Theorem: Co-area formula (13.1) HW

If $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ Lipschitz, $A \subset \mathbb{R}^n$ open then $t \longmapsto P(\{u > t\} : A)$ is a Borel function in $\mathbb{R}$ &

$$\int_A |\nabla u| = \int_{\mathbb{R}} P(\underbrace{\{u > t\}}_{E_t} \cap A)\, dt$$

Hints: **Claim 1**: If $u \geq 0$ $\quad u(x) = \int_0^\infty \chi_{E_t}(x)\, dt$

$u \leq 0$ $\quad u(x) = \int_{-\infty}^0 (\chi_{E_t}(x) - 1)\, dt$

Claim 2: $\int_A u \operatorname{div} \phi = \int_{-\infty}^{\infty} \left(\int_{E_t} \operatorname{div} \phi \, dx \right) dt$ $\quad \forall \phi \in C_c^1(A, \mathbb{R}^n)$, $|\phi| \leq 1$

$$\int_A |\nabla u| = \|Du\|(A) \leq \int_{-\infty}^{\infty} P(E_t ; A)\, dt$$

Claim 3 : $$m(t) = \int_{A \cap E_t} |Du|\,dx = \int_{A \cap \{u \le t\}} |Du|\,dx$$

is non-decreasing $\Rightarrow$ m' exists a.e

$$\text{If } m'(t) \ge -\int_{E_t} \operatorname{div} \phi \qquad \forall \phi \in C_c^1(A, \mathbb{R}^n),\ |\phi| \le 1$$

$$m'(t) \ge P(E_t; A)$$

$$\int_A |Du|\,dx = \int_{-\infty}^{\infty} m'(t)\,dt \ge \int_{-\infty}^{\infty} P(E_t : A)\,dt$$

∎

Theorem : (Approximation by smooth sets)

A Lebesgue measurable set $E \subset \mathbb{R}^n$ is of l.f.p. iff there exists a sequence $\{E_k\}_{k \in \mathbb{N}}$ of open sets with smooth boundary in $\mathbb{R}^n$, $\varepsilon_k \to 0^+$ such that

- $E_k \xrightarrow[loc]{} E$ $\quad (\chi_{E_k} \to \chi_E \text{ in } L^1_{loc}(\mathbb{R}^n))$
- $\sup_{k \in \mathbb{N}} P(E_k : B_R) < \infty \quad \forall R > 0$
- $|\mu_{E_k}| \longrightarrow |\mu_E|$ & $\partial E_k \subset (\partial E, \varepsilon_k)$

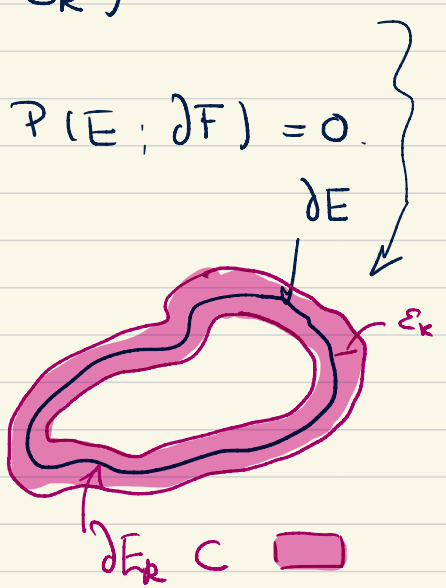
ε_k-nglbd of ∂E

In particular $P(E_k ; F) \to P(E ; F)$ whenever $P(E ; \partial F) = 0$.

Moreover

i) If $|E| < \infty$ then $|E_k| \to |E|$

(ii) If $P(E) < \infty$ then $P(E_k) \to P(E)$



Recall: approximation of BV functions by smooth functions.

