^2 \wedge \omega^3) + C_2(\omega^1 \wedge \omega^3) - C_3(\omega^1 \wedge \omega^2)$$

=>

$$i_t^* *_g(dt \wedge \alpha) = - C_1(\bar{\omega}^2 \wedge \bar{\omega}^3) + C_2(\bar{\omega}^1 \wedge \bar{\omega}^3) - C_3(\bar{\omega}^1 \wedge \bar{\omega}^2)$$

$$= - C_1 *_q \bar{\omega}^1 - C_2 *_q \bar{\omega}^2 - C_3 *_q \bar{\omega}^3$$

$$= - *_q \bar{\alpha}.$$

Given $A \in \Lambda^1(M)$, put

$$\begin{cases} \bar{A} = i_t^* A \\ \bar{\Pi} = *_q \circ i_t^* \circ *_g \circ dA. \end{cases}$$

[Note: In the setting of §64,

$$\begin{cases} \bar{A} \leftrightarrow A \\ \bar{\Pi} \leftrightarrow \Pi. \end{cases}$$]

65.7 LEMMA We have

$$\bar{\Pi} = - i_t^* \iota_{\partial/\partial t} dA.$$

PROOF Write

$$dA = C_{01}(\omega^0 \wedge \omega^1) + C_{02}(\omega^0 \wedge \omega^2) + C_{03}(\omega^0 \wedge \omega^3)$$

$$+ C_{12}(\omega^1 \wedge \omega^2) + C_{13}(\omega^1 \wedge \omega^3) + C_{23}(\omega^2 \wedge \omega^3).$$

Then

$$i_t^* \iota_{\partial/\partial t} dA = C_{01}\bar{\omega}^1 + C_{02}\bar{\omega}^2 + C_{03}\bar{\omega}^3.$$

On the other hand,

$$\bar{\Pi} = *_q \circ i_t^* \circ *_g \circ dA$$

$$= *_q \circ i_t^*(- C_{01}(\omega^2 \wedge \omega^3) + C_{02}(\omega^1 \wedge \omega^3) - C_{03}(\omega^1 \wedge \omega^2)$$

$$+ C_{12}(\omega^0 \wedge \omega^3) - C_{13}(\omega^0 \wedge \omega^2) + C_{23}(\omega^0 \wedge \omega^1))$$

$$= *_q(- C_{01}(\bar{\omega}^2 \wedge \bar{\omega}^3) + C_{02}(\bar{\omega}^1 \wedge \bar{\omega}^3) - C_{03}(\bar{\omega}^1 \wedge \bar{\omega}^2))$$

$$= - C_{01}\bar{\omega}^1 - C_{02}\bar{\omega}^2 - C_{03}\bar{\omega}^3$$

$$= - i_t^* \iota_{\partial/\partial t} dA.$$

§66. *ANALYSIS IN THE TEMPORAL GAUGE*

Let (M,g) be a globally hyperbolic spacetime which we shall assume is ultrastatic, thus (M,g) is spatially compact iff Σ is compact, a condition that we shall also assume to be in force.

[Note: The results set forth in §64 are therefore applicable. As regards the spatially noncompact situation, some of the formalities do go through but ultimately it is far more difficult to deal with (and the final word has yet to be written). The special case of Minkowski space is considered in §70.]

Functional Derivatives There is a pairing

$$\begin{cases} \Lambda_c^1(M) \times \Lambda_c^1(M) \to \underline{R} \\ (\alpha,\beta) \to \langle\alpha,\beta\rangle_g . \end{cases}$$

So, if

$$L:\Lambda_c^1(M) \to \underline{R},$$

then $\frac{\delta L}{\delta\alpha}$ is the element of $\Lambda_c^1(M)$ such that

$$\frac{d}{d\varepsilon} L(\alpha + \varepsilon\delta\alpha)\Big|_{\varepsilon=0} = \langle\delta\alpha, \frac{\delta L}{\delta\alpha}\rangle_g$$

for all $\delta\alpha \in \Lambda_c^1(M)$.

The Maxwell lagrangian is the functional

$$L_{MAX}:\Lambda_c^1(M) \to \underline{R}$$

defined by the prescription

$$L_{MAX}(\alpha) = \frac{1}{2} \int_M g(d\alpha, d\alpha) \mathrm{vol}_g.$$

66.1 <u>LEMMA</u> We have

$$\frac{\delta L_{MAX}}{\delta \alpha} = \delta_g d\alpha.$$

<u>PROOF</u> In fact,

$$\frac{1}{2} \int_M \frac{d}{d\varepsilon} g(d(\alpha + \varepsilon\delta\alpha), d(\alpha + \varepsilon\delta\alpha)) \Big|_{\varepsilon=0} \mathrm{vol}_g$$

$$= \int_M g(d\delta\alpha, d\alpha) \mathrm{vol}_g$$

$$= \int_M g(\delta\alpha, \delta_g d\alpha) \mathrm{vol}_g \qquad \text{(cf. 63.10)}$$

$$= \langle \delta\alpha, \delta_g d\alpha \rangle_g.$$

Therefore

$$\frac{\delta L_{MAX}}{\delta \alpha} = \delta_g d\alpha.$$

A <u>critical point</u> for L_{MAX} is an element $\alpha \in \Lambda_c^1(M)$ such that

$$\frac{\delta L_{MAX}}{\delta \alpha} = 0.$$

Accordingly, α is a critical point for L_{MAX} iff α is a solution to Maxwell's equation:

$$\delta_g d\alpha = 0.$$

Now change the notation: Write A for α and let $F = dA$ -- then

$$L_{MAX}(A) = \frac{1}{2} \int_M g(F,F)\text{vol}_g$$

$$= \frac{1}{2} \int_{\underline{R}} dt \int_\Sigma (q(\bar{F}_\Sigma,\bar{F}_\Sigma) - q(\bar{F}_0,\bar{F}_0))\text{vol}_q \qquad \text{(cf. 65.2)}.$$

Put $\bar{A} = i_t^* A$ -- then

$$\bar{F}_\Sigma = i_t^* F_\Sigma$$

$$= i_t^*(F - dt \wedge F_0)$$

$$= i_t^* F$$

$$= i_t^* dA$$

$$= d i_t^* A$$

$$= d\bar{A}.$$

Therefore

$$L_{MAX}(A)$$

$$= \frac{1}{2} \int_{\underline{R}} dt \int_\Sigma (q(d\bar{A},d\bar{A}) - q(\bar{F}_0,\bar{F}_0))\text{vol}_q.$$

Next

$$F_0 = \iota_{\partial/\partial t} F$$

$$= \iota_{\partial/\partial t} dA$$

$$= (L_{\partial/\partial t} - d \circ \iota_{\partial/\partial t}) A$$

$$= L_{\partial/\partial t} A - dA_0$$

=>

$$\bar{F}_0 = i_t^* F_0$$

$$= i_t^* L_{\partial/\partial t} A - d\bar{A}_0.$$

It remains to interpret

$$i_t^* L_{\partial/\partial t} A.$$

Given $\alpha \in \Lambda^p(M)$, put

$$\dot{\alpha} = \frac{d}{dt} i_t^* \alpha \quad (= \frac{d}{dt} \bar{\alpha}).$$

66.2 <u>LEMMA</u> We have

$$\dot{\alpha} = i_t^* L_{\partial/\partial t} \alpha.$$

<u>PROOF</u> First

$$i_{t+s} = \phi_s \circ i_t,$$

where ϕ_s is the flow attached to $\frac{\partial}{\partial t}$. Consequently,

$$\dot{\alpha} = \frac{d}{ds}\bigg|_{s=t} (i_s^* \alpha)$$

$$= \lim_{s \to 0} \frac{i^*{}_{t+s}\alpha - i_t^*\alpha}{s}$$

$$= \lim_{s \to 0} \frac{i_t^*\phi_s^*\alpha - i_t^*\alpha}{s}$$

$$= i_t^* \lim_{s \to 0} \frac{\phi_s^*\alpha - \alpha}{s}$$

$$= i_t^* \mathcal{L}_{\partial/\partial t}\alpha.$$

In view of this,

$$L_{MAX}(A)$$

$$= \frac{1}{2} \int_{\underline{R}} dt \int_\Sigma (q(d\bar{A},d\bar{A}) - q(\dot{\bar{A}} - d\bar{A}_0, \dot{\bar{A}} - d\bar{A}_0))\text{vol}_q.$$

66.3 <u>REMARK</u> To run a reality check on the definitions, write

$$0 = dF$$

$$= dt \wedge (\mathcal{L}_{\partial/\partial t}F_\Sigma - {}^3dF_0) + {}^3dF_\Sigma \quad \text{(cf. 65.1)}$$

=>

$$\left[\begin{array}{l} L_{\partial/\partial t}F_\Sigma - {}^3dF_0 = 0 \\ {}^3dF_\Sigma = 0. \end{array}\right.$$

Then $\forall$ t,

- $0 = i_t^*(L_{\partial/\partial t}F_\Sigma - {}^3dF_0)$

=>

$$i_t^* L_{\partial/\partial t}F_\Sigma = i_t^* {}^3dF_0$$

=>

$$\dot{\bar{F}}_\Sigma = i_t^*(dF_0 - dt \wedge L_{\partial/\partial t}F_0)$$

=>

$$d\dot{\bar{A}} = i_t^* dF_0 - i_t^* dt \wedge i_t^* L_{\partial/\partial t}F_0$$

$$= di_t^* F_0$$

$$= d\bar{F}_0$$

$$= d(\dot{\bar{A}} - d\bar{A}_0)$$

$$= d\dot{\bar{A}}.$$

- $0 = i_t^* dF_\Sigma$

$$= di_t^* F_\Sigma$$

$$= d\bar{F}_\Sigma$$

$$= dd\bar{A}$$

$$= 0.$$

Let $C = \Lambda^0(\Sigma) \times \Lambda^1(\Sigma)$ -- then

$$TC = C \times \Lambda^0(\Sigma) \times \Lambda^1(\Sigma)$$

is the <u>velocity phase space</u> of the theory.

[Note: Elements of C are pairs (A_0,A), where

$$\begin{cases} A_0 \in \Lambda^0(\Sigma) \\ A \in \Lambda^1(\Sigma), \end{cases}$$

and elements of TC are pairs of pairs $(A_0,A;\dot{A}_0,\dot{A})$, where

$$\begin{cases} \dot{A}_0 \in \Lambda^0(\Sigma) \\ \dot{A} \in \Lambda^1(\Sigma).] \end{cases}$$

The <u>lagrangian</u> of the theory is the function

$$L:TC \to \underline{R}$$

defined by the rule

$$L(A_0,A;\dot{A}_0,\dot{A}) = \frac{1}{2} \int_\Sigma (q(dA,dA) - q(\dot{A} - dA_0,\dot{A} - dA_0))\mathrm{vol}_q.$$

[Note: The variable $\dot{A}_0$ is not present.]

N.B. From the above,

$$L_{MAX}(A) = \int_{\underline{R}} L(\bar{A}_0, \bar{A}; 0, \dot{\bar{A}})dt.$$

Thinking of TC as the tangent bundle of C, put

$$T^*C = C \times \Lambda^0(\Sigma) \times \Lambda^1(\Sigma)$$

and call it the momentum phase space of the theory.

