Proposition: For $1 \le p < \infty$ the set of simple functions $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ where $\mu E_i < \infty$ if $a_i \neq 0$ is dense in $L^p(\mu)$.

Pf: $f \in L^p(\mu)$ Theo 2.10 $\exists \varphi_n$ simple functions s.t

$\varphi_n \to f$ μ a.e

$0 \le |\varphi_n| \le |\varphi_{n+1}| \le \dots \le |f| \implies |\varphi_n|^p \le |f|^p$

Moreover $|\varphi_n - f|^p \le 2^p |f|^p$ $\quad \Downarrow \quad \varphi_n \in L^p$

LDCT $\int |\varphi_n - f|^p \, d\mu \to 0$

$\varphi_n = \sum_{j=1}^{m} a_j^n \chi_{E_j^n}$ standard rep.

$$\int |\varphi_n|^p \, d\mu = \sum_{j=1}^{m} |a_j^n|^p \mu(E_j^n) \le \int |f|^p d\mu < \infty$$

$\uparrow$

if $a_j^n \neq 0$ then $\mu(E_j^n) < \infty$

Theorem: For $1 \le p < \infty$, $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Pf Let $S = \{ f : \mathbb{R}^n \to \mathbb{R} \quad f = \sum_{i=1}^{l} a_i \chi_{E_i}$ where $m E_i < \infty$ if $a_i \neq 0 \}$ is dense in $L^p(\mathbb{R}^n)$

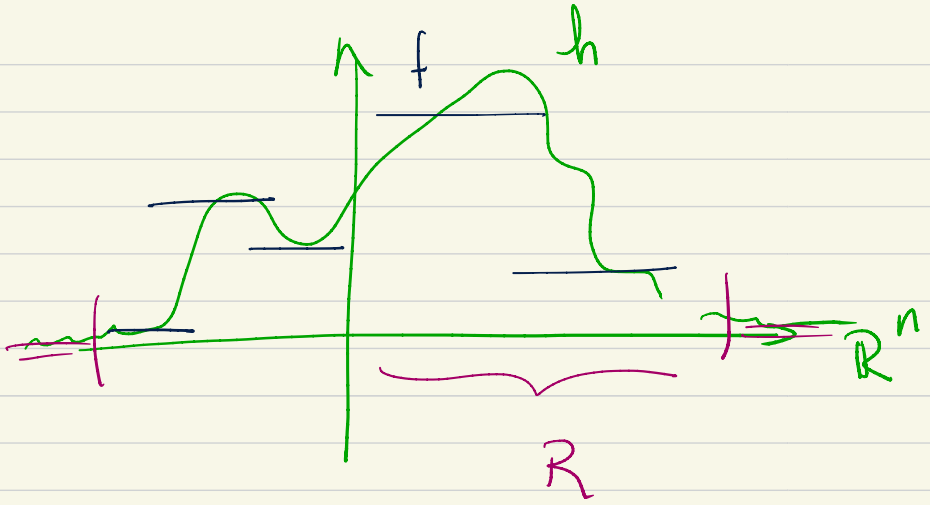
$h \in L^p(\mu)$ we want for every $\varepsilon > 0$ find $g_\varepsilon \in C_c(\mathbb{R}^n)$ s.t

$$\| f - g_\varepsilon \|_p < \varepsilon$$

Since $h \in L^p(\mathbb{R}^n)$ $\exists R > 0$ s.t $\int_{B_R^c} |h|^p \, dm < \varepsilon$

$\exists f \in S$ s.t $\int |f - h|^p \, dm < \varepsilon$, replace f by $f \chi_{B_R}$

$$\int |f \chi_{B_R} - h|^p \, dm = \int_{B_R} |f - h|^p \, dm + \int_{B_R^c} |h|^p \, dm < 2\varepsilon$$



