Density estimates for local perimeter minimizers

Definitions: (1) Given an open bounded set A, and a set of l.f.p $E \subset \mathbb{R}^n$ we say that E is **perimeter minimizer in A** if

(i) $\operatorname{spt} \mu_E = \operatorname{spt} |\mu_E| = \partial E$

(ii) whenever $E \Delta F \subset\subset A$ $\quad |\mu_E|(A) \leq |\mu_F|(A)$

($F = E$ outside A)

(2) We say that E is **a local perimeter minimizer** at scale $r_0 > 0$ in an open set A if

(i) $\operatorname{spt} \mu_E = \operatorname{spt} |\mu_E| = \partial E$

(ii) $|\mu_E|(A \cap B(x, r_0)) \leq |\mu_F|(A \cap B(x, r_0))$ whenever

$E \Delta F \subset\subset A \cap B(x, r_0) \quad \forall x \in A$

(if $A = \mathbb{R}^n \quad \forall x \in \mathbb{R}^n \quad \forall r_0 > 0$)

(i) non-restrictive assumption

Theorem: For $n \geq 2$ $\exists c(n) > 0$ s.t. if A is an open set in $\mathbb{R}^n$ & E is a local perimeter minimizer in A at scale $r_0 > 0$ then for every ball $B(x,r) \subset\subset A$ with $x \in \partial E \cap A$ & $r < \min\{r_0, d(x,\partial A)\}$ we have

(i) $$\frac{1}{2^n} \leq \frac{|E \cap B(x,r)|}{\omega_n r^n} \leq 1 - \frac{1}{2^n}$$

(ii) $$c(n) \leq \frac{|\mu_E|(B(x,r))}{r^{n-1}} \leq n\omega_n$$

in particular

$$\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0$$

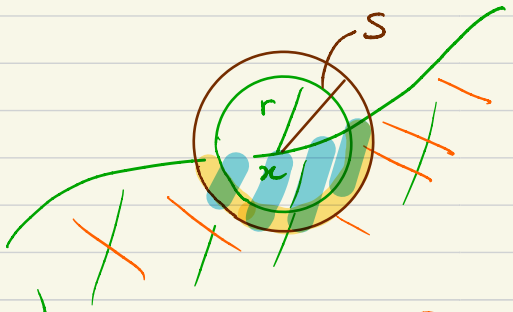
Pf since E local perimeter minimizer spt $|\mu_E| = \partial E$

thus $$0 < |E \cap B(x,r)| < \omega_n r^n \quad \forall x \in \partial E \quad \forall r > 0$$

let $x \in A \cap \partial E$ $\quad d = \min\{r_0, \text{dist}(x, \partial A)\}$

Since $\mathcal{H}^{n-1}(\partial^* E \cap B(x,d)) \le C(n)|\mu_E|(B(x,d)) < \infty$ then for a.e. $r < d$

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial B(x,r)) = 0$$



Note that

$$E \Delta F = E \cap (E \setminus B(x,r))^c \cup (E \setminus B(x,r) \cap E^c) \qquad F = E \setminus B(x,r)$$

$$= E \cap (E \cap B(x,r))^c = E \cap (E^c \cup B(x,r)^c)^c$$

$$= E \cap B(x,r) \subset\subset B(x,s) \subset\subset A \quad \text{for } 0 < r < s < d$$

Since E is a local perimeter minimizer

$$|\mu_E|(B(x,s)) \le |\mu_F|(B(x,s)) = |\mu_{E \setminus B(x,r)}|(B(x,s))$$

$$\le |\mu_E|(B(x,r)^{(0)} \cap B(x,s)) + |\mu_{B(x,r)}|(E^{(1)} \cap B(x,s))$$

$$+ \mathcal{H}^{n-1}(\{\nu_E = -\nu_{B(x,r)}\} \cap B(x,s))$$

$B(x,r)^{(0)}$: measure theoretic exterior

$E^{(1)}$: measure theoretic interior

$\{\nu_E = -\nu_{B(x,r)}\} \subset \partial^* E \cap \partial B(x,r)$

$$|\mu_E|(B(x,s)) \le |\mu_E|(B(x,r)^\circ \cap B(x,s)) + |\mu_{B(x,r)}|(E^{(1)} \cap B(x,s))$$
$$+ \mathcal{H}^{n-1}\{\nu_E = -\nu_{B(x,r)}\} \cap B(x,s)|$$
$$\subset \partial^* E \cap \partial B(x,r)$$

$$\le |\mu_E|(B(x,s) \setminus B(x,r)) + \mathcal{H}^{n-1}(\partial B(x,r) \cap E^{(1)} \cap B(x,s))$$
$$+ \mathcal{H}^{n-1}(\partial^* E \cap \partial B(x,r))$$ 0 (assumption)

$$\le |\mu_E|(B(x,s)) \setminus B(x,r)) + \mathcal{H}^{n-1}(\partial B(x,r) \cap E^{(1)})$$
+ theorem

letting $s \to r+$

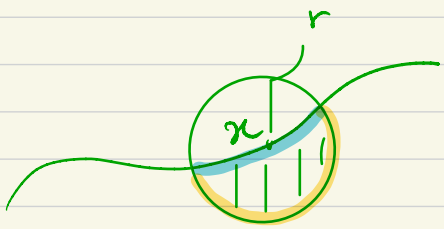
$$|\mu_E|(\overline{B(x,r)}) \le |\mu_E|(\partial B(x,r)) + \mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r))$$

