The local properties of sets of l.f.p are studied by introducing blow ups of E, for $x \in \mathbb{R}^n$ $r > 0$

$$E_{x,r} = \frac{E - x}{r} = \phi_{x,r}(E) = \left\{ y : \exists z \in E \cdot y = \frac{z - x}{r} \right\}$$

define

$$\Theta^n(E, x) = \lim_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} \quad \text{if it exists}$$

$\Theta^n(E, x) = 1$ iff

$\leftarrow$ measure theoretic interior

$$\frac{|E \cap B(x, rR)|}{\omega_n (rR)^n} = \frac{1}{\omega_n (Rr)^n} \int_{B(x, Rr) \cap E} dy =$$

$$z = \frac{y - x}{r}$$

$$= \frac{1}{\omega_n R^n} \int_{B(0,R) \cap E_{x,r}} dz = \frac{1}{\omega_n R^n} \int_{B(0,R)} \chi_{E_{x,r}} dz$$

$$\frac{|E \cap B(x, rR)|}{\omega_n (rR)^n} - 1 = \frac{1}{\omega_n R^n} \int_{B(0,R)} (\chi_{E_{x,r}} - 1) dz \leq 0$$

($\chi_{E_{x,r}} - 1 \leq 0$)

$r \to 0$

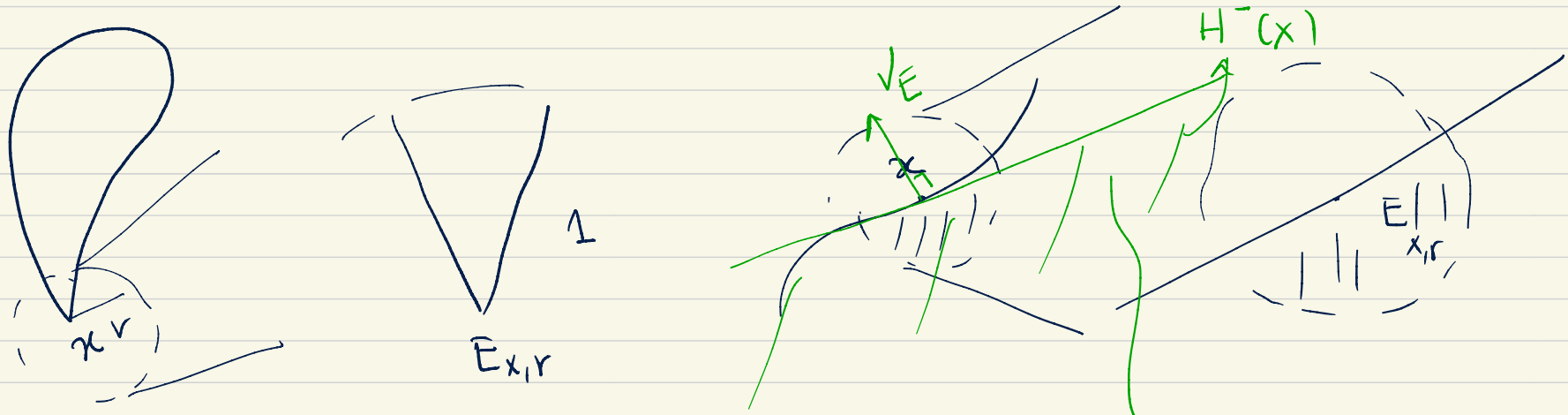
$\downarrow$

0

$$\Rightarrow \chi_{E_{x,r}} \to 1 \text{ in } L^1_{loc}(\mathbb{R}^n)$$

$\Theta^n(E,x)=0$ iff $\chi_{E_{x,r}} \longrightarrow \chi_\phi = 0$ in $L^1_{loc}(\mathbb{R}^n)$

measure theoretic exterior



Theorem: (Blow up of the reduced boundary)

