# The dual of $L^p$ when $1 \le p < \infty$

$(X, \mathcal{M}, \mu)$ complete $\sigma$-finite measure space

Suppose $1 \le p \le \infty$ & $\frac{1}{p} + \frac{1}{q} = 1$. Given $g \in L^q(\mu)$ consider

$\ell : L^p(\mu) \to \mathbb{R}$ defined by $\ell(f) = \int f g \, d\mu$

$\ell$ is a bounded linear functional. Recall

$$|\ell(f)| = \left| \int f g \, d\mu \right| \le \|g\|_q \|f\|_p \quad \text{thus}$$

$$\|\ell\| \le \|g\|_q$$

Thus $L^q(\mu) \subset (L^p(\mu))^*$ when $1 \le p \le \infty$.

Goal: Show that for $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

$$L^q(\mu) = (L^p(\mu))^*$$

not true for $p = \infty$!

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**Theorem**: Suppose $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $(L^p(\mu))^* = L^q(\mu)$

in the sense that for every bounded linear functional $l$ on $L^p(\mu)$ there exists a unique $g \in L^q(\mu)$ s.t. $\forall f \in L^p(\mu)$

$$l(f) = \int f g \, d\mu \quad \& \quad \|l\| = \|g\|_q$$

**Lemma**: Suppose $1 \leq p \leq \infty$ & $\frac{1}{p} + \frac{1}{q} = 1$

(i) If $g \in L^q(\mu)$ then

$$\|g\|_q = \sup \left\{ \left| \int f g \, d\mu \right| : \|f\|_p \leq 1 \right\}$$

(ii) Suppose $g$ is measurable &

$$M = \sup \left\{ \left| \int f g \, d\mu \right| : \|f\|_p \leq 1 \ \& \ f \text{ simple} \right\} < \infty$$

then $g \in L^q(\mu)$ & $\|g\|_q = M$.

Suppose $1 \le p \le \infty$ & $\frac{1}{p} + \frac{1}{q} = 1$

(i) If $g \in L^q(\mu)$ then

$$\|g\|_q = \sup \left\{ \left| \int fg \, d\mu \right| : \|f\|_p \le 1 \right\} =: \Lambda$$

$$\text{sign } x = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } 0 \end{cases}$$

if $g = 0$ ✓

$g \neq 0$ (a.e)

$\|g\|_q \neq 0$

By Hölder $\left| \int fg \, d\mu \right| \le \|f\|_p \|g\|_q \le \|g\|_q$

$\Lambda \le \|g\|_q$

find $f$ s.t $\|f\|_\infty = 1$

① $q = 1$ $p = \infty$ $\int fg \, d\mu = \int |g| \, d\mu$ ✓

$f(x) = \text{sign } g(x)$ $\|f\|_\infty = 1$

② $1 < p, q < \infty$ ( looking for $f$ s.t. $\int fg\,d\mu = \|g\|_q$ , $\|f\|_p \le 1$ )

$$f(x) = \operatorname{sign} g(x)\ \frac{|g(x)|^{q-1}}{\|g\|_q^{q-1}}$$

$$\int \operatorname{sign} g(x)\ \frac{|g(x)|^{q-1}}{\|g\|_q^{q-1}}\ g(x)\,d\mu = \int \frac{|g(x)|^q}{\|g\|_q^{q-1}} = \|g\|_q$$

$$\int |f|^p\,d\mu = \int \frac{|g(x)|^{(q-1)p}}{\|g\|_q^{(q-1)p}} = \int \frac{|g(x)|^q}{\|g\|_q^q} = 1$$

$(q-1)p = q$

③ $q = \infty$ & $p = 1$ , let $\varepsilon > 0$ there exists $E \in \mathcal{M}$

$\mu E > 0$ & $\mu E < \infty$ $\quad \forall x \in E \quad |g(x)| \ge \|g\|_\infty - \varepsilon$

↑ def $\|\ \|_\infty$ ↑ bec $\mu$ - $\sigma$ - finite

$$f(x) = \operatorname{sign} g(x) \frac{\chi_E}{\mu(E)} \qquad \int |f| d\mu = \frac{\mu(E)}{\mu(E)} = 1$$

$$\int fg\, d\mu = \int_E \frac{|g|\, d\mu}{\mu E} \geq \frac{(\|g\|_\infty - \varepsilon)\mu E}{\mu E} = \|g\|_\infty - \varepsilon$$

$$\sup_{\|f\|_1 \leq 1} \left\{ \left| \int fg\, d\mu \right| \right\} \geq \|g\|_\infty - \varepsilon \qquad \forall \varepsilon > 0.$$

ii) Suppose $g$ is measurable &

$$M = \sup \left\{ \left| \int fg\, d\mu \right| : \|f\|_p \le 1 \ \& \ f \text{ simple} \right\} < \infty$$

Then $g \in L^q(\mu)$ & $\|g\|_q = M$. / $X = \bigcup_{n=1}^{\infty} X_n$ — $\mu X_n < \infty$

$1 < p, q < \infty$ $\quad X_n \subset X_{n+1}$

$g$ measurable $\quad |g_n|^q \le |g_{n+1}|^q \le \dots \le |g|^q \quad g_n \sim g_n \chi_{X_n}$ $\quad \{g_n\}$ simple

Fatou $\quad \int |g|^q \, d\mu \le \liminf \int |g_n|^q d\mu$ $\quad \Big| \ |g_n| \to |g|$ $\mu$-a.e

MCT $\quad \int |g|^q \, d\mu = \lim \int |g_n|^q \, d\mu$

if we show that $\sup_n \int |g_n|^q \, d\mu < \infty$

$$f_n(x) = \operatorname{sign} g(x) \frac{|g_n(x)|^{q-1}}{\|g_n\|_q^{q-1}} \longleftrightarrow \begin{cases} f_n \text{ simple} \\ \|f_n\|_p = 1 \end{cases}$$

$$M \ge \int f_n(x) g(x) \, d\mu = \frac{1}{\|g_n\|_q^{q-1}} \int |g| \, |g_n|^{q-1} \, d\mu \ge \frac{\int |g_n|^q \, d\mu}{\|g_n\|_q^{q-1}}$$

$p=1 \quad q=\infty \qquad A_\varepsilon = \{x : |g(x)| > M+\varepsilon\}$

If $\mu A_\varepsilon > 0$, since $X = \bigcup_{n=1}^{\infty} X_n \qquad X_n \subset X_{n+1} \qquad \mu X_n < \infty$

$\exists\, B \subset A_\varepsilon \qquad 0 < \mu B < \infty$

$f = \operatorname{sign} g(x) \dfrac{\chi_B}{\mu B}$ simple $\qquad \|f\|_1 = 1$

$f = 0$ outside $B$

$$M \geq \int f g \, d\mu = \frac{1}{\mu B} \int |g| \, d\mu \geq M+\varepsilon \qquad *$$

$\mu A_\varepsilon = 0 \quad \forall \varepsilon > 0$

$\Rightarrow g \in L^\infty \qquad \|g\|_\infty \leq M$ by Hölder other ineq.

$$M \geq \int f_n g \, d\mu \geq \|g_n\|_q \Rightarrow \|g\|_q \leq M$$

By Hölder $M \leq \|g\|_q$.