Given $C_A \geq 1$ & $0 < r_0 < \infty$

$\mathcal{A}(C_A, r_0) = \{ E \subset \mathbb{R}^n$ set of locally finite perimeter s.t

$\operatorname{spt} \mu_E = \partial E$ & $\forall x \in \partial E \quad \forall r \leq r_0$

$$(\star) \qquad C_A^{-1} \leq \frac{|\mu_E|(B(x,r))}{r^{n-1}} \leq C_A \}$$

Goal : Understand the structure of the boundary of these sets
(class modeled by perimeter minimizers up to scale $r_0 > 0$)

Remarks : i) $(\star)$ means $|\mu_E|$ is Ahlfors regular up to scale r_0 with constant C_A.

ii) $\mathcal{H}^{n-1} \llcorner \partial E$ Ahlfors regular does not imply $(\star)$
(4 corner Cantor set)

iii) Is this class "compact" in some sense ?

Theorem: If $\{E_k\} \subset \mathcal{A}(C_A, r_0)$ with $0 \in \partial E_k$, $\forall k \geq 1$ there exists a subsequence $\{E_{n_k}\}$, a set E of l.f.p & a non-negative Radon measure μ such that

$$\chi_{E_{n_k}} \to \chi_E \quad L^1_{loc}(\mathbb{R}^n), \quad \mu_{E_{n_k}} \to \mu_E \quad \& \quad |\mu_{E_{n_k}}| \to \mu$$

(1) μ is Ahlfors regular up to scale r_0 & constant C_A

(2) $|\mu_E| \ll \mu$

(3) if $x \in \partial E$ $\exists x_{n_k} \in \partial E_{n_k}$ s.t $x_{n_k} \to x$

(4) if $x \in \operatorname{spt}\mu$ $\exists x_{n_k} \in \partial E_{n_k}$ s.t $x_{n_k} \to x$

(5) if $x_{n_k} \in \partial E_{n_k}$ & $x_{n_k} \to x$ then $x \in \operatorname{spt}\mu$

(3) & (4) facts about weak convergence of measures & their supports.

Pf: For $R \geq 1$ consider $\{E_k \cap B_{2R}\}_k$ sets of finite perimeter

$$|E_k \cap B_{2R}| \leq C R^n$$

$$|\mu_{E_k \cap B_{2R}}|(\mathbb{R}^n) \leq |\mu_{E_k}|(B_{2R}) + \mathcal{H}^{n-1}(\partial B_{2R} \cap E_k)$$

$$\leq |\mu_{E_k}|(B_{2R}) + C_n R^{n-1}$$

Cover $B_{2R} \cap \partial E_k$ by balls centered in $B_{2R} \cap \partial E_k$ and radius r_0 and such that the centers are at least r_0 apart. at most $C_n R^n / r_0^n$ such balls thus

$$|\mu_{E_k}|(B_{2R}) \leq C_n R^n / r_0^n \; C_A r_0^{n-1} = C_{n,A_0} \left(\frac{R}{r_0}\right) R^{n-1}$$

$$\| \chi_{E_k \cap B_{2R}} \|_{BV(\mathbb{R}^n)} = |E_k \cap B_{2R}| + |\mu_{E_k \cap B_{2R}}|(\mathbb{R}^n)$$

$$\leq C_{n, A_0, r_0} R^n$$

By compactness theorem for BV functions, there exists a subsequence

$\{E_{k_j}\}$ s.t $E_{k_j} \cap B_{2R} \to E_{2R} \subset B_{2R}$ E_{2R} set of finite perimeter

$\mu_{E_{k_j} \cap B_{2R}} \to \mu_{E_{2R}}$ $\operatorname{spt} \mu_{E_{2R}} = \partial E_{2R}$

Given a compact set $K \subset\subset \mathbb{R}^n$ $\exists M > 0$ st $K \subset B(0,M)$

and by same argument as before

$$\sup_{k_j} |\mu_{E_{k_j}}|(K) \leq \sup_{k_j} |\mu_{E_{k_j}}|(B(0,M)) \leq C_{n,A_0} \left(\frac{M}{r_0}\right)^{n-1} < \infty$$

