Problem 2 (R problem set 4)

Let $\mathcal{H}$ be a separable Hilbert space. Let $T \in \mathcal{L}(H, H)$ symmetric $T^* = T$ then

$$\|T\| \stackrel{?}{=} \sup\{|\langle Tf, f\rangle| : \|f\| = 1\} = M$$

$$(|\langle Th, h\rangle| \leq M\|h\|^2)$$

Recall

(1) $\|T\| = \sup\{|\langle Tf, g\rangle| : \|f\| \leq 1 \text{ & } \|g\| \leq 1\}$

(2) $\langle Tf, g\rangle = \langle f, T^* g\rangle = \langle f, Tg\rangle$ if $T = T^*$

Remark: $h \in \mathcal{H}$ $\quad \langle Th, h\rangle = \langle h, Th\rangle$

$$= \overline{\langle Th, h\rangle}$$

$\Rightarrow \langle Th, h\rangle \in \mathbb{R}$ $\quad \|f\| = 1$

$$|\langle Tf, f\rangle| \underset{\text{CS}}{\leq} \|Tf\|\,\|f\| \leq \|T\|\,\|f\|^2 = \|T\|$$

$\Rightarrow M \leq \|T\|$

"Polarization formula"

$\in \mathbb{R}$ (real part)

$$\langle Tf, g\rangle = \frac{1}{4}\Big[\langle T(f+g), f+g\rangle - \langle T(f-g), f-g\rangle + i\langle T(f+ig), f+ig\rangle - i\langle T(f-ig), f-ig\rangle\Big]$$

(imaginary part)

$$\langle Tf, g\rangle = |\langle Tf, g\rangle| e^{i\theta}$$

$$\langle Tf, \underbrace{e^{i\theta} g}_{\in \mathbb{R}}\rangle = e^{-i\theta}\langle Tf, g\rangle = |\langle Tf, g\rangle| \in \mathbb{R}$$

$$|\langle Tf, g\rangle| = \mathrm{Re}\,\langle Tf, e^{i\theta} g\rangle$$

polarization

$$= \frac{1}{4}\Big[\langle T(f+e^{i\theta} g); f+e^{i\theta} g\rangle - \langle T(f-e^{i\theta} g), f-e^{i\theta} g\rangle\Big]$$

$$\leq \frac{M}{4}\Big[\|f+e^{i\theta} g\|^2 + \|f-e^{i\theta} g\|^2\Big]$$

$$|\langle Tf, g\rangle| \le \frac{M}{4}\left[\|f+e^{i\theta}g\|^2 + \|f-e^{i\theta}g\|^2\right]$$ // law

$$\le \frac{M}{4}\left[2\|f\|^2 + 2\|e^{i\theta}g\|^2\right]$$

$$\le \frac{M}{4}\left[2\|f\|^2 + 2\|g\|^2\right]$$ $\|f\| \le 1$, $\|g\| \le 1$

$|\langle Tf, g\rangle| \le M$ taking sup on $f$, & $g$ ↑

$\Rightarrow \|T\| \le M$

Example of an unbounded linear operator on $L^2[0,1]$

"derivative operator unbounded"

$$f_n = \sqrt{2} \sin 2\pi n t \qquad n \geq 1$$

$$\int_0^1 f_n^2 \, dt = 2 \int_0^1 \frac{1 - \cos 4\pi n t}{2} \, dt = 1$$

$$Tf_n = 2\pi n \sqrt{2} \cos 2\pi n t$$

$$\|Tf_n\|_2^2 = (2\pi n)^2$$

$$\|Tf_n\| \xrightarrow[n \to \infty]{} \infty$$

$$\|f_n\| = 1$$

$T$ unbounded

# Radon - Nikodym Theorem

**Theorem** (Radon - Nikodym) Let $\mu$ be a $\sigma$-finite measure on $(X, \mathcal{M})$ and let $\nu$ be a $\sigma$-finite signed measure on $(X, \mathcal{M})$. There exist unique signed measures $\nu_a$ & $\nu_s$ on $\mathcal{M}$ such that