[Note: Elements of T^*C are pairs of pairs $(A_0, A; \Pi_0, \Pi)$, where

$$\begin{cases} \Pi_0 \in \Lambda^0(\Sigma) \\ \Pi \in \Lambda^1(\Sigma). \end{cases}$$]

66.4 REMARK If $(A_0, A) \in C$ and if

$$\begin{cases} X = (A_0, A; \dot{A}_0, \dot{A}) \in TC \\ \omega = (A_0, A; \Pi_0, \Pi) \in T^*C, \end{cases}$$

then the evaluation $\langle X, \omega \rangle$ is

$$\langle \dot{A}_0, \Pi_0 \rangle_q + \langle \dot{A}, \Pi \rangle_q.$$

Here

$$\langle \dot{A}_0, \Pi_0 \rangle_q = \int_\Sigma \dot{A}_0 \Pi_0 \text{vol}_q$$

and

$$\langle \dot{A}, \Pi \rangle_q = \int_\Sigma q(\dot{A}, \Pi) \text{vol}_q.$$

[Note: It is customary to write

$$\left[\begin{array}{l} X = \dot{A}_0\frac{\delta}{\delta A_0} + \dot{A}\frac{\delta}{\delta A} \\ \\ \omega = \Pi_0\delta A_0 + \Pi\delta A.] \end{array}\right.$$

The primary constraint submanifold of the theory is that subset C of T*C consisting of those points $(A_0,A;\Pi_0,\Pi)$ for which $\Pi_0 = 0$.

[Note: This definition is suggested by the fact that

$$\frac{\delta L}{\delta \dot{A}_0} = 0.]$$

We shall now pass to the hamiltonian of the theory, it being the function

$$H:C \rightarrow \underset{\sim}{R}$$

with the property that

$$H \circ FL(X) = \langle X,FL(X)\rangle - L(X).$$

Since

$$\frac{\delta L}{\delta \dot{A}} = - (\dot{A} - dA_0),$$

we have

$$FL(A_0,A;\dot{A}_0,\dot{A}) = (A_0,A;0, - (\dot{A} - dA_0)),$$

so

$$H(A_0,A;\dot{A} - dA_0)$$

$$= - \langle \dot{A}, \dot{A} - dA_0 \rangle_q - L(A_0, A; \dot{A}_0, \dot{A})$$

$$= - \langle \dot{A}, \dot{A} - dA_0 \rangle_q - \frac{1}{2} (\langle dA, dA \rangle_q - \langle \dot{A} - dA_0, \dot{A} - dA_0 \rangle_q)$$

$$= - \langle \dot{A} - dA_0 + dA_0, \dot{A} - dA_0 \rangle_q$$

$$+ \frac{1}{2} (\langle \dot{A} - dA_0, \dot{A} - dA_0 \rangle_q - \frac{1}{2} \langle dA, dA \rangle_q)$$

$$= - \frac{1}{2} \langle \dot{A} - dA_0, \dot{A} - dA_0 \rangle_q - \langle dA_0, \dot{A} - dA_0 \rangle_q - \frac{1}{2} \langle dA, dA \rangle_q.$$

I.e.: As a function on $C \times \Lambda^1(\Sigma)$,

$$H(A_0, A; \Pi)$$

$$= - \frac{1}{2} \langle \Pi, \Pi \rangle_q - \langle dA_0, \Pi \rangle_q - \frac{1}{2} \langle dA, dA \rangle_q.$$

The next step is to set the constraint algorithim into motion. One then finds that the secondary constraint submanifold of the theory is that subset C' of C consisting of those points $(A_0, A; \Pi)$ for which

$$\delta_q \Pi \ (= \frac{\delta H}{\delta A_0}) = 0.$$

[Note: There are no tertiary constraints.]

But we are still not out of the woods. Internal to the theory is the notion of gauge vector field, two points in C' being physically equivalent if they can be connected by an integral curve of a gauge vector field.

66.5 EXAMPLE Let

$$\begin{bmatrix} (A_0, A; \Pi) \\ \\ (A_0', A; \Pi) \end{bmatrix} \in C'$$

and put

$$\gamma(t) = (A_0 + t(A_0' - A_0), A; \Pi) \quad (0 \leq t \leq 1).$$

Then

$$\begin{bmatrix} \gamma(0) = (A_0, A; \Pi) \\ \\ \gamma(1) = (A_0', A; \Pi) \end{bmatrix}$$

and γ is an integral curve of the gauge vector field $\dot{A}_0 \frac{\delta}{\delta A_0}$ $(\dot{A}_0 = \frac{d}{dt}\gamma(t))$.

It follows that the A_0-component of a point in C' is physically irrelevant. One may therefore normalize the situation and take $A_0 = 0$. With this agreement, we shall view the final constraint submanifold of the theory as a subset $\bar{C}$ of $\Lambda^1(\Sigma) \times \Lambda^1(\Sigma)$, viz. the pairs (A,Π), where $\delta_q \Pi = 0$.

[Note: Put

$$\bar{H}(A,\Pi) = H(0,A;\Pi).$$

Then

$$\bar{H}(A,\Pi) = -\frac{1}{2} \langle \Pi,\Pi \rangle_q - \frac{1}{2} \langle dA,dA \rangle_q$$

is now the hamiltonian of the theory.]

The remaining gauge vector fields are parameterized by the $\phi \in C^{\infty}(\Sigma)$:

$$(d\phi)\frac{\delta}{\delta A}\ .$$

But this means that (A,Π) and $(A + d\phi,\Pi)$ are physically equivalent.

66.6 <u>SCHOLIUM</u> The physical phase space of the theory is

$$E = \{([A],\Pi):A,\Pi \in \Lambda^1(\Sigma),\delta_q\Pi = 0\},$$

in precise agreement with the earlier abstract considerations (cf. §64).

Dropping the supposition of compact support, take $A \in \Lambda^1(M)$ arbitrary, let $F = dA$, and put

$$\bar{\Pi} = *_q \circ i_t^* \circ *_g F.$$

Then

$$\bar{\Pi} = - i_t^* \iota_{\partial/\partial t} F \quad \text{(cf. 65.7)}$$

$$= - i_t^* F_0$$

$$= - \bar{F}_0 = - (\dot{\bar{A}} - d\bar{A}_0).$$

And it is clear that the assignment

$$t \to (\bar{A}_0, - \bar{A};0,\bar{\Pi})$$

is a path in $C \subset T^*C$.

Assume next that A satisfies Maxwell's equation, thus $\delta_g dA = 0$, which implies that $\delta_q \bar{\Pi} = 0$ (cf. 64.1), so the assignment

$$t \to (\bar{A}_0, -\bar{A}; 0, \bar{\Pi})$$

is a path in $C' \subset C$.

To proceed further, let us agree that A is in <u>temporal gauge</u> if

$$A_0 = \iota_{\partial/\partial t} A = 0.$$

66.7 <u>LEMMA</u> The gauge equivalence class $[A]$ contains an element A' in temporal gauge.

<u>PROOF</u> Define $f: M \to \underline{R}$ by

$$f(t,x) = -\int_0^t A_0(s,x)\,ds.$$

Put

$$A' = A + df.$$

Then

$$\begin{aligned} \iota_{\partial/\partial t} A' &= A_0 + \iota_{\partial/\partial t} df \\ &= A_0 + L_{\partial/\partial t} f \\ &= A_0 - A_0 \\ &= 0. \end{aligned}$$

Therefore A' is in temporal gauge.

[Note: If $A \in \Lambda_c^1(M)$, then, in general, $A' \notin \Lambda_c^1(M)$, hence passage to the temporal gauge may very well force one out of the compactly supported world.]

Maintaining the assumption that A satisfies Maxwell's equation, suppose further that A is in temporal gauge — then the assignment

$$t \to (-\bar{A},\bar{\Pi}) = (-\bar{A}, -\dot{\bar{A}})$$

is a path in $\bar{C}$.

To understand the evolutionary aspect of Maxwell's equation, we shall need a preliminary result which, in particular, leads to another proof of 64.1.

Define

$$^{3}*:\Lambda^*(M) \to \Lambda^*(M)$$

in the obvious way. E.g.:

$$^{3}*A = -*_g(dt \wedge A) \quad \text{(cf. 65.6)}.$$

66.8 <u>LEMMA</u> $\forall\ A \in \Lambda^1(M)$,

$$\delta_g F = *_g d *_g F \quad \text{(cf. 63.8)}$$

$$= L_{\partial/\partial t}F_0 + ({}^{3}*{}^{3}d{}^{3}*F_0)dt + {}^{3}*{}^{3}d{}^{3}*F_{\Sigma}.$$

<u>PROOF</u> First,

$$*_g F = *_g(dt \wedge F_0) + *_g F_{\Sigma}$$

$$= -{}^{3}*F_0 + dt \wedge {}^{3}*F_{\Sigma}.$$

But from the definitions,

$$({}^3{*}F_0)_0 = 0 \Rightarrow ({}^3{*}F_0)_\Sigma = {}^3{*}F_0$$

and

$$(dt \wedge {}^3{*}F_\Sigma)_0 = {}^3{*}F_\Sigma$$

$$\Rightarrow (dt \wedge {}^3{*}F_\Sigma) \;= dt \wedge {}^3{*}F_\Sigma - dt \wedge {}^3{*}F_\Sigma$$

$$= 0.$$

Therefore (cf. 65.1)

$$d{*}_g F = d(- {}^3{*}F_0) + d(dt \wedge {}^3{*}F_\Sigma)$$

$$= dt \wedge L_{\partial/\partial t} - {}^3{*}F_0 + {}^3d(- {}^3{*}F_0) + dt \wedge - {}^3d{}^3{*}F_\Sigma.$$

Applying $*_g$ one more time then leads to the stated formula.

It follows from this that

$$\delta_g F = 0$$

=>

$$\delta_q \bar{F}_0 = 0 \Rightarrow \delta_q \bar{\Pi} = 0 \quad \text{(cf. 64.1)}$$

and

$$\dot{\bar{F}}_0 + \delta_q \bar{F}_\Sigma = 0,$$

i.e.,

$$(\ddot{\bar{A}} - d\bar{A}_0) + \delta_q d\bar{A} = 0.$$

So, if A is in temporal gauge, then

$$\ddot{\bar{A}} + \delta_q d\bar{A} = 0.$$

Returning to 66.6, let us explicate $\Lambda^1(\Sigma)/\sim$. Thus write $\Lambda^1(\Sigma) = \text{Im } d \oplus \text{Ker } \delta_q$ (cf. 63.32) -- then a given $A \in \Lambda^1(\Sigma)$ admits a decomposition $A = d\phi + A^T$, where A^T is the <u>transverse</u> component of A.

66.9 <u>LEMMA</u> The map

$$\begin{cases} \Lambda^1(\Sigma)/\sim \to \text{Ker } \delta_q \\ [A] \longrightarrow A^T \end{cases}$$

is a welldefined bijection.

[If $A^T = B^T$, then

$$\begin{cases} A = d\phi + A^T \\ B = d\psi + B^T = d\psi + A^T \end{cases}$$

=>

$$A - B = d(\phi - \psi)$$

=>

$$A \sim B \Rightarrow [A] = [B].]$$

Put

$$\Lambda^{1,T}(\Sigma) = \text{Ker } \delta_q.$$

Then E can be realized as the direct sum

$$\Lambda^{1,T}(\Sigma) \oplus \Lambda^{1,T}(\Sigma),$$

or still, as the set of pairs (A,Π), where

$$\begin{cases} \delta_q A = 0 \\ \delta_q \Pi = 0. \end{cases}$$

Define

$$\sigma: E \times E \rightarrow \underline{R}$$

by

$$\sigma((A,\Pi),(A',\Pi')) = \langle A,\Pi'\rangle_q - \langle A',\Pi\rangle_q.$$

Then σ is nondegenerate (cf. 64.6), hence (E,σ) is a symplectic vector space.

The hamiltonian $\bar{H}$ passes to the quotient and defines a function on E, which again will be denoted by $\bar{H}$.

