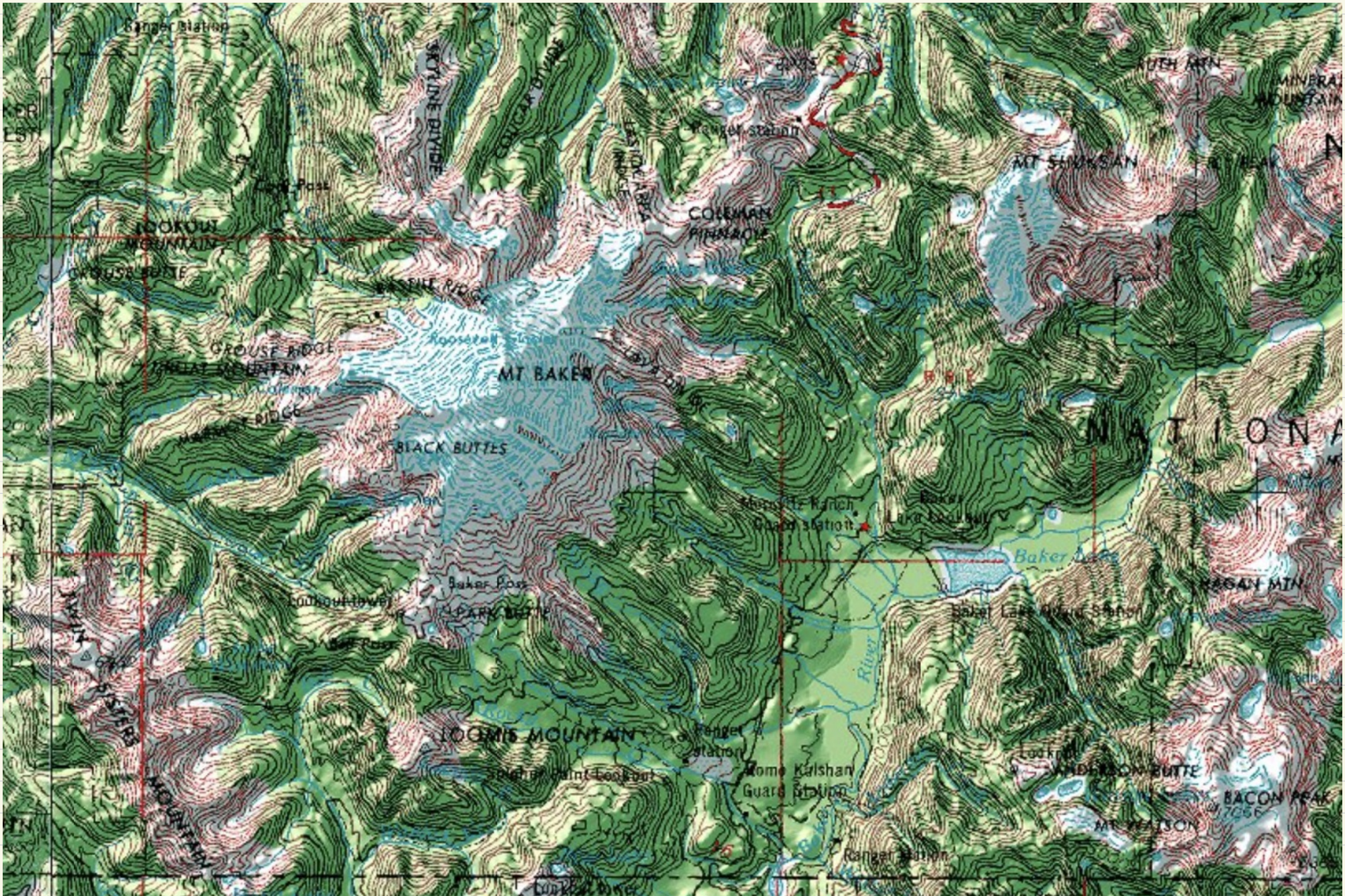
Goal : Show that if (X, d) is a metric space, there are plenty of continuous functions from X to $\mathbb{R}$