Assume $x \in \partial^* E$ then $\chi_{E_{x,r}} \xrightarrow{r\to 0} \chi_{H^-(x)}$ in $L^1_{loc}(\mathbb{R}^n)$

where

$$H^-(x) = \{ y \in \mathbb{R}^n : \langle y, \nu_E \rangle \leq 0 \} \qquad *$$

Moreover if $H(x) = \langle \nu_E(x) \rangle^\perp$ then

$$\mu_{E_{x,r}} = \nu_{E_{x,r}} |\mu_{E_{x,r}}| \xrightarrow[r\to 0]{} \nu_E(x)\, \mathcal{H}^{n-1} \llcorner H(x)$$

&

$$|\mu_{E_{x,r}}| \longrightarrow \mathcal{H}^{n-1} \llcorner H(x)$$

Rk : we know nothing about blow ups away from $\partial^* E$, measure theoretic interior & exterior.

Corollary : If $x \in \partial^* E$ then

$$\Theta^n(E,x) = \lim_{r \to 0} \frac{|E \cap B(x,r)|}{\omega_n r^n} = \frac{1}{2}$$

Pf : if $x \in \partial^* E$ since $\chi_{E_{x,r}} \to \chi_{H^-(x)}$ in $L^1_{loc}(\mathbb{R}^n)$

then $\dfrac{|E_{x,r} \cap B_1|}{\omega_n} = \dfrac{|E \cap B(x,r)|}{\omega_n r^n}$

$r \to 0$ ↓

$$\frac{|H^-(x) \cap B_1|}{\omega_n} = 1/2$$

↑ $\sqrt{E}$

$$|E_{x,r} \cap B_1| = \int_{B_1} \chi_{E_{x,r}} \, dy = \int_{B_1} \chi_E(ry+x)\,dy \overset{z = ry+x}{=} r^{-n} \int_{B_r(x)} \chi_E(z)\,dz$$

Pf of theorem: wlog $x = 0$ $\nu_E(0) = e_n$ $H(0) = \{y \in \mathbb{R}^n : y_n = 0\}$

$H^-(0) = \{y : y_n \leq 0\}$ $H^+(0) = \{y : y_n \geq 0\}$

Claim 1: Show that $E_{0,r} = E_r \xrightarrow[r \to 0]{} F$ a set of l.f.p with constant normal, $\nu_F = e_n$

Claim 2: F is a half space

Claim 3: $F = H^-(0)$

Note that it is enough to show that given any sequence $\{r_k\}$ $r_k \downarrow 0$ there exists a subsequence $\{s_j\} \subset \{r_k\}$ s.t

$\forall\, E_{s_j} = E_{0,s_j}$, $\chi_{E_{s_j}} \longrightarrow \chi_{H^-(0)}$ in $L^1_{loc}(\mathbb{R}^n)$

Fix $L > 0$ let $D_r = E_r \cap B_L(0) = E_r \cap B_L$

$g_r(y) = y/r$ then $\forall \phi \in C^1_c(\ , \mathbb{R}^n)$

similar to previous change of variable

$$\int_{D_r} \operatorname{div} \Phi \overset{\nearrow}{=} \frac{1}{r^{n-1}} \int_{E\cap B_{rL}} \operatorname{div}(\Phi\circ g_r)\, dy = \frac{1}{r^{n-1}} \int \Phi\circ g_r \cdot \nu_{E\cap B_{Lr}}\, d|\mu_{E\cap B_{rL}}|$$

$$\leq \frac{|\mu_{E\cap B_{rL}}|(\mathbb{R}^n)}{r^{n-1}} \leq C < \infty \quad \text{(by (vi) of previous lemma for } r \text{ small enough)}$$

$$= |\mu_{D_r}|(\mathbb{R}^n) \leq C < \infty \quad \text{for } r \text{ small enough } (r \leq 1), \text{ since}$$

