Some useful inequalities

Chebyshev's inequality: If $f \in L^p(\mu)$ $(0 < p < \infty)$ then for any $\alpha > 0$

$$\mu\{x : |f(x)| > \alpha\} \leq \left(\frac{\|f\|_p}{\alpha}\right)^p$$

Pf: $E_\alpha = \{x : |f(x)| > \alpha\}$

$$\alpha^p \mu(E_\alpha) \leq \int_{E_\alpha} |f|^p d\mu \leq \int |f|^p d\mu = \|f\|_p^p$$

$$\mu(E_\alpha) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p$$

Minkowski's inequality for integrals

Suppose $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces

$f : X \times Y \longrightarrow \overline{\mathbb{R}}$ $\quad \mathcal{M} \otimes \mathcal{N}$ measurable

a) If $f \geq 0$ $\quad 1 \leq p < \infty$ then

$$\left(\int \left(\int f(x,y)\, d\nu(y) \right)^p d\mu(x) \right)^{1/p} \leq \int \left(\int f(x,y)^p d\mu(x) \right)^{1/p} d\nu(y)$$

b) If $1 \leq p \leq \infty$ $\quad f(\cdot, y) \in L^p(\mu)$ ν a.e y & $y \longmapsto \| f(\cdot, y) \|_{L^p(\mu)}$ is in $L^1(\nu)$ then

$f(x, \cdot) \in L^1(\nu)$ μ a.e x & $x \longmapsto \underbrace{\int f(x,y)\, d\nu(y)}_{h(x)}$ is in $L^p(\mu)$ &

$$\left\| \int f(\cdot, y)\, d\nu(y) \right\|_{L^p(\mu)} \leq \int \| f(\cdot, y) \|_{L^p(\mu)}\, d\nu(y)$$

$\| h \|_{L^p(\mu)}$

a) $p = 1$ Tonelli

$$h(x) = \int f(x,y)\, d\nu(y) \qquad \|h\|_{L^p(\mu)} \leq \; ?$$

$1 < p < \infty$ $\qquad \frac{1}{p} + \frac{1}{q} = 1$ $\qquad g \in L^q(\mu)$

$$\|h\|_{L^p(\mu)} = \sup\left\{ \left| \int h g \, d\mu \right| : \|g\|_{L^q(\mu)} \leq 1 \right\}$$

$$\left| \int h g \, d\mu(x) \right| \leq \int h\,|g|\, d\mu = \int \left(\int f(x,y)\, d\nu(y) \right) |g(x)|\, d\mu(x)$$

$$\leq \int \left[\int |g(x)|\, f(x,y)\, d\mu(x) \right] d\nu(y) \qquad \text{Tonelli}$$

$$\leq \int \|g\|_{L^q(\mu)} \left(\int f(x,y)^p \, d\mu(x) \right)^{1/p} d\nu(y) \qquad \text{Hölder}$$

$$\left| \int h g \, d\mu \right| \le \|g\|_{L^q(\mu)} \int \left(\int f(x,y)^p \, d\mu(x) \right)^{1/p} d\nu(y)$$

$$\|h\|_{L^p(\mu)} \le$$

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b) $p = \infty$

want

$$\left\| \int f(\cdot, y) \, d\nu(y) \right\|_{L^\infty(\mu)} \le \int \|f(\cdot, y)\|_{L^\infty} \, d\nu(y)$$

$$|f(x, y)| \le \|f(\cdot, y)\|_{L^\infty(\mu)} \quad \mu\text{-a.e.}$$

$$\left\| \int |f(\cdot, y)| \, d\nu(y) \right\|_{L^\infty(\mu)} \le \int \|f(\cdot, y)\|_{L^\infty(\mu)} \, d\nu(y).$$

What is left? : Topology & Elements of Fourier Analysis

Topology

- Stone - Weierstrass
- Urysohn's lemma (partition of unity)

(X, d) compact metric space (compact Hausdorff space)

$$C(X) = \{ f : X \to \mathbb{R} \text{ continuous} \}$$

$\|f\|_u = \sup_{x \in X} |f(x)|$ is a norm, and

$(C(X), \|\ \|_u)$ complete metric space.

Definitions: ① A subset $\mathcal{A} \subset C(X)$ is said to **separate points** if for every $x, y \in X$, $x \neq y$ there exists $f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$

(2) $\mathcal{A}$ is called an **algebra** if it is a vector subspace of $C(X)$ s.t $fg \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$.

