Radon - Nikodym Theorem

Theorem (Radon - Nikodym) Let μ be a σ-finite measure on $(X, \mathcal{M})$ and let ν be a σ-finite signed measure on $(X, \mathcal{M})$.
There exist unique signed measures ν_a & ν_s on $\mathcal{M}$ such that

$$\nu_a \ll \mu, \quad \nu_s \perp \mu \quad \& \quad \nu = \nu_a + \nu_s \qquad d\nu = f d\mu + \nu_s$$

Moreover ν_a has the form $d\nu_a = f d\mu$ for some f s.t. either $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$ RN decomposition

Proof: due to Von Neumann

Case 1: μ & ν (non-negative) finite measures

Case 2: μ & ν σ-finite measures

Case 3: ν signed σ-finite measure

Case 1: μ & ν finite measures, $\rho = \mu + \nu$, $l : L^2(\rho) \to \mathbb{R}$

$l(\psi) = \int \psi \, d\nu$ is a bounded linear functional by

RRT $\exists!\ g \in L^2(\rho)$ s.t $\forall \psi \in L^2(\rho)$

$$l(\psi) = \int \psi \, d\nu = \int \psi g \, d\rho$$

Claim 1: $0 \le g \le 1$ ρ a.e $x \in X$

$$\boxed{\int \psi (1-g) \, d\nu = \int \psi g \, d\mu} \quad (**)$$

$\hookrightarrow \int \tilde{\psi} \chi_B g \, d\mu$

$= \int_B \psi \, d\mu$

$= 0$ since $\mu(B) = 0$

$A = \{x : 0 \le g(x) < 1\}$ $\quad B = \{g = 1\}$

$\nu_a(E) = \nu(E \cap A)$ $\quad \nu_s(E) = \nu(E \cap B)$

Claim 2 $\nu_s \perp \mu$ & $\nu_a \ll \mu$

$$\boxed{\int \psi\,(1-g)\,d\nu = \int \psi g\, d\mu} \qquad (\star\star)$$

$$A = \{\, 0 \le g < 1 \,\} \qquad \psi = \chi_E\,(1 + g + \dots + g^n)$$
$$E \in \mathcal{M}$$

$$\int_E (1+g+\dots+g^n)(1-g)\,d\nu = \int_E (1+g+\dots+g^n)\,g\,d\mu$$

$$\int_E (1-g^{n+1})\,d\nu = \int_E g\,(1+g+\dots+g^n)\,d\mu \qquad (\star\star\star)$$
$$n \to \infty \ ?$$

if $x \in B$ $\quad g(x) = 1 \quad 1 - g^{n+1}(x) = 0$

if $x \in A$ $\quad 1 \geq 1 - g^{n+1}(x) \xrightarrow[n \to \infty]{} 1$

$$\int_E (1-g^{n+1})\,d\nu \xrightarrow[n \to \infty]{}$$

$$\int_E (1-g^{n+1})\,d\nu = \int_{E\cap A} (1-g^{n+1})\,d\nu \longrightarrow \int_{E\cap A} d\nu = \nu(E\cap A) = \nu_a(E)$$

$$= \int_E g(1+\cdots g^n)\,d\mu \qquad (E'' = E\cap A \cup E\cap B)$$

$\mu(B)=0$

$$= \int_{E\cap A} g(1+\cdots +g^n)\,d\mu \xrightarrow[\text{MCT}]{n\to\infty} \int_{E\cap A} f\,d\mu = \int_E f\,d\mu$$

LDCT since ν is finite

$f=0$ on B, $\mu(B)=0$

$$0 \le g(1+g+\cdots+g^n) \le g(1+\cdots g^n+g^{n+1})$$

$0\le g<1$

$$\xrightarrow[n\to\infty]{} g\,\frac{1}{1-g} = \frac{g}{1-g} = f$$

$$\nu_a(E) = \int_E f\,d\mu$$

ν is finite $\quad f\ge 0 \quad 0 \le \int_X f\,d\mu = \nu_a(X) \le \nu(X) < \infty$

$\Rightarrow f\in L^1(\mu)$

Case 2 : μ & ν σ-finite $\exists \{E_j\} \subset \mathcal{M}$

$X = \bigcup_{j=1}^{\infty} E_j$ $\quad E_j \cap E_i = \emptyset$ $\quad i \neq j$ $\quad \mu E_i, \nu E_i < \infty$

