R problem 1

Due Wednesday January 13

Problem 1: Let (X, d) be a metric space. Show that $(\mathcal{B}(X), \| \ \|_u)$ is a complete metric space. Prove that $\mathcal{BC}(X)$ is a closed subspace of $\mathcal{B}(X)$ in the uniform norm. In particular $\mathcal{BC}(X)$ is complete.

Problem 2: Let (X, d) be a compact separable metric space. Let $\mathcal{D} \subset X$ be a countable dense set. Let $\mathcal{F} \subset \mathcal{C}(X)$ be equicontinuous and pointwise bounded. Let $\{f_n\}_n \subset \mathcal{F}$.

1. Prove that there exists a subsequence $\{f_{n_k}\}_k \subset \{f_n\}_n$ which converges for every $x \in \mathcal{D}$.
2. Prove that $\{f_{n_k}\}_k$ converges pointwise in X to a continuous function f.
3. Show that $\{f_{n_k}\}_k$ converges uniformly in X to f above.