**R problem set 5**

**Due Wednesday February 24**

Read Stein & Shakarchi Chapter 4, section 6: Compact operators.

Suppose $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis for a separable Hilbert space $\mathcal{H}$. A linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be **diagonalized** with respect to the basis $\{\varphi_k\}_{k=1}^{\infty}$ if

$$T(\varphi_k) = \lambda_k \varphi_k \quad \text{where} \quad \lambda_k \in \mathbb{C}.$$

In general a *non-zero* element $\varphi \in \mathcal{H}$ is an **eigenvector** of $T$ with **eingenvalue** $\lambda$ is $T\varphi = \lambda\varphi$. The sequence $\{\lambda_k\}_{k=1}^{\infty}$ is the **multiplier sequence** corresponding to $T$.

**Problem 1:** Let $\{\lambda_k\}_{k=1}^{\infty}$ is the multiplier sequence corresponding to $T$. Prove that

1. $\|T\| = \sup_k |\lambda_k|$
2. $T^* = T$ if and only if $\lambda_k \in \mathbb{R}$. Show that in this case eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. $T$ is unitary if and only if $|\lambda_k| = 1$.

Let $\mathcal{H}$ be a separable Hilbert space. A linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be **compact** if the image of unit ball by $T$

$$\{g \in \mathcal{H} : \ g = Tf \text{ for some } f \in \mathcal{H} \text{ with } \|f\| \leq 1\}$$

has a compact closure in $\mathcal{H}$.

**Problem 2:** Prove that $T$ is compact if and only if whenever $\{f_k\}_{k=1}^{\infty}$ is a bounded sequence in $\mathcal{H}$, there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ so that $Tf_{n_k}$ converges. Check that a compact operator is always automatically bounded. Prove that if $T$ has **finite rank**, that is, its range is finite-dimensional, then it is compact.

**Proposition:** Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$. Then

1. Let $\{T_n\}_n \subset \mathcal{L}(\mathcal{H}, \mathcal{H})$ be a family of compact operators with $\|T_n - T\| \to 0$ and $n \to \infty$, then $T$ is compact.

2. If $T$ is compact then there exists a sequence $\{T_n\}_n \subset \mathcal{L}(\mathcal{H}, \mathcal{H})$ of operators with finite rank such that $\|T_n - T\| \to 0$ and $n \to \infty$.

**Spectral Theorem:** Suppose $T$ is a compact symmetric operator on $\mathcal{H}$ . Then there exists an orthonormal basis $\{\varphi_k\}_k$ of $\mathcal{H}$ that consists of eigenvectors of $T$. Moreover if

$$T\varphi_k = \lambda_k \varphi_k$$

then $\lambda_k \in \mathbb{R}$ and $\lambda_k \to 0$ as $k \to \infty$.