$\mu_j(E) = \mu(E \cap E_j)$ $\quad \nu_j(E) = \nu(E \cap E_j)$

μ_j, ν_j finite measure $\xrightarrow[\text{case 1}]{\Rightarrow}$ $\nu_j = \nu_{j,a} + \nu_{j,s}$

$d\nu_{j,a} = f_j \, d\mu_j$ $\quad \nu_{j,s} \perp \mu$

f_j is supported in E_j

$f = \sum f_j$ $\quad \nu_s = \sum \nu_{j,s}$ $\quad \nu_a = \sum \nu_{j,a}$

Claim 3 $\quad \nu_s \perp \mu$ $\quad \nu_a \ll \mu$

$d\nu_a = f \, d\mu$

Case 3 : ν is signed σ-finite

$\nu = \nu^+ - \nu^-$ $\quad$ $\nu^\pm$ σ-finite measures

$$\nu^+ = \nu_a^+ + \nu_s^+$$
$$\nu^- = \nu_a^- + \nu_s^-$$

$$\nu = (\nu_a^+ - \nu_a^-) + (\nu_s^+ - \nu_s^-)$$

$\nu_a^\pm \ll \mu$ $\quad$ $\nu_s^\pm \perp \mu$

$$d\nu_a^+ = f^+ d\mu$$
$$d\nu_a^- = f^- d\mu$$

Uniqueness: $\nu = \nu_a + \nu_s = \nu_a' + \nu_s'$

$\nu_a, \nu_a' \ll \mu$ $\quad$ $\nu_s, \nu_s' \perp \mu$

$$\underbrace{\nu_a - \nu_a'}_{\ll \mu \text{ \& } \perp \mu} = \underbrace{\nu_s' - \nu_s}_{\ll \mu \text{ \& } \perp \mu} = 0 \Rightarrow$$

(do subtraction on each E_j above)

Theorem: Let μ be a Borel measure in $\mathbb{R}^d$ such that for any ball $B \subset \mathbb{R}^d$, $\mu B < \infty$ - Let

$d\mu = f\,dm + \lambda$ be the Radon - Nikodym representation (ie $\lambda \perp \mu$) - Then for m - a.e $x \in \mathbb{R}^d$

$$f(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = \lim_{r \to 0} \frac{\mu(B(x,r))}{\omega_d r^d}$$

To show this $\lim_{r \to 0} \frac{\lambda(B(x,r))}{m(B(x,r))} = 0$ m a.e

L^p spaces

Let $(X, \mathcal{M}, \mu)$ be a σ-finite complete measure space. For $1 \leq p < \infty$ let

$$L^p(\mu) = \{ f : X \longrightarrow \mathbb{R}(\mathbb{C}) \text{ measurable} : \int_X |f|^p d\mu < \infty \}$$

$$\|f\|_{L^p(\mu)} = \|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}$$

Remark : $\|f\|_p = 0$ $\Rightarrow$ $f = 0$ μ a.e

Remark: $L^p(\mu)$ is a vector space

$f, g \in L^p(\mu) \overset{?}{\Rightarrow} f+g \in L^p(\mu)$

$|f+g|^p \leq (2\max\{|f|,|g|\})^p$

$\int |f+g|^p d\mu \leq 2^p\left(\int |f|^p d\mu + \int |g|^p d\mu\right) < \infty$

$\leq 2^p \max\{|f|^p, |g|^p\}$

$\leq 2^p (|f|^p + |g|^p)$

Goal: Show that $\|\ \|_p$ is a norm

Lemma: If $a \geq 0$ & $b \geq 0$; $0 < \lambda < 1$ then

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

with equality iff $a = b$.

we want

$\frac{a^\lambda}{b^\lambda} \leq \lambda \frac{a}{b} + (1-\lambda)$

& $b = 0$ ✓ $b \neq 0$ divide by b

$t = \frac{a}{b}$ $t^\lambda \leq \lambda t + (1-t)$ show this

equality $t = 1$

to show $t^\lambda \le \lambda t + (1-\lambda)$

$f(t) = t^\lambda - \lambda t$

$f'(t) = \lambda t^{\lambda-1} - \lambda$

$f'(1) = 0$

$0 < t < 1 \qquad f'(t) > 0$

$t > 1 \qquad f'(t) < 0$

$f(t) \le f(1) \qquad \forall t$

$t^\lambda - \lambda t \le 1 - \lambda$