HOMEWORK 1

TI problems due Wednesday January 13 at 10pm PT

Problems from Folland: Chapter 3: 3, 9, 40.

Problem 1: Show that if F is of bounded variation in $[a, b]$, then:

$$\int_a^b |F'(x)|\, dx \leq T_a^b(F)$$

and

$$\int_a^b |F'(x)|\, dx = T_a^b(F)$$

if and only if F is absolutely continuous in $[a, b]$.

Problem 2: A bounded function $F : \mathbb{R} \to \mathbb{R}$ is said to be of bounded variation on $\mathbb{R}$ if F is of bounded variation on any finite interval $[a, b]$ and $\sup_{[a,b]} T_a^b(F) < \infty$. Prove that F enjoys the following properties:

1. $\int_{\mathbb{R}} |F(x + h) - F(x)|\, dx \leq A|h|$, for some constant A and all $h \in \mathbb{R}$.
2. $|\int_{\mathbb{R}} F(x)\varphi'(x)\, dx| \leq A$, where φ ranges over all C^1 functions with compact support satisfying $\sup_{x\in\mathbb{R}} |\varphi(x)| \leq 1$.

Problem 3 (Prelim 2018): Let F be an increasing right-continuous real-valued function on $[0, 1]$ with $F(0) = 0$. Prove that F can be written as

$$F = F_A + F_S + F_J,$$

where the functions F_A, F_S, and F_J are all increasing, $F_A(0) = F_S(0) = F_J(0) = 0$, and

1. F_A is absolutely continuous,
2. F_S is continuous and singular, that is, $F_S'(x) = 0$ for a.e. $x \in [0, 1]$, and
3. F_J is a right-continuous "jump function," meaning that there is an empty, finite, or countable subset $S \subset (0, 1]$ and positive constants $\{c_s\}_{s\in S}$ with $\sum_{s\in S} c_s < \infty$ for which

$$F_J(x) \;=\; \sum_{\{s\in S\,:\,s\leq x\}} c_s \quad \text{for all } x \in [0, 1],$$

where an empty sum is 0.

Moreover, show that these functions F_A, F_S, and F_J are uniquely determined by F.