Lemma (small excess implies flatness)

Given $C_A \geq 1$, $t_0 \in (0,1)$ $\exists\, w(n, t_0, C_A) > 0$ such that if $E \in \mathcal{C}(C_A, 4r_0)$ for some $r_0 > 0$, and $x_0 \in \partial E$ $r \leq 2r_0$ $\nu \in S^{n-1}$

$$e(E, x_0, r, \nu) = \frac{1}{r^{n-1}} \int_{C(x_0, r, \nu) \cap \partial^* E} \frac{|\nu_E - \nu|^2}{2} \, d\mathcal{H}^{n-1}$$

then if $e(E, x_0, 2r_0, \nu) \leq w(n, t_0, C_A)$ t.f.h

i) $|\langle x - x_0, \nu \rangle| < t_0 r_0 \quad \forall x \in C(x_0, r_0, \nu) \cap \partial E$

ii) $|\{x \in C(x_0, r_0, \nu) \cap E : \langle x - x_0; \nu \rangle > t_0 r_0\}| = 0$

(iii) $|\{x \in C(x_0, r_0, \nu) \cap E^c : \langle x - x_0, \nu \rangle < -t_0 r_0\}| = 0$

Small oscillation of the unit normal in $L^2 \Rightarrow \partial E$ is flat (note i) measures the height of ∂E in $C(x_0, r_0, \nu)$).

Pf: Contradiction & compactness argument.

Assume $\exists t_0 \in (0,1)$ $\{E_k\}_{k\in\mathbb{N}} \subset \partial(C_A, 4r_k)$ $r_k > 0$ $x_k \in \partial E_k$ $\nu_k \in S^{n-1}$

$$e(E_k, x_k, 2r_k, \nu_k) < 2^{-k}$$

and at least one of the following conditions hold for infinitely many k.

(i) $\{x \in C_k \cap \partial E_k : t_0 r_k \le |q_k(x)| \le r_k\} \neq \emptyset$

(ii) $|\{x \in C_k \cap E_k : q_k(x) > t_0 r_k\}| > 0$

(iii) $|\{x \in C_k \setminus E_k : q_k(x) < -t_0 r_k\}| > 0$

where

$$C_k = C(x_k, r_k, \nu_k) \quad \&$$

$$q_k(x) = \langle x - x_k, \nu_k \rangle$$

Modulo translation, dilation & rotation we have a family

$F_k \in \mathcal{A}(C_A, 4)$ & $e(F_k, 0, 2, e_n) = e_n(F_k, 0, 2) < 2^{-k}$

and at least one of the following holds for infinitely many k's.

(i) $\{x \in C \cap \partial F_k : t_0 \leq |q(x)| \leq 1\} \neq \phi$ where

(ii) $|\{x \in C \cap F_k : q(x) > t_0\}| > 0$

(iii) $|\{x \in C \setminus F_k : q(x) < -t_0\}| > 0$

$q(x) = \langle x, e_n \rangle$ & $C = C(0, 2, e_n)$

By the compactness result $\exists F$ set of l.f.p, $0 \in \partial F$

modulo relabeling $\chi_{F_k} \to \chi_F$ in $L^1_{loc}(\mathbb{R}^n)$ & $\mu_{F_k} \to \mu_F$

$|\mu_{F_k}| \to \mu$ ahlfors regular up to scale 4 with constant C_A.

$$\left|\frac{\nu_{F_k} - e_n}{2}\right|^2 = \frac{1}{2}\left(|\nu_k|^2 + |e_n|^2 - 2\langle \nu_k, e_n\rangle\right) \qquad \begin{matrix} |\nu_k| = 1 \ \mu_{F_k}\text{ a.e} \\ |e_n| = 1 \end{matrix}$$

thus

$$(1) \qquad \theta_n(F_k, 0, 2) = \frac{1}{2^{n-1}} \int_{\partial^* F_k \cap C_2} \overbrace{\left(1 - \langle \nu_k, e_n\rangle\right)}^{\geq 0} d\mathcal{H}^{n-1} \leq 2^{-k}$$

Given U open set such that $\mu(\partial U) = 0 \Rightarrow |\mu_F|(\partial U) = 0$

since $\mu_{F_k} = \nu_k \, \mathcal{H}^{n-1} \llcorner \partial^* F_k \to \mu_F$ &

$|\mu_{F_k}| = \mathcal{H}^{n-1} \llcorner \partial^* F_k \to \mu$ we have

letting $k \to \infty$ in (1) for $U \subset C_2$

$$0 = \lim_{k\to\infty} e_n(F_k, 0, 2) \geq \frac{1}{2^{n-1}} \lim_{k\to\infty} \int_{\partial^* F_k \cap U} (1 - \langle \nu_k, e_n \rangle)\, d\mathcal{H}^{n-1} .$$

$$\geq \frac{1}{2^{n-1}} \lim_{k\to\infty} \left(|\mu_{F_k}|(U) - \langle e_n, \mu_{F_k}(U) \rangle \right) \geq 0$$

$$\geq \frac{1}{2^{n-1}} \left(\mu(U) - \langle e_n, \mu_F(U) \rangle \right) \geq 0$$

