Spectral Theorem: Suppose T is a compact symmetric operator on a separable Hilbert space. Then there exists an o.n. basis $\{\varphi_k\}_k \subset \mathcal{H}$ that consists of eigenvectors of T. Moreover if

$$T\varphi_k = \lambda_k \varphi_k \quad \text{then} \quad \lambda_k \in \mathbb{R} \ \& \ \lambda_k \to 0 \ \text{as} \ k \to \infty$$

(1) Since T is symmetric, its eigenvalues are real. If f_1, f_2 are eigenvectors corresponding to different eigenvalues, then f_1 and f_2 are orthogonal.

(2) If $\lambda \neq 0$ (eigenvalue) $\dim (T - \lambda I)^{-1}\{0\}$ is finite (contradiction uses compactness)

(3) T has a countable set of eigenvalues $\{\lambda_k\}$, with $\lambda_k \to 0$
$\forall \mu > 0$, the dimension of linear space spanned by eigenvectors corresponding to λ_k with $|\lambda_k| > \mu$ is finite dimensional (contradiction uses compactness)

④ either $\|T\|$ or $-\|T\|$ is an eigenvalue of T

Proof of Spectral theorem

$\mathcal{S}$ = closure of the linear space spanned by all eigenvectors of T (by ④ $\mathcal{S} \neq \phi$)

either $\mathcal{S} = \mathcal{H}$ ✓ or $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$ & $\mathcal{S}^\perp \neq \phi$

note if $f \in \mathcal{S}$ then $Tf \in \mathcal{S}$

if $g \in \mathcal{S}^\perp$ then $\langle Tg, f \rangle = \langle g, Tf \rangle = 0$ ($f \in \mathcal{S}$)

$\Downarrow$

$Tg \in \mathcal{S}^\perp$

$T_1 = T|_{\mathcal{S}^\perp}$ $\mathcal{S}^\perp$ Hilbert space, T_1 compact & symmetric

T_1 has a non-zero eigenvector in $\mathcal{S}^\perp$, thus this is an eigenvector of T (which belongs to $\mathcal{S}^\perp$) ×

Let $\mathcal{H} = L^2(\mathbb{R}^d)$; for $f \in L^2(\mathbb{R}^d)$ define

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)\, f(y)\, dy$$; T is an integral operator, K is its associated kernel

If $K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ T is called a Hilbert - Schmidt operator

Proposition: Let T be a Hilbert - Schmidt operator in $L^2(\mathbb{R}^d)$ with kernel K.

i) If $f \in L^2(\mathbb{R}^d)$, for a.e x the function $y \to K(x,y) f(y)$ is integrable

ii) T is bounded for $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$ and

$$\|T\| \leq \|K\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$$

iii) T^* has kernel $\overline{K(y,x)}$

iv) T is compact ⇝ how do you approximate T by finite rank operators ?

$\{\varphi_k\}$ o.n. for $L^2(\mathbb{R}^d)$ then the collection $\{\varphi_k(x)\varphi_\ell(y)\}_{k,\ell\geq 1}$ o.n. basis for $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Let

$$K(x,y) = \sum_{k,\ell=1}^{\infty} a_{k\ell}\,\varphi_k(x)\,\varphi_\ell(y)$$

$$T_n f(x) = \int_{\mathbb{R}^d} K_n(x,y)\,f(y)\,dy \qquad \text{where}$$

$$K_n(x,y) = \sum_{k,\ell=1}^{n} a_{k\ell}\,\varphi_k(x)\,\varphi_\ell(y) \quad \text{then } T_n \text{ has finite-dimensional range}$$

$$\|K - K_n\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} = \sum_{k\geq n \text{ or } \ell \geq n} |a_{k\ell}|^2 \quad \text{as} \quad n \longrightarrow \infty$$

$$\|T - T_n\| \leq \|K - K_n\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$$