R problem set 8

Due Wednesday March 10

Let $f, g : \mathbb{R}^n \to \mathbb{R}$, and let $y \in \mathbb{R}^n$, define

$$\tau_y f(x) = f(x - y)$$

Note that

1. For $1 \leq p \leq \infty$ $\|\tau_y f\|_p = \|f\|_p$.
2. f is uniformly continuous if and only if $\|\tau_y f - f\|_u \to 0$ as $y \to 0$.

The **convolution** of f and g is the function $f * g$ defined by

$$f * g(x) = \int f(x - y)g(y)\, dy.$$

Problem 1: Prove that if $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ then $f * g(x)$ exists for almost every $x \in \mathbb{R}^n$, $f * g \in L^p(\mathbb{R}^n)$, and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Problem 2: Prove that if p and q are conjugate exponents, and if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g$ exists for every x, $f * g$ is bounded and uniformly continuous, and

$$\|f * g\|_u \leq \|f\|_p \|g\|_q.$$

Consider the function $\psi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\psi(x) = \exp\left(\frac{-1}{1 - |x|^2}\right) \chi_{B(0,1)}(x).$$

Let

$$\phi(x) = \frac{\psi(x)}{\int \psi(y)\, dy},$$

and for $t > 0$ define

$$\phi_t(x) = t^{-n}\phi(\frac{x}{t}).$$

Note that

$$\int_{\mathbb{R}^n} \phi(x)\, dx = \int_{\mathbb{R}^n} \phi_t(x)\, dx = 1.$$

Problem 3: Show that

1. If $f \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ then $f * \phi_t \to f$ in the $L^p(\mathbb{R}^n)$ norm as $t \to 0$.
2. If $f \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ then $f * \phi_t \to f$ uniformly on compact subsets of $\mathbb{R}^n$ as $t \to 0$.
3. Show that in either case $f * \phi_t \in C^\infty(\mathbb{R}^n)$ for any $t > 0$.

Problem 4: Show that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$.

Problem 5: Show that if $K \subset U \subset \mathbb{R}^n$, where K is compact and U is open, there exists a function $f \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq f \leq 1$, $f = 1$ on K and $f = 0$ on U^c.