Theorem : Stone - Weierstrass

Let X be a compact metric space. Let $\mathcal{A} \subset C(X)$ be a closed algebra in $C(X)$ that separates points then either

i) $\mathcal{A} = C(X)$

ii) $\mathcal{A} = \{ f \in C(X) : f(x_0) = 0 \}$ for some $x_0 \in X$

The first alternative holds iff $\mathcal{A}$ contains the constant functions.

Corollary 1 : Suppose $\mathcal{B} \subset C(X)$ is an algebra that separates points and contains constants then $\mathcal{B}$ is dense in $C(X)$.

Corollary 2 : Every continuous functions on a compact subset $K \subset \mathbb{R}^d$ can be uniformly approximated on K by polynomials in the coordinates.

Example: (1) $\mathcal{A} \subset C(C[0,1])$ closed algebra that separates points

$\mathcal{A}$ = generated $\{\sin nx\}_{n\in\mathbb{N}} \cup \{\cos nx\}_{n\in\mathbb{N}}$ $\quad 0\in\mathbb{N}$

$1 \in \mathcal{A}$ $\quad (n=0 \quad \cos 0 = 1)$ $\quad\quad \mathcal{A} = C[0,1]$

$\widetilde{\mathcal{A}}$ = generated by $\{\sin nx\}_{n\in\mathbb{N}}$ $\quad\quad \widetilde{\mathcal{A}} = \{f \in C[0,1] : f(0)=0\}$

(2) $X = \{a, b\}$ $\quad d(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$

$C(X) = \mathbb{R}^2$ $\quad\quad f : X \to \mathbb{R}$ $\quad (f(a), f(b))$

One way to prove Stone - Weierstrass in this case is to understand all the subalgebras in $\mathbb{R}^2$.

Lemma 1: Consider $\mathbb{R}^2$ as an algebra under coordinate wise addition & multiplication. The only subalgebras of $\mathbb{R}^2$ are $\mathbb{R}^2$, $\{(0,0)\}$ and the linear spans of $(0,1)$, $(1,0)$ or $(1,1)$.

Q: Which ones separate points in the previous examples?

span $\{(0,1)\}$, span $\{(1,0)\}$, $\mathbb{R}^2$ separate points

span $\{(0,0)\} = \{(0,0)\}$; span $\{(1,1)\}$ do not separate points

$$\mathcal{A} = \{ f \in C(X) : (f(a), f(b)) \in \text{span}\{(0,1)\} \}$$

Pf of lemma 1: $\mathcal{A} \subset \mathbb{R}^2$ $\quad \mathcal{A} \neq \{(0,0)\}$ ✓

$\exists (a,b) \in \mathcal{A}$ $\quad (a,b) \neq (0,0)$ & $(a^2, b^2) \in \mathcal{A}$

Want to understand if (a,b) & (a^2, b^2) are l.i. or not?

if $a = 0$, $b \neq 0$ $\quad \mathcal{A} = \text{span}\{(0,1)\}$ ✓

if $a \neq 0$ $\quad b = 0$ $\quad \mathcal{A} = \text{span}\{(1,0)\}$ ✓

if $\exists (a,b) \in \mathcal{A}$ s.t $a \neq 0$ & $b \neq 0$ then
(a,b) & (a^2, b^2) are l.i or not. $(\alpha(a,b) + \beta(a^2,b^2) = 0)$

$\alpha a + \beta a^2 = 0 \Rightarrow a(\alpha + \beta a) = 0 \Rightarrow \alpha + \beta a = 0$

$\alpha b + \beta b^2 = 0 \Rightarrow b(\alpha + \beta b) = 0 \Rightarrow \alpha + \beta b = 0$

$\Rightarrow$ $\begin{cases} \alpha = \beta = 0 \Rightarrow (a,b) \& (a^2,b^2) \text{ l.i} \quad \mathcal{A} = \mathbb{R}^2 \checkmark \\ a = b = -\alpha/\beta \Rightarrow \quad \mathcal{A} = \text{span}\{(1,1)\} \checkmark \end{cases}$

↑ if this happens $\forall (a,b) \in \mathcal{A}$

Lemma 2: $\forall \varepsilon > 0$ there exists a polynomial P on $\mathbb{R}$ such that

