Theorem: Assume $x \in \partial^* E$. Let $H_x^{\pm} = \{y : \pm \langle y - x, \nu_E(x) \rangle \geq 0\}$

(i) $$\lim_{r \to 0} \frac{|B(x,r) \cap E \cap H_x^+|}{r^n} = 0 \qquad H_x^{\pm} = H^{\pm}(x) + x$$

(ii) $$\lim_{r \to 0} \frac{|B(x,r) \setminus E \cap H_x^-|}{r^n} = 0$$

(iii) $$\lim_{r \to 0} \frac{|\mu_E|(B(x,r))}{\omega_{n-1} r^{n-1}} = 1$$

Pf

$$\frac{|E \cap B(x,r) \cap (H^+(x) + x)|}{r^n} = |E_{x,r} \cap B_1 \cap H^+(x)|$$

$$\xrightarrow{r \to 0} |H^-(x) \cap B_1 \cap H^+(x)| = 0$$

(iii) wlog $x = 0$ by (2) $\quad |\mu_{E_r}|(B_1) = \frac{1}{r^{n-1}} |\mu_E|(B_r)$

since $|\mu_{H^-(0)}|(\partial B_1) = 0$ $\quad \downarrow \; r \to 0$

$$|\mu_{H^-(0)}(B_1)| = \omega_{n-1}$$

Lemma: There exists a constant $c>0$ depending on n s.t for any Borel set $B \subset \partial^* E$

$$\mathcal{H}^{n-1}(B) \leq c\,|\mu_E|(B)$$

Pf: let $\varepsilon, \delta > 0$ if $B \subset \partial^* E$ since $|\mu_E|$ is a Radon measure for $E \subset \partial^* B$ Borel $\exists U$ open st

$$|\mu_E|(U) \leq |\mu_E|(B) + \varepsilon$$

By lemma

$$\liminf_{r \to 0} \frac{|\mu_E|(B(x,r))}{r^{n-1}} \geq A_3 > 0$$

$$\mathcal{F} = \{ B(x,r) : x \in B,\ B(x,r) \subset U,\ r < \delta/10 \text{ and } |\mu_E|(B(x,r)) \geq A_3/2\, r^{n-1} \}$$

By Vitali's covering lemma there exists a disjoint family $\{B(x_i, r_i)\} \subset \mathcal{F}$ s.t

$$B \subset \bigcup_i B(x_i, 5r_i) \qquad \& \qquad \operatorname{diam} B(x_i, 5r_i) \le \delta$$

$$\mathcal{H}^{n-1}_\delta(B) \le \sum_i \omega_{n-1} (5r_i)^{n-1} = 5^{n-1} \sum_{i=1}^{n} \omega_{n-1} r_i^{n-1}$$

$$\le c_n \sum_i |\mu_E|(B(x_i, r_i)) \le c_n |\mu_E|(U)$$

$$\le c_n \left(|\mu_E|(B) + \varepsilon \right) \qquad \varepsilon \to 0$$

$$\mathcal{H}^{n-1}_\delta(B) \le c_n |\mu_E|(B) \qquad \& \qquad \delta \to 0$$

$$\mathcal{H}^{n-1}(B) \le c_n |\mu_E|(B)$$

De Giorgi's structure theorem

Assume E is a set of locally finite perimeter in $\mathbb{R}^n$, then

(i) $\partial^* E \subset \bigcup_{k=1}^{\infty} K_k \cup N$ where $|\mu_E|(N) = 0$ &

K_k is a compact subset of a C^1 hypersurface S_k

(ii) For $x \in \partial^* E \cap K_k$ $\quad \nu_E(x) = \nu_{S_k}(x)$

(iii) $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E \Rightarrow \mathcal{H}^{n-1}(N) = 0$

Pf: wlog E of finite perimeter (since $|\mu_E|$ Radon $R_j \nearrow \infty$ $|\mu_E|(\partial B_{R_j}) = 0$), $E \cap B_{R_{j+1}} \setminus B_{R_j}$ of finite perimeter

$$\partial^* E = \bigcup_j \left(\partial^*(E \cap B_{R_{j+1}} \setminus B_{R_j})\right) \cap \operatorname{int}(B_{R_{j+1}} \setminus B_{R_j})$$

$\cup N_0$ s.t. $|\mu_E|(N_0) = 0$

Recall that for $x \in \partial^* E$

$$\lim_{r \to 0} \frac{|B(x,r) \cap E \cap H_x^+|}{r^n} = 0 = \lim_{r \to 0} \frac{|B(x,r) \setminus E \cap H_x^-|}{r^n} \quad (1)$$

By Egoroff's theorem given $i \in \mathbb{N}$ $\exists F_i \subset \partial^* E$ s.t. (1) holds uniformly on F_i &

$$|\mu_E|(\partial^* E \setminus F_i) < 2^{-i} \quad / \quad |\mu_E|(F_i) < \infty$$

E finite perimeter

thus $|\mu_E|\left(\partial^* E \setminus \bigcup_{i=1}^{\infty} F_i\right) \leq |\mu_E|(\partial^* E \setminus F_i) < 2^{-i} \quad \forall i$

$\Rightarrow$

$$\begin{cases} |\mu_E|(\partial^* E \setminus \cup F_i) = 0 \; ; \; |\mu_E|(F_i) < \infty \\ \text{convergence in (1) uniform on each } F_i \text{ \& } \\ \nu_E \text{ is measurable on } F_i \end{cases}$$

By Lusin's theorem for each i $\exists \{E_i^j\}$ disjoint compact sets $E_i^j \subset F_i$

$$|\mu_E|\left(F_i \setminus \bigcup_j E_i^j\right) = 0 \quad \& \quad \nu_{E|E_i^j} \text{ is continuous.}$$

Relabeling the set $\{E_i^j\}_{i,j}$ by $\{K_\ell\}$ we have

(2)
$$\begin{cases} \partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N & |\mu|(N) = 0 \\ \text{the convergence in (1) is uniform on } K_k \\ \nu_E|_{K_k} \text{ is continuous for each } k \in \mathbb{N} \end{cases}$$

Q: the unit normal to K_k is continuous but how do we know that there exists a C^1 submanifold S_k containing K_k?

