HOMEWORK 7

TI problems due Wednesday February 24 at 10pm PT

Problems from Folland: Chapter 6, section 1: problems 2, 3, 10.

Problem 1: Let $\mathcal{B}$ be a separable Banach space. Let $\{T_n\}_n \subset \mathcal{B}^*$ be such that $\sup_n \|T_n\| = M < \infty$. Show that there exists a subsequence $\{T_{n_k}\}_{n_k}$ such that the limit

$$Tx = \lim_{k\to\infty} T_{n_k}(x)$$

exists for every $x \in \mathcal{B}$. Moreover T is linear and $\|T\| \leq M$. T is the *weak limit* of $\{T_{n_k}\}_{n_k}$.

Problem 2: (Prelim 2018) Let $1 < p < \infty$, and $\{f_n\}$ be a sequence of functions in $C^1((0,1)) \cap C([0,1])$, satisfying

$$\int_0^1 |f_n(x)|^p \, dx + \int_0^1 |f_n'(x)|^p \, dx \leq 1 \quad \text{for every } n \geq 1.$$

1. Show that there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in L^p([0,1]) \cap C([0,1])$ such that
$$f_{n_k} \to f \text{ uniformly in } [0,1], \text{ and } f_{n_k} \to f \text{ in } L^p([0,1]), \text{ as } k \to \infty.$$

2. Show that there exist a sub-subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ and $h \in L^p([0,1])$ such that for every $\varphi \in C_c((0,1))$ (that is, φ is continuous with compact support in $(0,1)$),
$$\lim_{j\to\infty} \int_0^1 f_{n_{k_j}}'(x)\varphi(x)\, dx = \int_0^1 h(x)\varphi(x)\, dx.$$

3. Show that for every $\varphi \in C_c^1((0,1))$ (that is, φ is C^1 with compact support in $(0,1)$),
$$\int_0^1 h(x)\varphi(x)\, dx = -\int_0^1 f(x)\varphi'(x)\, dx.$$