By compactness theorem for Radon measures there is a further subsequence $\{E_{k_j'}\}$ of a Radon measure μ^R st

$|\mu_{E_{k_j'}}| \to \mu^{2R}$ (this subsequence depends on R a priori)

choose $R = 2^l$ applying a diagonal argument obtain a subsequence

$\{E_{n_k}\}$, a set E of l.f.p of a Radon measure μ

$\chi_{E_{n_k}} \longrightarrow \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$ $\left(|E \Delta B_{2^l} \cap E_{2^l}| = 0 \right)$

$\mu_{E_{n_k}} \longrightarrow \mu_E$ $\quad |\mu_{E_{n_k}}| \longrightarrow \mu$ $\quad$ & $\quad \mu|_{B_{2^l}} = \mu_{B^{2^l}}$

By lower semi-continuity of the perimeter measure for U open s.t $\mu(\partial U) = 0$

$$|\mu_E|(U) \leq \liminf_{n_k} |\mu_{E_{n_k}}|(U) = \lim_{n_k \to \infty} |\mu_{E_{n_k}}|(U) = \mu(U)$$

Since given x for a.e $r > 0$ $\mu(\partial B(x,r)) = 0$ then

$$|\mu_E|(B(x,r)) \leq \mu(B(x,r)) \qquad (\clubsuit)$$

Then $|\mu_E| \ll \mu$ • For $A \subset \mathbb{R}^n$ s.t $\mu A = 0$ given $\varepsilon > 0$ $\exists\, U_\varepsilon$ open s.t $A \subset U_\varepsilon$ $\mu A < \varepsilon$ - Cover E

by balls $\{B(x,r) : x \in A,\ B(x,r) \subset U_\varepsilon\}$ (fine cover)

by corollary to Besicovitch $\exists \{B(x_i, r_i)\}$ disjoint collection s.t

$|\mu_E|(A \cap U \setminus \bigcup_i B(x_i, r_i)) = 0$ thus

$$|\mu_E|(A) \leq \sum_i |\mu_E|(B(x_i, r_i)) \leq \sum_i \mu(B(x_i, r_i))$$

$$\lesssim \mu(U_\varepsilon) < \varepsilon \implies |\mu_E|(A) < \varepsilon \quad \forall\, \varepsilon > 0$$

$\Rightarrow |\mu_E|(A) = 0$

$(*) \Rightarrow d|\mu_E| / d\mu \leq 1$ (2) ✓

(3) & (4) Claim: If $x \in \partial E$ or $x \in \operatorname{spt}\mu$

$\forall \varepsilon > 0$, $\exists\, n_k(\varepsilon)$ s.t $\forall\, n_k \geq n_k(\varepsilon)$

$$\operatorname{spt}\mu_{E_{n_k}} \cap B(x, \varepsilon) = \operatorname{spt}|\mu_{E_{n_k}}| \cap B(x, \varepsilon) \neq \emptyset$$

assume not: $\exists \varepsilon_0 > 0$ s.t $\forall\, n_k$ $\exists\, \tilde{n}_k > n_k$ s.t

$$\operatorname{spt}\mu_{E_{\tilde{n}_k}} \cap B(x, \varepsilon_0) = \operatorname{spt}|\mu_{E_{\tilde{n}_k}}| \cap B(x, \varepsilon_0) = \emptyset$$

then $|\mu_E|(B(x,\varepsilon_0)) \leq \liminf_{n_k \to \infty} |\mu_{E_{n_k}}|(B(x,\varepsilon_0)) = 0$ bec

$\mu_{E_{n_k}} \to \mu_E$

$\mu(B(x,\varepsilon_0)) \leq \liminf_{n_k \to \infty} |\mu_{E_{n_k}}|(B(x,\varepsilon_0)) = 0$ bec

$|\mu_{E_{n_k}}| \to \mu$