66.10 <u>REMARK</u> Thus

$$\bar{H}(A,\Pi) = -\frac{1}{2}\langle\Pi,\Pi\rangle_q - \frac{1}{2}\langle dA,dA\rangle_q.$$

To be in agreement with the usual conventions, jettison the minus signs and stipulate that the hamiltonian of the theory is

$$\bar{H}(A,\Pi) = \frac{1}{2}\langle\Pi,\Pi\rangle_q + \frac{1}{2}\langle dA,dA\rangle_q.$$

Observe that this would have been the outcome if we had worked from the beginning with

$$- L_{MAX}(\alpha) = - \frac{1}{2} \int_M g(d\alpha, d\alpha)\, \mathrm{vol}_g$$

and, of course

$$\delta_g d\alpha = 0 \iff - \delta_g d\alpha = 0.$$

66.11 <u>LEMMA</u> The hamiltonian vector field

$$X_{\bar{H}} : E \to E$$

attached to $\bar{H}$ is given by

$$X_{\bar{H}}(A,\Pi) = \left(\frac{\delta \bar{H}}{\delta \Pi}, - \frac{\delta \bar{H}}{\delta A}\right).$$

But

$$\begin{cases} \dfrac{\delta \bar{H}}{\delta \Pi} = \Pi \\[2ex] \dfrac{\delta \bar{H}}{\delta A} = \delta_q dA. \end{cases}$$

Accordingly, if

$$\gamma(t) = (A(t),\Pi(t))$$

is an integral curve for $X_{\bar{H}}$, then

$$\dot{\gamma}(t) = X_{\bar{H}}(A(t),\Pi(t))$$

$$= (\Pi(t), -\delta_q dA(t))$$

=>

$$\begin{cases} \dot{A}(t) = \Pi(t) \\ \dot{\Pi}(t) = -\delta_q dA(t) \end{cases}$$

=>

$$\ddot{A}(t) + \delta_q dA(t) = 0.$$

Put

$$\tilde{\Lambda}^{1,T}(\Sigma) = \delta(\Lambda^2(\Sigma)),$$

so that

$$\Lambda^{1,T}(\Sigma) = \tilde{\Lambda}^{1,T}(\Sigma) \oplus \underline{H}^1.$$

Then

$$E = E_o \oplus E_f,$$

where

$$\begin{cases} E_0 = \tilde{\Lambda}^{1,T}(\Sigma) \oplus \tilde{\Lambda}^{1,T}(\Sigma) \\ E_f = \underline{H}^1 \oplus \underline{H}^1. \end{cases}$$

• E_o is the "oscillating" sector of E. In it, the equations of motion are

$$\begin{cases} \dot{A}(t) = \Pi(t) \\ \dot{\Pi}(t) = \Delta_q A(t) \end{cases}$$

and formally, the integral curve $\gamma(t) = (A(t),\Pi(t))$ passing through (A,Π) at $t = 0$ is

$$\gamma(t) = \begin{bmatrix} \cos(t(-\bar{\Delta}_q)^{1/2}) & (-\bar{\Delta}_q)^{-1/2}\sin(t(-\bar{\Delta}_q)^{1/2}) \\ -(-\bar{\Delta}_q)^{1/2}\sin(t(-\bar{\Delta}_q)^{1/2}) & \cos(t(-\bar{\Delta}_q)^{1/2}) \end{bmatrix} \begin{bmatrix} A \\ \Pi \end{bmatrix}.$$

- E_f is the "free" sector of E. In it, the equations of motion are

$$\begin{cases} \dot{A}(t) = \Pi(t) \\ \dot{\Pi}(t) = 0 \end{cases}$$

and formally, the integral curve $\gamma(t) = (A(t),\Pi(t))$ passing through (A,Π) at $t = 0$ is

$$\gamma(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ \Pi \end{bmatrix}.$$

Specialize and assume that $\underline{H}^1 = 0$, hence $E = E_o$. Taking $\Lambda_q^{2,1}(\Sigma)$ over $\underline{C}$, define a real linear map

$$k:E \to \Lambda_q^{2,1}(\Sigma)$$

by

$$k(A,\Pi) = -\sqrt{-1}\,(-\bar{\Delta}_q)^{1/4}A + (-\bar{\Delta}_q)^{-1/4}\Pi.$$

[Note: Since $\underline{H}^1 = 0$, $-\bar{\Delta}_q$ is positive and has a bounded inverse.]

Now apply an evident variant of the Deutsch-Najmi construction and define

$$\mu_M : E \times E \to \underline{R}$$

by

$$\mu_M((A,\Pi),(A',\Pi')) = \langle A, (-\bar{\Delta}_q)^{1/2} A' \rangle_q + \langle \Pi, (-\bar{\Delta}_q)^{-1/2} \Pi' \rangle_q .$$

Then $\mu \in \mathrm{IP}(E,\sigma)$ and is pure.

Definition The Maxwell state is the pure state on $\mathcal{W}(E,\sigma)$ determined by μ_M.

§67. *THE LAPLACIAN IN $\underline{R}^3$*

Recall that the domain of

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

is $W^{2,2}(\underline{R}^3)$ (cf. 1.15).

67.1 <u>LEMMA</u> Let $\phi \in W^{2,2}(\underline{R}^3)$ -- then ϕ is a bounded continuous function and $\exists\ C > 0$, independent of ϕ, such that

$$||\phi||_\infty \leq ||\Delta\phi||_2 + C||\phi||_2.$$

67.2 <u>LEMMA</u> Let $\phi \in L^2(\underline{R}^3)$. Assume: ϕ is harmonic, i.e., $\Delta\phi = 0$ -- then $\phi = 0$ (cf. 63.21).

<u>PROOF</u> In fact, $\phi \in W^{2,2}(\underline{R}^3)$, hence is bounded. But the bounded harmonic functions on $\underline{R}^3$ are the constants.

[Note: Here is a different proof: $\phi \in L^2(\underline{R}^3) \Rightarrow \hat{\phi} \in L^2(\underline{R}^3)$, so

$$\Delta\phi = 0 \Rightarrow |\xi|^2\hat{\phi}(\xi) = 0 \Rightarrow \hat{\phi} = 0 \Rightarrow \phi = 0.]$$

Let

$$G = -\frac{1}{4\pi r} \quad (r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}).$$

Then G is a distribution and

$$\Delta G = \delta.$$

Therefore

$$\Delta(G*f) = \Delta G*f = \delta*f = f \quad (f \in C_c^\infty(\underline{R}^3)).$$

67.3 <u>REMARK</u> G is a tempered distribution with Fourier transform

$$\hat{G}(\xi) = -\frac{1}{(2\pi)^{3/2}} \frac{1}{|\xi|^2}.$$

The convolution G*f is automatically C^∞ and

$$G*f\Big|_x = -\frac{1}{4\pi} \int_{\underline{R}^3} \frac{f(y)}{|x-y|} dy$$

$$= -\frac{1}{(2\pi)^{3/2}} \int_{\underline{R}^3} \frac{\hat{f}(\xi)}{|\xi|^2} e^{\sqrt{-1}\, x\xi} d\xi.$$

67.4 <u>RAPPEL</u> Let (X, M, μ) be a σ-finite measure space. Suppose that $f: X \to \underline{R}$ is measurable. Define

$$\lambda_f :]0,\infty[\to [0,\infty]$$

by

$$\lambda_f(t) = \mu(\{x : |f(x)| > t\}).$$

Then for any p $(0 < p < \infty)$,

$$\int_X |f|^p d\mu = p\int_0^\infty t^{p-1}\lambda_f(t)dt.$$

Put

$$||f||_{p,w} = (\sup_{t > 0} t^p \lambda_f(t))^{1/p}.$$

Then f is said to be in <u>weak L^p</u>, written $f \in L_w^p(X,\mu)$, if $||f||_{p,w} < \infty$. While $||\cdot||_{p,w}$ is not a norm (the triangle inequality fails), one does have $||\cdot||_{p,w} \leq ||\cdot||_p$, so

$$L^p(X,\mu) \subset L_w^p(X,\mu).$$

67.5 <u>EXAMPLE</u> Take $X = \underline{R}^3$, $\mu = dx$, and let $f = r^{-3/p}$ -- then $\lambda_f(t) = \frac{4}{3} \pi t^{-p}$, thus $f \in L_w^p(\underline{R}^3)$ but $f \notin L^p(\underline{R}^3)$.

67.6 <u>LEMMA</u> Let $f \in C_c^\infty(\underline{R}^3)$ -- then $G*f \in L^p(\underline{R}^3)$ $(p > 3)$.

<u>PROOF</u> Since

$$G \in L_w^3(\underline{R}^3),$$

the generalized Young inequality gives

$$||G*f||_p \leq C||G||_{3,w}||f||_q \leq C'||f||_q \quad (p > 1, q > 1),$$

where

$$\frac{1}{3} + \frac{1}{q} = 1 + \frac{1}{p}.$$

Let $1 < q < \frac{3}{2}$ -- then $3 < p < \infty$ and the result follows.

Write

$$G_i(x) = \partial_i G(x) = \frac{1}{4\pi} \frac{x_i}{r^3}.$$

67.7 <u>LEMMA</u> Let $f \in C_c^\infty(\underline{R}^3)$ -- then

$$G_i * f \in L^p(\underline{R}^3) \quad (p > \tfrac{3}{2}).$$

<u>PROOF</u> Since

$$G_i \in L_w^{3/2}(\underline{R}^3),$$

the generalized Young inequality gives

$$||G_i * f||_p \leq C||G_i||_{3/2,w}||f||_q \leq C'||f||_q \quad (p > 1, q > 1),$$

where

$$\frac{2}{3} + \frac{1}{q} = 1 + \frac{1}{p}.$$

Let $1 < q < 3$ -- then $\frac{3}{2} < p < \infty$ and the result follows.

In particular:

$$G_i * f \in L^2(\underline{R}^3)$$

=>

$$\text{grad } G * f \in L^2(\underline{R}^3; \underline{R}^3).$$

67.8 <u>REMARK</u> We have

$$G_i * f \in W^{2,k}(\underline{R}^3) \quad (k = 1,2,\ldots).$$

Indeed,

$$\partial_j (G_i * f) = G_i * \partial_j f \in L^2(\underline{R}^3),$$

so one can proceed from here by iteration.

The condition on f can, of course, be relaxed. To be specific, let us assume that $f \in L^2(\underline{R}^3)$ and is compactly supported -- then it makes sense to consider $G*f$, which is thus harmonic in the exterior of $\{x:|x| \leq R\}$ for R sufficiently large and

$$\lim_{|x| \to \infty} (G*f)(x) = 0.$$

67.9 <u>REMARK</u> Suppose that $f \in L^2_{loc}(\underline{R}^3)$ and

$$\int_{\underline{R}^3} \frac{|f(x)|}{1 + |x|} \, dx < \infty.$$

Then it is still possible to define $G*f$ but, in general, $G*f$ need not tend to zero at infinity.

[Note: Obviously,

$$|f(x)| \leq \frac{C}{|x|^{2+\varepsilon}} \qquad (|x| >> 0)$$

=>

$$\int_{\underline{R}^3} \frac{|f(x)|}{1 + |x|} \, dx < \infty.]$$

§68. *VECTOR FIELDS*

Given $X \in C^\infty(\underline{R}^3;\underline{R}^3)$, write

$$X = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + f_3 \frac{\partial}{\partial x_3}$$

and put

$$\omega_X = f_1 dx_1 + f_2 dx_2 + f_3 dx_3.$$

Then the map

$$X \rightarrow \omega_X$$

from

$$C^\infty(\underline{R}^3;\underline{R}^3) \text{ to } \Lambda^1(\underline{R}^3)$$

is bijective.