$$\nu_a \ll \mu, \quad \nu_s \perp \mu \quad \& \quad \nu = \nu_a + \nu_s$$

Moreover $\nu_a$ has the form $d\nu_a = f\,d\mu$ for some $f$ s.t. either

$$\int f^+ \, d\mu < \infty \quad \text{or} \quad \int f^- \, d\mu < \infty$$

**Proof**: due to Von Neumann

**Case 1**: $\mu$ & $\nu$ (non-negative) finite measures

**Case 2**: $\mu$ & $\nu$ $\sigma$-finite measures

**Case 3**: $\nu$ signed $\sigma$-finite measure

Case 1: $\mu$, $\nu$ finite measure. Consider $\rho = \mu + \nu$ finite measure

$$l : L^2(\rho) \to \mathbb{R} \qquad l(\psi) = \int \psi \, d\nu$$

Claim: $l$ is a bounded linear functional

$\forall E \in \mathcal{M} \qquad \rho(E) = \mu(E) + \nu(E) \geq \nu(E)$

$$|l(\psi)| \leq \int |\psi| \, d\nu \leq \int |\psi| \, d\rho$$

$$\overset{CS}{\leq} \left( \int |\psi|^2 \, d\rho \right)^{1/2} \left( \int 1 \, d\rho \right)^{1/2}$$

$$\leq \rho(X)^{1/2} \, \|\psi\|_{L^2(\rho)}$$

By RRT $\exists \, g \in L^2(\rho)$ s.t $\forall \psi \in L^2(\rho)$

$$l(\psi) = \int \psi \, d\nu = \int \psi g \, d\rho$$

$$\ell(\psi) = \boxed{\int \psi \, d\nu = \int \psi g \, d\rho} \qquad (*)$$

If $E \in \mathcal{M}$ s.t $\rho(E) > 0$ $\quad \psi = \chi_E$

$$\rho(E) \geq \nu(E) = \int \chi_E \, d\nu = \int \chi_E g \, d\rho = \int_E g \, d\rho \geq 0$$

$$\boxed{0 \leq \frac{1}{\rho(E)} \int_E g \, d\rho \leq 1}$$

Claim: $0 \leq g(x) \leq 1$ $\quad \rho$ a.e $x \in X$

$E \in \mathcal{M}$ $\quad \rho(E) > 0$ $\quad \int_E g \, d\rho \geq 0 \Rightarrow g \geq 0$ a.e.

(contrapositive)

$E = \{x : g(x) > 1\}$ goal show $\rho(E) = 0$

assume $\rho(E) > 0$ $\quad \exists \varepsilon_0 > 0$ s.t $E' = \{x : g(x) > 1 + \varepsilon_0\}$

$\rho(E') > 0$

$E' = \{x : g(x) > 1+\varepsilon_0\}$ $\qquad \rho(E') > 0$ $\qquad E = \{g > 1\}$

$$\frac{\varepsilon_0 \rho(E')}{\rho(E)} \le \frac{1}{\rho(E)} \int_{E'} (g-1)\, d\rho \le \frac{1}{\rho(E)} \int_{E} (g-1)\, d\rho \le 0$$

$$\uparrow$$

$$\frac{1}{\rho(E)} \int_E g\, d\rho \le 1$$

$$\frac{\varepsilon_0 \rho(E')}{\rho(E)} \le 0 \qquad \not\in$$

From $(\star)$

$$\int \psi\, d\nu = \int \psi g\, (d\mu + d\nu)$$

$$\boxed{\int \psi (1-g)\, d\nu = \int \psi g\, d\mu} \qquad \forall\, \psi \in L^2(\rho)$$

$A = \{x : 0 \le g(x) < 1\}$ $\qquad B = \{x : g(x) = 1\}$

$$\nu_a(E) = \nu(E \cap A) \qquad \nu_s(E) = \nu(E \cap B)$$

Claim : $\nu_a \ll \mu$ $\quad$ & $\quad \nu_s \perp \mu$

Claim $\nu_s \perp \mu$ $\qquad B = \{g = 1\}$

$$(\star\star) \quad \boxed{\int \psi(1-g)\,d\nu = \int \psi g\,d\mu}$$

$\psi = \chi_B$ $\qquad \int_B (1-g)\,d\nu = 0 = \int_B d\mu = \mu(B)$

$\mu(E) = \mu(E \cap B^c)$ $\qquad \nu_s \perp \mu$