First variation of perimeter / Mean curvature

Fundamental idea from calculus of variations, to look at necessary conditions of minimality, examine "curves" of competitors that "pass through" a given candidate for minimizer then derive the Euler Lagrange equation (1st order nec. minimality conditions) and stability inequalities (2nd order nec. minimality conditions). (Calculus analogy)

A **one parameter family of diffeomorphisms** of $\mathbb{R}^n$ is a smooth function

$$f : \mathbb{R}^n \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n \qquad \text{s.t. for each } |t| < \varepsilon$$
$$(x,t) \longmapsto f(x,t) = f_t(x) \qquad f_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ is a diffeomorphism}$$

Given an open set A $\{f_t\}_{|t|<\varepsilon}$ is a **local variation** in A if it defines a one parameter family of diffeomorphisms s.t.

- $f_0(x) = x \qquad \forall x \in \mathbb{R}^n$
- $\{x \in \mathbb{R}^n : f_t(x) \neq x\} \subset\subset A \qquad \forall |t| < \varepsilon$ (compactly contained)

$$\Rightarrow \quad f_t(E) \,\Delta\, E \subset\subset A \qquad \forall E \subset \mathbb{R}^n$$

The following Taylor expansion holds uniformly on $\mathbb{R}^n$

$$f_t(x) = x + t\,T(x) + O(t^2) \;; \quad \nabla f_t(x) = id + t\,\nabla T(x) + O(t^2)$$

where $T \in C^\infty_c(A, \mathbb{R}^n)$ is the initial velocity of $\{f_t\}_{|t|<\varepsilon}$

$$T(x) = \frac{\partial f_t}{\partial t}(x,0) \qquad \forall x \in \mathbb{R}^n$$

If E is a perimeter minimizer in A

$|\mu_E|(A) \leq |\mu_{f_t(E)}|(A)$ thus up to differentiability questions the necessary conditions for minimality are.

(1) $\frac{d}{dt} |\mu_{f_t(E)}|(A) = 0$ (E stationary set for perimeter)

&

(2) $\frac{d^2}{dt^2} |\mu_{f_t(E)}|(A) \geq 0$

Proposition (Diffeomorphic images of sets of finite perimeter)
Let $E \subset \mathbb{R}^n$ s.l.s.p, $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ diffeomorphism, $f^{-1} = g$ then $f(E)$ s.l.s.p with

(1) $\mathcal{H}^{n-1}(f(\partial^* E) \Delta \partial^* f(E)) = 0$ $\forall \varphi \in C_c(\mathbb{R}^n)$

(2) $\int_{\partial^* f(E)} \varphi \, \nu_{f(E)} \, d\mathcal{H}^{n-1} = \int_{\partial^* E} (\varphi \circ f) \, Jf \, (\nabla g \circ f)^* \nu_E \, d\mathcal{H}^{n-1}$

(3) $\mathcal{H}^{n-1}(F \cap \partial^* f(E)) = \int_{g(F) \cap \partial^* E} Jf \, |(\nabla g \circ f)^* \nu_E| \, d\mathcal{H}^{n-1}$

Taylor's expansion of the determinant close to the identity

Lemma: (1) If $z \in \mathbb{R}^n \otimes \mathbb{R}^n$ $\quad Id = Id_{\mathbb{R}^n}$

$$(Id + tz)^{-1} = Id - tz + t^2 z^2 + O(t^3) \qquad \text{for } t \text{ small}$$

(2) If $A, B \in \mathbb{R}^n \otimes \mathbb{R}^n$

$$\det(Id + tA + t^2 B) = 1 + t \operatorname{trace} A + t^2 \left(\operatorname{trace} B + \frac{1}{2}(\operatorname{trace} A)^2 - \frac{1}{2}\operatorname{trace} A^2\right) + O(t^3)$$