(3) $\mathcal{A} \subset C(X)$ is called a **lattice** if whenever $f, g \in \mathcal{A}$

$$f \vee g = \max\{f, g\} \in \mathcal{A}$$

$$f \wedge g = \min\{f, g\} \in \mathcal{A}$$

Remark: If $\mathcal{A}$ is an algebra or a lattice so is $\overline{\mathcal{A}}$.

Theorem: Stone - Weierstrass

Let X be a compact metric space. Let $\mathcal{A} \subset C(X)$ be a closed algebra in $C(X)$ that separates points then either

i) $\mathcal{A} = C(X)$

ii) $\mathcal{A} = \{f \in C(X) : f(x_0) = 0\}$ for some $x_0 \in X$

The first alternative holds iff $\mathcal{A}$ contains the constant functions.

Corollary 1: Suppose $\mathcal{B} \subset C(X)$ is an algebra that separates points and contains constants then $\mathcal{B}$ is dense in $C(X)$.

Corollary 2: Every continuous functions on a compact subset $K \subset \mathbb{R}^d$ can be uniformly approximated on K by polynomials in the coordinates.

$\Rightarrow$ for any closed ball B polynomials with rational coefficients are dense in $C(B)$, in the uniform norm. Since $C(B)$ is dense in $L^p(B)$ for any $1 \leq p < \infty$, so are the polynomials with rational coefficients (why?).
Thus polynomials with rational coefficients are a countable dense set in $L^p(B)$ & $L^p(B)$ is separable. This implies that $L^p(\mathbb{R}^d)$ is separable (why?)

B = closed ball

Polynomials with $\mathbb{Q}$ coefficients $\rightsquigarrow$ $C(B)$ $\rightsquigarrow$ $L^p(B)$

dense in $\|\ \|_u$ (Stone-Weierstrass)

$1 \le p < \infty$, dense in $\|\ \|_p$

dense in $\|\ \|_p$

$$\left(\int_B |f-g|^p \, dm \right)^{1/p} \le \|f-g\|_u \, m(B)^{1/p} \qquad f, g \in C(B)$$

Claim $L^p(\mathbb{R}^n)$ is also separable

$$\mathbb{R}^n = \bigcup_{m=1}^{\infty} B_m \qquad m \in \mathbb{N}$$

$B_m = B(0,m)$

φ_m =



$$\mathcal{D} = \overline{\{ h = \varphi_m p \}}_{m,\, p \in \{\text{Polynomial with rational coefficients}\}}$$

countable & dense in $L^p(\mathbb{R}^n)$

$f \in L^p(\mathbb{R}^n)$ $\quad \forall \varepsilon > 0 \quad \int_{B_m^c} |f|^p \, dx < \varepsilon$

$\exists$ Q polynomial @ coefficients

(*) $\int_{B_{2m}} |f - Q|^p \, dx < \varepsilon \Rightarrow \int_{B_m} |f - Q|^p < \varepsilon$

$\left(\int_{\mathbb{R}^n} |f - Q\varphi_m|^p \right)^{1/p} \leq \underbrace{\left(\int_{B_m} |f - Q\varphi_m|^p \right)^{1/p}}_{< \varepsilon^{1/p}} + \left(\int_{B_{2m} \setminus B_m} |f - Q\varphi_m|^p \right)^{1/p}$

$\varphi_m \leq 1$ $\quad + \left(\int_{B_{2m}^c} |f|^p \, dx \right)^{1/p}$

$\left(\int_{B_{2m} \setminus B_m} |f - Q\varphi_m|^p \right)^{1/p} \leq \left(\int_{B_{2m} \setminus B_m} |f|^p \right)^{1/p} + \left(\int_{B_{2m} \setminus B_m} |Q|^p \right)^{1/p}$

$|\varphi_m| \leq 1$ $\qquad \sim \varepsilon^{1/p}$ $\qquad$ (*) $\Rightarrow \leq 2\varepsilon^{1/p}$

$$\left(\int_{B_{2m} \setminus B_m} |Q|^p \, dx \right)^{1/p} \leq \underbrace{\left(\int_{B_{2m} \setminus B_m} |f - Q|^p \, dx \right)^{1/p}}_{\leq \varepsilon^{1/p}} + \underbrace{\left(\int_{B_{2m} \setminus B_m} |f|^p \, dx \right)^{1/p}}_{\leq \varepsilon^{1/p}}$$