Goal use Whitney's extension theorem which allows us to extend a continuous function with certain properties to a C^1 function.

To show that we can apply Whitney's extension theorem we need to "verify hypotheses".

For $\delta > 0$ define

$$\rho_k(\delta) = \sup \left\{ \frac{\langle \nu_E(x) ; y - x \rangle}{|y - x|} : 0 < |x-y| < \delta \; ; \; x, y \in K_k \right\}$$

Claim: for each $k \in \mathbb{N}$, $\rho_k(\delta) \to 0$ as $\delta \to 0$

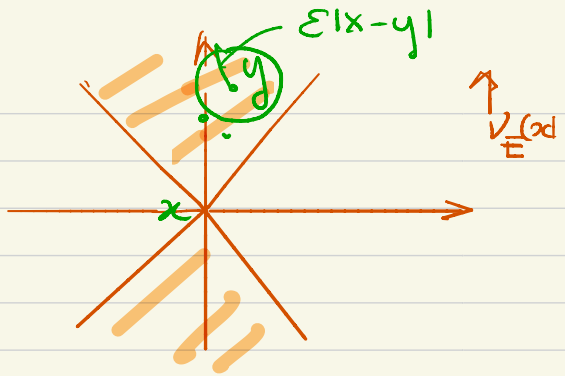
Pf Assume $k = 1$ $0 < \varepsilon < 1$ by (1) & (2) $\exists \delta \in (0,1)$ s.t if $z \in K_1$, $r < 2\delta$ then

$$(3) \quad \begin{cases} |E \cap B(z,r) \cap H_z^+| < \dfrac{\varepsilon^n}{2^{n+2}} \omega_n r^n \\ |B(z,r) \cap E \cap H_z^-| > \left(\dfrac{1}{2} - \dfrac{\varepsilon^n}{2^{n+2}} \right) \omega_n r^n \end{cases}$$

Subclaim: $\forall x, y \in K_1$, $0 < |x-y| \leq \delta$

$$|\langle \nu_E(x) ; y - x \rangle| \leq \varepsilon |x-y|$$

Assume not: i.e. $\exists x, y \in K_1$ with $0 < |x-y| \leq \delta$ s.t



case 1 : $\langle V_E(x), y-x\rangle \geq \varepsilon\,|x-y|$

case 2 : $\langle V_x(x), y-x\rangle \leq -\varepsilon|x-y|$

We address case 1 (2 very similar)

(4) $\boxed{B(y, \varepsilon|x-y|) \subset H_x^+ \cap B(x, 2|x-y|)}$

because if $z \in B(y, \varepsilon|x-y|)$ then $z = y+w$ with $|w| < \varepsilon|x-y|$

$$\langle V_E(x); z-x\rangle = \langle V_E(x); y-x\rangle + \langle V_E(x); z-y\rangle$$
$$= \langle V_E(x); y-x\rangle + \langle V_E(x); w\rangle$$
$$\geq \varepsilon|x-y| - |w| \geq 0 \Rightarrow z \in H_x^+$$

Since $x \in K_1$ and $2|x-y| < 2\delta$ by (3)

(5) $|E \cap B(x, 2|x-y|) \cap H_x^+| < \varepsilon^n/2^{n+2}\,\omega_n\, 2^n |x-y|^n = \dfrac{\varepsilon^n |x-y|^n \omega_n}{4}$

since $y \in K_1$ $\varepsilon|x-y| < \delta$ by (3)

(6) $|E \cap B(y, \varepsilon|x-y|)| \geq |E \cap B(y, \varepsilon|x-y|) \cap H_y^-| \geq \dfrac{\varepsilon^n \omega_n |x-y|^n}{2}\left(1 - \dfrac{\varepsilon^n}{2^{n+1}}\right)$

$\wedge$ *

Combining (4), (5) & (6) yield a contradiction, thus for $\forall x, y \in K_1$

$$\langle V_E(x) ; y - x \rangle \leq \varepsilon |x - y|$$

case 2 done similarly !

Thus $\forall \varepsilon > 0$ $\exists \delta > 0$ such that if $|x - y| < \delta$, $x, y \in K_1$

$$f_{K_1}(\delta) < \varepsilon$$ (similarly for all k!)

Whitney's extension theorem

Let $C \subset \mathbb{R}^n$ be a closed set $f : C \to \mathbb{R}$, $d : C \to \mathbb{R}^n$

(i) $R(x,y) = \dfrac{f(y) - f(x) - \langle d(x), y-x \rangle}{|y-x|}$ $\quad x, y \in C \quad x \neq y$

(ii) Let $K \subset C$ compact and $\delta > 0$ s.t

$$\rho_K(\delta) = \sup \{ |R(x,y)| : 0 < |x-y| < \delta \; ; \; x, y \in K \}$$

Theorem: Assume f, d are continuous, and for each compact set $K \subset C$; $\rho_K(\delta) \to 0$ as $\delta \to 0$ then there exists $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ s.t

(i) $\bar{f}$ is C^1

(ii) $\bar{f} = f$ and $D\bar{f} = d$ on C