Proposition: If E set of l.f.p in $\mathbb{R}^n$ then

$$(\mu_E)_\varepsilon = \mu_E * \eta_\varepsilon = -\nabla(\underbrace{\chi_E * \eta_\varepsilon}_{u_\varepsilon})\, d\mathcal{H}^n \qquad \forall \varepsilon > 0$$

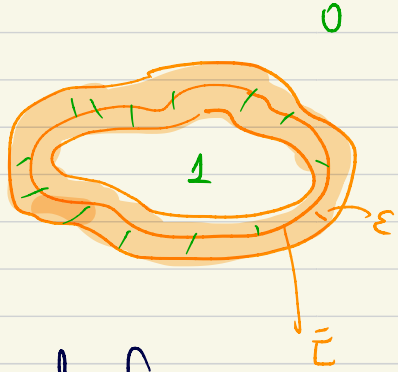
$$-\nabla(\chi_E * \eta_\varepsilon)\, \mathcal{H}^n \longrightarrow \mu_E \quad \text{as} \quad \varepsilon \to 0$$

$$|\nabla(\chi_E * \eta_\varepsilon)|\, \mathcal{H}^n \longrightarrow |\mu_E| \quad \text{as} \quad \varepsilon \to 0 \qquad (\blacktriangle)$$

Conversely if $E \subset \mathbb{R}^n$ Lebesgue measurable s.t

$$(*) \qquad \limsup_{\varepsilon \to 0} \int_K |\nabla(\chi_E * \eta_\varepsilon)|\, dx < \infty$$

for every compact set $K \subset\subset \mathbb{R}^n$ then E set of l.f.p.



Remarks: ① If $|E| < \infty$ & $P(E) < \infty$ we can assume each E_R is bounded

② E bounded Borel ; $u, v \in L^1_{loc}(\mathbb{R}^n)$ then

$$\int_E |u - v| \, dx = \int_{\mathbb{R}} |E \cap \{u > t\} \Delta \{v > t\}| \, dt$$

Fubini

$$\int_{E \cap \{u > v\}} |u - v| \, dx = \int_{\bar{E}} \overline{\chi_{\{u > v\}}}(x) \overbrace{\int_{\mathbb{R}} \chi_{\{u > t\}}(x) \, \chi_{\{v \le t\}}(x) \, dt}^{\int_{v(x)}^{u(x)} dt}$$

$$= |\{(x,t) : x \in \bar{E} : u(x) > t \ge v(x)\}|$$

$$= \int_{\mathbb{R}} |E \cap \{u > t\} \setminus \{v > t\}|$$

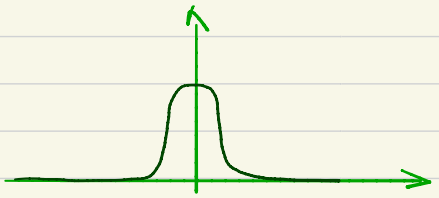
similarly

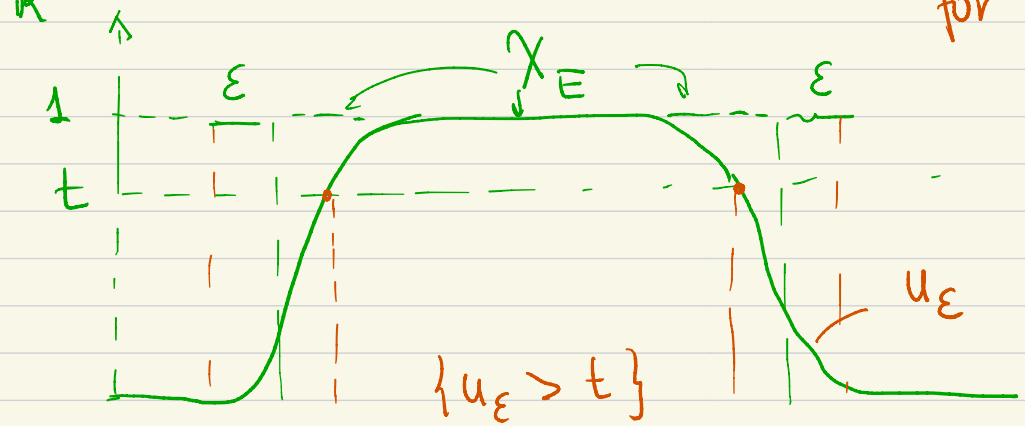
$$\int_{E \cap \{u < v\}} |u - v| \, dx$$