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial B(x,r)) = 0$$

(A) $$|\mu_E|(\overline{B(x,r)}) \le \mathcal{H}^{n-1}(E^{(1)} \cap \partial B(x,r)) \le \sigma_{n-1} r^{n-1}$$

$$|\mu_E|(\overline{B(x,r)}) \le n\omega_n r^{n-1}$$ (upper bound is)

Since $|\mu_{E\cap B(x,r)}|(\mathbb{R}^n) \le |\mu_E|(B(x,r)) + \mathcal{H}^{n-1}(E^{(1)}\cap\partial B(x,r))$



by (★) $\le 2\,\mathcal{H}^{n-1}(E^{(1)}\cap\partial B(x,r))$

Combining with the Euclidean isoperimetric inequality

$$n\omega_n^{1/n}\,|E\cap B(x,r)|^{(n-1)/n} \le 2\,\mathcal{H}^{n-1}(E^{(1)}\cap B(x,r)) \qquad (★★)$$

By the co-area formula the function $m:(0,s)\to[0,\infty)$ defined by $m(s) = |E^{(1)}\cap B(x,s)| \overset{\text{Lebesgue density theorem}}{=} |E\cap B(x,s)|$ is absolutely continuous with

$$m'(s) = \mathcal{H}^{n-1}(E^{(1)}\cap\partial B(x,s)) \quad \text{a.e } s>0$$

by (★★)

$$n\omega_n^{1/n}\, m(r)^{(n-1)/n} \le 2\,m'(r) \quad \text{a.e } r<d.$$

since $m(0+) = \lim_{r\to 0} m(r) = 0$ & $0 < m(r) < \omega_n r^n$

thus

$$n\,\omega_n^{1/n} \le 2\,m'(r)\,m(r)^{1-n/n} = 2m'(r)\,m(r)^{1/n-1}$$

$$\omega_n^{1/n} \le 2\frac{d}{dr}\,m(r)^{1/n} \rightsquigarrow \omega_n^{1/n}\,r \le 2\,m(r)^{1/n} \quad \forall r < d$$

integrating between 0 & r

$$\Rightarrow \quad m(r) \ge \frac{1}{2^n}\,\omega_n r^n \Rightarrow \frac{|E\cap B(x,r)|}{\omega_n r^n} \ge \frac{1}{2^n}.$$

Note that if E is a local perimeter minimizer in A at scale $r_0>0$ then so is E^c, in fact if $F\Delta E^c \subset\subset A\cap B(x,r_0)$

$$F\Delta E^c = F\setminus E^c \cup E^c\setminus F = F\cap E \cup E^c\cap F^c \subset\subset A\cap B(x,r_0)$$

$$= E\Delta F^c \subset\subset A\cap B(x,r_0)$$

E local minimizer

$$|\mu_{E^c}|(A\cap B(x,r_0)) \le |\mu_E|(B(x,r_0)\cap A) \le |\mu_{F^c}|(B(x,r_0)\cap A)$$

$$\le |\mu_F|(B(x,r_0)\cap A)$$

thus similarly we obtain $\frac{|E^c\cap B(x,r)|}{\omega_n r^n} \ge \frac{1}{2^n} \Rightarrow$

$$\frac{|E \cap B(x,r)|}{\omega_n r^n} \leq 1 - \frac{1}{2^n}$$

By (i) using the relative isoperimetric inequality

$$|\mu_E|(B(x,r)) \geq c(n) \min\{|E \cap B(x,r)|, |B(x,r) \setminus E|\}^{n-1/n}$$

$$\geq \tilde{c}(n) r^{n-1} \quad \text{(lower bound of ii))}$$

Note that (i) $\Rightarrow$ if $x \in \partial E \cap A \Rightarrow x \in \partial_* E$ thus $\partial E = \partial_* E$ in A and

$$\mathcal{H}^{n-1}((\partial_* E \setminus \partial^* E) \cap A) = \mathcal{H}^{n-1}((\partial E \setminus \partial^* E) \cap A) = 0$$

□

Given $C_A \geq 1$ & $0 < r_0 < \infty$

$\mathcal{A}(C_A, r_0) = \{ E \subset \mathbb{R}^n$ set of locally finite perimeter s.t

$\operatorname{spt} \mu_E = \partial E$ & $\forall x \in \partial E \quad \forall r \leq r_0$

$$(\star) \qquad C_A^{-1} \leq \frac{|\mu_E|(B(x,r))}{r^{n-1}} \leq C_A \}$$

Goal : Understand the structure of the boundary of these sets
(class modeled by perimeter minimizers up to scale $r_0 > 0$)

Remarks : i) $(\star)$ means $|\mu_E|$ is Ahlfors regular up to scale r_0 with constant C_A.

ii) $\mathcal{H}^{n-1} \llcorner \partial E$ Ahlfors regular does not imply $(\star)$
(4 corner Cantor set)

iii) Is this class "compact" in some sense ?