**Solutions to the first midterm**

**Math 134**

**Problem 1.** Let $r$ be a rational number, $r > 1$. Let $f$ be defined by

$$f(x) = (1+x)^r - (1+rx) \quad \text{for} \quad x \geq -1.$$

**1.1** Find the critical points of $f$. Determine the intervals where $f$ is increasing and where $f$ is decreasing.

**Solution:** $f$ is differentiable on $(-1, +\infty)$ and

$$f'(x) = r(1+x)^{r-1} - r \quad \text{for} \quad x > -1.$$

If $f'(x) = 0$ then $(1+x)^{r-1} - 1 = 0$ which implies that $x = 0$ is a critical point. For $-1 < x < 0$, $0 < (1+x)^{r-1} < 1$, therefore $f'(x) < 0$, and $f$ is decreasing in $(-1, 0)$. For $x > 0$, $(1+x)^{r-1} > 1$, thus $f'(x) > 0$ and $f$ is increasing on $(0, +\infty)$. Hence $x = 0$ is a minimum of $f$.

**1.2** Determine the concavity of the graph of $f$.

**Solution:** $f'$ is differentiable in $(-1, +\infty)$ and

$$f''(x) = r(r-1)(1+x)^{r-2} \quad \text{for} \quad x > -1.$$

For $x > -1$, $f''(x) > 0$ and $f$ is concave up on $(-1, +\infty)$.

**1.3** Determine the value of $f$ at $-1$. Determine the behavior of $f(x)$ as $x$ approaches $+\infty$.

**Solution:** $f(-1) = r - 1$. In order to determine the behavior of $f(x)$ as $x$ approaches $+\infty$ note that for $x > 0$

$$f(x) = x^r \left(1 + \frac{1}{x}\right)^r - x\left(r + \frac{1}{x}\right) = x^r \left[\left(1 + \frac{1}{x}\right)^r - \frac{1}{x^{r-1}}\left(r + \frac{1}{x}\right)\right].$$

Note that

$$\left(1 + \frac{1}{x}\right)^r \to 1 \quad \text{as} \quad x \to +\infty,$$

and since $r - 1 > 0$, then

$$\frac{1}{x^{r-1}}\left(r + \frac{1}{x}\right) \to 0 \quad \text{as} \quad x \to +\infty.$$

Thus

$$x^r \left[ \left(1 + \frac{1}{x}\right)^r - \frac{1}{x^{r-1}} \left(r + \frac{1}{x}\right) \right] \to +\infty \quad \text{as} \quad x \to +\infty.$$

**Problem 2.** Let $f : [0,1] \to \mathbf{R}$. Suppose that the function $f$ has the following property

$$|f(x) - f(y)| \leq (x-y)^2 \quad \text{for all} \quad x, y \in [0,1].$$

**2.1** Show that the function $f$ is continuous on [0,1].

**Solution:** Option1: Let $c \in [0,1]$. Given $\epsilon > 0$, let $\delta = \sqrt{\epsilon}/\sqrt{2}$ then for $y \in [0,1]$ such that $|c-y| < \delta$ then

$$|f(c) - f(y)| \leq (c-y)^2 < \delta^2 = \frac{\epsilon}{2} < \epsilon.$$

This proves that $f$ is continuous at each point $c \in [0,1]$.

Option2: Let $c \in (0,1)$, then by hypothesis

$$-(c-y)^2 \leq f(c) - f(y) \leq (c-y)^2.$$

Hence

$$f(c) - (c-y)^2 \leq f(y) \leq f(c) + (c-y)^2.$$

Note that

$$\lim_{y \to c} \left( f(c) - (c-y)^2 \right) = \lim_{y \to c} \left( f(c) + (c-y)^2 \right) = f(c).$$

Therefore by the pinching theorem

$$\lim_{y \to c} f(y) = f(c).$$

This implies that $f$ is continuous in $(0,1)$. If $c = 0$ then for $y \in [0,1]$ we have

$$f(0) - y^2 \leq f(y) \leq f(0) + y^2.$$

Note that

$$\lim_{y \to 0^+} f(0) - y^2 = \lim_{y \to 0^+} f(0) + y^2 = f(0).$$

Therefore by the pinching theorem

$$\lim_{y \to 0^+} f(y) = f(0).$$

This implies that $f$ continuous to the right at 0. A similar proof shows that $f$ is continuous to the left at 1. Hence $f$ is continuous on [0,1].

Option3: Do problem 2.2 first then argue that every differentiable function is continuous. This would prove continuity on the open interval (0,1). The continuity at the end points can be dealt with by one of the arguments above.

**2.2** Show that the function $f$ is differentiable in (0,1), and that $f'(x) = 0$ for every $x \in (0,1)$.

**Solution:** Let $x \in (0,1)$ consider the ratio $\frac{f(x+h)-f(x)}{h}$ for $h$ small enough. The hypothesis above ensures that for $|h| > 0$ and small enough

$$\left|\frac{f(x+h)-f(x)}{h}\right| \leq \frac{h^2}{|h|},$$

which implies that

$$0 \leq \left|\frac{f(x+h)-f(x)}{h}\right| \leq |h|.$$

Since $\lim_{h\to 0} |h| = 0$, the pinching theorem implies that

$$\lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| = 0.$$

Thus

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = 0. \qquad \blacksquare$$

**Problem 3.**

**3.2** Let $f$ be a differentiable function on $(0,+\infty)$. Assume that $f'(x) \to 0$ as $x \to +\infty$. Let $g(x) = f(x+1) - f(x)$ show that $g(x) \to 0$ as $x \to +\infty$.

**Solution:** Since $f$ is continuous in any interval of the form $[x, x+1]$ for $x > 0$, and $f$ is differentiable in the open interval $(x, x+1)$, the mean value theorem ensures that for each such $x$ there is $c_x \in (x, x+1)$ so that

$$g(x) = f(x+1) - f(x) = f'(c_x)(x+1-x) = f'(c_x).$$

Note that as $x \to +\infty$, since $c_x > x$ then $c_x \to +\infty$. Hence as $x \to +\infty$, $f'(c_x) \to 0$ (by hypothesis) and so $g(x) \to 0$. $\blacksquare$