68.1 <u>LEMMA</u> We have

$$\begin{cases} \omega_{\text{grad } f} = \omega_{\nabla f} = df \\ \omega_{\text{curl } X} = \omega_{\nabla \times X} = *d\omega_X. \end{cases}$$

On $\Lambda^1(\underline{R}^3)$,

$$\delta = (-1)^{3+3+1} *d* = - *d*.$$

Therefore

$$\text{div } X = -\delta\omega_X = *d*\omega_X.$$

And

$$\omega_{\nabla(\nabla\cdot X)} = \omega_{\nabla(\text{div } X)}$$

$$= d(\text{div } X)$$

$$= - d\delta(\omega_X).$$

On $\Lambda^2(\underline{R}^3)$,

$$\delta = (-1)^{6+3+1} *d* = *d*.$$

Therefore

$$\delta d\omega_X = *d*d\omega_X$$

$$= *d\omega_{\nabla \times X}$$

$$= \omega_{\nabla \times (\nabla \times X)}.$$

68.2 <u>LEMMA</u> We have

$$\Delta X = \nabla(\nabla\cdot X) - \nabla \times (\nabla \times X).$$

<u>PROOF</u> In view of what has been said above,

$$\Delta\omega_X = - (d \circ \delta + \delta \circ d)\omega_X$$

$$= \omega_{\nabla(\nabla\cdot X)} - \omega_{\nabla \times (\nabla \times X)}.$$

On the other hand,

$$\Delta\omega_X = \omega_{\Delta X}.$$

Let $X \in C^{\infty}(\underline{R}^3;\underline{R}^3)$ -- then X is said to be

$$\begin{cases} \underline{\text{longitudinal}} & \text{if } \nabla \times X = 0 \\ \underline{\text{transverse}} & \text{if } \nabla \cdot X = 0. \end{cases}$$

68.3 LEMMA If X is both longitudinal and transverse, then $\Delta X = 0$.
[This is immediate (cf. 68.2).]

Assume now that $X \in C_c^{\infty}(\underline{R}^3;\underline{R}^3)$. Put

$$\begin{cases} X_{||} = \text{grad}(G*\text{div } X) \\ X^T = -\text{curl}(G*\text{curl } X). \end{cases}$$

Then

$$\begin{cases} X_{||} \in C^{\infty}(\underline{R}^3;\underline{R}^3) \\ X^T \in C^{\infty}(\underline{R}^3;\underline{R}^3). \end{cases}$$

Since

$$\begin{cases} \text{curl} \circ \text{grad} = 0 \\ \text{div} \circ \text{curl} = 0, \end{cases}$$

it follows that $X_{||}$ is longitudinal and X^T is transverse. In addition, $X_{||}$ and X^T are square integrable (cf. 67.7) and mutually orthogonal:

$$\langle X_{||}, X^T \rangle$$

$$= \int_{\underline{R}^3} \langle \text{grad}(G*\text{div } X), -\text{curl}(G*\text{curl } X) \rangle dx$$

$$= \int_{\underline{R}^3} (G*\text{div } X)(\text{div curl}(G*\text{curl } X)) dx$$

$$= 0.$$

68.4 LEMMA We have

$$\begin{cases} \text{div } X_{||} = \text{div } X \\ \text{curl } X^T = \text{curl } X. \end{cases}$$

PROOF

- $\text{div } X_{||} = \text{div grad}(G*\text{div } X)$

$$= \Delta(G*\text{div } X)$$

$$= \Delta G*\text{div } X$$

$$= \delta*\text{div } X$$

$$= \text{div } X.$$

- $\text{curl } X^T = -\text{curl curl}(G*\text{curl } X)$

$$= -\nabla \times \nabla(G*(\nabla \times X))$$

$$= \Delta(G*(\nabla \times X)) - \nabla(\nabla\cdot(G*(\nabla \times X))) \quad (\text{cf. } 68.2)$$

$$= \Delta G*(\nabla \times X) - \nabla(\nabla\cdot(\nabla \times G*X))$$

$$= \delta*(\nabla \times X)$$

$$= \nabla \times X$$

$$= \text{curl } X.$$

68.5 LEMMA $\forall\ X \in C_c^\infty(\underline{R}^3;\underline{R}^3)$,

$$X = X_{||} + X^T.$$

PROOF Consider the difference

$$X - (X_{||} + X^T).$$

Then (cf. 68.4)

$$\begin{cases} \text{div}(X - (X_{||} + X^T)) = 0 \\ \text{curl}(X - (X_{||} + X^T)) = 0 \end{cases}$$

=>

$$\Delta(X - (X_{||} + X^T)) = 0 \quad \text{(cf. 68.3)}.$$

But

$$X - (X_{||} + X^T) \in L^2(\underline{R}^3;\underline{R}^3).$$

Therefore

$$X = X_{||} + X^T \quad \text{(cf. 67.2)}.$$

Recall that

$$X = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + f_3 \frac{\partial}{\partial x_3} .$$

This said, denote by grad X (or ∇X) the associated triple of triples, viz.

$$(\nabla f_1, \nabla f_2, \nabla f_3).$$

68.6 LEMMA If $X \in C_c^\infty(\underline{R}^3;\underline{R}^3)$, then

$$\int_{\underline{R}^3} |\text{grad } X|^2 dx$$

$$= \int_{\underline{R}^3} (|\text{div } X|^2 + |\text{curl } X|^2) dx.$$

PROOF Write

$$\int_{\underline{R}^3} |\nabla f_i|^2 dx = - \int_{\underline{R}^3} f_i \Delta f_i dx \quad (i = 1,2,3).$$

Then

$$\int_{\underline{R}^3} |\text{grad } X|^2 dx$$

$$= \sum_{i=1}^{3} \int_{\underline{R}^3} |\nabla f_i|^2 dx$$

$$= - \sum_{i=1}^{3} \int_{\underline{R}^3} f_i \Delta f_i dx$$

$$= - \int_{\underline{R}^3} \langle X, \Delta X \rangle dx$$

$$= - \int_{\underline{R}^3} \langle X, \nabla(\nabla\cdot X)\rangle dx + \int_{\underline{R}^3} \langle X, \nabla \times (\nabla \times X)\rangle dx$$

$$= \int_{\underline{R}^3} (\nabla\cdot X)^2 dx + \int_{\underline{R}^3} \langle \nabla \times X, \nabla \times X\rangle dx$$

$$= \int_{\underline{R}^3} (|\mathrm{div}\ X|^2 + |\mathrm{curl}\ X|^2) dx.$$

[Note: Needless to say, the supposition that the

$$f_i \in C_c^\infty(\underline{R}^3) \qquad (i = 1,2,3)$$

can obviously be weakened.]

§69. HELMHOLTZ'S THEOREM

It is understood that derivatives are taken in the sense of distributions.

69.1 LEMMA Let $F \in L^2(\underline{R}^3;\underline{R}^3)$. Assume: $\nabla \cdot F = 0$ and $\nabla \times F = 0$ -- then $F = 0$.

PROOF The hypotheses imply that $\Delta F = 0$ (cf. 68.2). Now apply 67.2.

69.2 LEMMA Let $F \in L^2(\underline{R}^3;\underline{R}^3)$. Assume: $\nabla \cdot F \in L^2(\underline{R}^3)$ and $\nabla \times F \in L^2(\underline{R}^3;\underline{R}^3)$ -- then

$$\int_{\underline{R}^3} |\text{grad } F|^2 dx$$

$$= \int_{\underline{R}^3} (|\text{div } F|^2 + |\text{curl } F|^2) dx$$

$$< \infty \quad \text{(cf. 68.6)}.$$

[Note: Accordingly, if $F = (F_1, F_2, F_3)$, then

$$\nabla F_i \in L^2(\underline{R}^3;\underline{R}^3) \quad (i = 1,2,3).$$

Therefore

$$F \in W^{2,1}(\underline{R}^3;\underline{R}^3).]$$

Put

$$L^2_{||}(\underline{R}^3;\underline{R}^3) = \{F \in L^2(\underline{R}^3;\underline{R}^3) : \nabla \times F = 0\}.$$

69.3 <u>LEMMA</u> $L^2_{||}(\underline{R}^3;\underline{R}^3)$ is the closure in $L^2(\underline{R}^3;\underline{R}^3)$ of $\{\nabla f: f \in C_c^\infty(\underline{R}^3)\}$.

<u>PROOF</u> Suppose that $F \in L^2_{||}(\underline{R}^3;\underline{R}^3)$ and

$$F \perp \{\nabla f: f \in C_c^\infty(\underline{R}^3)\}.$$

Then $\forall\ f \in C_c^\infty(\underline{R}^3)$,

$$\int_{\underline{R}^3} (\nabla\cdot F) f dx = - \int_{\underline{R}^3} \langle F, \nabla f\rangle dx = 0$$

$$\Rightarrow$$

$$\nabla\cdot F = 0.$$

Therefore $F = 0$ (cf. 69.1).

Put

$$L^2(\underline{R}^3;\underline{R}^3)^T = \{F \in L^2(\underline{R}^3;\underline{R}^3): \nabla\cdot F = 0\}.$$

69.4 <u>LEMMA</u> $L^2(\underline{R}^3;\underline{R}^3)^T$ is the closure in $L^2(\underline{R}^3;\underline{R}^3)$ of $\{\nabla \times X: X \in C_c^\infty(\underline{R}^3;\underline{R}^3)\}$.

<u>PROOF</u> Suppose that $F \in L^2(\underline{R}^3;\underline{R}^3)^T$ and

$$F \perp \{\nabla \times X: X \in C_c^\infty(\underline{R}^3;\underline{R}^3)\}.$$

Then $\forall\ X \in C_c^\infty(\underline{R}^3;\underline{R}^3)$,

$$\int_{\underline{R}^3} \langle\nabla \times F, X\rangle dx = \int_{\underline{R}^3} \langle F, \nabla \times X\rangle dx = 0$$

$$\Rightarrow$$

$$\nabla \times F = 0.$$

Therefore $F = 0$ (cf. 69.1).

69.5 THEOREM (Helmholtz) There is an orthogonal decomposition

$$L^2(\underline{R}^3;\underline{R}^3) = L^2_{||}(\underline{R}^3;\underline{R}^3) \oplus L^2(\underline{R}^3;\underline{R}^3)^T.$$

PROOF It is clear that

$$L^2_{||}(\underline{R}^3;\underline{R}^3) \perp L^2(\underline{R}^3;\underline{R}^3)^T.$$

On the other hand, if F is orthogonal to

$$\{\nabla f : f \in C_c^\infty(\underline{R}^3)\}$$

and

$$\{\nabla \times X : X \in C_c^\infty(\underline{R}^3;\underline{R}^3)\},$$

then by the above, $\nabla \cdot F = 0$ and $\nabla \times F = 0$, hence $F = 0$ (cf. 69.1). Therefore $L^2_{||}(\underline{R}^3;\underline{R}^3)$ and $L^2(\underline{R}^3;\underline{R}^3)^T$ span $L^2(\underline{R}^3;\underline{R}^3)$.

