Lemma (Excess measure)

Let $E \subset \mathbb{R}^n$ s.t. $\int_p$ in $\mathbb{R}^n$, $0 \in \partial E$ s.t. for some $t_0 \in (0,1)$

(1) $|q(x)| < t_0 \quad \forall\, x \in C \cap \partial E$

(2) $|\{x \in C \cap E : q(x) > t_0\}| = 0$

(3) $|\{x \in C \setminus E : q(x) < -t_0\}| = 0$

then if $M = C \cap \partial^* E$ we have for $\forall\, G \subset D$ Borel

i) $\mathcal{H}^{n-1}(G) \leq \mathcal{H}^{n-1}(M \cap p^{-1}(G))$

ii) $\mathcal{H}^{n-1}(G) = \int_{M \cap p^{-1}(G)} \langle \nu_E, e_n \rangle \, d\mathcal{H}^{n-1}$

(iii) $\int_D \varphi \, d\mathcal{H}^{n-1} = \int_M \varphi(p(x)) \langle \nu_E(x), e_n \rangle \, d\mathcal{H}^{n-1} \quad \forall \varphi \in C_c(D)$

(iv) $\int_{E_t \cap D} \varphi \, d\mathcal{H}^{n-1} = \int_{M \cap \{q(x) > t\}} \varphi(p(x)) \langle \nu_E(x), e_n \rangle \, d\mathcal{H}^{n-1}$

for $t \in (-1,1)$ where

$$E_t = \{ z \in \mathbb{R}^{n-1} : (z,t) \in E \}$$

The set function (excess measure of E over D)

$$\xi(G) = |\mu_E|(C \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G)$$ defines a Radon measure in D.

Remarks: (1) $\xi(D) = e_n(1) = e(E,0,1,e_n)$

(2) For $G \subset D$ Borel $\xi(G) \leq \xi(D)$

$$\mathcal{H}^{n-1}(G) \overset{(i)}{\leq} \mathcal{H}^{n-1}(M \cap p^{-1}(G)) = \mathcal{H}^{n-1}(\partial^* E \cap C \cap p^{-1}(G))$$

$$\leq |\mu_E|(C \cap p^{-1}(G)) \leq \mathcal{H}^{n-1}(G) + \xi(G)$$

similarly

(3) $$\mathcal{H}^{n-1}(D \cap \bar{E}_t) \leq \mathcal{H}^{n-1}(M \cap \{q(x) > t\}) \leq \mathcal{H}^{n-1}(D \cap \bar{E}_t) + e_n(1)$$

Pf: By standard approximation techniques (iv) $\Rightarrow$ ii) $(t=-1)$

ii) $\Rightarrow$ i). We prove (iii) & (iv) - Using density we can assume $\varphi \in C^1_c(D)$ - Recall that for a.e $r \in (0,1)$

a) $\mathcal{H}^{n-1}(\partial^* E \cap (\partial D_r \times \mathbb{R})) = 0 \qquad D_r = B_r^{n-1} \times \{0\}$

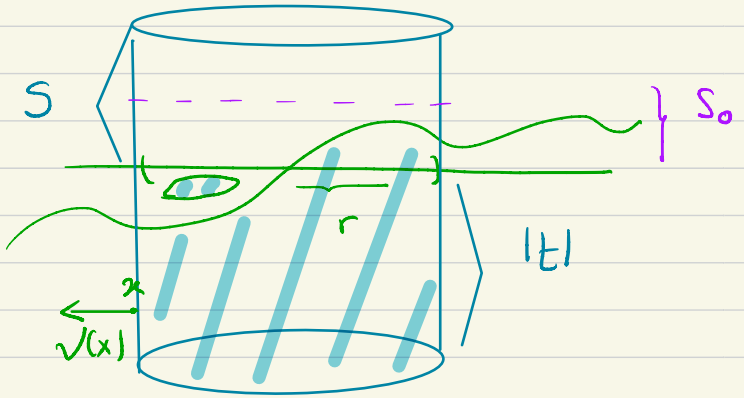
Using Fubini in (2) & (3) we have

b) $\mathcal{H}^{n-1}(E \cap D \times \{s\}) = 0$ a.e $s \in (t_0, 1)$

c) $\mathcal{H}^{n-1}(E \cap D \times \{t\}) = \mathcal{H}^{n-1}(D)$ a.e $t \in (-1, -t_0)$

Let r as in a), s as in b) $t \in (-1, -t_0)$ with t as in c)

$F = E \cap (D_r \times (t,s))$



$$\mu_F = \mu_E \llcorner (D_r \times (t,s)) + \mu_{D_r \times (t,s)} \llcorner E$$

Set $\nu(x) = \dfrac{px}{|px|}$ if $p(x) \neq 0$

$$\mu_{D_r \times (t,s)} = e_n \, \mathcal{H}^{n-1} \llcorner (D_r \times \{s\}) + \nu \, \mathcal{H}^{n-1} \llcorner (\partial D_r \times (s,t))$$

$$- e_n \, \mathcal{H}^{n-1} \llcorner (D_r \times \{t\})$$

Note $\langle \nu(x), e_n \rangle = 0$ for $x \in \mathbb{R}^n$ if $p(x) \neq 0$

$$e_n \cdot \mu_F = \langle e_n, \nu_E \rangle \, \mathcal{H}^{n-1} (\partial^* E \cap D_r \times (t,s))$$

$$+ \underbrace{\cancel{\mathcal{H}^{n-1} (E \cap D_r \times \{s\})}}_{0} - \mathcal{H}^{n-1} (E \cap D_r \times \{t\})$$

$$\underbrace{e_n \cdot \mu_F}_{e_n \cdot \nu_F \, \mathcal{H}^{n-1} \llcorner \partial^* F} = \langle e_n, \nu_E \rangle \, \mathcal{H}^{n-1} (\partial^* E \cap D_r \times (t,s)) \qquad (A)$$

$$- \mathcal{H}^{n-1} (E \cap D_r \times \{t\})$$

If $\varphi \in C^1_c(D)$, let $T(x) = \varphi(p(x)) e_n$; $T \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\operatorname{div} T = \frac{\partial}{\partial x_n} (\varphi(p(x)) = 0 \qquad \text{independent of } x_n$$

Hence

$$(**) \quad 0 = \int_F \operatorname{div} T = \int_{\partial^* F} T(x) \cdot \nu_F \, d\mathcal{H}^{n-1} = \int_{\partial^* F} \varphi(p(x)) \langle e_n, \nu_F \rangle \, d\mathcal{H}^{n-1}$$

Combining (*) with (**)

$$\int_{\partial^* E \cap (D_r \times (t,s))} \langle \nu_E, e_n \rangle \varphi(p(x))\, d\mathcal{H}^{n-1} = \int_{E \cap D_r \times \{t\}} \varphi(p(x))\, d\mathcal{H}^{n-1}$$

let $r \to 1^-$ $\quad s \to 1^-$

$$\int_{\partial^* E \cap (D \times (t,1))} \langle \nu_E, e_n \rangle \varphi(p(x))\, d\mathcal{H}^{n-1} = \int_{E \cap (D \times \{t\})} \varphi(p(x))\, d\mathcal{H}^{n-1}$$

$\partial^* E \cap C \cap \{q(x) > t\}$

$M \cap \{q(x) > t\}$ iv) ✓

letting $t \to (-1)^+$

$$\int_M \langle \nu_E(x), e_n \rangle \varphi(p(x))\, d\mathcal{H}^{n-1} = \int_P \varphi(p(x))\, d\mathcal{H}^{n-1}$$ iii) ✓