$$|\chi_{D_r}|_{L^1(\mathbb{R}^n)} = |D_r| \leq |B_L| < \infty$$

$$\sup_{0\leq r\leq 1} \|\chi_{D_r}\|_{BV(\mathbb{R}^n)} \leq C' < \infty$$

by the compactness theorem for BV functions $\exists \{s_j\}$ subsequence of $\{r_k\}$ s.t. if $E_j^L = E_{s_j} \cap B_L$, $\chi_{E_j^L} \to f_L$ in $L^1(\mathbb{R}^n)$

wlog $f_L = \chi_{F_L}$, choosing a further subsequence $\{\tilde{s}_j\}$; $\forall L > 0$ (diagonal argument)

$E_j \to \chi_F$ $(= \chi_{F_L}$ in $B_L)$, F set of l. f. p.

If $\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_F \operatorname{div}\phi \, dy = \int_{\mathbb{R}^n} \phi \cdot \nu_F \, d|\mu_F| \qquad \text{with } |\nu_F| = 1 \text{ a.e}$$

Claim 1 : $\nu_F = \nu_n$ $|\mu_F|$ a.e

If $\nu_j = \nu_{E_j}$ $(E_j = E_{E_j})$ then for $\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \phi \cdot \nu_j \, d|\mu_{E_j}| = \int_{E_j} \operatorname{div}\phi$$

$j \to \infty$ ↓ ↓

$$\int_{\mathbb{R}^n} \phi \cdot \nu_F \, d|\mu_F| = \int_F \operatorname{div}\phi \Rightarrow \nu_{E_j} d|\mu_{E_j}| \to \nu_F \, d|\mu_F|$$

If $|\mu_F|(\partial B_L) = 0$ then

$$\boxed{\int_{B_L} \nu_j \, d|\mu_j| \longrightarrow \int_{B_r} \nu_F \, d|\mu_F|} \qquad (1)$$

Since

$$\int_{\mathbb{R}^n} \Phi \cdot \nu_j \, d|\mu_j| = \frac{1}{\tau_j^{n-1}} \int_E \operatorname{div}(\Phi \circ g_{\tau_j}) \, dy = \frac{1}{\tau_j^{n-1}} \int_{\mathbb{R}^n} \Phi \cdot g_{\tau_j} \cdot \nu_E \, d|\mu_E|$$

taking sup Φ $\quad \Phi \in C_c^1(B_L, \mathbb{R}^n)$ $\quad \|\Phi\|_\infty \le 1$ $\quad \left(g_{\tau_j}(z) = \frac{z}{\tau_j} \right)$

$$\boxed{|\mu_j|(B_L) = \frac{1}{\tau_j^{n-1}} |\mu_E|(B_{L\tau_j})} \qquad (2)$$

Moreover letting $\Phi \to \chi_{B_L} e_l$ $\quad l = 1 \dots n$ (each coordinate)

$$\boxed{\int_{B_L} \nu_j \, d|\mu_j| = \frac{1}{\tau_j^{n-1}} \int_{B_{L\tau_j}} \nu_E \, d|\mu_E|} \qquad (3)$$

thus (2) & (3)

$$\int_{B_L} \nu_j \, d|\mu_j| = \int_{B_{L\tau_j}} \nu_E \, d|\mu_E| \Rightarrow \lim_{j \to \infty} \int_{B_{L\tau_j}} \nu_E \, d|\mu_E| = \nu_E(0)$$