69.6 EXAMPLE Let $X \in C_c^\infty(\underline{R}^3;\underline{R}^3)$ -- then

$$\begin{cases} X_{||} \in L^2_{||}(\underline{R}^3;\underline{R}^3) \\ X^T \in L^2(\underline{R}^3;\underline{R}^3)^T \end{cases} \quad \text{(cf. §68).}$$

69.7 REMARK Identify $L^2(\underline{R}^3;\underline{R}^3)$ with $\Lambda_g^{2,1}(\underline{R}^3)$ (g = usual metric) -- then

69.5 is a special case of 63.23. Indeed,

$$\Lambda_g^{2,1}(\underline{R}^3) = \overline{\delta\Lambda_c^2(\underline{R}^3)} \oplus \overline{d\Lambda_c^0(\underline{R}^3)},$$

the space $\underline{H}^1$ of harmonic 1-forms being trivial. Obviously,

$$d\Lambda_c^0(\underline{R}^3) \longleftrightarrow \{\nabla f : f \in C_c^\infty(\underline{R}^3)\}.$$

As for $\delta\Lambda_c^2(\underline{R}^3)$, take an $\alpha \in \Lambda_c^2(\underline{R}^3)$ and define $X \in C_c^\infty(\underline{R}^3;\underline{R}^3)$ by $*\alpha = \omega_X$ -- then

$$\delta\alpha = *d*\alpha = *d\omega_X = \omega_{\nabla \times X} \quad \text{(cf. 68.1)}.$$

Therefore

$$\delta\Lambda_c^2(\underline{R}^3) \longleftrightarrow \{\nabla \times X : X \in C_c^\infty(\underline{R}^3;\underline{R}^3)\}.$$

[Note: Let

$$\text{Dom}(\nabla) = \{f \in C^\infty(\underline{R}^3) \cap L^2(\underline{R}^3) : \nabla f \in L^2(\underline{R}^3;\underline{R}^3)\}.$$

Then ∇ admits closure and

$$L_{||}^2(\underline{R}^3;\underline{R}^3) = \overline{\text{Im}\,\bar{\nabla}} \quad \text{(cf. 63.25)}.$$

Still, Im $\bar{\nabla}$ itself is not closed.]

The decomposition

$$L^2(\underline{R}^3;\underline{R}^3) = L_{||}^2(\underline{R}^3;\underline{R}^3) \oplus L^2(\underline{R}^3;\underline{R}^3)^T$$

can also be approached via Fourier transforms. Thus given $F \in L^2(\underline{R}^3;\underline{R}^3)$, write

$$\hat{F} = \hat{F}_{||} + \hat{F}^T,$$

where

$$\hat{F}_{||}(\xi) = \frac{\xi}{|\xi|}\left(\frac{\xi}{|\xi|} \cdot \hat{F}(\xi)\right)$$

and

$$\hat{F}^T(\xi) = \hat{F}(\xi) - \hat{F}_{||}(\xi).$$

Then

$$\begin{cases} ||\hat{F}_{||}||_{L^2} \leq ||\hat{F}||_{L^2} = ||F||_{L^2} \\ ||\hat{F}^T||_{L^2} \leq ||\hat{F}||_{L^2} = ||F||_{L^2}. \end{cases}$$

In addition,

$$\hat{F}_{||} \cdot \hat{F}^T = 0$$

and

$$\begin{cases} \sqrt{-1}\ \xi \times \hat{F}_{||}(\xi) = 0 \\ \sqrt{-1}\ \xi \cdot \hat{F}^T(\xi) = 0. \end{cases}$$

Denote the inverse transforms by $F_{||}$ and F^T -- then $F = F_{||} + F^T$ and

$$\begin{aligned} \langle F_{||}, F^T \rangle &= \int_{\underline{R}^3} F_{||} \cdot F^T dx \\ &= \int_{\underline{R}^3} \hat{F}_{||} \cdot \hat{F}^T d\xi \\ &= 0. \end{aligned}$$

And

$$\begin{cases} \nabla \times F_{||} = 0 \\ \nabla \cdot F^T = 0. \end{cases}$$

69.8 <u>LEMMA</u> The maps

$$\begin{cases} F \rightarrow F_{||} \\ F \rightarrow F^T \end{cases}$$

are the orthogonal projections of $L^2(\underline{R}^3;\underline{R}^3)$ onto

$$\begin{cases} L^2_{||}(\underline{R}^3;\underline{R}^3) \\ L^2(\underline{R}^3;\underline{R}^3)^T. \end{cases}$$

69.9 <u>REMARK</u> By definition,

$$(\widehat{F^T})_i(\xi) = \sum_j (\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2})\hat{F}_j(\xi)$$

or still,

$$(F^T)_i(x) = \sum_j \int_{\underline{R}^3} \delta^T_{ij}(x-y)F_j(y)dy,$$

where

$$\delta^T_{ij}(x) = \frac{1}{(2\pi)^3} \int_{\underline{R}^3} e^{\sqrt{-1}\, x\cdot\xi}(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2})d\xi$$

$$= \delta_{ij}\delta(x) + \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{(2\pi)^3} \int_{\underline{R}^3} e^{\sqrt{-1}\, x\cdot\xi} \frac{1}{|\xi|^2} d\xi$$

$$= \delta_{ij}\delta(x) + \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} .$$

Therefore

$$(F^T)_i(x) = \sum_j \delta_{ij}(\delta * F_j)(x) - \sum_j \frac{\partial^2}{\partial x_i \partial x_j} G * F_j(x)$$

$$= F_i(x) - \sum_j \frac{\partial^2}{\partial x_i \partial x_j} G * F_j(x).$$

Let $G_j = \partial_j G$ -- then the generalized Young inequality implies that

$$G_j * F_j \in L^6(\underline{R}^3) \qquad \text{(cf. 67.7)}.$$

Thus

$$\partial_i G_j * F_j = \partial_i (G_j * F_j),$$

so

$$(F^T)_i(x) = F_i(x) - \partial_i (\sum_j G_j * F_j)(x).$$

[Note: Without further ado, some authorities write

$$\sum_j G_j * F_j = \sum_j G * \partial_j F_j$$

$$= G * \sum_j \partial_j F_j$$

$$= G * \text{div } F.$$

But such a move requires justification and is a priori valid only under certain restrictions on the F_j.]

§70. *BEPPO LEVI SPACES*

Write $BL(\underline{R}^3)$ for the closure of $C_c^\infty(\underline{R}^3)$ w.r.t. the norm

$$||f||_{BL} = \left(\sum_{i=1}^{3} \int_{\underline{R}^3} \left|\frac{\partial f}{\partial x_i}\right|^2 dx\right)^{1/2}.$$

Then $BL(\underline{R}^3)$ is called the Beppo Levi space of level 1.

70.1 REMARK Write $BL_k(\underline{R}^3)$ for the closure of $C_c^\infty(\underline{R}^3)$ w.r.t. the norm

$$||f||_{BL_k} = \left(\sum_{|\alpha|=k} \int_{\underline{R}^3} |\partial^\alpha f|^2 dx\right)^{1/2}.$$

Then $BL_k(\underline{R}^3)$ is called the Beppo Levi space of level k.

[Note: As usual,

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial x_3}\right)^{\alpha_3},$$

and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.]

In what follows, we shall deal exclusively with the case $k = 1$.

N.B. By construction, $BL(\underline{R}^3)$ is a Hilbert space and

$$u \in BL(R^3) \Rightarrow \nabla u \in L^2(\underline{R}^3;\underline{R}^3).$$

70.2 LEMMA (Sobolev) $\exists\, C > 0$ such that $\forall\, f \in C_c^\infty(\underline{R}^3)$,

$$\int_{\underline{R}^3} f^6 dx \le C(\int_{\underline{R}^3} \nabla f \cdot \nabla f dx)^3.$$

Therefore

$$BL(\underline{R}^3) \subset L^6(\underline{R}^3).$$

70.3 REMARK Let T be a distribution on $\underline{R}^3$. Assume: $\frac{\partial T}{\partial x_i} \in L^2(\underline{R}^3)$ $(i = 1,2,3)$ — then $T \in L^6_{loc}(\underline{R}^3)$.

[Note: No global conclusion is possible (take T to be a constant).]

70.4 LEMMA If $u \in BL(\underline{R}^3)$ and if $\Delta u = 0$, then $u = 0$.

PROOF In fact,

$$\Delta(\frac{\partial u}{\partial x_i}) = \frac{\partial}{\partial x_i} \Delta u = 0 \quad (i = 1,2,3).$$

But

$$\frac{\partial u}{\partial x_i} \in L^2(\underline{R}^3) \Rightarrow \frac{\partial u}{\partial x_i} = 0 \quad \text{(cf. 67.2)}.$$

Therefore $\nabla u = 0$, thus u is a constant. However, $||u||_6 < \infty$, so $u = 0$.

70.5 LEMMA Let $U \in BL(\underline{R}^3;\underline{R}^3)$. Assume: $\nabla \cdot U = 0$ and $\nabla \times U = 0$ — then $U = 0$.

PROOF The hypotheses imply that $\Delta U = 0$ (cf. 68.2), hence $U = 0$ (cf. 70.4).

It has been shown above that $BL(\underline{R}^3)$ is contained in $L^6(\underline{R}^3)$ but more can be said.

70.6 LEMMA The Beppo Levi space $BL(\underline{R}^3)$ coincides with

$$\{u \in L^6(\underline{R}^3): \frac{\partial u}{\partial x_i} \in L^2(\underline{R}^3) \quad (i = 1,2,3)\}.$$

PROOF Denote the set in question by E and put

$$||u|| = ||u||_6 + (\sum_{i=1}^{3} \int_{\underline{R}^3} \left| \frac{\partial u}{\partial x_i} \right|^2 dx)^{1/2}.$$

Then $||\cdot||$ is a norm on E. Moreover, E is a Banach space containing $C_c^\infty(\underline{R}^3)$ as a dense subspace, which implies that $E = BL(\underline{R}^3)$.

70.7 RAPPEL We have

$$W^{2,1}(\underline{R}^3) \subset L^p(\underline{R}^3) \quad (2 \leq p \leq 6).$$

In particular:

$$W^{2,1}(\underline{R}^3) \subset L^6(\underline{R}^3).$$

Consequently, in view of 70.6,

$$W^{2,1}(\underline{R}^3) \subset BL(\underline{R}^3).$$

[Note: To argue directly, let $f \in W^{2,1}(\underline{R}^3)$ -- then $\nabla f \in L^2_{||}(\underline{R}^3;\underline{R}^3)$, hence $\exists$ a sequence $f_n \in C_c^\infty(\underline{R}^3)$ such that $\nabla f_n \to \nabla f$ in $L^2(\underline{R}^3;\underline{R}^3)$ (cf. 69.3). Meanwhile, $\exists\ u \in BL(\underline{R}^3):f_n \to u$. And $\nabla f = \nabla u \Rightarrow f = u + c$, c a constant. But

$$f,u \in L^6(\underline{R}^3) \Rightarrow c = 0 \Rightarrow f = u \Rightarrow f \in BL(\underline{R}^3).]$$

70.8 <u>REMARK</u> Let T be a distribution on $\underline{R}^3$. Assume: $\frac{\partial T}{\partial x_i} \in L^2(\underline{R}^3)$ $(i = 1,2,3)$ -- then $T \in L^6_{loc}(\underline{R}^3)$ (cf. 70.3) and, in light of the preceding considerations, $\exists\ u \in BL(\underline{R}^3)$:

$$T = u + c,$$

where c is some constant.

[Note: u and c are unique.]

Given $f \in L^2(\underline{R}^3)$, write U_f for the convolution

$$U_f(x) = \int_{\underline{R}^3} \frac{f(y)}{|x-y|^2}\,dy.$$

70.9 <u>LEMMA</u> $\forall\ f \in L^2(\underline{R}^3)$,

$$U_f \in L^6(\underline{R}^3).$$

<u>PROOF</u> Thanks to 67.5,

$$\frac{1}{r^2} \in L_w^{3/2}(\underline{R}^3).$$

So, upon application of the generalized Young inequality, we conclude that

$$||U_f||_6 \le C||f||_2 \quad \text{(cf. 67.7)}.$$

Given $f \in L^2(\underline{R}^3)$, define

$$R_i f \quad (i = 1,2,3)$$

by

$$R_i f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi^2} \int_{|x-y| \ge \varepsilon} \frac{x_i - y_i}{|x - y|^4} f(y) dy.$$

Then the R_i are bounded linear operators on $L^2(\underline{R}^3)$ and

$$\frac{\partial}{\partial x_i} U_f = - 2\pi^2 R_i f \quad (i = 1,2,3).$$

Therefore

$$U_f \in BL(\underline{R}^3) \quad \text{(cf. 70.6)}.$$

70.10 <u>LEMMA</u> $\forall f \in L^2(\underline{R}^3)$,

$$f = (\sqrt{-1})^2 \left(\sum_{i=1}^{3} R_i^2 f \right).$$

[Take Fourier transforms on both sides.]