Pf: (1) $C = Id + tz$ invertible $\quad Id = CC^{-1} = C^{-1} + tzC^{-1}$

$$C^{-1} = Id - tzC^{-1} = Id - tz(Id - tzC^{-1})$$
$$= Id - tz + t^2 z^2 C^{-1} = Id - tz + t^2 z^2 (Id - tzC^{-1})$$
$$= Id - tz + t^2 z^2 - t^3 z^3 C^{-1}$$

(2) $B = 0$ $\quad A$ symmetric (real in Maggi)

$$h(t) = \det(Id + tA) = \det\left(Id + t\sum_{i=1}^{n} \lambda_i v_i \otimes v_i\right)$$ bec A diagonalizable

$$= \prod_{i=1}^{n} (1 + t\lambda_i)$$

$$= 1 + t\sum_{i=1}^{n} \lambda_i + t^2 \sum_{i \neq j} \lambda_i \lambda_j + O(t^3)$$

$$\text{trace } A = \sum_{i=1}^{n} \lambda_i \qquad \text{trace } A^2 = \sum_{i=1}^{n} \lambda_i^2$$

$$(\text{trace } A)^2 - \text{trace } A^2 = \left(\sum_{i=n}^{n} \lambda_i\right)^2 - \sum_{i=1}^{n} \lambda_i^2$$

$$= 2 \sum_{i \neq j} \lambda_i \lambda_j$$

$$h(t) = 1 + t \text{ trace } A + \frac{t^2}{2}\left[(\text{trace } A)^2 - \text{trace } A^2\right]$$

Gauss - Green theorem on C^2 hypersurfaces of $\mathbb{R}^n$

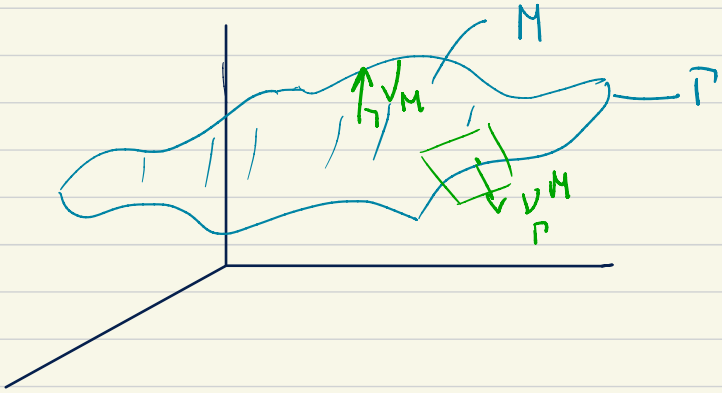
If $M \subset \mathbb{R}^n$ is a C^2 hypersurface with boundary Γ there exists a normal vector field $\vec{H}_M \in C(M, \mathbb{R}^n)$ to M ($\vec{H}_M(x) \in (T_x M)^\perp$) & a normal vector field $\nu_\Gamma^M \in C^1(\Gamma, S^{n-1})$ to Γ s.t $\forall \varphi \in C_c^1(\mathbb{R}^n)$

$$(\star) \qquad \int_M \nabla^M \varphi \, d\mathcal{H}^{n-1} = \int_M \varphi \cdot \vec{H}_M \, d\mathcal{H}^{n-1} + \int_\Gamma \varphi \, \nu_\Gamma^M \, d\mathcal{H}^{n-2}$$

$\vec{H}_M$: mean curvature vector

Moreover if $T \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ is normal to M then

$$\langle T, \nu_\Gamma^M \rangle = 0 \quad \text{on } \Gamma \implies \langle \nu_M, \nu_\Gamma^M \rangle = 0$$



$$\& \int_M \operatorname{div}_M T \, d\mathcal{H}^{n-1} = \int_M T \cdot \vec{H}_M \, d\mathcal{H}^{n-1}$$

Given $T \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ the tangential divergence of T on M is

$$\operatorname{div}^M T = \operatorname{div} T - \langle \nabla T(\nu_M), \nu_M \rangle = \operatorname{trace} \nabla^M T$$

