Theorem : Stone - Weierstrass

Let X be a compact metric space. Let $\mathcal{A} \subset C(X)$ be a closed algebra in $C(X)$ that separates points then either

i) $\mathcal{A} = C(X)$

ii) $\mathcal{A} = \{ f \in C(X) : f(x_0) = 0 \}$ for some $x_0 \in X$

The first alternative holds iff $\mathcal{A}$ contains the constant functions.

Proof : Claim : $\mathcal{A}$ is a lattice ✓

$x, y \in X$ $\quad \mathcal{A}_{xy} = \{ (f(x), f(y)) : f \in \mathcal{A} \} \subset \mathbb{R}^2$ is an algebra, that

$\mathcal{A}_{xy} = \mathbb{R}^2, \{(0,0)\}, \text{span}\{(1,0)\}, \text{span}\{(0,1)\}$

~~$\text{span}\{(1,1)\}$~~

Case 1 $\exists x_0 \in X$ s.t $\forall g \in \mathcal{A}$ $g(x_0) = 0$

$\mathcal{A}_{x_0 y} = \text{span}\{(0,1)\}$ $y \neq x_0$

Case 2: $\forall x \in X$ $\exists g_x = g \in \mathcal{A}$, $g(x) \neq 0$

$\mathcal{A}_{xy} \neq \text{span}\{(0,1)\}$, $\forall y \in X$ $\exists h \in \mathcal{A}$

$h(y) \neq 0 \Rightarrow \mathcal{A}_{x,y} = \mathbb{R}^2$

Claim for case 1: $\mathcal{A} = \{f \in C(X) : f(x_0) = 0\}$

Claim for case 2 $\mathcal{A} = C(X)$ $\mathcal{A}_{xy} = \mathbb{R}^2$ $(f(x), f(y)) \in \mathbb{R}^2$

$f \in C(X)$, $\mathcal{A}_{xy} = \mathbb{R}^2$, $\forall$ $\exists g_{xy} \in \mathcal{A}$ s.t

$g_{xy}(x) = f(x)$ & $g_{xy}(y) = f(y)$

$\varepsilon > 0$

$$U_{xy} = \{ z \in X : f(z) < g_{xy}(z) + \varepsilon \}$$ open (continuity of f & g_{xy})

$$V_{xy} = \{ z \in X : f(z) > g_{xy}(z) - \varepsilon \}$$ "

$x, y \in U_{xy} \cap V_{xy}$

Fix $y \in X$ $\{U_{xy}\}_{x \in X}$ is an open cover of X,

by compactness $\exists \{U_{x_i y}\}_{i=1}^{n}$ s.t. $X \subset \bigcup_{i=1}^{n} U_{x_i y}$

$g_y = g_{x_1 y} \vee \dots \vee g_{x_n y} \in \mathcal{A}$ (bec $\mathcal{A}$ is a lattice)

$\forall z \in X$ $\exists i$ $z \in U_{x_i y}$ $f(z) < g_{x_i y}(z) + \varepsilon \leq g_y(z) + \varepsilon$

$y \in V_y = \bigcap_{i=1}^{n} V_{x_i y}$ open ; if $z \in V_y$

$$f(z) \geq g_{x_i y}(z) - \varepsilon \geq g_y(z) - \varepsilon$$

$\uparrow$ $i = 1, \dots n$

$\{V_y\}_{y \in X}$ is an open cover of X, $\exists \{V_{y_i}\}_{i=1}^{m}$ finite subcover
(X compact)

(*) $g = g_{y_1} \wedge \cdots \wedge g_{y_m} \in \mathcal{A}$ (lattice

$\Rightarrow \forall z \in X \quad f(z) \leq g(z) + \varepsilon$ (1)

$\forall z \in X \quad \exists z \in V_{y_i}$ for some $i = 1 \ldots m$

$\Rightarrow \quad f(z) \geqslant g_{y_i}(z) - \varepsilon \geqslant g(z) - \varepsilon$ (2)

$\forall \varepsilon > 0 \quad \exists g \in \mathcal{A}$ s.t. $\|f - g\|_u \leq \varepsilon$ (1) & (2)

$\Rightarrow f \in \overline{\mathcal{A}} = \mathcal{A}$

Problem 1: Let (X, d) be a metric space. Show that:

1. If $x, y \in X$ and $x \neq y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$. Every metric space is Hausdorff or T_2

2. If $A, B \subset X$ are disjoint closed sets there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Every metric space is normal or T_4.

Problem 2: Let (X, d) and (Y, ρ) be a metric spaces. Let $f, g : X \to Y$ be continuous maps. Show that:

1. $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.

2. If $f = g$ on a dense subset of X, then $f = g$ on all of X.