70.11 <u>LEMMA</u> Let $f,g \in L^2(\underline{R}^3)$. Assume: $U_f = U_g$ -- then $f = g$.

<u>PROOF</u> For

$$U_f = U_g \Rightarrow \frac{\partial}{\partial x_i} U_f = \frac{\partial}{\partial x_i} U_g$$

$$\Rightarrow R_i f = R_i g$$

$$\Rightarrow R_i^2 f = R_i^2 g$$

$$\Rightarrow f = g \quad \text{(cf. 70.10)}.$$

Therefore the map

$$\begin{cases} L^2(\underline{R}^3) \to BL(\underline{R}^3) \\ f \to U_f \end{cases}$$

is injective.

Given $u \in BL(\underline{R}^3)$, put

$$Du = \sum_{i=1}^{3} R_i\left(\frac{\partial u}{\partial x_i}\right).$$

Then $Du \in L^2(\underline{R}^3)$.

70.12 LEMMA $\forall\ f \in C_c^\infty(\underline{R}^3)$,

$$R_i(Df) = -\frac{\partial f}{\partial x_i} \quad (i = 1,2,3).$$

[Take Fourier transforms on both sides.]

70.13 LEMMA The map

$$\begin{cases} L^2(\underline{R}^3) \to BL(\underline{R}^3) \\ f \to U_f \end{cases}$$

is bijective.

PROOF The issue is surjectivity. Fix $u \in BL(\underline{R}^3)$ and let

$$f = \frac{1}{2\pi^2} Du.$$

Then $f \in L^2(\underline{R}^3)$ and the claim is that $U_f = u$. With the understanding that $i = 1,2,3,$ choose a sequence $f_n \in C_c^\infty(\underline{R}^3)$:

$$\frac{\partial f_n}{\partial x_i} \xrightarrow[L^2]{} \frac{\partial u}{\partial x_i}$$

and in the relation

$$R_i(Df_n) = - \frac{\partial f_n}{\partial x_i} \qquad \text{(cf. 70.12)},$$

let $n \to \infty$ to get

$$2\pi^2 R_i f = - \frac{\partial u}{\partial x_i} .$$

But

$$\frac{\partial}{\partial x_i} U_f = - 2\pi^2 R_i f.$$

Therefore

$$\frac{\partial}{\partial x_i} U_f = \frac{\partial u}{\partial x_i}$$

or still,

$$\nabla(U_f - u) = 0$$

$$\Rightarrow$$

$$U_f = u.$$

70.14 REMARK Inspection of the foregoing shows that $\exists\ C > 0 : \forall\ f \in L^2(\underline{R}^3)$,

$$C^{-1}||f||_2 \leq ||U_f||_{BL} \leq C||f||_2.$$

70.15 LEMMA (Stein-Weiss) $\forall\ f \in L^2(\underline{R}^3)$,

$$\int_{\underline{R}^3} \frac{|U_f(x)|^2}{(1 + |x|)^2}\, dx < \infty.$$

[Note: Suppose that $f \in C_c^\infty(\underline{R}^3)$ -- then

$$2 \int_{\underline{R}^3} \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(x) f(x) \frac{x_i}{|x|^2}\, dx$$

$$= \int_{\underline{R}^3} \sum_{i=1}^{3} \frac{\partial f^2}{\partial x_i}(x) \frac{x_i}{|x|^2}\, dx$$

$$= - \int_{\underline{R}^3} \frac{f(x)^2}{|x|^2}\, dx$$

$\Rightarrow$

$$\int_{\underline{R}^3} \frac{f(x)^2}{|x|^2}\, dx$$

$$\leq 2\left(\int_{\underline{R}^3} \frac{f(x)^2}{|x|^2} \sum_{i=1}^{3} \frac{x_i^2}{|x|^2}\, dx\right)^{1/2} \left(\int_{\underline{R}^3} \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i}\right)^2 dx\right)^{1/2}$$

=>

$$\int_{\underline{R}^3} \frac{f(x)^2}{|x|^2} dx \le 4 \int_{\underline{R}^3} \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i}\right)^2 dx.]$$

Denote now by $W_{-1}^1(\underline{R}^3)$ the set of locally integrable functions f on $\underline{R}^3$ such that

$$\frac{f}{(1 + r^2)^{1/2}} \in L^2(\underline{R}^3)$$

and

$$\frac{\partial f}{\partial x_i} \in L^2(\underline{R}^3) \quad (i = 1,2,3).$$

Then $W_{-1}^1(\underline{R}^3)$ is a so-called weighted Sobolev space (see below) and

$$BL(\underline{R}^3) \subset W_{-1}^1(\underline{R}^3).$$

70.16 LEMMA (Lizorkin) We have

$$BL(\underline{R}^3) = W_{-1}^1(\underline{R}^3).$$

PROOF Let $f \in W_{-1}^1(\underline{R}^3)$ -- then $\exists\ u \in BL(\underline{R}^3)$ and a constant c:

$$f = u + c \quad (\text{cf. } 70.8).$$

But

$$\int_{\underline{R}^3} \frac{|f(x) - u(x)|^2}{(1 + |x|)^2} dx < \infty$$

$$\Rightarrow c = 0.$$

Therefore $f \in BL(\underline{R}^3)$.

70.17 REMARK Let

$$\sigma(x) = (1 + |x|)^2 \quad (x \in \underline{R}^3).$$

Fix $k \in \underline{Z}_{\geq 0}$, $\delta \in \underline{R}$ — then the weighted Sobolev space $W_\delta^k(\underline{R}^3)$ attached to k,δ is the Hilbert space consisting of those locally integrable functions $f:\underline{R}^3 \to \underline{R}$ possessing locally integrable distributional derivatives up to order k such that

$$||f||_{W_\delta^k} = \Big(\sum_{|\alpha| \leq k} \int_{\underline{R}^3} \sigma^{2(\delta + |\alpha|)} |\partial^\alpha f|^2 dx\Big)^{1/2} < \infty.$$

[Note: $C_c^\infty(\underline{R}^3)$ is dense in $W_\delta^k(\underline{R}^3)$.]

70.18 LEMMA (Poincaré Inequality) Suppose that $\delta > -\frac{3}{2}$ — then $\exists\ C > 0$ such that $\forall\ f \in W_\delta^1(\underline{R}^3)$,

$$\int_{\underline{R}^3} |f|^2 \sigma^{2\delta} dx \leq C \int_{\underline{R}^3} |\text{grad } f|^2 \sigma^{2(\delta + 1)} dx.$$

[Note: Take $\delta = -1$ to get

$$\int_{\underline{R}^3} |f|^2 \sigma^{-2} dx \leq C \int_{\underline{R}^3} |\text{grad } f|^2 dx.]$$

Our next objective will be to establish an analog of 69.5 with $L^2(\underline{R}^3;\underline{R}^3)$ replaced by $BL(\underline{R}^3;\underline{R}^3)$.

Put

$$BL_{||}(\underline{R}^3;\underline{R}^3) = \{U \in BL(\underline{R}^3;\underline{R}^3) : \nabla \times U = 0\}.$$

70.19 <u>LEMMA</u> $BL_{||}(\underline{R}^3;\underline{R}^3)$ is the closure in $BL(\underline{R}^3;\underline{R}^3)$ of $\{\nabla f: f \in C_c^\infty(\underline{R}^3)\}$.

<u>PROOF</u> Suppose that $U \in BL_{||}(\underline{R}^3;\underline{R}^3)$ and

$$U \perp \{\nabla f: f \in C_c^\infty(\underline{R}^3)\}.$$

Then $\forall\ f \in C_c^\infty(\underline{R}^3)$,

$$0 = \langle \text{grad } U, \text{grad } \nabla f\rangle$$

or still,

$$0 = \int_{\underline{R}^3} (\nabla\cdot U)(\nabla\cdot(\nabla f))dx$$

$$+ \int_{\underline{R}^3} \langle\nabla \times U, \nabla \times \nabla f\rangle dx \quad \text{(cf. 69.2)}$$

or still,

$$0 = \int_{\underline{R}^3} (\nabla\cdot U)(\nabla\cdot(\nabla f))dx$$

or still,

$$0 = \int_{\underline{R}^3} \langle U, \nabla(\nabla\cdot(\nabla f))\rangle dx$$

or still,

$$0 = \int_{\underline{R}^3} \langle U, \nabla \times (\nabla \times \nabla f) + \Delta\nabla f\rangle dx \quad \text{(cf. 68.2)}$$

or still,

$$0 = \int_{\underline{R}^3} \langle\nabla \times U, \nabla \times \nabla f\rangle dx$$

$$+ \int_{\underline{R}^3} \langle U, \Delta\nabla f\rangle dx$$

or still,

$$0 = \int_{\underline{R}^3} \langle U, \Delta\nabla f\rangle dx$$

or still,

$$0 = \int_{\underline{R}^3} \langle U, \nabla\Delta f\rangle dx$$

or still,

$$0 = \int_{\underline{R}^3} (\nabla\cdot U)\Delta f dx$$

$$\Rightarrow$$

$$\Delta(\nabla\cdot U) = 0$$

$$\Rightarrow$$

$$\nabla\cdot U = 0 \quad (\text{cf. } 67.2).$$

Therefore $U = 0$ (cf. 70.5).

Put

$$BL(\underline{R}^3;\underline{R}^3)^T = \{U \in BL(\underline{R}^3;\underline{R}^3) : \nabla\cdot U = 0\}.$$

70.20 LEMMA $BL(\underline{R}^3;\underline{R}^3)^T$ is the closure in $BL(\underline{R}^3;\underline{R}^3)$ of $\{\nabla \times X : X \in C_c^\infty(\underline{R}^3;\underline{R}^3)\}$.

PROOF Suppose that $U \in BL(\underline{R}^3;\underline{R}^3)^T$ and

$$U \perp \{\nabla \times X : X \in C_c^\infty(\underline{R}^3;\underline{R}^3)\}.$$

Then $\forall\, X \in C_c^\infty(\underline{R}^3;\underline{R}^3)$,

$$0 = \langle \text{grad } U, \text{grad}(\nabla \times X)\rangle$$

or still,

$$0 = \int_{\underline{R}^3} (\nabla\cdot U)(\nabla\cdot(\nabla \times X))dx$$

$$+ \int_{\underline{R}^3} <\nabla \times U, \nabla \times (\nabla \times X)>dx \quad \text{(cf. 69.2)}$$

or still,

$$0 = \int_{\underline{R}^3} <\nabla \times U, \nabla \times (\nabla \times X)>dx$$

or still,

$$0 = \int_{\underline{R}^3} <\nabla \times U, \nabla(\nabla\cdot X) - \Delta X>dx \quad \text{(cf. 68.2)}$$

or still,

$$0 = \int_{\underline{R}^3} <U, \nabla \times \nabla(\nabla\cdot X)>dx$$

$$- \int_{\underline{R}^3} <\nabla \times U, \Delta X>dx$$

or still,

$$0 = \int_{\underline{R}^3} <\nabla \times U, \Delta X>dx$$

=>

$$\Delta(\nabla \times U) = 0$$

=>

$$\nabla \times U = 0 \quad \text{(cf. 67.2)}.$$

Therefore $U = 0$ (cf. 70.5).

