$(X, \mathcal{M}, \mu)$ complete σ-finite measure space

Theorem: Suppose $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $(L^p(\mu))^* = L^q(\mu)$

in the sense that for every bounded linear functional l on $L^p(\mu)$ there exists a unique $g \in L^q(\mu)$ s.t. $\forall f \in L^p(\mu)$

$$l(f) = \int f g \, d\mu \qquad \text{and} \qquad \|l\| = \|g\|_q$$

Pf: Step 1: μ finite Step 2: μ σ-finite

$l \in (L^p(\mu))^*$ $E \in \mathcal{M}$ $\chi_E \in L^p(\mu)$

$$\nu(E) = l(\chi_E)$$

Claim: ν is a signed measure, ν is finite, $\nu \ll \mu$

then by Radon-Nikodym $\exists g \in L^1(\mu)$

$$d\nu = g \, d\mu \qquad l(\chi_E) = \int_E g \, d\mu = \int \chi_E g \, d\mu$$

ν signed measure, finite, $\nu \ll \mu$

(*) $\boxed{|\nu(E)| = |\ell(\chi_E)| \le \|\ell\| \|\chi_E\|_p = \|\ell\| \mu(E)^{1/p}}$ $\Rightarrow$ ν is finite

ν is a signed measure (*) $\nu \ll \mu$

i) $\nu(\phi) = \ell(\chi_\phi) = \ell(0) = 0$ (ℓ linear)

ii) $\{E_j\}_{j} \subset \mathcal{A}$ $\quad E_j \cap E_i = \phi$ $\quad E = \bigcup_{j=1}^{\infty} E_j = \underbrace{\bigcup_{j=1}^{n} E_j \cup E_n^*}$

where $E_n^* = \bigcup_{j=n+1}^{\infty} E_j$ $\quad \chi_E = \sum_{j=1}^{n} \chi_{E_j} + \chi_{E_n^*}$ finite sum

$$\nu(E) = \ell(\chi_E) = \sum_{j=1}^{n} \ell(\chi_{E_j}) + \ell(E_n^*) = \sum_{j=1}^{n} \nu(E_j) + \underbrace{\nu(E_n^*)}_{\downarrow n\to\infty \; 0}$$

(ℓ linear)

$$\mu(E) = \sum_{j=1}^{n} \mu(E_j) + \mu(E_n^*) \le \mu(X) < \infty$$

μ is finite

$$\mu(E) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j)$$

$\Rightarrow \mu(E_n^*) \xrightarrow[n\to\infty]{} 0$ $\quad \overset{(*)}{\Rightarrow}$ $\quad |\nu(E_n^*)| \le \|\ell\| \mu(E_n^*)^{1/p} \xrightarrow[n\to\infty]{} 0$ (*)

By RN $g \in L^1(\mu)$ s.t $l(\chi_E) = \nu(E) = \int_E g\,d\mu \quad \forall E \in \mathcal{A}$

For all simple function $f \in L^p(\mu)$

$(**)$ $l(f) = \int f g\,d\mu \qquad \left|\int g f\,d\mu\right| \le \|l\| \|f\|_p$

$M = \sup \left\{ \left|\int f g\,d\mu\right| : f \in L^p(\mu) \text{ simple } \|f\|_p \le 1 \right\} \le \|l\|$

(ii) $g \in L^q(\mu) \qquad \|g\|_q = M$

Using $(**)$ & $g \in L^q(\mu)$ $l(f) = \int f g\,d\mu \Rightarrow \|l\| = \|g\|_q$

Case 2 : μ is σ-finite $\quad X = \bigcup_{n=1}^{\infty} E_n \quad E_n \subset E_{n+1} \quad \mu E_n < \infty$

$$L^p(E_n, \mu) = \{ f \in L^p(\mu) : f = 0 \text{ on } E_n^c \}$$

$$L^q(E_n, \mu) = \{ g \in L^q(\mu) : g = 0 \text{ on } E_n^c \}$$

$\exists \; l \in (L^p(\mu))^* \Rightarrow l \in (L^p(E_n, \mu))^*$

$\Rightarrow g_n \in L^q(\mu)$ s.t. $\forall f \in L^p(E_n, \mu)$

$$l(f) = \int f g_n \, d\mu$$

Claims : ① $\|g_n\|_q \le \|l\|$

② $g_n = g_m$ on $E_n \quad n \le m$

③ define $g = g_n$ on $E_n \Rightarrow g \in L^q(\mu)$, $\|g\|_q = \|l\|$

④ $l(f) = \int f g \, d\mu$

Corollary: If $1 < p < \infty$, $L^p(\mu)$ is reflexive

Definition: If $1 \leq p < \infty$, $\{f_n\} \subset L^p(\mu)$ we say that f_n converges weakly to f; $f_n \rightharpoonup f$ if for every $g \in L^q(\mu)$, with $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\int f_n g \, d\mu \xrightarrow[n \to \infty]{} \int f g \, d\mu$$