R problem 2

Due Monday January 25

A normed vector space $\mathcal{X}$ is **strictly convex** if whenever $x \neq y$ and $\|x\| = \|y\| = 1$ then for all $0 < t < 1$,

$$\|tx + (1-t)y\| < 1.$$

A normed vector space $\mathcal{X}$ is **uniformly convex** if for every $\epsilon > 0$ there exists $\delta \in (0,1)$ so that if

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \epsilon \text{ then } \|\frac{x+y}{2}\| < 1 - \delta.$$

Problem 0: What does it mean geometrically for a space to be *stricly convex* and *uniformly convex*?

Problem 1: Let $\mathcal{X}$ be a normed vector space. The duality map F is defined for every $x \in \mathcal{X}$ by

$$F(x) = \{f \in \mathcal{X}^* : \|f\| = \|x\| \text{ and } f(x) =< f, x >= \|x\|^2\}.$$

1. Prove that

$$F(x) = \{f \in \mathcal{X}^* : \|f\| \leq \|x\| \text{ and } f(x) =< f, x >= \|x\|^2\}.$$

   Deduce that $F(x)$ is nonempty, closed and convex.

2. Prove that if $\mathcal{X}^*$ is strictly convex then $F(x)$ contains a single point. Show that in this case given $x \in \mathcal{X}$, $x \neq 0$ there exists a unique $g \in \mathcal{X}^*$ so that $\|g\| = 1$ and $g(x) = \|x\|$.

Problem 2: Let $\mathcal{X}$ be a uniformly convex normed vector space. Show that $\mathcal{X}$ is strictly convex.