70.21 <u>THEOREM</u> (Helmholtz) There is an orthogonal decomposition

$$BL(\underline{R}^3;\underline{R}^3) = BL_{||}(\underline{R}^3;\underline{R}^3) \oplus BL(\underline{R}^3;\underline{R}^3)^T.$$

PROOF It is clear that

$$BL_{||}(\underline{R}^3;\underline{R}^3) \perp BL(\underline{R}^3;\underline{R}^3)^T.$$

On the other hand, if U is orthogonal to

$$\{\nabla f : f \in C_c^\infty(\underline{R}^3)\}$$

and

$$\{\nabla \times X : X \in C_c^\infty(\underline{R}^3;\underline{R}^3)\},$$

then by the above, $\nabla \cdot U = 0$ and $\nabla \times U = 0$, hence $U = 0$ (cf. 70.5). Therefore $BL_{||}(\underline{R}^3;\underline{R}^3)$ and $BL(\underline{R}^3;\underline{R}^3)^T$ span $BL(\underline{R}^3;\underline{R}^3)$.

Let $\mathcal{T}$ be the set of distributions T on $\underline{R}^3$ such that $\frac{\partial T}{\partial x_i} \in L^2(\underline{R}^3)$ $(i = 1,2,3)$ (cf. 70.3).

70.22 LEMMA The image of $\mathcal{T}$ under the arrow

$$\begin{cases} \mathcal{T} \to L^2(\underline{R}^3;\underline{R}^3) \\ T \to \nabla T \end{cases}$$

is $L^2_{||}(\underline{R}^3;\underline{R}^3)$.

[Note: Therefore

$$\text{grad } BL(\underline{R}^3) = L^2_{||}(\underline{R}^3;\underline{R}^3) \quad \text{(cf. 70.8)}.$$

Observe too that

$$\text{grad}: BL(\underline{R}^3) \to L^2_{||}(\underline{R}^3;\underline{R}^3)$$

is norm perserving.]

70.23 RAPPEL Define $\rho \in C_c^\infty(\underline{R}^3)$ by

$$\rho(x) = \begin{cases} C \exp(-\frac{1}{1 - |x|^2}) & (|x| < 1) \\ 0 & (|x| \geq 1). \end{cases}$$

Here

$$C = (\int_{|x|<1} \exp(-\frac{1}{1 - |x|^2}) dx)^{-1},$$

thus

$$\int_{\underline{R}^3} \rho(x) dx = 1.$$

Given $t > 0$, define

$$\rho_t : \underline{R}^3 \to \underline{R}$$

by

$$\rho_t(x) = t^{-3} \rho(\frac{x}{t}).$$

Then

$$\operatorname{spt} \rho_t = \{x \in \underline{R}^3 : |x| \leq t\}$$

and

$$\int_{\underline{R}^3} \rho_t(x) dx = 1.$$

Passing to the proof of 70.22, suppose that

$$\int_{\underline{R}^3} \langle F, \Pi \rangle dx = 0$$

for all $\Pi \in L^2(\underline{R}^3;\underline{R}^3)^T$. Specialize and take

$$\Pi = \nabla \times (\rho_t * X) \quad (= \rho_t * (\nabla \times X)),$$

where $X \in C_c^\infty(\underline{R}^3;\underline{R}^3)$ -- then

$$0 = \int_{\underline{R}^3} \langle F, \rho_t * (\nabla \times X) \rangle dx$$

$$= \int_{R^3} \langle \rho_t * F, \nabla \times X \rangle dx$$

$$= \int_{R^3} \langle \nabla \times (\rho_t * F), X \rangle dx$$

=>

$$\nabla \times (\rho_t * F) = 0,$$

X being arbitrary. Now define ϕ_t by the line integral

$$\phi_t(x) = \int_0^x \rho_t * F$$

to get

$$\text{grad } \phi_t = \rho_t * F.$$

Consideration of the limit as $t \to 0$ finishes the argument.

70.24 LEMMA The image of $\mathcal{T}^3$ under the arrow

$$\begin{cases} \mathcal{T}^3 \to L^2(\underline{R}^3;\underline{R}^3) \\ T \to \nabla \times T \end{cases}$$

is $L^2(\underline{R}^3;\underline{R}^3)^T$.

PROOF Given $F \in L^2(\underline{R}^3;\underline{R}^3)^T$, put

$$\mathcal{F}_F(\xi) = |\xi|^{-2}(\sqrt{-1}\,\xi \times \hat{F}(\xi)).$$

Then

$$\nabla\cdot F = 0 \Rightarrow \sqrt{-1}\,\xi\cdot\hat{F}(\xi) = 0$$

=>

$$|\mathcal{F}_F(\xi)|^2 = |\xi|^{-4}\,|\sqrt{-1}\,\xi \times \hat{F}(\xi)|^2$$

$$= |\xi|^{-4}(|\sqrt{-1}\,\xi|^2|\hat{F}(\xi)|^2 - (\sqrt{-1}\,\xi\cdot\hat{F}(\xi))^2)$$

$$= |\xi|^{-2}|\hat{F}(\xi)|^2$$

=>

$$|\mathcal{F}_F(\xi)| = |\xi|^{-1}|\hat{F}(\xi)|$$

=>

$$\begin{cases} \int_{|\xi|\le 1} |\mathcal{F}_F(\xi)|\,d\xi < \infty \\ \int_{|\xi|>1} |\mathcal{F}_F(\xi)|^2 d\xi < \infty \end{cases}$$

=>

$$\mathcal{F}_F = \hat{T}_F,$$

where T_F is tempered.

• $\sqrt{-1}\ \xi\cdot\hat{T}_F(\xi)$

$$= \sqrt{-1}\ \xi\cdot|\xi|^{-2}(\sqrt{-1}\ \xi \times \hat{F}(\xi))$$

$$= |\xi|^{-2}(\sqrt{-1}\ \xi \times \sqrt{-1}\ \xi)\cdot\hat{F}(\xi)$$

$$= 0.$$

• $\sqrt{-1}\ \xi \times \hat{T}_F(\xi)$

$$= |\xi|^{-2}(\sqrt{-1}\ \xi \times (\sqrt{-1}\ \xi \times \hat{F}(\xi)))$$

$$= |\xi|^{-2}((\sqrt{-1}\ \xi\cdot\hat{F}(\xi))\sqrt{-1}\ \xi - (\sqrt{-1}\ \xi\cdot\sqrt{-1}\ \xi)\hat{F}(\xi))$$

$$= \hat{F}(\xi).$$

Therefore

$$|\hat{F}(\xi)|^2 = \overline{\xi \times \hat{T}_F(\xi)}\cdot\xi \times \hat{T}_F(\xi)$$

$$= (\xi\cdot\xi)(\overline{\hat{T}_F(\xi)}\cdot\hat{T}_F(\xi)) - (\xi\cdot\hat{T}_F(\xi))(\overline{\hat{T}_F(\xi)}\cdot\xi)$$

$$= (\xi\cdot\xi)(\overline{\hat{T}_F(\xi)}\cdot\hat{T}_F(\xi)).$$

So, if

$$T_F = (T_{F,1}, T_{F,2}, T_{F,3}),$$

then

$$\xi_i \hat{T}_{F,j} \in L^2(\underline{R}^3)$$

=>

$$\frac{\partial}{\partial x_i} T_{F,j} \in L^2(\underline{R}^3)$$

=>

$$T_F \in \mathcal{T}^3.$$

And

$$\begin{cases} \nabla \cdot T_F = 0 \\ \nabla \times T_F = F. \end{cases}$$

The construction in 70.24 defines a linear map

$$\begin{cases} L^2(\underline{R}^3;\underline{R}^3)^T \rightarrow \mathcal{T}^3 \\ \qquad\qquad F \rightarrow T_F. \end{cases}$$

Determine

$$\begin{cases} U_F \in BL(\underline{R}^3;\underline{R}^3) \\ c_F \in \underline{R}^3 \end{cases}$$

per 70.8 (thus $T_F = U_F + c_F$) -- then

$$\nabla \cdot U_F = 0 \Rightarrow U_F \in BL(\underline{R}^3;\underline{R}^3)^T$$

and, of course

$$\nabla \times U_F = F.$$

So we have an arrow

$$\begin{cases} L^2(\underline{R}^3;\underline{R}^3)^T \to BL(\underline{R}^3;\underline{R}^3)^T \\ F \to U_F \end{cases}$$

which is norm preserving and surjective:

$$U \in BL(\underline{R}^3;\underline{R}^3)^T \Rightarrow \nabla \times U \in L^2(\underline{R}^3;\underline{R}^3)^T.$$

And

$$U_{\nabla \times U} = U.$$

In fact,

$$\begin{aligned} F_{\nabla \times U}(\xi) &= |\xi|^{-2}(\sqrt{-1}\ \xi \times (\nabla \hat{\times} U)(\xi)) \\ &= |\xi|^{-2}(\sqrt{-1}\ \xi \times (\sqrt{-1}\ \xi \times \hat{U}(\xi))) \\ &= |\xi|^{-2}((\sqrt{-1}\ \xi \cdot \hat{U}(\xi))\sqrt{-1}\ \xi - (\sqrt{-1}\ \xi \cdot \sqrt{-1}\ \xi)\hat{U}(\xi)) \\ &= \hat{U}(\xi). \end{aligned}$$

70.25 <u>THEOREM</u> (Schmidt) There is an orthogonal decomposition

$$L^2(\underline{R}^3;\underline{R}^3) = \text{grad } BL(\underline{R}^3) \oplus \text{curl } BL(\underline{R}^3;\underline{R}^3)^T.$$

[Combine 69.5, 70.22, 70.24, and subsequent discussion.]

70.26 REMARK This result implies the L^2-version of the Poincaré lemma.

- Let $F \in L^2(\underline{R}^3;\underline{R}^3)$. Assume: $\nabla \times F = 0$ -- then $\exists\, u \in BL(\underline{R}^3)$ such that

$$\text{grad } u = F.$$

- Let $F \in L^2(\underline{R}^3;\underline{R}^3)$. Assume: $\nabla \cdot F = 0$ -- then $\exists\, U \in BL(\underline{R}^3;\underline{R}^3)$ such that

$$\nabla \times U = F.$$

70.27 LEMMA We have

$$\text{div } BL(\underline{R}^3;\underline{R}^3) = L^2(\underline{R}^3).$$

PROOF The image

$$\text{div } BL(\underline{R}^3;\underline{R}^3)$$

is a closed subspace of $L^2(\underline{R}^3)$. If

$$f_0 \perp \text{div } BL(\underline{R}^3;\underline{R}^3),$$

then $\forall\, f \in C_c^\infty(\underline{R}^3)$,

$$0 = \int_{\underline{R}^3} f_0 (\text{div } \nabla f)\, dx$$

$$= \int_{\underline{R}^3} f_0 (\Delta f)\, dx$$

=>

$$\Delta f_0 = 0$$

=>

$$f_0 = 0 \quad (\text{cf. } 67.2).$$

• In

$$L^2(\underline{R}^3;\underline{R}^3)^T \oplus L^2(\underline{R}^3;\underline{R}^3)^T,$$

let

$$A = \begin{bmatrix} 0 & \nabla \times \\ & \\ -\nabla \times & 0 \end{bmatrix}$$

with Dom(A) consisting of the pairs (F_1,F_2):

$$(\nabla \times F_2, -\nabla \times F_1) \in L^2(\underline{R}^3;\underline{R}^3)^T \oplus L^2(\underline{R}^3;\underline{R}^3)^T.$$

• In

$$BL(\underline{R}^3;\underline{R}^3)^T \oplus L^2(\underline{R}^3;\underline{R}^3)^T,$$

let

$$X = \begin{bmatrix} 0 & I \\ & \\ \Delta & 0 \end{bmatrix}$$

with Dom(X) consisting of the pairs (U,F):

$$(F,\Delta U) \in BL(\underline{R}^3;\underline{R}^3)^T \oplus L^2(\underline{R}^3;\underline{R}^3)^T.$$