$\mu(U) = \langle e_n, \mu_F(U) \rangle$ for U open $\quad \mu(\partial U) = 0$

Check $\Rightarrow \quad \mu = \langle e_n, \mu_F \rangle$. Thus

$$|\mu_F| \leq \mu \leq |\mu_F| = \mathcal{H}^{n-1} \llcorner \partial^* F \quad \Rightarrow \quad |\mu_F| = \mu$$

$$\operatorname{spt} \mu = \partial F = \operatorname{spt} |\mu_F|$$

Let $r \in \left(\frac{4}{3}, 2\right)$ such that $\mu(\partial C_{4/3}) = 0$ then

$$\int_{C_r \cap \partial^* F} (1 - \underbrace{\langle \nu_F, e_n \rangle}_{\geq 0})\, d\mathcal{H}^{n-1} = 0 \quad \Rightarrow \quad 1 = \langle \nu_F, e_n \rangle \ \ \mathcal{H}^{n-1}\text{ a.e}$$

$$C_{4/3} \cap \partial^* E$$

$0 \in \partial F \implies F \cap C_{4/3} = \{x : q(x) \le 0\} \cap C_{4/3}$

(as in the proof that for $x \in \partial^* E$ the blow-up of E is a lower half-plane)

Assume that

(i) $\{x \in C \cap \partial F_k : t_0 \le |q(x)| \le 1\} \neq \phi$ holds for infinitely many k's

Then up to extracting a subsequence we can assume (i) holds for all $k \in \mathbb{N}$, $\exists\, x_k \in \partial F_k \cap C$ s.t. $t_0 < |q(x_k)| \le 1$

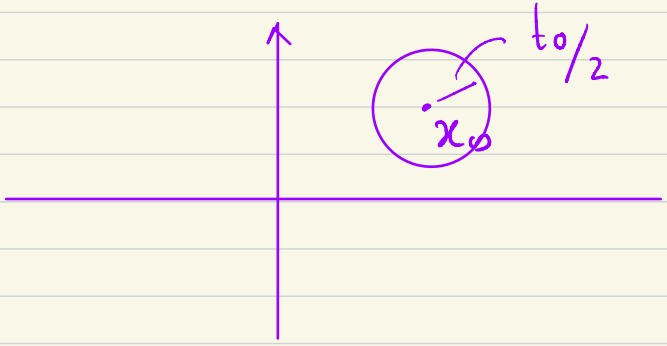
$x_k \to x_\infty \in \bar{C} \cap \partial F \subset C_{4/3} \cap \partial F$ & $|q(x_\infty)| \ge t_0$

which contradicts the fact that

$$F \cap C_{4/3} = \{x : q(x) \le 0\}$$

if $q(x_\infty) \ge t_0$ $\quad |F \cap B(x_\infty, t_0/2)| = 0$

$\implies x_\infty \notin \partial F$ *



similarly $q(x_\infty) \le -t_0$

$|F \cap B(x_\infty, -t_0/2)| = \omega_n \left(\frac{-t_0}{2}\right)^n \implies x_\infty \notin \partial F$ *

Thus $\exists k_0 \in \mathbb{N}$ $\quad \forall k \geq k_0$

(i) $\quad \{ x \in C \cap \partial F_k : t_0 \leq |q(x)| \leq 1 \} = \phi$

$$|\mu_{C \cap F_k}| = \mu_C \llcorner F_k^{(1)} + |\mu_{F_k}| \llcorner \{ C \cup \{ V_{E_k} = V_C \} \}$$

$$|\mu_{C \cap F_k}| \left(\{ x \in C : t_0 < |q(x)| < 1 \} \right) = 0 \qquad \forall k \geq k_0$$

$$\left(\sup \left\{ \int_{C \cap F_k} \operatorname{div} \varphi \; : \; \varphi \in C_c^1 \left(\{ x \in C : t_0 < |q(x)| < 1 \} : \mathbb{R}^n \right) \right. \right.$$
$$\left. \left. |\varphi|_\infty \leq 1 \right\} \right)$$

$\Rightarrow$ $\chi_{C \cap F_k}$ is constant in each connected component of $\{ x \in C : t_0 < |q(x)| < 1 \}$ since $F_k \longrightarrow F = \{ q(x) \leq 0 \}$

$\Rightarrow$ $\chi_{C \cap F_k} = 0$ a.e $k \geq k_0$ on $\{ t_0 < q(x) < 1 \}$

$\chi_{C \cap F_k} = 1$ a.e $k \geq k_0$ on $\{ -1 < q(x) < -t_0 \}$

$*$

Corollary: Under the previous assumption (since (i), (ii) & (iii) hold) then

$$\{ x \in C(x_0, r_0, \nu) \cap E : \langle x - x_0, \nu \rangle > t_0 r_0 \} = \emptyset$$

&

$$\{ x \in C(x_0, r_0, \nu) \setminus E : \langle x - x_0, \nu \rangle < -t_0 r_0 \} = \emptyset$$

Pf: In fact if $\bar{x} \in C(x_0, r_0, \nu) \cap E$ & $\langle \bar{x} - x_0, \nu \rangle > t_0 r_0$

$\Rightarrow \bar{x} \notin \partial E$ (because of (i)) - since

$$\left| \{ x \in C(x_0, r_0, \nu) \cap E^c : \langle x - x_0, \nu \rangle > t_0 r_0 \} \right|$$

$$= \left| \{ x \in C(x_0, r_0, \nu) : \langle x - x_0, \nu \rangle > t_0 r_0 \} \right|$$

then $\bar{x}$ is an accummulation point of elements in E^c

$\Rightarrow \bar{x} \in E \cap \overline{E}^c \Rightarrow x \in \partial E$ $\not\!\!\!