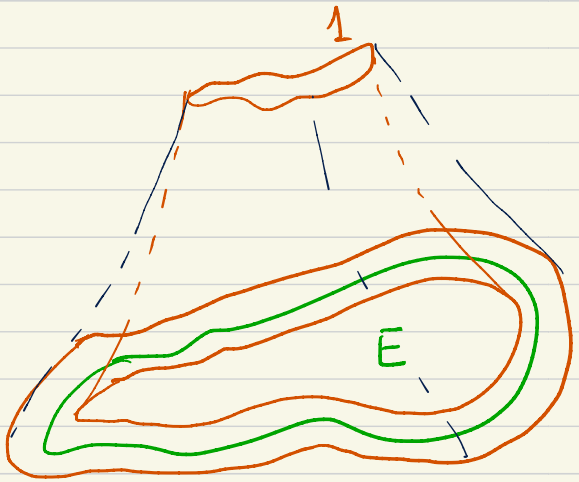
Pf of the theorem : given $\varepsilon > 0$, $\varepsilon_k \to 0^+$ $t \in (0,1)$, η standard mollifier

$u_\varepsilon = \chi_E * \eta_\varepsilon$ $\quad u_k = u_{\varepsilon_k}$

$E_k^t = \{u_k > t\}$ by Morse-Sard lemma for a.e $t \in (0,1)$ the open set $\{u_\varepsilon > t\}$ has smooth boundary







Q: Can you do this construction be done with $E_{\varepsilon_k}^t \subset E$?
Useful in PDE.

Since $\chi_E \in L^1_{loc}(\mathbb{R}^n)$, $u_k \xrightarrow[(*)]{} \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$ then by remark (2) above

$$\chi_{E_k^t} = \chi_{\{u_k > t\}} \longrightarrow \chi_{\{\chi_E > t\}} \text{ in } L^1_{loc}(\mathbb{R}^n)$$

$$\left(\int_B |u_k - \chi_E| = \int_{\mathbb{R}} |B \cap \{u_k > t\} \Delta \underbrace{\{\chi_E > t\}}_{E}| \right) \quad (*) \quad \text{for } t \in (0,1)$$

$$= \int_0^1 \int_B |\chi_{\{u_k > t\}} - \chi_{\{\chi_E > t\}}| \, dx \, dt$$

$$= \int_0^1 \int_B |\chi_{E_k^t} - \chi_E| \, dx \, dt \Rightarrow \chi_{E_k^t} \to \chi_E \text{ in } L^1_{loc}(\mathbb{R}^n)$$

By l.s.c $\quad \forall A \subset \mathbb{R}^n$ open $\quad \forall t \in (0,1)$

$$(*) \quad \begin{cases} P(E;A) \leq \liminf_{k \to \infty} P(E_k^t; A) \\ \leq \int_0^1 \liminf_{k \to \infty} P(E_k^t; A) \, dt \leq \liminf_{k \to \infty} \int_0^1 P(E_k^t; A) \, dt \end{cases}$$

(Fatou)

Moreover for every $t \in (0,1)$

$$\partial E_k^t = \{u_k = t\} \subset (\partial E : \varepsilon_k)$$

In fact note that $\{0 < u_\varepsilon < 1\} \subset (\partial E ; \varepsilon)$ because if $x \notin (\partial E ; \varepsilon)$ then $B(x,\varepsilon) \cap \partial E = \emptyset$ thus either $B(x,\varepsilon) \subset \operatorname{int} E$ or $B(x,\varepsilon) \subset \mathbb{R}^n \setminus \overline{E}$ thus either