Given $F \in L^2(\underline{R}^3;\underline{R}^3)^T$, put

$$\zeta(F) = U_F.$$

Then

$$\zeta : L^2(\underline{R}^3;\underline{R}^3)^T \to BL(\underline{R}^3;\underline{R}^3)^T$$

is an isometric isomorphism such that

$$\begin{cases} \nabla \times \zeta(F) = F \\ \zeta(\nabla \times U) = U. \end{cases}$$

70.28 <u>LEMMA</u> The arrow

$$\zeta \oplus I: L^2(\underline{R}^3;\underline{R}^3)^T \oplus L^2(\underline{R}^3;\underline{R}^3)^T \rightarrow BL(\underline{R}^3;\underline{R}^3)^T \oplus L^2(\underline{R}^3;\underline{R}^3)^T$$

sends Dom(A) onto Dom(X) and

$$(\zeta \oplus I)A(\zeta \oplus I)^{-1} = X.$$

<u>PROOF</u> Suppose that $(F_1,F_2) \in \text{Dom}(A)$. Let $U_1 = \zeta F_1$ -- then $F_1 = \nabla \times U_1$ and we claim that

$$\begin{cases} \Delta U_1 \in L^2(\underline{R}^3;\underline{R}^3)^T \\ F_2 \in BL(\underline{R}^3;\underline{R}^3)^T. \end{cases}$$

In fact,

$$\Delta U_1 = \nabla(\nabla \cdot U_1) - \nabla \times (\nabla \times U_1) \quad \text{(cf. 68.2)}$$

$$= - \nabla \times (\nabla \times U_1) \quad (\nabla \cdot U_1 = 0)$$

$$= - \nabla \times F_1 \in L^2(\underline{R}^3;\underline{R}^3)^T.$$

On the other hand,

$$\left[\begin{array}{l} F_2 \in L^2(\underline{R}^3;\underline{R}^3)^T \\ \nabla \times F_2 \in L^2(\underline{R}^3;\underline{R}^3)^T \\ \nabla \cdot F_2 = 0 \end{array}\right.$$

=>

$$\int_{\underline{R}^3} |\text{grad } F_2|^2 dx < \infty \quad \text{(cf. 69.2)}$$

=>

$$F_2 \in W^{2,1}(\underline{R}^3;\underline{R}^3)^T \subset BL(\underline{R}^3;\underline{R}^3)^T.$$

Therefore

$$(\zeta \oplus I)\text{Dom}(A) \subset \text{Dom}(X).$$

That

$$(\zeta \oplus I)\text{Dom}(A) = \text{Dom}(X)$$

follows upon reversing the steps. Finally,

$$\begin{aligned} &(\zeta \oplus I)A(\zeta \oplus I)^{-1}(U,F) \\ &\quad = (\zeta \oplus I)A(\nabla \times U,F) \\ &\quad = (\zeta \oplus I)(\nabla \times F, - \nabla \times \nabla \times U) \\ &\quad = (F,\Delta U) \\ &\quad = X(U,F). \end{aligned}$$

70.29 <u>REMARK</u>

- If

$$(F_1(t),F_2(t)) \in L^2(\underline{R}^3;\underline{R}^3)^T \oplus L^2(\underline{R}^3;\underline{R}^3)^T,$$

then Maxwell's equations are encoded by

$$\begin{bmatrix} \dot{F}_1 \\ \dot{F}_2 \end{bmatrix} = \begin{bmatrix} 0 & \nabla \times \\ -\nabla \times & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

- If

$$(U(t),F(t)) \in BL(\underline{R}^3;\underline{R}^3)^T \oplus L^2(\underline{R}^3;\underline{R}^3)^T,$$

then the wave equation is encoded by

$$\begin{bmatrix} \dot{U} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} U \\ F \end{bmatrix}.$$

Therefore 70.28 provides a connection between Maxwell's equations and the wave equation.

According to 70.21, there is an orthogonal decomposition

$$BL(\underline{R}^3;\underline{R}^3) = BL_{||}(\underline{R}^3;\underline{R}^3) \oplus BL(\underline{R}^3;\underline{R}^3)^T.$$

Let $U_1,U_2 \in BL(\underline{R}^3;\underline{R}^3)$ -- then U_1,U_2 are said to be <u>gauge equivalent</u>, written $U_1 \sim U_2$, if $U_1 - U_2 \in BL_{||}(\underline{R}^3;\underline{R}^3)$.

70.30 <u>LEMMA</u> The map

$$\begin{cases} BL(R^3;R^3)/\sim \to BL(R^3;R^3)^T \\ [U] \to U^T \end{cases}$$

is a welldefined bijection (cf. 66.9).

<u>Definition</u> The <u>physical phase space</u> of Maxwell theory in $\underline{R}^3$ is

$$BL(\underline{R}^3;\underline{R}^3)^T \oplus L^2(\underline{R}^3;\underline{R}^3)^T.$$

[Note: The underlying hamiltonian is the function

$$(U,F) \to \frac{1}{2}\int_{\underline{R}^3} (||F||^2 + ||\nabla \times U||^2)dx.]$$

APPENDIX: HERMITE POLYNOMIALS

There is no universally agreed to convention for their definition, so it's necessary to make a choice and stick with it.

Put

$$H_0(x) = 1$$

and

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \quad (n \geq 1).$$

<u>Generating Function</u>

$$e^{zx - \frac{1}{2} z^2} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x)$$

<u>Explicit Formulas</u>

$$\begin{bmatrix} H_n(x) = n! \displaystyle\sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{2^k k!(n-2k)!} \\ x^n = n! \displaystyle\sum_{k=0}^{[n/2]} \frac{H_{n-2k}(x)}{2^k k!(n-2k)!} \end{bmatrix}$$

<u>Recursion Relation</u>

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x) \quad (n \geq 1)$$

<u>Derivative</u>

$$H_n'(x) = nH_{n-1}(x) \quad (n \geq 1)$$

Differential Equation

$$H_n''(x) - xH_n'(x) + nH_n(x) = 0$$

Multiplication Formula

$$H_m(x)H_n(x) = \sum_{k=0}^{m\wedge n} k! \binom{m}{k} \binom{n}{k} H_{m+n-2k}(x)$$

Algebraic Relations

$$\left[\begin{array}{l} H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^k H_{n-k}(y) \\ H_n(\lambda x) = n! \sum_{k=0}^{[n/2]} \frac{(\lambda^2-1)^k \lambda^{n-2k}}{2^k k! (n-2k)!} H_{n-2k}(x) \end{array}\right.$$

Orthogonality

$$\frac{1}{\sqrt{2\pi}} \int_{\underline{R}} \frac{H_m(x)}{\sqrt{m!}} \frac{H_n(x)}{\sqrt{n!}} e^{-x^2/2}\, dx = \delta_{mn}$$

Integral Representation

$$H_n(x) = \frac{1}{\sqrt{2\pi}} \int_{\underline{R}} (x \pm \sqrt{-1}\, y)^n e^{-y^2/2}\, dy$$

Mehler Kernel Formula

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)H_n(y)$$

$$= \frac{1}{\sqrt{1-t^2}} \exp\left(- \frac{t^2x^2-2txy+t^2y^2}{2(1-t^2)}\right) \quad (|t| < 1)$$

Let $d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ — then the polynomials $\frac{H_k(x)}{\sqrt{k!}}$ $(k \geq 0)$ are an orthonormal basis for $L^2(\underline{R},\gamma)$. In the applications, it is also important to consider

$$d\mu(z) = \frac{1}{\pi} e^{-|z|^2}\, dz \quad (= \frac{1}{\pi} e^{-x^2-y^2}\, dxdy)$$

and this leads to the introduction of another set of polynomials which then form an orthonormal basis for $L^2(\underline{C},\mu)$.

Put

$$H_{0,0}(z,\bar{z}) = 1$$

and

$$H_{a,b}(z,\bar{z}) = (-1)^{a+b} e^{z\bar{z}} \frac{\partial^{a+b}}{\partial\bar{z}^a \partial z^b} e^{-z\bar{z}} \quad (a \geq 0, b \geq 0, a\wedge b \geq 1).$$

<u>Generating Function</u>

$$e^{-\zeta\bar{\zeta} + \zeta\bar{z} + \bar{\zeta}z} = \sum_{a,b=0}^{\infty} \frac{\bar{\zeta}^a \zeta^b}{a!b!} H_{a,b}(z,\bar{z})$$

<u>Explicit Formula</u>

$$H_{a,b}(z,\bar{z}) = \sum_{k=0}^{a\wedge b} (-1)^k \frac{a!b!}{k!(a-k)!(b-k)!} z^{a-k} \bar{z}^{b-k}$$

In particular:

$$\begin{cases} H_{a,0}(z,\bar{z}) = z^a \\ H_{0,b}(z,\bar{z}) = \bar{z}^b. \end{cases}$$

Conjugation Relation

$$H_{a,b}(z,\bar{z}) = \overline{H_{b,a}(z,\bar{z})}$$

Recursion Relation

$$\begin{cases} H_{a+1,b}(z,\bar{z}) = zH_{a,b}(z,\bar{z}) - bH_{a,b-1}(z,\bar{z}) \\ H_{a,b+1}(z,\bar{z}) = \bar{z}H_{a,b}(z,\bar{z}) - aH_{a-1,b}(z,\bar{z}) \end{cases}$$

Derivative

$$\begin{cases} \frac{\partial}{\partial z} H_{a,b}(z,\bar{z}) = aH_{a-1,b}(z,\bar{z}) \\ \frac{\partial}{\partial \bar{z}} H_{a,b}(z,\bar{z}) = bH_{a,b-1}(z,\bar{z}) \end{cases}$$

Differential Equation

$$\begin{cases} \frac{\partial^2}{\partial z \partial \bar{z}} H_{a,b}(z,\bar{z}) - \bar{z} \frac{\partial}{\partial \bar{z}} H_{a,b}(z,\bar{z}) + bH_{a,b}(z,\bar{z}) = 0 \\ \frac{\partial^2}{\partial \bar{z} \partial z} H_{a,b}(z,\bar{z}) - z \frac{\partial}{\partial z} H_{a,b}(z,\bar{z}) + aH_{a,b}(z,\bar{z}) = 0 \end{cases}$$

Orthogonality

$$\frac{1}{\pi} \int_{\underline{C}} \frac{1}{\sqrt{a!b!}} \overline{H_{a,b}(z,\bar{z})} \frac{1}{\sqrt{c!d!}} H_{c,d}(z,\bar{z}) e^{-|z|^2} dz = \delta_{ac}\delta_{bd}$$

Integral Representation

$$H_{a,b}(z,\bar{z}) = \frac{1}{\pi} \int_{\underline{C}} (z + \sqrt{-1}\, w)^a \, (\bar{z} + \sqrt{-1}\, \bar{w})^b \, e^{-|w|^2} \, dw$$

REMARK The H_n and the $H_{a,b}$ are connected by the following identities:

- $$H_{a,b}(z,\bar{z}) = \frac{1}{2^{a+b}} \sum_{\ell=0}^{a+b} \sum_{k=0 \vee (\ell-b)}^{a \wedge \ell} (-1)^k \, (\sqrt{-1})^{\ell}$$

$$\times \binom{a}{k} \binom{b}{\ell-k} H_{a+b-\ell}(x) \, H_{\ell}(y);$$

- $$H_a(x) H_b(y) = \sum_{\ell=0}^{a+b} \sum_{k=(\ell-a) \vee 0}^{\ell \wedge a} (-1)^k \, (\sqrt{-1})^{\ell}$$

$$\times \binom{a}{\ell-k} \binom{b}{k} H_{\ell, a+b-\ell} (z,\bar{z}).$$

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