Theorem (Hölder's inequality)

conjugate exponents

Suppose $1 < p < \infty$ & $1 < q < \infty$ s.t $\frac{1}{p} + \frac{1}{q} = 1$. Then

if $f \in L^p(\mu)$ & $g \in L^q(\mu)$ then $fg \in L^1(\mu)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \qquad (*)$$

Recall: if $a \geq 0$, $b \geq 0$ $0 < \lambda < 1$

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

Pf: $\|f\|_p = 0$ or $\|g\|_q = 0 \Rightarrow f = 0$ μ a.e or $g = 0$ μ a.e ✓

$\|f\|_p \neq 0$ & $\|g\|_q \neq 0$

$F = \frac{f}{\|f\|_p}$ $G = \frac{g}{\|g\|_q}$

$\|F\|_p = \|G\|_q = 1$

enough to show $\|FG\|_1 \leq 1 \Rightarrow (*)$

$a = |F(x)|^p \qquad b = |G(x)|^q \qquad a^\lambda b^{1-\lambda} \quad ?$

$\lambda = 1/p \qquad 1-\lambda = 1-1/p = 1/q$

$$a^\lambda b^{1-\lambda} = |F(x)||G(x)| \underset{\uparrow \text{lemma}}{\leq} \frac{1}{p}|F(x)|^p + \frac{1}{q}|G(x)|^q$$

$$\int |F(x)G(x)|\,d\mu \leq \frac{1}{p}\int |F(x)|^p d\mu + \frac{1}{q}\int |G(x)|^q d\mu$$

$$\leq \frac{1}{p} + \frac{1}{q} = 1$$

Theorem (Minkowski's inequality)

If $1 \le p < \infty$ and $f, g \in L^p(\mu)$ then $f+g \in L^p(\mu)$

&

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

$$(|f+g|^p \le 2^p(|f|^p + |g|^p))$$

Pf $p=1$ ✓ $p>1$ $\quad |f+g|^p \le |f+g|^{p-1}(|f|+|g|)$

$\frac{1}{p}+\frac{1}{q}=1$ $\quad \int |f+g|^p \, d\mu \le \int |f+g|^{p-1}|f| + \int |f+g|^{p-1}|g|$

Hölder's ineq. (r.h.s integrals) (p to $|f|$, q in $|f+g|^{p-1}$)

$$\int |f+g|^p \, d\mu \le \left(\int |f+g|^{\overbrace{(p-1)q}^{p}} \, d\mu \right)^{1/q} \left(\int |f|^p \, d\mu \right)^{1/p}$$

$$+ \left(\int |f+g|^{\overbrace{(p-1)q}^{p}} \, d\mu \right)^{1/q} \left(\int |g|^p \, d\mu \right)^{1/p}$$

$$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \Rightarrow (p-1)q = p$$

$$\int |f+g|^p \, d\mu \leq \left(\int |f+g|^p \, d\mu\right)^{1/q} \left[\left(\int |f|^p \, d\mu\right)^{1/p} + \left(\int |g|^p \, d\mu\right)^{1/p}\right]$$

$$\|f+g\|_p^p \leq \|f+g\|_p^{p/q} \left(\|f\|_p + \|g\|_p\right) \longleftarrow$$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

$$\Rightarrow \|f+g\|^{p-p/q} \leq \|f\|_p + \|g\|_p$$

$$p - p/q = p\left(1 - \frac{1}{q}\right) = p\,\frac{1}{p} = 1$$

Theorem: The space $L^p(\mu)$ is a complete normed vector space in the norm $\| \ \|_p$. $(1 \le p < \infty)$

Pf (Thm 5.1) enough to show absolutely convergent series converges in $L^p(\mu)$ $\quad \{f_k\} \subset L^p(\mu)$

$$B = \sum_{k=1}^{\infty} \|f_k\|_p < \infty$$

$$G_n = \sum_{k=1}^{n} |f_k| \in L^p(\mu) \qquad G = \sum_{k=1}^{\infty} |f_k| \in L^p(\mu)$$

$$0 \le G_n \le G_{n+1} \le \dots \le G \qquad G_n \nearrow G$$

$$\|G_n\|_p \le \sum_{k=1}^{n} \|f_k\|_p \le B < \infty$$

Claim: $G \in L^p(\mu)$: $\quad 0 \le G_n^p \le G_{n+1}^p \le \dots \le G^p$

MCT $\quad G_n^p \nearrow G^p$

$$\int G^p \, d\mu = \lim_{n\to\infty} \int G_n^p \le B^p < \infty$$

$G \in L^p(\mu) \Rightarrow G^p(x) < \infty$ μ-a.e $\quad G(x) < \infty$ μ-a.e

$\Rightarrow \sum_{k=1}^{\infty} |f_k|$ converges μ-a.e.

$\Rightarrow \sum_{k=1}^{\infty} f_k$ converges μ-a.e $\quad F(x) = \sum_{k=1}^{\infty} f_k(x)$

We want $\int |F(x) - \sum_{k=1}^{n} f_k|^p \, d\mu \longrightarrow 0$ $\quad (*)$
$n \to \infty$

$|F| \le G$ & $\sum_{k=1}^{n} f_k \longrightarrow F$ μ-a.e.

$|F - \sum_{k=1}^{n} f_k|^p \le (|F| + G_n)^p \le (|F| + G)^p \le \underbrace{(2G)^p}_{\in L^1(\mu)}$

LDCT $(*)$ holds.