R problem set 7

Due Wednesday March 10

Problem 1: Consider $\mathbb{R}^d$ with the topology given by the Euclidean metric.

1. Let $K \subset U \subset \mathbb{R}^d$, where K is compact and U is open. Prove that there exists an open set V such that $\overline{V}$ is compact and $K \subset V \subset \overline{V} \subset U$.

2. Let $K \subset U \subset \mathbb{R}^d$, where K is compact and U is open. Prove that there exists $f \in C(\mathbb{R}^d, [0,1])$ such that $f = 1$ on K and $f = 0$ outside a compact subset of U.

3. Let K be a compact subset of $\mathbb{R}^d$, and let $\{U_j\}_{j=1}^n$ be an open cover of K. Prove that there exists a family of compactly supported continuous functions $\{h_j\}_{j=1}^n \subset C(\mathbb{R}^d, [0,1])$ such that the support of h_j is contained in U_j and $\sum_{j=1}^n h_j(x) = 1$ for all $x \in K$. (The family $\{h_j\}_{j=1}^n$ is a partition of unity on K subordinated to the open cover $\{U_j\}_{j=1}^n$.)