HOMEWORK 9

TI problems due Wednesday March 10 at 10pm PT

Problems from Folland: Chapter 6, section 3: problems 31 and 34

Problem 1 (Prelim 2011): Suppose $f \in L^1([0,1])$. Show that if

$$\int_0^1 f(x)(\sin x)^n \, dx = 0, \qquad \text{for all } n = 1, 2, \cdots$$

then $f = 0$ a.e.

Problem 2 (Prelim 2004): Let f be a fixed but arbitrary Lebesgue measurable function on $\mathbb{R}$. For $p \in \mathbb{R}$ define

$$F(p) = \int_{\mathbb{R}} |f(t)|^p \, dt,$$

where we use the convention $0^0 = 0$. Show that

$$I = \{p : F(p) < \infty\}$$

is a connected subset of $\mathbb{R}$ and that $\log F$ is a convex function on this set.

Problem 3 (Prelim 2014): Suppose that $K \in L^p([0,1] \times [0,1])$ for some $1 < p < \infty$. Let q be the conjugate exponent of p, that is $1/p + 1/q = 1$.

1. For $f \in L^q([0,1])$ let
$$(Af)(x) = \int_0^1 K(x,y) f(y) \, dy.$$
Show that $(Af)(x)$ exists for almost every (with respect to the 1-dimensional Lebesgue measure) $x \in [0,1]$. Prove that A is a bounded linear map from $L^q([0,1])$ into $L^p([0,1])$, i.e. $A \in \mathcal{L}(L^q([0,1]), L^p([0,1]))$.

2. Suppose that for every $f \in L^q([0,1])$, $(Af)(x) = 0$ for almost every (with respect to the 1-dimensional Lebesgue measure) $x \in [0,1]$. Show that $K = 0$ for almost every (with respect to the 2-dimensional Lebesgue measure) $(x,y) \in [0,1] \times [0,1]$.