HOMEWORK 6

TI problems due Wednesday February 17 at 10pm PT

Problems from Folland: Chapter 5: problem 56.

Problem 1: Let $\mathcal{H}$ be a separable Hilbert space. Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{H})$. Let A^* be A's adjoint. Show that for every pair $a, b \in \mathcal{H}$, the system of equations

$$\begin{cases} x + A^* y = a \\ Ax - y = b \end{cases}$$

has a unique solution $x, y \in \mathcal{H}$ provided that the range of $I + A^*A$ is closed.

Problem 2: Let P be the orthogonal projection associated with the closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, that is

$$P(f) = f \text{ if } f \in \mathcal{S} \quad \text{and} \quad P(f) = 0 \text{ if } f \in \mathcal{S}^{\perp}.$$

1. Show that $P \circ P = P^2 = P$ and $P^* = P$.
2. Conversely if P is a bounded linear operator satisfying $P \circ P = P^2 = P$ and $P^* = P$, prove that P is the orthogonal projection for some closed subspace of $\mathcal{H}$.

Problem 3: (Prelim 2011) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ a one-to-one bounded linear map whose range $T(\mathcal{X})$ is closed in $\mathcal{Y}$. Show that for each bounded linear functional ϕ on $\mathcal{X}$ there is a bounded linear functional ψ on $\mathcal{Y}$ such that $\phi = \psi \circ T$, and there is a constant C (independent of ϕ) such that ψ can be chosen to satisfy $\|\psi\| \leq C\|\phi\|$. Here $\|\cdot\|$ denotes the operator norm.

Problem 4: (Prelim 2017) Suppose $\mathcal{H}$ is a Hilbert space and $y_n \in \mathcal{H}$ are such that the sequence $\langle x, y_n \rangle$ converges, as $n \to \infty$, for every $x \in \mathcal{H}$. Prove that there exists $y \in \mathcal{H}$ such that $\lim_{n\to\infty} \langle x, y_n \rangle = \langle x, y \rangle$ for every $x \in \mathcal{H}$.