$0 \in \partial^* E$

$= e_n$

thus $$\lim_{j\to\infty} \int_{B_L} v_j \, d|\mu_j| = e_n \quad \text{if} \quad |\mu_F|(\partial B_L) = 0$$

thus $$\lim_{j\to\infty} \int_{B_L} e_n \cdot v_j \, d|\mu_j| = \lim_{j\to\infty} \frac{1}{|\mu_j|(B_L)} \int_{B_L} e_n \cdot v_F \, d|\mu_j|$$

l.s.c $\downarrow$ $\qquad\qquad$ $\updownarrow$

$$|\mu_F|(B_L) \le \liminf_{j\to\infty} |\mu_{E_j}|(B_L) \le \liminf_{j\to\infty} \int_{B_L} e_n \cdot v_j \, d|\mu_j|$$

$$\le \int_{B_L} e_n \cdot v_F \, d|\mu_F| \qquad \text{since } |v_F| = 1 \ |\mu_F| \text{ a.e}$$

$$\Rightarrow \quad e_n = v_F \quad |\mu_F| \text{ a.e in } B_L$$

$$\Rightarrow \quad |\mu_F|(B_L) = \lim_{j\to\infty} |\mu_{E_j}|(B_L)$$

if $|\mu_F|(\partial B_L) = 0$

$$v_j \, d|\mu_j| \longrightarrow e_n \, d|\mu_F|$$

Claim 2: F is a half space,

by claim 1 $\int_F \operatorname{div}\phi = \int_F e_n \cdot \phi \, d|\mu_F|$

fix $\varepsilon > 0$ $\quad f^\varepsilon = \chi_F * \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ $\quad \eta$ usual mollifier

$$\int_{\mathbb{R}^n} f^\varepsilon \operatorname{div}\phi = \int_F \operatorname{div}(\phi * \eta_\varepsilon) = \int_{\mathbb{R}^n} \eta_\varepsilon * \phi \cdot e_n \, d|\mu_F|$$

$$\| \qquad\qquad -\int_{\mathbb{R}^n} \nabla f^\varepsilon \cdot \phi$$

let $\phi = \xi e_j$ $\quad \xi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} \eta_\varepsilon * \xi \langle e_j, e_n \rangle \, d|\mu_F| = -\int_{\mathbb{R}^n} (\nabla f^\varepsilon \cdot e_j)\, \xi \, dx = -\int_{\mathbb{R}^n} \frac{\partial f^\varepsilon}{\partial x_j} \xi \, dx$$

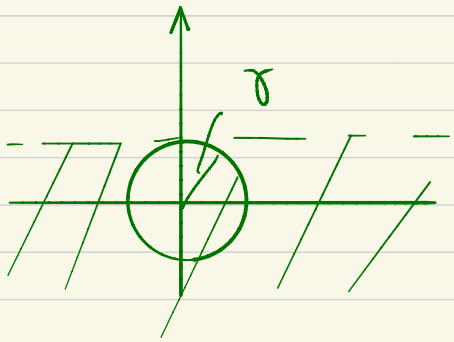
if $\xi \geq 0 \implies f^\varepsilon_{x_n} \leq 0$ & $f^\varepsilon_{x_j} = 0$ $\quad j = 1, \dots n-1$

as $f^\varepsilon \to \chi_F$ in $L^1_{loc}(\mathbb{R}^n)$ as $\varepsilon \to 0$ we conclude that

$$F = \{ y \in \mathbb{R}^n : y_n \le \gamma \} \text{ for some } \gamma \in \mathbb{R}.$$

Claim 3 : $\gamma = 0$ ie $F = H^-(0)$

If $\gamma > 0$ since $\chi_{E_j} \to \chi_F$ in $L^1_{loc}(\mathbb{R}^n)$



$$\omega_n \gamma^n = |B(0,\gamma) \cap F| = \lim_{j \to \infty} |E_j \cap B(0,\gamma)|$$

$$= \lim_{j \to \infty} \frac{|E \cap B(0, \tau_j \gamma)|}{\tau_j^n} \quad \not\to \quad 0 \in \partial^* E \text{ (previous lemma i) \& ii))}$$

similarly $\gamma < 0$

■

Remark $|\mu_{E_j}| \to |\mu_F| = \gamma^{n-1} \, \mathcal{L} \lfloor e_n^\perp$; $\nu_j |\mu_j| \to \nu_F |\mu_F|$