$|B(x,\varepsilon) \cap E| = |B(x,\varepsilon)|$ or $|B(x,\varepsilon) \cap E| = 0$

$u_\varepsilon(x) = 1$ or $u_\varepsilon(x) = 0$

Note that if $t < 0$ $E_k^t = \{u_k > t\} = \mathbb{R}^n$

$t > 1$ $E_k^t = \{u_k > t\} = \emptyset$ $\Rightarrow |\mu_{E_k^t}| = 0$

By the co-area formula if $|\mu_E|(\partial A) = 0$

$$\int_A |\nabla u_k| = \int_{\mathbb{R}} P(E_k^t ; A)\, dt .$$ By (▲)

(▲▲)

$$P(E ; A) = |\mu_E|(A) = \lim_{k\to\infty} \int_A |\nabla u_k| = \lim_{k\to\infty} \int_{\mathbb{R}} P(E_k^t, A)\, dt$$

By (A) & (AA)

$$P(E;A) = \int_{\mathbb{R}} \liminf_{k\to\infty} P(E_k^t;A)\,dt \quad \text{if } |\mu_E|(\partial A)=0$$

also $$P(E;A) = \liminf_{k\to\infty} P(E_k^t;A) \quad \text{if } |\mu_E|(\partial A)=0$$

Let $\{r_i\}$ be an increasing subsequence such that

$$|\mu_E|(\partial B_{r_i}) = 0 \quad \text{(exists because } |\mu_E| \text{ Radon)}$$

$$P(E;B_{r_i}) = \liminf_{k\to\infty} P(E_k^t;B_{r_i}) \quad \forall i\in\mathbb{N}$$

For each $k\in\mathbb{N}$, $\exists\,\Lambda_k \subset (0,1)$ s.t Λ_k.
$\forall\, t\notin\Lambda_k$, E_k^t is open & smooth; thus if
$t\notin \bigcup_k \Lambda_k$ $\forall\, k$ E_k^t open & smooth

By a diagonal argument $\exists\, t\notin\bigcup_k\Lambda_k$ and a subsequence $\{k_j\}$, s.t $k_j\to\infty$; s.t

$E_j = E_{k_j}^t$; $\{E_j\}_j$ open smooth $\chi_{E_j} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$

$\partial E_j \subset (\partial E, \varepsilon_{k_j})$ $\Rightarrow \mu_{E_j} \to \mu$

$$\underbrace{P(E, B_{r_i})}_{E \text{ l.f.p} \Rightarrow} = \underbrace{\liminf_{j\to\infty} P(E_j, B_{r_i})}_{< \infty} \quad \forall i \in \mathbb{N}$$

$\forall\, R > 0 \quad (R < r_i)$

Exercise 4.31 If $\mu_{E_j} \to \mu_E$ and $r_i \to \infty$ s.t

$$\liminf_{j\to\infty} |\mu_{E_j}|(B_{r_i}) = |\mu_E|(B_{r_i}) \quad \text{then} \quad |\mu_{E_j}| \to |\mu_E|$$

It remains to show i) & ii)

i) If $|E| < \infty$ then $u_k \to \chi_E$ in $L^1(\mathbb{R}^n)$ &

$\chi_{E_k^t} \to \chi_E$ in $L^1(\mathbb{R}^n)$ a.e t smooth

ii) if $P(E) < \infty$, $P(E) = \liminf_{k\to\infty} P(E_k^t)$ / diagonal arg. simplified yields result ii) ∎

Remark: If $|E| < \infty$ & $P(E) < \infty$ there is a sequence as in the theorem with $\{E_k\}$ bounded open smooth sets.

Q: What would you do?

$\exists\ R_i \to \infty$ s.t $E \cap B_{R_i} \to E$

$P(E \cap B_{R_i}) \to P(E)$

apply this to $E \cap B_{R_i}$ & diagonal argument.