By Lusin's theorem

$\exists g \in C(B_{2R})$ s.t

$$m\{x \in B_{2R} : g(x) \neq \chi_{B_R} f(x)\} < \varepsilon$$

w.l.o.g. $g \equiv 0$ on B_{2R}^c

$$\left(\int |g-h|^p \, dm\right)^{1/p} \leq \left(\int |g - f\chi_{B_R}|^p\right)^{1/p} + \left(\int |f\chi_{B_R} - h|^p dm\right)^{1/p}$$

$$< (2\varepsilon)^{1/p} + M\varepsilon^{1/p} \qquad \leq (2\varepsilon)^{1/p} + M\, m\{g \neq \chi_{B_R} f\}^{1/p}$$

Lusin's ✓ $\quad |g| \leq \sup |f\chi_R|$

$$|g - \chi_{B_R} f| \leq |g| + |f\chi_{B_R}| \leq |g| + \sup |f\chi_{B_R}|$$

$$\leq 2 \sup |f\chi_{B_R}|$$

M

$(X, \mathcal{M}, \mu)$ complete measure space

$$L^\infty(\mu) = \{ f : X \to \mathbb{R}(\mathbb{C}) : \exists M > 0 \text{ s.t. } |f(x)| \le M \ \ \mu \text{ a.e } x \in X \}$$

$$\|f\|_\infty = \inf \{ M \in \mathbb{R} : \mu\{x \in X : |f(x)| > M\} = 0 \}$$

Remark : Note that $|f(x)| \le \|f\|_\infty$ μ a.e $x \in X$

$$E = \{x : |f(x)| > \|f\|_\infty\} = \bigcup_n \underbrace{\{x : |f(x)| \ge \|f\|_\infty + \tfrac{1}{n}\}}_{E_n}$$

$$\Rightarrow \mu E_n = 0 \quad (\text{by def } \|f\|_\infty) \Rightarrow \mu E = 0$$

Theorem : $(L^\infty(\mu), \|\ \|_\infty)$ is a Banach space.

Note that if $p = 1$ & $q = \infty$ then $\frac{1}{p} + \frac{1}{q} = 1$ then if $f \in L^1(\mu)$ & $g \in L^\infty(\mu)$ then

$fg \in L^1(\mu)$ and

$$\int |fg| \, d\mu \le \|g\|_\infty \int |f| \, d\mu = \|g\|_\infty \|f\|_1$$

Proposition : If $f \in L^\infty(\mu)$ is supported on a set of finite measure. Then $f \in L^p(\mu)$ for $1 \le p < \infty$ and

$$\|f\|_p \longrightarrow \|f\|_\infty \quad \text{as} \quad p \longrightarrow \infty.$$

Pf $E \in \mathcal{A}$ $f = 0$ on E^c & $\mu E < \infty$

If $\mu E = 0$ $f = 0$ μ-a.e ✓ $\|f\|_\infty = \|f\|_p = 0$

$0 < \mu E < \infty$ $\|f\|_p = \left(\int_E |f|^p d\mu\right)^{1/p} \le \|f\|_\infty \, \mu(E)^{1/p}$

$p \to \infty$ $\quad \limsup_{p \to \infty} \|f\|_p \le \|f\|_\infty$ ✓

Given $\varepsilon > 0$ $\exists \delta > 0$ $\varepsilon \le \|f\|_\infty / 2$

$$\mu\{\underbrace{x : |f(x)| \ge \|f\|_\infty - \varepsilon}_{A}\} = \delta > 0$$

$$\left(\int |f|^p d\mu\right)^{1/p} \ge \left(\int_A |f|^p d\mu\right)^{1/p} \ge \left[\mu(A)\,(\|f\|_\infty - \varepsilon)^p\right]^{1/p} = (\|f\|_\infty - \varepsilon)\,\mu(A)^{1/p}$$

$$\left(\int |f|^p \, d\mu \right)^{1/p} \geq \delta^{1/p} \left(\|f\|_\infty - \varepsilon \right)$$

$p \to \infty$

$$\liminf_{p \to \infty} \|f\|_p \geq \|f\|_\infty - \varepsilon$$

$\forall \varepsilon > 0$

$$\liminf_{p \to \infty} \|f\|_p \geq \|f\|_\infty$$