(*) can be reformulated as

$$(**) \qquad \int_M \operatorname{div}^M T \, d\mathcal{H}^{n-1} = \int_M T \cdot \vec{H}_M \, d\mathcal{H}^{n-1} + \int_\Gamma \langle T, \nu_\Gamma^M \rangle \, d\mathcal{H}^{n-2}$$

if T is tangent to M by Stokes theorem ($T(x) \in T_x M$)

$$\int_M \operatorname{div}^M T = \int_\Gamma \langle T, \nu_\Gamma^M \rangle \, d\mathcal{H}^{n-2}$$

if T is normal to M ($T(x) \in (T_x M)^\perp$) $T = \varphi \nu_M$ $\varphi \in C^1_c(\mathbb{R}^n)$ & $\{\tau_i\}_{i=1}^{n-1}$ o.n basis for TM then

$$\operatorname{div}^M T = \sum_{k=1}^{n-1} \tau_k \cdot \nabla T(\tau_k) = \sum_{k=1}^{n-1} \tau_k \left(\partial_k \varphi \, \nu_M + \varphi \, \nabla \nu_M(\tau_k) \right)$$

$$= \sum_{k=1}^{n-1} \varphi \langle \nabla_M(\tau_k), \tau_k \rangle = \varphi H_M = \langle T, \vec{H}_M \rangle$$

where $H_M = \sum_{k=1}^{n-1} \langle \nabla \nu_M(\tau_k), \tau_k \rangle$ - If $T \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$

then $T = T'' + T^\perp$

↑ tangent ↑ normal

$T^\perp = \langle T, \nu_M \rangle \nu_M$

$T'' = T - \langle T, \nu_M \rangle \nu_M$

$$\int_M \operatorname{div}^M T = \int_M \operatorname{div}^M T'' + \int_M \operatorname{div}^M T^\perp$$

$$= \int_\Gamma \underbrace{\langle T'', \nu_\Gamma^M \rangle}_{\langle T, \nu_\Gamma^M \rangle} d\mathcal{H}^{n-2} + \int_M \underbrace{\langle T^\perp, \vec{H}_M \rangle}_{\langle T, \vec{H}_M \rangle} d\mathcal{H}^{n-1}$$

(**) holds

Theorem (FIRST VARIATION OF PERIMETER)

If $A \subset \mathbb{R}^n$ is an open set, E a set of locally finite perimeter, and $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A with initial velocity T for $\{f_t\}_{|t|<\varepsilon}$ then

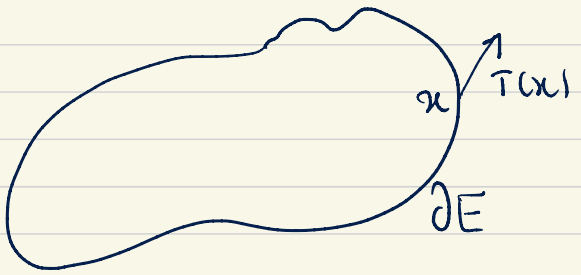
$$|\mu_{f_t(E)}|(A) = |\mu_E|(A) + t \int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} + o(t^2)$$

where $\operatorname{div}_E T : \partial^* E \longrightarrow \mathbb{R}$

$$\operatorname{div}_E T(x) = \operatorname{div} T(x) - \nu_E(x) \cdot \nabla T(x) \nu_E(x)$$

↑ Boundary divergence of T on E.

Remark: If E is a C^2 domain in $\mathbb{R}^n$



$$\operatorname{div}_E T = \operatorname{div}^{\partial E} T$$

$$\int_{\partial E} \operatorname{div}^{\partial E} T \, d\mathcal{H}^{n-1} = \int_{\partial E} \langle T, \vec{H}_{\partial E} \rangle \, d\mathcal{H}^{n-1} \qquad (\partial(\partial E) = \emptyset)$$

Thus

$$\frac{d}{dt}\left|\mu_{f_t(E)}\right|(A)\Big|_{t=0} = \int_{\partial E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial E} \langle T, \vec{H}_{\partial E} \rangle \, d\mathcal{H}^{n-1}$$