$P(0) = 0$ and

$$\big||s| - P(s)\big| < \varepsilon \qquad \forall\, s \in [-1,1]$$

Pf: Mac Laurin series for $t \in (-1,1)$

$$(1-t)^{1/2} = 1 + \sum_{n=1}^{\infty} -\frac{1}{2}\left(\frac{1}{2}\right)\cdots\left(\frac{2n-3}{2}\right)\frac{t^n}{n!}$$

$$= 1 - \sum_{n=1}^{\infty} c_n t^n \qquad c_n > 0$$

By the ratio test this series conv. unif for $|t| < 1$ (unif on compact subsets of $(-1,1)$)

$$\sum_{n=1}^{\infty} c_n = \lim_{t \to 1^-} \sum_{n=1}^{\infty} c_n t^n = \left(1 - \lim_{t \to 1^-} (1-t)^{1/2}\right)$$

↑ by MCT

$$\mu = \sum \delta_{\{n\}} \qquad f_t(n) = c_n t^n \qquad t < s < 1 \qquad f_t(n) \le f_s(n)$$

$t_k \to 1 \qquad t_k \le t_{k+1}$

$$f_k(n) = c_n t_k^n \qquad f_k \le f_{k+1}$$

$$f_k(n) \le c_n \qquad f_k(n) \xrightarrow[k\to\infty]{} c_n$$

by MCT

$$\sum_{n=1}^{\infty} c_n = \lim_{k\to\infty} \sum_{n=1}^{\infty} c_n t_k^n = 1$$

Thus $1 - \sum_{n=1}^{\infty} c_n t^n$ converges unif on $[0,1]$

$$0 \le \sum_{n=k}^{\infty} c_n t^n \le \sum_{n=k}^{\infty} c_n \longrightarrow 0$$

$t \le 1$ $\qquad k \to \infty$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t. $\forall t \in [0,1]$

$$\left| (1-t)^{1/2} - \left(1 - \sum_{n=1}^{N} c_n t^n\right) \right| < \varepsilon/2$$

$$\left|(1-t)^{1/2} - \left(1 - \sum_{n=1}^{N} c_n t^n\right)\right| < \varepsilon/2 \qquad \forall \varepsilon > 0 \;\; \exists N \;\; \forall t \in [0,1]$$

$t = 1 - s^2 \qquad s \in [-1,1]$

$$R(s) = 1 - \sum_{n=1}^{N} c_n (1-s^2)^n \qquad (1-t)^{1/2} = (s^2)^{1/2} = |s|$$

$$|\,|s| - R(s)| < \varepsilon/2 \qquad \overset{s=0}{\frown} \qquad |R(0)| < \varepsilon/2$$

$P(s) = R(s) - R(0)$

$$|\,|s| - P(s)| < \varepsilon \qquad \forall s \in [-1,1]$$

Proof of Stone - Weierstrass

Claim: If $\mathfrak{A} \subset C(X)$ is a closed algebra then $\mathfrak{A}$ is a lattice

Pf: $f \in \mathfrak{A}$ assume $\|f\|_u \leq 1$ by L2 given $\varepsilon > 0$

$\exists P_\varepsilon$ polynomial s.t $P_\varepsilon(0) = 0$ $\quad (*) \quad \big| |f| - P_\varepsilon(f) \big| < \varepsilon \quad (*)$

Since $\mathfrak{A}$ is an algebra, $\forall f \in \mathfrak{A}$ then $P_\varepsilon(f) \in \mathfrak{A}$

$\forall (*) \Longrightarrow |f| \in \overline{\mathfrak{A}} = \mathfrak{A}$. If $g \in \mathfrak{A}$ $\quad g \not\equiv 0$

$$f = g / \|g\|_u \in \mathfrak{A} \Longrightarrow |f| \in \mathfrak{A} \Longrightarrow |g| \in \mathfrak{A}$$

$$\|f\|_u = 1$$

$(*)$ $\forall x \in X$ $\quad |f(x)| \leq \|f\|_u \leq 1$ $\quad \big| |f(x)| - P_\varepsilon(f(x)) \big| < \varepsilon$

$$\big\| |f| - P_\varepsilon(f) \big\|_u \leq \varepsilon$$

if $f, g \in \mathcal{A}$

$$f \vee g = \max\{f, g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g| \in \mathcal{A}$$

$$f \wedge g = \min\{f, g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g| \in \mathcal{A}$$