Lemma: Suppose (X, d) a metric space, A, B closed & disjoint sets in X. Let

$$\Delta = \{ k2^{-n} : n \geq 1, \ 0 < k < 2^n \}$$ dyadic rationals

There is a family $\{U_r\}_{r \in \Delta}$ of open sets in X such that

$$A \subset U_r \subset B^c \quad \forall r \in \Delta \ \&$$

$$\overline{U}_r \subset U_s \quad \text{for} \quad r < s$$

A, B closed $A \cap B = \emptyset$ $\exists U, W$ open s.t

$A \subset U$, $B \subset W$ & $U \cap W = \emptyset$

$$A = \overline{U}_0 \subset U \subset \underbrace{W^c}_{\text{closed}} \subset B^c = U_1$$

let $U_{1/2} = U$ open $\quad \overline{U}_0 \subset U_{1/2} \subset \overline{U}_{1/2} \subset W^c \subset U_1$

Induction on n for $r = k2^{-n}$ $\underbrace{0 < k < 2^{-n}}$

Suppose we have constructed U_r ↓ $n \leq m-1$

$r < s$

$$A \subset U_r \subset \overline{U}_r \subset U_s \subset B^c$$

we need to construct sets corresponding to $k2^{-m}$,
we already have $2j\,2^{-m}$ ones we need to construct
$(2j+1)\,2^{-m}$

$$\overline{\underbrace{U_{j\,2^{-m+1}}}} = \overline{U_{2j\,2^{-m}}} \subset \underbrace{U_{(j+1)\,2^{-m+1}}} = U_{2(j+1)2^{-m}}$$

$$\overline{\underbrace{U_{j2^{-m+1}}}_{\overline{U}_0}} \cap \underbrace{U^c_{(j+1)\,2^{-m+1}}}_{U_1^c} = \emptyset$$

same construction ↓

$\exists\ U_{(2j+1)2^{-m}}$ open set (as we constructed $U_{1/2}$)

$$\overline{U_{j2^{-m+1}}} \subset U_{(2j+1)2^{-m}} \subset \overline{U}_{(2j+1)2^{-m}} \subset U_{(j+1)2^{-m+1}}$$

Urysohn's lemma: Let (X,d) be a metric space. Let A, B be disjoint closed sets in X. There exists $f \in C(X, [0,1])$ such that $f = 0$ on A & $f = 1$ on B

Remark: Since for $x \in X$ $\{x\}$ is closed there are as many continuous functions as there are pairs of distinct points.

Pf: $\{U_r : r \in \Delta\}$ $\quad A \subset U_r \subset \bar{U}_r \subset U_s \subset B^c$ $\quad r < s$

Define $U_1 = X$, $x \in X$ $\quad \Delta = \{k2^{-n} : 0 < k < 2^{-n}, n \geq 1\}$

$f(x) = \inf\{r : x \in U_r\}$ define on X

$f(x) = 0$ in A, $f(x) = 1$ on B $\quad 0 \leq f \leq 1$

Claim: f is continuous

enough to show that $f^{-1}(-\infty, \alpha)$ & $f^{-1}(\alpha, +\infty)$ are open

a) $f(x) < \alpha$ iff $\underbrace{x \in U_r \text{ for some } r < \alpha}$

$$f^{-1}(-\infty, \alpha) = x \in \bigcup_{r<\alpha} U_r \quad \text{open set}$$

b) $f(x) > \alpha$ iff $\exists s > \alpha$ s.t $x \notin U_s$

iff $\exists s > \alpha$ st $x \in U_s^c$

iff $\exists r \in \Delta$ $s > r > \alpha$ s.t $(\bar{U}_r \subset U_s)$

$x \in U_s^c \subset (\bar{U}_r)^c$

$$f^{-1}(\alpha, +\infty) = \bigcup_{r>\alpha} (\bar{U}_r)^c \quad \text{open set}$$

f is continuous!